# Pacific Journal of Mathematics


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Volume 214 No. 2 April 2004

PACIFIC JOURNAL OF MATHEMATICS
Vol. 214, No. 2, 2004

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**Some non-existence result is established for bounded solutions of a Neumann problem on the upper half space. Based on this non-existence result, precise asymptotic behavior is given for the principal eigenvalue of some linear eigenvalue problem in bounded $C^1$ domains, and this answers a question that appeared in Lacey et al, 1998.**

## 1. Introduction.

For any $\gamma > 0$, set

$$\Lambda(\gamma) = \sup_{u \in H^1(\Omega)\setminus\{0\}} \frac{\gamma \int_{\partial\Omega} u^2 - \int_\Omega |\nabla u|^2}{\int_\Omega u^2}, \tag{1}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega$. It is straightforward to show that the supremum of (1) is attained by some positive function $u_\gamma \in H^1(\Omega)$, which is a weak solution of

$$\Delta u = \Lambda(\gamma) u \quad \text{in} \quad \Omega, \qquad \frac{\partial u}{\partial \nu} = \gamma u \quad \text{on} \quad \partial\Omega, \tag{2}$$

where $\nu$ is the outward unit normal vector on $\partial\Omega$; $\nu$ exists a.e. for Lipschitz domains. The goal of this paper is to understand the asymptotic behavior of $\Lambda(\gamma)$ as $\gamma \to \infty$ when $\partial\Omega \in C^1$. Since $\Lambda(\gamma) \to \infty$ when $\gamma \to \infty$, (2) can be viewed as a singularly perturbed linear eigenvalue problem.

The asymptotic behavior of $\Lambda(\gamma)$ was first studied by Lacey, Ockendon and Sabina in [**3**], where they investigated some reaction-diffusion model in which distributed nonlinear absorption mechanisms compete with nonlinear boundary sources. In order to describe the long time behaviors of solutions to this reaction-diffusion model, it is important to understand the asymptotic behavior of $\Lambda(\gamma)$ as $\gamma \to \infty$ (see [**3**] and the references therein). Among other things, Lacey, Ockendon and Sabina showed in [**3**] that

$$\lim_{\gamma\to\infty} \frac{\Lambda(\gamma)}{\gamma^2} = 1 \tag{3}$$

if $\partial\Omega$ is $C^2$ and is differentially equivalent to the unit sphere. On the other hand, when $\Omega$ is a planar domain and $\partial\Omega$ is piecewise $C^1$, they proved that

$$\liminf_{\gamma\to\infty} \frac{\Lambda(\gamma)}{\gamma^2} \geq \operatorname{cosec}^2\alpha \geq 1,$$

where $\alpha$ is the smallest interior semiangle on $\partial\Omega$. These considerations indicate that the asymptotic behavior of $\Lambda(\gamma)$ is strongly affected by the smoothness of the boundary. In this connection, we prove:

**Theorem 1.1.** (3) *holds for any bounded $C^1$ domain.*

**Remark 1.1.** Similar result can be established for the problem

$$\Delta u = \Lambda(\gamma)u \quad \text{in} \quad \Omega, \qquad \frac{\partial u}{\partial \nu} = \gamma b(x)u \quad \text{on} \quad \partial\Omega.$$

More precisely, if $b(x) \in C(\partial\Omega)$ is positive somewhere, then

$$\lim_{r\to\infty} \frac{\Lambda(\gamma)}{\gamma^2} = \max_{\partial\Omega}(b_+)^2.$$

In the following we briefly sketch our approach: Since (2) is a singularly perturbed problem, it is natural to "blow up" $u_\gamma$, the solution of (2), near its maximum which must be attained on $\partial\Omega$ (via the Maximum Principle). That is, straightening out the boundary and rescaling $u_\gamma$ suitably, by passing to the limit we are led to the following Neumann problem on the upper half space:

$$\Delta u = au \quad \text{in} \quad \mathbb{R}^n_+, \qquad \frac{\partial u}{\partial x_n} = -u \quad \text{on} \quad \partial\mathbb{R}^n_+, \tag{4}$$

where $a$ is the limit of $\Lambda(\gamma)/\gamma^2$ (subject to a subsequence) as $\gamma \to \infty$. By adequate choice of test function in (1), one can show that $a \geq 1$. On the other hand, using some similar ideas as the sliding method developed in [**1**], we are able to show:

**Theorem 1.2.** *If $a > 1$,* (4) *has no bounded nontrivial solutions.*

By some non-degeneracy result in Section 3, the solution of (4) obtained via the blowup process is indeed nontrivial. Hence, Theorem 1.2 ensures that $a = 1$, which in turn yields Theorem 1.1.

**Remark 1.2.** It turns out that Theorem 1.2 is sharp: For every $a \leq 1$, (4) has bounded nontrivial solutions of the form $w(x')e^{-x_n}$, where $x = (x', x_n)$ and $w$ is a solution of

$$\Delta w = (a-1)w \quad \text{in} \quad R^{n-1}; \tag{5}$$

Theorem 1.2 also fails without the boundedness condition: (4) has positive (unbounded) solutions of the form $w(x')e^{-x_n}$, where $w$ is a positive solution of (5) for $a > 1$. We refer to [**2**] for the classification of positive solutions to (5).

The plan of this paper is as follows: Theorem 1.2 is established in Section 2. In Section 3, we first derive the relation between $\lim_{\gamma\to\infty}\Lambda(\gamma)/\gamma^2$ and (4), and then use it to complete the proof of Theorem 1.1.

## 2. The proof of Theorem 1.2.

We prove Theorem 1.2 in this section. Our idea is to construct some super-solution of (4), by employing some similar ideas as the sliding method of Berestycki, Caffarelli and Nirenberg (see, e.g., [**1**]).

Throughout this section, we assume that $u(x)$ is a bounded solution to (4) and $a > 0$; without loss of generality, we may assume that $\sup_{\mathbb{R}^n_+} u > 0$.

**Lemma 2.1.** $\sup_{\mathbb{R}^n_+} u = \sup_{\partial\mathbb{R}^n_+} u$.

*Proof.* Let $\{x^j\}_{j=1}^{\infty}$ be a sequence of points with $u(x^j) \to \sup_{\mathbb{R}^n_+} u$. Denote $x^j_n$ the last component of $x^j$. We first show that $x^j_n \to 0$ as $j \to \infty$. If not, then there is a $\delta > 0$, such that $x^j_n > \delta$ (after passing to some subsequence). We consider $u^j(x) = u(x + x^j)$ for $|x| \le \delta$. Notice that $u^j$ is bounded in $|x| \le \delta$. By standard elliptic estimates, we know that after passing to some subsequence, $u^j \to u_0$ in $C^2(B_{\delta/2}(0))$, where $u_0$ satisfies

$$\Delta u_0 = a u_0 \quad \text{in} \quad B_{\delta/2}(0),$$

and $u_0$ assumes its positive maximum at the origin. This is clearly impossible.

Again, standard elliptic estimates yield that $|\nabla u| \le C$ for some positive constant $C$. Therefore, $\sup_{\mathbb{R}^n_+} u = \sup_{\partial\mathbb{R}^n_+} u$.

We normalize $u(x)$ so that $\sup_{\mathbb{R}^n_+} u = 1$. By Lemma 2.1,

$$\sup_{\partial\mathbb{R}^n_+} u = 1. \tag{6}$$

Define

$$\begin{aligned}
\Omega_h &= \{(x', x_n) \mid x' \in R^{n-1},\ 0 < x_n < h\},\\
\Omega_{h,r} &= \{(x', x_n) \mid |x'| < r,\ \ 0 < x_n < h\},\\
\Gamma_{1,h,r} &= \partial\Omega_{h,r} \cap \{x_n > 0\},\\
\Gamma_{2,h,r} &= \partial\Omega_{h,r} \cap \{x_n = 0\}.
\end{aligned}$$

We first state a lemma concerning the sub- and super-solution method. Consider

$$\begin{aligned}
-\Delta u &= f(x, u) && \text{in} \quad \Omega_{h,r},\\
u &= g(x) && \text{on} \quad \Gamma_{1,h,r},\\
\tfrac{\partial u}{\partial \nu} &= h(x, u) && \text{on} \quad \Gamma_{2,h,r},
\end{aligned} \tag{7}$$

where $f(x,u)$, $h(x,u)$ are Carathéodory functions, $g(x)$ is continuous.

A function $u \in H^1(\Omega_{h,r})$ is called a sub-solution to (7) if $u \le g(x)$ on $\Gamma_{1,h,r}$ and

$$\int_{\Omega_{h,r}} \left[\nabla u \nabla \eta - f(x,u)\eta\right] - \int_{\Gamma_{2,h,r}} h(x,u)\eta \le 0,$$

for all $\eta \in C^\infty(\Omega_{h,r})$, $\eta \ge 0$, and $\eta = 0$ on $\Gamma_{1,h,r}$. Similarly $u \in H^1(\Omega_{h,r})$ is called a super-solution to (7) if the above inequalities are reversed. We refer the proof of the following result to [**5**]:

**Lemma 2.2.** *Suppose that $\overline{u}, \underline{u} \in H^1(\Omega_{h,r})$ are both bounded, $\overline{u} \ge \underline{u}$, and they are super-solution, sub-solution of* (7), *respectively. Then there is a solution $u \in H^1(\Omega_{h,r})$ to* (7) *such that $\underline{u} \le u \le \overline{u}$ holds a.e. in $\Omega_{h,r}$.*

We now apply Lemma 2.2 to construct a super-solution of (4).

**Lemma 2.3.** *For any fixed $h$, $r > 0$, there is a unique solution to*

$$\begin{aligned} &\Delta\psi = a\psi && \text{in} \quad \Omega_{h,r}, \\ &\psi = 1 && \text{on} \quad \Gamma_{1,h,r}, \\ &\tfrac{\partial\psi}{\partial\nu} = 1 && \text{on} \quad \Gamma_{2,h,r}. \end{aligned} \tag{8}$$

*Proof.* It is easy to check that $\underline{u} = 0$ is a sub-solution and $\overline{u} = 1 + h - x_n$ is a super-solution to (8). The existence follows from Lemma 2.2, whereas the uniqueness follows from the Maximum Principle.

**Lemma 2.4.** *Let $u(x)$ be a solution to* (4) *and $\psi_{h,r}$ be the unique solution to* (8). *Then $\psi_{h,r} \ge u$ in $\Omega_{h,r}$.*

*Proof.* Set $w = \psi_{h,r} - u$. Then $w$ satisfies

$$\begin{aligned} &\Delta w = aw && \text{in} \quad \Omega_{h,r}, \\ &w \ge 0 && \text{on} \quad \Gamma_{1,h,r}, \\ &\tfrac{\partial w}{\partial\nu} = 1 - u && \text{on} \quad \Gamma_{2,h,r}. \end{aligned}$$

If $w(x_0) = \min_{\overline{\Omega}_{h,r}} w(x) < 0$, then $x_0 \in \Gamma_{2,h,r} \setminus \{|x'| = r\}$. By the Hopf Boundary Lemma, we know that $\partial w/\partial\nu(x_0) < 0$, which contradicts

$$\partial w/\partial\nu(x_0) = 1 - u(x_0) \ge 0.$$

This proves Lemma 2.4.

Now consider

$$\begin{aligned} &\Delta\psi = a\psi && \text{in} \quad \Omega_{h,r}, \\ &\psi = 1 && \text{on} \quad \Gamma_{1,h,r}, \\ &\tfrac{\partial\psi}{\partial\nu} = 0 && \text{on} \quad \Gamma_{2,h,r}. \end{aligned} \tag{9}$$

It is easy to see that $\underline{u} = 0$ and $\overline{u} = 1$ are sub- and super-solutions to (9), respectively. Thus there is a unique solution to (9). We denote it by $\psi_{1,h,r}$. Decompose $\psi_{h,r}$ as

$$\psi_{h,r} = \psi_{1,h,r} + \psi_{2,h,r},$$

where $\psi_{2,h,r}$ is the unique solution to

$$\begin{aligned} &\Delta\psi = a\psi && \text{in} \quad \Omega_{h,r}, \\ &\psi = 0 && \text{on} \quad \Gamma_{1,h,r}, \\ &\tfrac{\partial\psi}{\partial\nu} = 1 && \text{on} \quad \Gamma_{2,h,r}. \end{aligned}$$

It again follows from the Maximum Principle that

$$0 < \psi_{1,h,r} < 1, \qquad 0 < \psi_{2,h,r} < 1 + h, \quad \forall x \in \Omega_{h,r}.$$

Furthermore, we can show:

**Lemma 2.5.**

a) $\psi_{1,h,r}$ *is non-increasing in* $r$. *For any fixed* $h > 0$, *as* $r \to \infty$, $\psi_{1,h,r}$ *converges monotonically to* $\psi_{1,h}$, *where* $\psi_{1,h}(x)$ *is a function of* $x_n$ *alone and satisfies*

$$\begin{aligned} &\Delta\psi = a\psi && \textit{in} \quad \Omega_h, \\ &\psi = 1 && \textit{on} \quad \{x_n = h\}, \\ &\tfrac{\partial\psi}{\partial\nu} = 0 && \textit{on} \quad \{x_n = 0\}. \end{aligned} \tag{10}$$

b) $\psi_{2,h,r}$ *is non-decreasing in* $r$. *For any fixed* $h > 0$, *as* $r \to \infty$, $\psi_{2,h,r}$ *converges monotonically to* $\psi_{2,h}$, *where* $\psi_{2,h}(x)$ *is a function of* $x_n$ *alone and satisfies*

$$\begin{aligned} &\Delta\psi = a\psi && \textit{in} \quad \Omega_h, \\ &\psi = 0 && \textit{on} \quad \{x_n = h\}, \\ &\tfrac{\partial\psi}{\partial\nu} = 1 && \textit{on} \quad \{x_n = 0\}. \end{aligned} \tag{11}$$

*Proof.* We only give the proof of Part (a) since Part (b) can be established in the same spirit. For any $r' > r$, set $w = \psi_{1,h,r'} - \psi_{1,h,r}$ in $\Omega_{h,r}$. Then $w$ satisfies

$$\begin{aligned} &\Delta w = aw && \text{in} \quad \Omega_{h,r}, \\ &w \le 0 && \text{on} \quad \Gamma_{1,h,r}, \\ &\tfrac{\partial w}{\partial\nu} = 0 && \text{on} \quad \Gamma_{2,h,r}. \end{aligned} \tag{12}$$

It follows from the Maximum Principle and the Hopf Boundary Lemma that $w \le 0$ in $\overline{\Omega}_{h,r}$. This proves the monotonicity of $\psi_{1,h,r}$.

Therefore, for any fixed $h > 0$, as $r \to \infty$, $\psi_{1,h,r}$ monotonically converges to $\psi_{1,h}$ in $\mathbb{R}^n_+$ and $\psi_{1,h}$ satisfies (10).

We still need to show that $\psi_{1,h}$ is a function of $x_n$ only. For any $P \in \mathbb{R}^{n-1}$ and $r',\ r > 0$ with $r' - r > |P|$, consider the difference $w_1 = \psi_{1,h,r'}(x' + P, x_n) - \psi_{1,h,r}(x', x_n)$ in $\Omega_{h,r}$. It is easy to see that $w_1$ satisfies (12), thus $w_1 \le 0$ in $\Omega_{h,r}$. That is,

$$\psi_{1,h,r'}(x' + P, x_n) \le \psi_{1,h,r}(x', x_n), \qquad \forall x \in \Omega_{h,r}, \qquad \forall P \in R^{n-1}.$$

Sending $r, r' \to \infty$, we have

$$\psi_{1,h}(x' + P, x_n) \le \psi_{1,h}(x), \qquad \forall x \in \Omega_h.$$

Hence $\psi_{1,h}(x) = \psi_{1,h}(x_n)$.

*Proof of Theorem* 1.2. Denote $\psi_h = \psi_{1,h} + \psi_{2,h}$. Then by Lemmas 2.4 and 2.5, $\psi_h(x) \ge u(x)$ in $\Omega_h$ and $\psi_h$ satisfies

$$\begin{aligned} &\psi'' - a\psi = 0 \qquad \text{in} \quad (0,\ h), \\ &\psi'(0) = -1, \qquad \psi(h) = 1. \end{aligned}$$

Direct calculation shows that

$$\psi_h(x_n) = c_1 e^{\sqrt{a} x_n} + c_2 e^{-\sqrt{a} x_n},$$

where

$$\begin{aligned} c_1 &= \frac{1 - \frac{1}{\sqrt{a}} e^{-\sqrt{a} h}}{e^{\sqrt{a} h} + e^{-\sqrt{a} h}}, \\ c_2 &= \frac{1 + \frac{1}{\sqrt{a}} e^{\sqrt{a} h}}{e^{\sqrt{a} h} + e^{-\sqrt{a} h}}. \end{aligned}$$

Sending $h \to \infty$, we have $\psi_h(x_n) \to \frac{1}{\sqrt{a}} e^{-\sqrt{a} x_n}$. Thus by Lemma 2.4,

$$u(x) \le \frac{1}{\sqrt{a}} e^{-\sqrt{a} x_n}, \qquad \forall x \in \mathbb{R}^n_+.$$

It follows from (6) that

$$1 = \sup_{\partial \mathbb{R}^n_+} u(x) \le \frac{1}{\sqrt{a}},$$

i.e., $a \le 1$. This proves Theorem 1.2.

## 3. Asymptotic behaviors of eigenvalues.

We prove Theorem 1.1 in this section. For every piecewise smooth domain $\Omega$, it was proved in [**3**] that

$$\underline{\lim}_{\gamma \to \infty} \frac{\Lambda(\gamma)}{\gamma^2} \ge 1.$$

To prove Theorem 1.1, we need to show that when $\partial\Omega$ is $C^1$,

$$\overline{\lim}_{\gamma \to \infty} \frac{\Lambda(\gamma)}{\gamma^2} \le 1. \tag{13}$$

*Proof of Theorem* 1.1. For any $\gamma > 1$, let $u_\gamma$ be a positive solution of (2) and $u_\gamma$ attains its maximum at $x_\gamma$. By the Maximum Principle, we know that $x_\gamma \in \partial\Omega$. After normalization we can assume that $\max_{\overline{\Omega}} u_\gamma = 1$ and $x_\gamma \to 0 \in \partial\Omega$. Further, we can assume that there is a $C^1$ function $\phi$ such that $\partial\Omega \cap B_2(0)$ can be represented by $x_n = \phi(x')$ for $|x'| \le 2$ with $\phi(0) = 0$ and $\partial\phi/\partial x_i(0) = 0$ for $i = 1, \dots, n-1$.

For any $\eta \in C_0^\infty(B_2(0))$, $u_\gamma$ satisfies

$$\int_\Omega \nabla u_\gamma \cdot \nabla\eta + \Lambda(\gamma)\int_\Omega u_\gamma\eta - \gamma\int_{\partial\Omega} u_\gamma\eta = 0. \tag{14}$$

Now we flatten $\partial\Omega$ near the origin. Let $y = \Phi(x) : \quad \Omega \cap B_2(0) \to \Omega_\Phi \equiv \Phi(\Omega \cap B_2(0))$, be such that

$$\begin{aligned} &\Phi_i(x) = x_i, \qquad i = 1, 2, \dots, n-1,\\ &\Phi_n(x) = x_n - \phi(x'). \end{aligned}$$

Denote the inverse of $y = \Phi(x)$ by $x = \Psi(y)$. Then (14) can be rewritten as

$$\begin{aligned} \sum_{k,l=1}^{n} \int_{\Omega_\Phi} \frac{\partial u_\gamma}{\partial y_k}\frac{\partial \eta}{\partial y_l}\frac{\partial \Phi_k}{\partial x_i}(\Psi(y))\frac{\partial \Phi_l}{\partial x_i}(\Psi(y))|D\Psi|dy \\ + \Lambda(\gamma)\int_{\Omega_\Phi} u_\gamma\eta|D\Psi|dy - \gamma\int_{\partial\Omega_\Phi} u_\gamma\eta\sqrt{1+|\nabla\phi(y')|^2}dy' = 0, \end{aligned}$$

where $|D\Psi|$ is the derterminant of $D\Psi$. Notice that $|\nabla\phi| = o(1)$ as $x' \to 0$. Thus $D\Psi \to I$ as $|y| \to 0$, where $I$ is the $n \times n$ identity matrix. We now consider two different cases.

*Case* 1.

$$\overline{\lim}_{\gamma\to\infty}\frac{\Lambda(\gamma)}{\gamma^2} = a < +\infty.$$

Without loss of generality, we may assume that

$$\frac{\gamma\int_{\partial\Omega} u_\gamma^2 - \int_\Omega |\nabla u_\gamma|^2}{\gamma^2\int_\Omega u_\gamma^2} = \sup_{u \in H^1(\Omega)\setminus\{0\}} \frac{\gamma\int_{\partial\Omega} u^2 - \int_\Omega |\nabla u|^2}{\gamma^2\int_\Omega u^2} \to a,$$

and $u_\gamma(x_\gamma) = \max_\Omega u_\gamma(x) = 1$, $x_\gamma \to 0$. We let $z = \gamma(y - y_\gamma)$, where $y_\gamma = (x'_\gamma, 0)$, and set $v_\gamma(z) = u_\gamma(y)$. Then for any $R > 0$ and $\eta$ with compact support in $B_{2R}$, as $\gamma$ becomes sufficiently large, $v_\gamma$ satisfies

$$
\begin{aligned}
(15)\qquad & \sum_{k,l=1}^{n} \int_{B_{2R}^+} \frac{\partial v_\gamma}{\partial z_k}\frac{\partial \eta}{\partial z_l} \\
&\quad \cdot \frac{\partial \Phi_k}{\partial x_i}\left(\Psi\left(y_\gamma+\frac{z}{\gamma}\right)\right)\frac{\partial \Phi_l}{\partial x_i}\left(\Psi\left(y_\gamma+\frac{z}{\gamma}\right)\right)|D\Psi|\left(y_\gamma+\frac{z}{\gamma}\right)dz \\
&\quad + \frac{\Lambda(\gamma)}{\gamma^2}\int_{B_{2R}^+} v_\gamma \eta |D\Psi|\left(y_\gamma+\frac{z}{\gamma}\right)dz \\
&\quad - \int_{z_n=0} v_\gamma \eta \sqrt{1+|\nabla\phi(y'_\gamma+z'/\gamma)|^2}dz' = 0.
\end{aligned}
$$

Since for $z \in B_{2R}^+$, $y_\gamma + z/\gamma \to 0$ as $\gamma \to \infty$, we know that for sufficiently large $\gamma$,

$$
\left(\sum_{i=1}^{n} \frac{\partial \Phi_k}{\partial x_i}\left(\Psi\left(y_\gamma+\frac{z}{\gamma}\right)\right)\frac{\partial \Phi_l}{\partial x_i}\left(\Psi\left(y_\gamma+\frac{z}{\gamma}\right)\right)\right)_{kl} > \frac{I}{2}.
$$

Let $\eta_R$ be a cutoff function satisfying $\eta_R = 1$ in $B_R$ with compact support in $B_{2R}$ and $|\nabla \eta_R| \le C$. Choosing $\eta = u \cdot \eta_R$ in (15), we have,

$$
\|\nabla v_\gamma\|_{L^2(B_R)} \le C\|v_\gamma\|_{L^\infty} = C.
$$

Therefore, after passing to a subsequence, $v_\gamma \to v_0$ weakly in $H^1_{\mathrm{loc}}(\mathbb{R}^n_+)$ as $\gamma \to \infty$, where $v_0 \in H^1_{\mathrm{loc}}(\mathbb{R}^n_+)$ satisfies

$$
\begin{aligned}
\Delta v_0 = a v_0 &\qquad \text{in } \mathbb{R}^n_+, \\
\tfrac{\partial v_0}{\partial \nu} = v_0 &\qquad \text{on } \partial\mathbb{R}^n_+, \\
0 \le v_0 \le 1 &\qquad \text{in } \mathbb{R}^n_+.
\end{aligned}
$$

To show that $v_0$ is nontrivial, we claim that there is a constant $C > 0$ such that

$$
(16)\qquad 1 = \|v_\gamma\|_{L^\infty(B_1^+)} \le C\left(\|v_\gamma\|_{L^2(\partial\mathbb{R}^n_+\cap\{|x'|<2\})} + \|v_\gamma\|_{L^{\frac{2n}{n-1}}(B_2^+(0))}\right).
$$

Since the embeddings from $W^{1,2}(B_2^+(0))$ to $L^{\frac{2n}{n-1}}(B_2^+(0))$ and $L^2(\partial\mathbb{R}^n_+ \cap \{|x'| < 2\})$ are both compact, we know that

$$
1 \le C\left(\|v_0\|_{L^2(\partial\mathbb{R}^n_+\cap\{|x'|<1\})} + \|v_0\|_{L^{\frac{2n}{n-1}}(B_1^+(0))}\right),
$$

from which it follows that $v_0 \ne 0$. From Theorem 1.2 we see that $a \le 1$. Hence it suffices to establish (16).

Inequality (16) can be obtained via Moser iteration. Though it seems to be a standard result (see, e.g., [**4**]), we include a proof here for the completeness.

Direct calculation shows that

$$\begin{aligned}
&\int_{B_2^+} \nabla v_\gamma \cdot \nabla (v_\gamma^k \xi^2) \\
&= \frac{4k}{(k+1)^2} \int_{B_2^+} \left| \nabla \left( v_\gamma^{\frac{k+1}{2}} \xi \right) \right|^2 + \frac{k-1}{(k+1)^2} \int_{B_2^+} v_\gamma^{k+1} \Delta \xi^2 \\
&\quad - \frac{4k}{(k+1)^2} \int_{B_2^+} v_\gamma^{k+1} |\nabla \xi|^2 - \frac{k-1}{(k+1)^2} \int_{\{x_n=0, \quad |x'|\le 2\}} v_\gamma^{k+1} \nabla \xi^2 \cdot \nu.
\end{aligned}$$

Choosing $\eta = v_\gamma^k \xi^2$ in (15) with $k > 1$ and $\xi$ having compact support in $B_2^+$, we have

(17)

$$\begin{aligned}
&\int_{B_2^+} \left| \nabla \left( v_\gamma^{\frac{k+1}{2}} \xi \right) \right|^2 \\
&\le \frac{k-1}{4k} \int_{B_2^+} v_\gamma^{k+1} \Delta \xi^2 + \int_{B_2^+} v_\gamma^{k+1} |\nabla \xi|^2 + \frac{k-1}{4k} \int_{\{x_n=0, \quad |x'|\le 2\}} v_\gamma^{k+1} \nabla \xi^2 \cdot \nu \\
&\quad + \frac{(k+1)^2 C}{4k} \int_{\{x_n=0, \quad |x'|\le 2\}} v_\gamma^{k+1} \xi + \frac{(k+1)^2 C}{4k} \int_{B_2^+} v_\gamma^{k+1} \xi.
\end{aligned}$$

Let

$$r_i = 1 + \frac{1}{2^{i-1}}, \qquad i = 1, 2, \ldots,$$

and choose $\xi_i$ satisfying

$$\begin{aligned}
&\xi_i = 1, \quad |x| \le r_{i+1}; \\
&\xi_i = 0, \quad |x| > r_i; \\
&|\nabla \xi_i| \le 2 \cdot 2^i, \quad |\nabla^2 \xi_i| \le 4 \cdot 4^i.
\end{aligned}$$

Replacing $\xi$ by $\xi_i$ in (17) and using the Sobolev inequality and the trace inequality, we have, for $n \ge 3$,

$$\begin{aligned}
&\left( \int_{B_{r_i}^+} \left( v_\gamma^{\frac{k+1}{2}} \xi_i \right)^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} + \int_{\partial B_{r_i}^+ \cap \{x_n=0\}} \left( v_\gamma^{\frac{k+1}{2}} \xi_i \right)^{\frac{2(n-1)}{n-2}} \\
&\le C \cdot \left( 4^i + \frac{(k+1)^2}{4k} \right) \int_{B_{r_i}^+} v_\gamma^{k+1} + C \cdot \left( 4^i + \frac{(k+1)^2}{4k} \right) \int_{\partial B_{r_i}^+ \cap \{x_n=0\}} v_\gamma^{k+1},
\end{aligned}$$

where $C$ is some universal constant. By

$$\int_{B_{r_i}^+} v_\gamma^{k+1} \le C \left( \int_{B_{r_i}^+} v_\gamma^{(k+1)\cdot \frac{n}{n-1}} \right)^{\frac{n-1}{n}},$$

we arrive at

$$\left(\int_{B_{r_i}^+}\left(v_\gamma^{\frac{k+1}{2}}\xi_i\right)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}+\int_{\partial B_{r_i}^+\cap\{x_n=0\}}\left(v_\gamma^{\frac{k+1}{2}}\xi_i\right)^{\frac{2(n-1)}{n-2}} \tag{18}$$
$$\leq C\cdot\left(4^i+\frac{(k+1)^2}{4k}\right)\left(\int_{B_{r_i}^+}v_\gamma^{(k+1)\cdot\frac{n}{n-1}}\right)^{\frac{n-1}{n}}$$
$$+C\cdot\left(4^i+\frac{(k+1)^2}{4k}\right)\int_{\partial B_{r_i}^+\cap\{x_n=0\}}v_\gamma^{k+1}.$$

Define $\beta=(n-1)/(n-2)$, $q_0=2$, $q_{i+1}=\beta q_i$ and $p_i=nq_i/(n-1)$ for $i=0,1,\ldots$; choosing $k=q_i-1$ in (18), we have

$$\|v_\gamma\|^{q_i}_{L^{p_{i+1}}(B^+_{r_{i+1}})}+\|v_\gamma\|^{q_i}_{L^{q_{i+1}}(\Gamma_{i+1})}$$
$$\leq C\cdot\left(4^i+\frac{q_i^2}{q_i-1}\right)\cdot\left(\|v_\gamma\|^{q_i}_{L^{p_i}(B^+_{r_i})}+\|v_\gamma\|^{q_i}_{L^{q_i}(\Gamma_i)}\right),$$

where $\Gamma_i=\partial B^+_{r_i}\cap\{x_n=0\}$ for $i=0,1,\ldots$; since $\beta>1$, $(a^\beta+b^\beta)^{1/\beta}\leq a+b$. It follows that

$$\left(\|v_\gamma\|^{q_{i+1}}_{L^{p_{i+1}}(B^+_{r_{i+1}})}+\|v_\gamma\|^{q_{i+1}}_{L^{q_{i+1}}(\Gamma_{i+1})}\right)^{1/q_{i+1}} \tag{19}$$
$$\leq\left(C\cdot\left(4^i+\frac{q_i^2}{q_i-1}\right)\right)^{1/q_i}\left(\|v_\gamma\|^{q_i}_{L^{p_i}(B^+_{r_i})}+\|v_\gamma\|^{q_i}_{L^{q_i}(\Gamma_i)}\right)^{1/q_i}.$$

Since $q_i=2\beta^i$, it is easy to see that

$$\left(C4^i+\frac{Cq_i^2}{q_i-1}\right)^{1/p_i}\leq\left[C(4^i+2\beta^i)\right]^{1/(2\beta^i)}\leq C^{1/(2\beta^i)}(4+\beta)^{i/(2\beta^i)}.$$

Thus

$$\prod_{i=1}^{\infty}\left(4^iC+\frac{q_i^2C}{q_i-1}\right)^{1/q_i}\leq C<\infty.$$

It follows that

$$\|v_\gamma\|_{L^{p_{i+1}}(B^+_{r_{i+1}})}\leq C\left(\|v_\gamma\|_{L^2(\partial\mathbb{R}^n_+\cap\{|x'|<2\})}+\|v_\gamma\|_{L^{\frac{2n}{n-1}}(B_2^+(0))}\right).$$

Sending $i\to\infty$, we obtain (16). For $n=2$, we can obtain (16) in the same spirit. We thereby complete the proof of Theorem 1.1 in Case 1.

*Case* 2.

$$\overline{\lim}_{\gamma\to\infty}\frac{\Lambda(\gamma)}{\gamma^2}=\infty.$$

We will also rule out this possibility. Let $u_\gamma$ be the sequence of positive functions such that

$$\frac{\gamma\int_{\partial\Omega}u_\gamma^2-\int_\Omega|\nabla u_\gamma|^2}{\gamma^2\int_\Omega u_\gamma^2}=\sup_{u\in H^1(\Omega)\backslash\{0\}}\frac{\gamma\int_{\partial\Omega}u^2-\int_\Omega|\nabla u|^2}{\gamma^2\int_\Omega u^2}=a(\gamma)\to\infty$$

as $\gamma\to\infty$, and $u_\gamma(x_\gamma)=\max_{\overline{\Omega}}u(x)=1$. Define $z=\sqrt{a(\gamma)}\gamma(y-y_\gamma)$ and $v_\gamma(z)=u_\gamma(y)$. Then for any $R>0$ and $\eta$ with compact support in $B_{2R}^+$, as $\gamma$ becomes sufficiently large, $v_\gamma$ satisfies

$$\begin{aligned}\sum_{k,l=1}^n\int_{B_{2R}^+}&\frac{\partial v_\gamma}{\partial z_k}\frac{\partial\eta}{\partial z_l}\frac{\partial\Phi_k}{\partial x_i}\left(\Psi\left(y_\gamma+\frac{z}{\gamma}\right)\right)\\&\cdot\frac{\partial\Phi_l}{\partial x_i}\left(\Psi\left(y_\gamma+\frac{z}{\gamma}\right)\right)|D\Psi|\left(y_\gamma+\frac{z}{\gamma}\right)dz\\&+\frac{\Lambda(\gamma)}{a(\gamma)\gamma^2}\int_{B_{2R}^+}v_\gamma\eta|D\Psi|\left(y_\gamma+\frac{z}{\gamma}\right)dz\\&-\frac{1}{\sqrt{a(\gamma)}}\int_{z_n=0}v_\gamma\eta\sqrt{1+|\nabla\phi(y'_\gamma+z'/\gamma)|^2}\,dz'=0.\end{aligned}$$

Similarly as in Case 1, we can show that $v_\gamma\to v_0$ weakly in $H^1_{\text{loc}}(B_R^+)$, where $0\le v_0\le 1$, $v_0\ne 0$, and $v_0\in H^1_{\text{loc}}(\mathbb{R}^n_+)$ is a weak solution of

$$\begin{array}{lll}\Delta v_0=v_0 & \text{in} & \mathbb{R}^n_+,\\ \frac{\partial v_0}{\partial\nu}=0 & \text{on} & \partial\mathbb{R}^n_+.\end{array}\tag{20}$$

Using even reflection, from (20) we know that there is a positive bounded function satisfying

$$\Delta v_0=v_0\quad\text{in}\quad\mathbb{R}^n.$$

On the other hand, it is well-known that there is no nontrivial positive solution to the above equation. This finishes the proof of Theorem 1.1.

## References


[1] H. Berestycki, L. Caffarelli and L. Nirenberg, *Symmetry for elliptic equations in a half space*, in 'Boundary value problems for partial differential equations and applications', volume dedicated to E. Magenes, J.L. Lions et al., ed., Masson, Paris, 1993, 27-42, MR 1260436, Zbl 0793.35034.

[2] L. Caffarelli and W. Littman, *Representation formulas for solutions to* $\Delta u-u=0$ *in* $R^n$, Studies in partial differential equations, 249-263, MAA Stud. Math., **23**, Math. Assoc. America, Washington, D.C., 1982, MR 0716508, Zbl 0533.35017.

[3] A. Lacey, J. Ockendon and J. Sabina, *Multidimensional reaction diffusion equations with nonlinear boundary conditions*, SIAM J. Appl. Math., **58**(**5**) (1998), 1622-1647, MR 1637882, Zbl 0932.35120.

[4] Y.Y. Li and M. Zhu, *Sharp Sobolev trace inequality on Riemannian manifolds with boundary*, Comm. Pure Appl. Math., **50** (1997), 449-487 , MR 1443055, Zbl 0869.58054.

[5] M. Struwe, *Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*, Ergebnisse Math. Grenz., **34**, Springer-Verlag, Berlin, 2000, MR 1736116, Zbl 0939.49001.

Received December 15, 2000.

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