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For every positively graded algebra A, we show that its categories of linear complexes of projectives and almost injectives (see definition below) are both naturally equivalent to the category of graded modules over the quadratic dual algebra $A^!$. In case $A = \Lambda$ is a graded factor of a path algebra with Yoneda algebra Γ, we show that the category $\mathcal{L}c_\Gamma$ of linear complexes of finitely generated right projectives over Γ is dual to the category of locally finite graded left modules over the quadratic algebra $\widetilde{\Lambda}$ associated to Λ. When Λ is Koszul and Γ is graded right coherent, we also prove that the suspended category $\overline{gr}_\Lambda$ has a (triangulated) stabilization $S(\overline{gr}_\Lambda)$ which is triangle-equivalent to the bounded derived category of the 'category of tails' $fpgr_\Gamma/L_\Gamma$.

1. Introduction and terminology.

The interest on Koszul equivalences and dualities arises mainly in the context of derived categories and, specially, dealing with Koszul algebras (see definitions below). In case Λ is a graded Koszul algebra with Yoneda algebra Γ, Beilinson, Ginzburg and Soergel ([**1**]) showed the existence of an equivalence between certain full triangulated subcategories of the derived categories $\mathcal{D}({}_\Lambda Gr)$ and $\mathcal{D}({}_\Gamma Gr)$. When composing with the canonical duality defined by $\mathrm{Hom}_{\Lambda_0}(-, \Lambda_0)$, one gets a duality between suitable subcategories of $\mathcal{D}({}_\Lambda Gr)$ and $\mathcal{D}({}_\Gamma Gr)$. The aim of this paper is to show that Koszul equivalences and dualities also appear naturally between some nice abelian categories associated to positively graded algebras. In this context, no restriction is needed a priori on the graded algebras, although quadratic algebras will play a predominant role as in the context of derived categories. On one side, our results generalize those of Yoshino ([**18**]) for symmetric and exterior algebras and, on the other, they show that the above mentioned triangulated equivalences of [**1**] already live in an abelian context.

Throughout the paper, K will be a field and, for every K-algebra R, we shall denote by Mod_R (resp. mod_R) the category of all right (resp. finitely generated right) R-modules and by ${}_R\mathrm{Mod}$ (resp. ${}_R\mathrm{mod}$) its left-right symmetric version. The term **positively graded algebra** will stand for a graded K-algebra $A = \oplus_{n\geq 0} A_n$ such that A_0 is a K-algebra isomorphic to a finite direct product of copies of K and $\dim_K A_1 <\propto$. We shall say that such an algebra is **locally finite** when $\dim_K A_n <\propto$, for all $n \in \mathbf{Z}$. A particular case comes as follows: Let Q be a finite oriented graph or quiver and give KQ a grading by assigning positive degrees to the arrows. Then, for every ideal I of KQ, homogeneous with respect to that grading and contained in the ideal generated by the paths of length 2, the algebra $A = KQ/I$ is positively graded and locally finite. Every graded algebra isomorphic to one of this form will be called a **generalized graded factor of a path algebra**, reserving the term **graded factor of a path algebra**, for the case when the grading on KQ is the classical one, i.e., obtained by assigning degree 1 to all arrows. We shall distinguish this latter case by putting $\Lambda \cong KQ/I$, reserving letters A, B for general positively graded algebras.

When Q is a finite quiver, we shall identify $Q_0 = \{1, \dots, n\}$ with the set of vertices and will denote by Q_n the set of paths in Q of length n while KQ_n will be the vector subspace of KQ generated by Q_n. When p is a path $i \to \dots \to j$ in Q, we shall put $i = o(p)$ and $j = t(p)$ for the **origin** and **terminus** of p. We write paths $\alpha_1 \dots \alpha_n$ convening that $t(\alpha_i) = o(\alpha_{i+1})$, for all $i = 1, \dots, n-1$. The idempotent of KQ given by $i \in Q_0$ will be denoted by e_i. The opposite quiver Q^{op} of Q has $Q_0^{\mathrm{op}} = Q_0$ and is obtained from Q by reversing the orientation of the arrows. Whenever $p = \alpha_1 \dots \alpha_n \in Q_n$, we shall put $p^o = \alpha_n^o \dots \alpha_1^o$ and then, clearly, $Q_n^{\mathrm{op}} = \{p^o : p \in Q_n\}$.

Notice that if A is a positively graded algebra, then the subalgebra $\overline{A}$ of A generated by the subspace $A_0 \oplus A_1$ is a graded factor of a path algebra. Indeed, there is a uniquely determined (up to isomorphism) finite quiver Q such that $KQ_0 \cong A_0$, as K-algebras, and $KQ_1 \cong A_1$, as $KQ_0 - KQ_0$-bimodules. Then Q will be called the **quiver of** A, although A may not be a graded factor algebra of KQ. The isomorphism $KQ_0 \oplus KQ_1 \cong A_0 \oplus A_1$ extends to a homomorphism of graded algebras $\pi_A : KQ \longrightarrow A$ with image $\overline{A}$, where the grading on KQ is the classical one. If $I = \mathrm{Ker}(\pi_A)$ and $I_2 = \{x \in I : x \text{ is homogeneous of degree } 2\}$, then we denote by $\langle I_2 \rangle$ the homogeneous ideal of KQ generated by I_2 and $\widetilde{A} = KQ/\langle I_2 \rangle$ will be called the **quadratic algebra associated to** A. We identify $KQ_0 = A_0$ all through the paper and unspecified tensors are tensors over A_0. The canonical duality $D = \mathrm{Hom}_{A_0}(-, A_0) = \mathrm{Hom}_{KQ_0}(-, KQ_0) : {}_{A_0}\mathrm{mod} \longrightarrow {}_{A_0}\mathrm{mod} = \mathrm{mod}_{A_0}$ is 'inverse to itself'. If Q^{op} denotes the opposite quiver of Q, then we have canonical isomorphisms of $KQ_0 - KQ_0$-bimodules $KQ_n^{\mathrm{op}} \cong D(KQ_n)$, for all $n \geq 0$. When W is a $KQ_0 - KQ_0$-subbimodule of either KQ_n or KQ_n^{op}, we shall denote by $W^\perp$ its orthogonal with respect to the usual duality $KQ_n^{\mathrm{op}} \otimes KQ_n \cong D(KQ_n) \otimes KQ_n \longrightarrow KQ_0$. Notice that there are actually two dualities, namely, one for the case when KQ_n is considered as a left KQ_0-module and one for the case when it is considered as a right KQ_0-modules. They map $p^o \otimes q$ onto $\delta_{pq} e_{t(q)}$ and $\delta_{pq} e_{o(q)}$, respectively, where

δ_{pq} is the Kronecker symbol. Nonetheless, $W^{\perp}$ is the same for both dualities. In the above situation, the algebra $A^! = KQ^{\mathrm{op}} / \langle I_2^{\perp} \rangle$ is called the **quadratic dual algebra** of A. We shall put ${}^!A = (A^!)^{\mathrm{op}}$ for the opposite algebra, which is then a graded factor of KQ. Up to graded isomorphism, $\widetilde{A}$ and $A^!$ do not depend on the presentation of A, i.e., do not depend on the choice of the graded homomorphism $\pi_A : KQ \longrightarrow A$. If A and B are positively graded algebras, we shall say that they are **orthogonal** when $A^! \cong \widetilde{B}$ and that they are **quadratically equivalent** when $\widetilde{A} \cong \widetilde{B}$ (isomorphisms as graded algebras in both cases).

We will be concerned with the category Gr_A of $\mathbf{Z}$-graded right A-module and its full subcategories $lfrg_A$, gr_A and $fpgr_A$ consisting of locally finite (i.e., $\dim_K M_i <\propto$, for all $i \in \mathbf{Z}$), finitely generated and finitely presented graded right A-modules, respectively. Of course, ${}_AGr$, ${}_Alfgr$, ${}_Agr$ and ${}_Afpgr$ will stand for the left-right symmetric versions. To some of these categories, and also to some categories of cochain complexes that will eventually appear in the paper, we will add a superindex $+$ or $-$ meaning that we consider the corresponding full subcategory of lower or upper bounded objects (e.g., $lfgr_A^+$ will be the full subcategory of $lfgr_A$ with objects $M = \oplus_{n\in\mathbf{Z}} M_n$ such that $M_n = 0$, for all $n \ll 0$).

An object $M = \oplus_{n\in\mathbf{Z}} M_n$ of Gr_A will be called **generated in degree** j when M_j generates M as a graded A-module. Dually, M will be called **cogenerated in degree** j when M_j cogenerates M as a graded module, i.e., when $M = M_{\leq j} = \oplus_{n\leq j} M_n$ and $N \cap M_j \neq 0$, for every nonzero graded submodule N of M. For every $k \in \mathbf{Z}$, the k-shifting $M[k]$ of M coincides with M as an ungraded A-module, but its grading is given by $M[k]_n = M_{k+n}$, for all $n \in \mathbf{Z}$. In general, given any cocomplete abelian category $\mathcal{A}$ and $X \in Ob(\mathcal{A})$, we shall denote by $\mathrm{Add}\,(X)$ the full subcategory of $\mathcal{A}$ with objects the direct summands of direct sums of copies of X. For instance, when $\mathcal{A} = Gr_A$ and $X = \oplus_{k\in\mathbf{Z}} A[k]$, $\mathrm{Add}\,(X)$ is just the class of projective objects in Gr_A.

The canonical duality D extends to a contraviant functor $D : {}_AGr \longrightarrow Gr_A$ (resp. $Gr_A \longrightarrow {}_AGr$), for if $M = \oplus_{n\in\mathbf{Z}} M_n$ is an object of ${}_AGr$ then $D(M) =: \oplus_{n\in\mathbf{Z}} D(M)_n$, where $D(M)_n = D(M_{-n})$ for all $n \in \mathbf{Z}$, is a graded right A-module with multiplication $f \cdot a : x \longrightarrow f(ax)$, for all $a \in A_m$ and $f \in D(M)_n$. Clearly, D restricts to a duality $D : {}_Alfgr \longrightarrow lfgr_A$ 'inverse to itself'. The objects of $\mathrm{Add}\,(\oplus_{k\in\mathbf{Z}} D(A)[k])$ (full subcategory of Gr_A) will be called **almost injective** graded A-modules. They need not be injective objects of Gr_A, but they are so when A is right Noetherian. We shall denote by $\mathrm{Proj}\,{}_A^k = \mathrm{Add}\,(A[k])$ and $\mathrm{Inj}\,{}_A^k = \mathrm{Add}\,(D(A)[k])$ the full subcategories of Gr_A consisting of projective graded A-modules generated in degree $-k$ and almost injective graded A-modules cogenerated in degree $-k$, respectively.

In our situation, every $M \in Gr_A$ has a projective cover in Gr_A (cf. [**5**, Prop. 2.6]), $\epsilon_M : P(M) \longrightarrow M$. We define inductively $\Omega^0 M = M$, $\Omega^1 M = \Omega M = \operatorname{Ker} \epsilon_M$ and, then, $\Omega^n M = \Omega(\Omega^{n-1} M)$, for all $n > 0$. The category Mod_{A_0} (resp. mod_{A_0}) can be identified with the category of (finitely generated) semisimple graded right A-modules. When $X \in \mathrm{Mod}_{A_0}$ and $M \in Gr_A$, we have an isomorphism $\mathrm{Ext}^n_A(M, X) \cong \mathrm{Hom}_A(\Omega^n M, X)$ (extensions and homomorphisms as ungraded right A-modules!), for all $n \geq 0$. In the particular case when $M = X = \Lambda_0$, we can consider the Yoneda algebra of A, $\Gamma = \oplus_{n \geq 0} \mathrm{Ext}^n_A(A_0, A_0)$. It is a graded algebra with the Yoneda product as multiplication. It is positively graded in our sense only in case $\mathrm{Ext}^1_A(A_0, A_0)$ is finite dimensional, something which always happens when A is a generalized graded factor of a path algebra. More restrictively, when $A = \overline{A} = \Lambda$ is a graded factor of a path algebra, the quiver of Γ is Q^{op}. Indeed, $\Gamma_0 = \mathrm{End}_{\Lambda_0}(\Lambda_0) \cong \Lambda_0 \cong KQ_0$ and, from the projective presentation of Λ_0 as a left Λ-module, $0 \to \Lambda_{\geq 1} \longrightarrow \Lambda \longrightarrow \Lambda_0 \to 0$, one immediately gets that $\Gamma_1 = \mathrm{Ext}^1_\Lambda(\Lambda_0, \Lambda_0) \cong \mathrm{Hom}_\Lambda(\Lambda_{\geq 1}, \Lambda_0) \cong \mathrm{Hom}_{\Lambda_0}(\Lambda_1, \Lambda_0) \cong D(KQ_1)$. Then $\Gamma_0 \oplus \Gamma_1$ can be identified with $KQ_0^{\mathrm{op}} \oplus KQ_1^{\mathrm{op}}$.

A positively graded algebra A is a **Koszul algebra** in case $\Omega^n(A_0)$ is locally finite and generated in degree n, for all $n \geq 0$. In that case, $A = \Lambda$ is a graded factor of a path algebra and $\Gamma \cong A^!$.

The organization of the paper goes as follows: Let $A = \oplus_{n \geq 0} A_n$ be a positively graded algebra with quiver Q. In Section 2 we show that the graded versions of $- \otimes A$ and $\mathrm{Hom}_{A_0}(A, -)$ embed $_{KQ}Gr$ in two different ways as a full subcategory of the category $Gr_{A[X]}$ of $\mathbf{Z} \times \mathbf{Z}$-graded modules over $A[X]$. That induces by restriction equivalences of categories between $_{^!A}Gr = Gr_{A^!}$ and the categories $\mathcal{LC}_A$ and $\mathcal{LC}^*_A$ of linear complexes of projective and almost injective graded A-modules, respectively (Theorems 2.4 and 2.10). In Section 3 we show that in the case when $A = \Lambda$ is a graded factor of a path algebra, Λ is orthogonal to its Yoneda algebra Γ and then there is an induced duality between $_{\widetilde{\Lambda}}lfgr$ and the category $\mathcal{L}c_\Gamma$ of linear complexes of finitely generated projective graded modules over Γ (Theorem 3.3). Among the consequences of these results, we characterize quadratic algebras in categorical terms (Corollary 3.4) and show that the categories of linear complexes of projective (resp. almost injective) graded modules are equivalent for quadratically equivalent algebras. In case the algebras are quadratically equivalent graded factors of path algebras, the categories of linear complexes of finitely generated projective graded modules over their Yoneda algebras are also equivalent (Corollary 3.5). In the final Section 4, somewhat independent from the rest, we extend some equivalences of derived categories obtained by Bernstein, Gelfand and Gelfand (cf. [**4**] and [**9**]) in the classification of algebraic vector bundles over the projective space.

2. Koszul equivalences.

All throughout this section $A = \oplus_{n\geq 0}A_n$ will be a positively graded algebra with quiver Q. We fix a homomorphism $\pi_A : KQ \longrightarrow A$ of graded K-algebras and put $\bar{p} = \pi_A(p)$, for every path p in Q. One-sided modules over $A_0 = KQ_0$ will be considered indistinctly as left or right modules, with the same action of A_0 on both sides. It is convenient now to make some comments concerning the canonical duality D. Suppose ${}_{A_0}S_{A_0}$ is a bimodule and $\mathcal{B}$ is a K-basis of S satisfying the following property: For every $b \in \mathcal{B}$ there exist (necessarily unique) $i, j \in Q_0$ such that $u = e_iue_j$. We put $i = o(b)$ and $j = t(b)$. For each $b \in \mathcal{B}$, we denote by b^* the homomorphism of right A_0-modules defined by the rule $b^*(c) = \delta_{bc}e_{t(c)}$, where δ_{bc} is the Kronecker symbol. It is clear that $\mathcal{B}^* = \{b^* : b \in \mathcal{B}\}$ is a K-linearly independent subset of $D(S)$, which is a basis when S is finite dimensional. A symmetric argument works when we consider homomorphisms of left A_0-module, but then $b^*(c) = \delta_{bc}e_{o(c)}$. We shall call $\mathcal{B}^*$ the **dual basis** of $\mathcal{B}$, the side of the A_0-homomorphisms being clear from the context. The following remark and the next two lemmas will be very useful in the sequel.

Remark 2.1. Let S be finite dimensional in the above situation and let X_{A_0} (resp. ${}_{A_0}X$) be an A_0-module. For each $b \in \mathcal{B}$ and each $x \in Xe_{t(b)}$ (resp $x \in e_{o(b)}X$), we consider the A_0-homomorphisms $xb^*(-) : S \longrightarrow X$ (resp. $b^*(-)x : S \longrightarrow X$), mapping $s \longrightarrow xb^*(s)$ (resp. $s \longrightarrow b^*(s)x$). Then the set $\{xb^*(-) : b \in \mathcal{B},\ x \in Xe_{t(b)}\}$ (resp. $\{b^*(-)x : b \in \mathcal{B},\ x \in e_{o(b)}X\}$) generates $\mathrm{Hom}_{A_0}(S, X)$ as a K-vector space

Proof. Straightforward consequence of the isomorphism $X \otimes D(S) \cong \mathrm{Hom}_{A_0}(S, X)$ (resp. $D(S) \otimes X \cong \mathrm{Hom}_{A_0}(S, X)$), which maps $x \otimes b^*$ onto $xb^*(-)$ (resp. $b^* \otimes x$ onto $b^*(-)x$), for all $x \in X$, $b \in \mathcal{B}$. □

Lemma 2.2. *The assignment $X \longrightarrow X \otimes A$ extends to a fully faithful covariant exact functor $T : \mathrm{Mod}_{A_0} \longrightarrow Gr_A$ with essential image $\mathrm{Proj}\,^0_A$. In particular, it induces an equivalence of categories $\mathrm{Mod}_{A_0} \cong \mathrm{Proj}\,^0_A$.*

Proof. It is clear that the assignment extends to a covariant functor $T = -\otimes A : \mathrm{Mod}_{A_0} \longrightarrow Gr_A$ with essential image contained in $\mathrm{Proj}\,^0_A$. Moreover, since ${}_{A_0}A$ is projective, the functor is clearly exact. We also have that $\mathrm{Mod}_{A_0} = \mathrm{Add}\,((A_0)_{A_0})$, $\mathrm{Proj}\,^0_A = \mathrm{Add}\,(A_A)$, T preserves direct sums and $T(A_0) \cong A$. From that it follows that $\mathrm{Proj}\,^0_A \subseteq \mathrm{Im}\,(T)$, and hence equality. It also follows that the fully faithful condition reduces to check that the functorial map $\mathrm{Hom}_{A_0}(A_0, A_0) \longrightarrow \mathrm{Hom}_{Gr_A}(A, A)$, $\lambda \longrightarrow T(\lambda)$ is bijective. That is straightforward. □

The isomorphisms of next lemma and Lemma 2.9 can be derived from apropriate adjunction settings, but we give their explicit definition for they are used in the proofs or our theorems.

Lemma 2.3. *Let X, Y be A_0-modules. The map* $\varphi : \mathrm{Hom}_{KQ_0}(KQ_1 \otimes X, Y) \longrightarrow \mathrm{Hom}_{A_0}(X, Y \otimes A_1)$ *taking μ onto* $\varphi(\mu) : x \longrightarrow \sum_{\alpha \in Q_1} \mu(\alpha \otimes x) \otimes \overline{\alpha}$ *is an isomorphism of K-vector spaces. Moreover, if* $\mu \in \mathrm{Hom}_{KQ_0}(KQ_1 \otimes X, Y)$, $\mu' \in \mathrm{Hom}_{KQ_0}(KQ_1 \otimes X', Y')$ *and* $f : X \longrightarrow X'$, $g : Y \longrightarrow Y'$ *are A_0-homomorphisms, then one of the following diagrams commutes iff the other does:*

$$\begin{array}{ccc} KQ_1 \otimes X & \xrightarrow{\mu} & Y \\ {\scriptstyle 1\otimes f}\downarrow & & \downarrow{\scriptstyle g} \\ KQ_1 \otimes X' & \xrightarrow{\mu'} & Y' \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\varphi(\mu)} & Y \otimes A_1 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g\otimes 1} \\ X' & \xrightarrow{\varphi(\mu')} & Y' \otimes A_1. \end{array}$$

Proof. Since $\{\overline{\alpha} : \alpha \in Q_1\}$ is a basis of A_1, every element of $Y \otimes A_1$ can be written as a sum $\sum_{\alpha \in Q_1} y_\alpha \otimes \overline{\alpha}$, where $y_\alpha \in Ye_{o(\alpha)}$, for all $\alpha \in Q_1$. In particular, if $f \in \mathrm{Hom}_{A_0}(X, Y \otimes A_1)$ then it maps x onto a sum $\sum_{\alpha \in Q_1} f_\alpha(x) \otimes \overline{\alpha}$, with $f_\alpha(x) \in Ye_{o(\alpha)}$ for all $\alpha \in Q_1$. Moreover, if $x \in Xe_i$ then the fact that f is a morphism in Mod_{A_0} implies that we can take $f_\alpha(x) = 0$ whenever $i \neq t(\alpha)$. Hence, we get a uniquely determined family of K-linear maps $\{f_\alpha : Xe_{t(\alpha)} \longrightarrow Ye_{o(\alpha)} : \alpha \in Q_1\}$ such that $f(x) = \sum_{\alpha \in Q_1} f_\alpha(x) \otimes \overline{\alpha}$. We now define $\xi : \mathrm{Hom}_{A_0}(X, Y \otimes A_1) \longrightarrow \mathrm{Hom}_{KQ_0}(KQ_1 \otimes X, Y)$ by the rule $\xi(f)(\alpha \otimes x) = f_\alpha(x)$. The choice of the f_α guarantees that $\xi(f)$ is a morphism in ${}_{A_0}\mathrm{Mod}$. We leave as an easy exercise to check that φ and ξ are mutually inverse. The rest of the proof is then routinary. $\square$

Let $(\mathcal{A}_k)_{k \in \mathbf{Z}}$ be a family of categories. We shall denote by $\prod_{k \in \mathbf{Z}} \mathcal{A}_k$ the corresponding product category. Its objects are the families $(U_k)_{k \in \mathbf{Z}}$ such that $U_k \in \mathcal{A}_k$, for every $k \in \mathbf{Z}$. Its morphisms are families of morphisms $(f_k : U_k \longrightarrow V_k)_{k \in \mathbf{Z}}$, with f_k a morphism in $\mathcal{A}_k$, for all $k \in \mathbf{Z}$. The composition of morphisms is defined pointwise. In particular, we shall denote by $\mathcal{A}^{\mathbf{Z}}$ the category $\prod_{k \in \mathbf{Z}} \mathcal{A}_k$, where $\mathcal{A}_k = \mathcal{A}$, for all $k \in \mathbf{Z}$. If $U \in \mathcal{A}^{\mathbf{Z}}$ and $n \in \mathbf{Z}$ then the object $U\{n\}$ of $\mathcal{A}^{\mathbf{Z}}$ is defined by the rule $U\{n\}_k = U_{n+k}$ for all $k \in \mathbf{Z}$. If $f : U \longrightarrow V\{n\}$ is a morphism in $\mathcal{A}^{\mathbf{Z}}$, we shall write $f : U \xrightarrow{n} V$ and shall say that f is a morphism of degree n from U to V.

We are mainly interested in the cases when $\mathcal{A} = {}_{KQ_0}\mathrm{Mod} = \mathrm{Mod}_{A_0}$ and $\mathcal{A} = Gr_A$ in the above situation. For technical reasons, we shall still keep subindices for the first case, while we shall use superindices for the second case (e.g., an object of $Gr_A^{\mathbf{Z}}$ will be denoted by $P^{\cdot} = (P^k)_{k \in \mathbf{Z}}$, where $P^k \in Gr_A$ for all k). We introduce now a new (Grothendieck) category $Gr_{A[X]}$ as follows: Its objects are pairs $(P^{\cdot}, d^{\cdot})$, where $P^{\cdot} \in Gr_A^{\mathbf{Z}}$ and $d^{\cdot} : P^{\cdot} \xrightarrow{+1} P^{\cdot}$ is a morphism in $Gr_A^{\mathbf{Z}}$ of degree $+1$. A morphism $f^{\cdot} : (P^{\cdot}, d^{\cdot}) \longrightarrow (Q^{\cdot}, \delta^{\cdot})$ in $Gr_{A[X]}$ is just a morphism $f^{\cdot} : P^{\cdot} \longrightarrow Q^{\cdot}$ in $Gr_A^{\mathbf{Z}}$ such that $f^{\cdot} \circ d^{\cdot} = \delta^{\cdot} \circ f^{\cdot}$. The notation $Gr_{A[X]}$ makes sense. Indeed, we can provide the polynomial algebra $A[X]$ with a $\mathbf{Z}\times\mathbf{Z}$-grading by putting $A[X]_{(m,n)} = A_m X^n$, whenever $m,n \geq 0$, and $A[X]_{(m,n)} = 0$ otherwise. If $M = \oplus M_{(m,n)}$ is a $\mathbf{Z}\times\mathbf{Z}$-graded right $A[X]$-module, then $M^n = \oplus_{m\in\mathbf{Z}} M_{(m,n)}$ is an object of Gr_A and multiplication by X yields morphisms in Gr_A, $d^n : M^n \longrightarrow M^{n+1}$, for all $n \in \mathbf{Z}$. In that way, we get an object of $Gr_{A[X]}$ and the category of $\mathbf{Z}\times\mathbf{Z}$-graded right $A[X]$-modules is identified with $Gr_{A[X]}$. We shall pay attention to the full subcategory $\mathcal{LG}_A$ of $Gr_{A[X]}$ consisting of those pairs $(P^{\cdot}, d^{\cdot})$ such that $P^{\cdot} \in \prod_{k\in\mathbf{Z}} \mathrm{Proj}\,^k_A$, i.e., such that P^k is a projective object of Gr_A generated in degree $-k$, for all $k \in \mathbf{Z}$. Inside $\mathcal{LG}_A$ we consider the full subcategory $\mathcal{LC}_A$ consisting of those $(P^{\cdot}, d^{\cdot})$ which are cochain complexes, i.e., such that $d^{\cdot} \circ d^{\cdot} = 0$. The objects of $\mathcal{LC}_A$ are called **linear complexes of projectives**. The full subcategory of $\mathcal{LC}_A$ with objects $(P^{\cdot}, d^{\cdot})$ such that P^k is finitely generated, for all $k \in \mathbf{Z}$, will be denoted $\mathcal{L}c_A$.

Our main results in the section concern the category ${}_{KQ}Gr$. We point out that an object of that category can be identified with a pair (M, μ), where $M = (M_k)$ is an object of ${}_{KQ_0}\mathrm{Mod}^{\mathbf{Z}}$ and $\mu = (\mu_k : KQ_1 \otimes M_k \longrightarrow M_{k+1})$ is a family of morphisms in ${}_{KQ_0}\mathrm{Mod}$. In that vein, a morphism $f : (M, \mu) \longrightarrow (N, \mu')$ in ${}_{KQ}Gr$ is identified with a morphism $f = (f_k)_{k\in\mathbf{Z}}$ in ${}_{KQ_0}\mathrm{Mod}^{\mathbf{Z}} = \mathrm{Mod}^{\mathbf{Z}}_{A_0}$ such that $f_{k+1} \circ \mu_k = \mu'_k \circ (1_{KQ_1} \otimes f_k)$, for all $k \in \mathbf{Z}$. We shall indistinctly use this and the classical interpretation of the category ${}_{KQ}Gr$.

When $\Lambda \cong KQ/I$ is a graded factor of a path algebra (e.g., $\Lambda =^! A$ in our case), the category ${}_{\Lambda}Gr$ can be identified with the full subcategory of ${}_{KQ}Gr$ consisting of graded left KQ-modules annihilated by I. That is the sense of the word 'restriction' in our next theorem.

Theorem 2.4. *Let $A = \oplus_{n\geq 0} A_n$ be a positively graded algebra with quiver Q. There is a fully faithful exact functor $\psi = \psi_A : {}_{KQ}Gr \longrightarrow Gr_{A[X]}$ which induces by restriction equivalences of categories ${}_{{}^!A}Gr = Gr_{A^!} \xrightarrow{\cong} \mathcal{LC}_A$ and ${}_{{}^!A}lfgr = lfgr_{A^!} \xrightarrow{\cong} \mathcal{L}c_A$.*

Proof. By Lemma 2.2, the composition $\mathrm{Mod}_{A_0} \xrightarrow{T} Gr_A \xrightarrow{-[k]} Gr_A$, $X \rightsquigarrow X \otimes A[k]$, is a fully faithful covariant exact functor, which we denote by T_k and induces an equivalence of categories $\mathrm{Mod}_{A_0} \xrightarrow{\cong} \mathrm{Proj}\,^k$, for every $k \in \mathbf{Z}$. As a consequence the product $\hat{T} = \prod_{k\mathbf{Z}} T_k : \mathrm{Mod}^{\mathbf{Z}}_{A_0} \longrightarrow Gr^{\mathbf{Z}}_A$ is a fully faithful exact functor inducing an equivalence of categories

$$(*) \qquad \hat{T} : \mathrm{Mod}^{\mathbf{Z}}_{A_0} \cong \prod_{k\in\mathbf{Z}} \mathrm{Proj}\,^k.$$

With the above interpretation of the objects in ${}_{KQ}Gr$ and $Gr_{A[X]}$, we are ready to define a functor $\psi : {}_{KQ}Gr \longrightarrow Gr_{A[X]}$ verifying the requirements. Using Lemma 2.3, to every $(M, \mu) \in_{KQ} Gr$ we can assign a family $(\varphi(\mu_k) : M_k \longrightarrow M_{k+1} \otimes A_1)_{k\in\mathbf{Z}}$ of morphisms in Mod_{A_0}. But we have a K-linear isomorphism $\mathrm{Hom}_{A_0}(M_k, M_{k+1} \otimes A_1) \cong \mathrm{Hom}_{Gr_A}(M_k \otimes A[k], M_{k+1} \otimes A[k+1])$, for every $k \in \mathbf{Z}$. Hence, the family $\mu = (\mu_k)$ induces a uniquely determined family $\varphi(\mu) = (\varphi(\mu_k) : \hat{T}(M)^k = M_k \otimes A[k] \longrightarrow M_{k+1} \otimes A[k+1] = \hat{T}(M)^{k+1})_{k \in \mathbf{Z}}$ of morphisms in Gr_A. That is, we get a morphism $\varphi(\mu) : \hat{T}(M) \xrightarrow{+1} \hat{T}(M)$ in $Gr_A^{\mathbf{Z}}$ of degree $+1$. We then define $\psi : {}_{KQ}Gr \longrightarrow Gr_{A[X]}$ on objects by taking the pair (M, μ) onto $(\hat{T}(M), d^{\cdot})$, with $d^{\cdot} = \varphi(\mu)$. Suppose now that (M, μ) and (N, μ') are objects of ${}_{KQ}Gr$ and let $g : \hat{T}(M) \longrightarrow \hat{T}(N)$ be a morphism in $Gr_A^{\mathbf{Z}}$. Since $\hat{T}(M)$ and $\hat{T}(N)$ belong to $\prod_{k \in \mathbf{Z}} \mathrm{Proj}\,_A^k$, the above equivalence $(*)$ gives a uniquely determined morphism $f : M \longrightarrow N$ in $\mathrm{Mod}_{A_0}^{\mathbf{Z}}$ such that $\hat{T}(f) = g$. Then $g^k = f_k \otimes 1_{A[k]}$, for all $k \in \mathbf{Z}$. We claim that f is a morphism $(M, \mu) \longrightarrow (N, \mu')$ in ${}_{KQ}Gr$ iff g is a morphism $(\psi(M), \varphi(\mu)) \longrightarrow (\psi(N), \varphi(\mu'))$ in $Gr_{A[X]}$. If that is proved it will follow that, defining $\psi(f) = \hat{T}(f)$ for every morphism f in ${}_{KQ}Gr$, one obtains a fully faithful exact functor $\psi : {}_{KQ}Gr \longrightarrow Gr_{A[X]}$ with essential image $\mathcal{LG}_A$. Let us prove our claim. We know that f is a morphism in ${}_{KQ}Gr$ iff $f_{k+1} \circ \mu_k = \mu'_k \circ (1 \otimes f_k)$, for all $k \in \mathbf{Z}$. By Lemma 2.3, that is equivalent to say that $(f_{k+1} \otimes 1_{A_1}) \otimes \varphi(\mu_k) = \varphi(\mu'_k) \circ f_k$, for all $k \in \mathbf{Z}$. This is in turn equivalent to say that $g^{k+1} \circ d^k = d^k \circ g^k$, for all $k \in \mathbf{Z}$, where $d^{\cdot} = \varphi(\mu)$. That occurs iff g is a morphism in $Gr_{A[X]}$, thus proving our claim.

In the final part of the proof, we come back to the classical interpretation of objects in ${}_{KQ}Gr = Gr_{KQ^{\mathrm{op}}}$, which will be looked at as graded right KQ^{op}-modules. With the equality $\mathcal{LG}_A = \mathrm{Im}\psi$ at hand, the rest of the proof reduces to check that $\psi(M)$ is a cochain complex iff $M_k \cdot I_2^{\perp} = 0$ for every $k \in \mathbf{Z}$, where $I = \mathrm{Ker}(\pi_A)$. From that the equivalences of the last part of the theorem will follow. On one hand, $\psi(M)$ is a cochain complex iff the composition $M_k \otimes A[k] \xrightarrow{d^k} M_{k+1} \otimes A[k+1] \xrightarrow{d^{k+1}} M_{k+2} \otimes A[k+2]$ is zero, for each $k \in \mathbf{Z}$. But, since $M_k \otimes A[k]$ is generated by $M_k \otimes A_0 \cong M_k$, that is equivalent to say that $d^{k+1} \circ d^k$ vanish on M_k. Direct calculation shows that $(d^{k+1} \circ d^k)(x) = \sum_{p \in Q_2} px \otimes \bar{p} = \sum_{p \in Q_2} xp^o \otimes \bar{p}$, for all $x \in M_k$. In particular, $(d^{k+1} \circ d^k)(M_k) \subseteq M_{k+2} \otimes A_1^2$, where $A_1^2 = A_1 \cdot A_1$. On the other hand, the obvious sequences $0 \to I_2 \longrightarrow KQ_2 \longrightarrow A_1^2 \to 0$ and $0 \to I_2^{\perp} \longrightarrow KQ_2^{\mathrm{op}} \longrightarrow D(I_2) \to 0$ (in ${}_{A_0}\mathrm{Mod}$ and Mod_{A_0}, respectively) are dual to each other. Hence, we have an isomorphism $A_1^2 \cong D(I_2^{\perp})$ which maps $\bar{p}$ onto the restriction of p^{o*} to $I_2^{\perp}$, where $\{p^{o*} : p \in Q_2\}$ is the dual basis in $D(KQ_2^{\mathrm{op}})$ of Q_2^{op}. Taking now the composition of $d^{k+1} \circ d^k : M_k \longrightarrow M_{k+2} \otimes A_1^2$ followed by the canonical isomorphism $M_{k+2} \otimes A_1^2 \cong M_{k+2} \otimes D(I_2^{\perp}) \cong \mathrm{Hom}_{A_0}(I_2^{\perp}, M_{k+2})$, we get a map $\delta^k : M_k \longrightarrow \mathrm{Hom}_{A_0}(I_2^{\perp}, M_{k+2})$. In a routinary way, one checks that $\delta^k(x) : I_2^{\perp} \longrightarrow M_{k+2}$ is the restriction to $I_2^{\perp}$ of the map $f_x = \sum_{p \in Q_2} (xp^o) p^{o*}(-) : KQ_2^{\mathrm{op}} \longrightarrow M_{k+2}$ (with notation as in Remark 2.1). The latter maps q^o onto xq^o, for every $q \in Q_2$, so that $\delta^k(x)(a) = xa$ for all $a \in I_2^\perp$. Therefore $d^{k+1} \circ d^k$ vanish on M_k iff $M_k \cdot I_2^\perp = 0$ as desired. □

Remark 2.5. The equivalences of the above theorem restrict to the corresponding full subcategories of upper or lower bounded objects. For instance, the equivalence ${}_{!A}lfgr = lfgr_{A^!} \xrightarrow{\cong} \mathcal{L}c_A$ restricts to an equivalence ${}_{!A}lfgr^- = lfgr_{A^!}^- \xrightarrow{\cong} \mathcal{L}c_A^-$.

In case $\Lambda = KQ/I$ is a graded factor of a path algebra, the category $Gr\Lambda$ can be seen in a canonical way as a subcategory of $Gr_{\widetilde{\Lambda}}$. In particular, for every positively graded algebra A which is orthogonal to Λ, swapping the roles of Q and $Q^{\rm op}$, Theorem 2.4 yields a fully faithful exact embedding $\psi_A : Gr_\Lambda \longrightarrow Gr_{A[X]}$ such that $\psi_A(M)$ is a cochain complex, for all $M \in Gr_\Lambda$. We then have the following consequence:

Corollary 2.6. *Let $\Lambda = KQ/I$ be a graded factor of a path algebra. For every $M \in Gr_\Lambda^-$ and every $j \in \mathbf{Z}$, the following assertions are equivalent:*

1) *M is cogenerated in degree j.*
2) *For every positively graded algebra $A = \oplus_{n\geq 0} A_n$ orthogonal to Λ, the cohomology graded A-module $H^k(\psi_A(M))$ is generated in degrees $> -k$, for all $k \neq j$.*
3) *There is a positively graded algebra A orthogonal to Λ satisfying 2).*

Proof. Suppose $M = \oplus_{k\leq k_0} M_k$, with $M_{k_0} \neq 0$. Then M is cogenerated in degree j implies $k_0 = j$. On the other hand, $H^k(\psi_A(M)) = 0$, for all $k > k_0$ and $H^{k_0}(\psi_A(M))_{-k_0} \cong M_{k_0} \otimes A_0 \cong M_{k_0} \neq 0$. Hence, $j = k_0$ in 1), 2) or 3). Now M is cogenerated in degree j iff for every $k < j$ and for every $0 \neq x \in M_k$, $x\Lambda_{j-k} \neq 0$. But, in our case, $\Lambda_n = \Lambda_1^n = \Lambda_1 \overset{n}{\cdots} \Lambda_1$ for all $n > 0$. We then get that M is cogenerated in degree j iff for every $k < j$ and for every $0 \neq x \in M_k$, $x\Lambda_1 \neq 0$. On the other hand, given any positively graded algebra A orthogonal to Λ, the graded A-module $H^k(\psi_A(M))$ has support contained in $\{n \in \mathbf{Z} : n \geq -k\}$, for all $k \in \mathbf{Z}$. Moreover, the homogeneous component of degree $-k$ of $H^k(\psi_A(M))$, denoted $H^k(\psi_A(M))_{-k}$, is the kernel of the map $d_k : M_k \cong M_k \otimes A_0 \longrightarrow M_{k+1} \otimes A_1$. This map, after the suitable adaptation from Theorem 2.2 due to the swapping of roles of Q and $Q^{\rm op}$, takes the form $x \longrightarrow \sum_{\alpha\in Q_1} x\alpha \otimes \overline{\alpha}^o$. We now compose this latter map with the canonical isomorphism $M_{k+1} \otimes A_1 \cong M_{k+1} \otimes D(KQ_1) \cong \mathrm{Hom}_{A_0}(KQ_1, M_{k+1}) \cong \mathrm{Hom}_{A_0}(\Lambda_1, M_{k+1})$. The resulting map $M_k \longrightarrow \mathrm{Hom}_{A_0}(\Lambda_1, M_{k+1})$ takes x onto $\sum_{\alpha\in Q_1}(x\alpha)\overline{\alpha}^*(-) : \overline{\beta} \longrightarrow x\beta$ (with the notation of Remark 2.1). Then $H^k(\psi_A(M))_{-k} \cong \{x \in M_k : x\Lambda_1 = 0\}$ and the desired equivalence of 1), 2) and 3) follows. □

We leave as an exercise the proof of the following lemma, which will be useful in the proof of our next theorem:

Lemma 2.7. *Let $f, g : M \longrightarrow N$ be two morphisms in Gr_A and assume that N is cogenerated in degree j. Then $f = g$ if, and only if, $f_j = g_j$.*

Given a A_0-module X and viewing each A_i as a right A_0-module, the vector space $\mathcal{H}om_{A_0}(A, X) = \oplus_{k \in \mathbf{Z}} \mathrm{Hom}_{A_0}(A_{-k}, X)$ gets a canonical structure of graded right A-module by defining $\mathcal{H}om_{A_0}(A, X)_k = \mathrm{Hom}_{A_0}(A_{-k}, X)$ and $(fa)(x) = f(ax)$ whenever $a \in A_j$, $f \in \mathcal{H}om_{A_0}(A, X)_k$ and $x \in A_{-(j+k)}$. In the particular case when $X = A_0$, we get $\mathcal{H}om_{A_0}(A, A_0) = D(A)$. We have now:

Lemma 2.8. *Assume that A is locally finite. The assignment $X \longrightarrow \mathcal{H}om_{A_0}(A, X)$ extends to a fully faithful covariant exact functor $H : \mathrm{Mod}_{A_0} \longrightarrow Gr_A$ with essential image $\mathrm{Inj}\,^0_A$. In particular, it induces an equivalence of categories $\mathrm{Mod}_{A_0} \cong \mathrm{Inj}\,^0_A$.*

Proof. We define $H(f) = f_* : u \longrightarrow f \circ u$, for every composable morphisms f and u in Mod_{A_0}. Then we clearly get a covariant functor $H : \mathrm{Mod}_{A_0} \longrightarrow Gr_A$ such that $H(X) = \mathcal{H}om_{A_0}(A, X)$, for every $X \in \mathrm{Mod}_{A_0}$. The functor is exact because A_{A_0} is projective. Since each A_k is a finitely generated A_0-module, H preserves direct sums. Notice that $\mathrm{Mod}_{A_0} = \mathrm{Add}\,(A_{0A_0})$ and $\mathrm{Add}\,(H(A_0)) = \mathrm{Add}\,(D(A)_A) = \mathrm{Inj}\,^0_A$. Then, using the preservation of direct sums, we conclude that our task reduces to check that H induces a bijection

$$(*) \quad \mathrm{Hom}_{A_0}(A_0, A_0) \to \mathrm{Hom}_{Gr_A}(H(A_0), H(A_0)) = \mathrm{Hom}_{Gr_A}(D(A), D(A)).$$

Let us prove this. By the canonical duality $D : {}_A lfgr \longrightarrow lfgr_A$, every morphism $g : D(A) \longrightarrow D(A)$ in Gr_A is of the form $D(\rho_a)$, where $\rho_a : A \longrightarrow A$ is right multiplication by a, for a uniquely determined $a \in A_0$. We take the left multiplication by a, $\lambda_a : A_0 \longrightarrow A_0$, which is a morphism in Mod_{A_0}. Then, for every $x \in A_k$ and $u \in \mathrm{Hom}_{A_0}(A_k, A_0) = H(A_0)_{-k}$, we have $[H(\lambda_a)(u)](x) = (\lambda_a \circ u)(x) = au(x)$. Then commutativity of A_0 yields $au(x) = u(x)a = u(xa) = [D(\rho_a)(u)](x) = (g(u))(x)$, from which we get $g = H(\lambda_a)$, for a uniquely determined $a \in A_0$. This proves that the map $(*)$ is bijective, thus ending the proof. □

Lemma 2.9. *Let X, Y be A_0-modules. The map $\eta : \mathrm{Hom}_{KQ_0}(KQ_1 \otimes X, Y) \longrightarrow \mathrm{Hom}_{A_0}(\mathrm{Hom}_{A_0}(A_1, X), Y)$ defined by $\eta(\mu)(u) = \sum_{\alpha \in Q_1} \mu(\alpha \otimes u(\overline{\alpha}))$, for all $u \in \mathrm{Hom}_{A_0}(A_1, X)$ and all $\mu \in \mathrm{Hom}_{KQ_0}(KQ_1 \otimes X, Y)$, is an isomorphism of K-vector spaces. Moreover, if $\mu \in \mathrm{Hom}_{KQ_0}(KQ_1 \otimes X, Y)$, $\mu' \in \mathrm{Hom}_{KQ_0}(KQ_1 \otimes X', Y')$ and $f : X \longrightarrow X'$, $g : Y \longrightarrow Y'$ are A_0-homomorphisms, then one of the following diagrams commutes iff the other does:*

$$
\begin{array}{ccccccc}
KQ_1 \otimes X & \xrightarrow{\mu} & Y & \qquad & \mathrm{Hom}_{A_0}(A_1, X) & \xrightarrow{\eta(\mu)} & Y \\
{\scriptstyle 1\otimes f}\downarrow & & \downarrow{\scriptstyle g} & & {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle g} \\
KQ_1 \otimes X' & \xrightarrow{\mu'} & Y' & & \mathrm{Hom}_{A_0}(A_1, X') & \xrightarrow{\eta(\mu')} & Y'.
\end{array}
$$

Proof. We define $\rho : \mathrm{Hom}_{A_0}(\mathrm{Hom}_{A_0}(A_1, X), Y) \longrightarrow \mathrm{Hom}_{KQ_0}(KQ_1 \otimes X, Y)$ by the rule $\rho(h)(\alpha \otimes x) = h(x\overline{\alpha}^*(-))$, where $x\overline{\alpha}^*(-) : A_1 \longrightarrow X$ is as in Remark 2.1, using $\{\overline{\alpha} = \pi_A(\alpha) : \alpha \in Q_1\}$ as basis of A_1. The fact that η and ρ are mutually inverse follows easily using that $u = \sum_{\alpha \in Q_1} u(\overline{\alpha})\overline{\alpha}^*(-)$ (with the same terminology of Remark 2.1), for every morphism $u : A_1 \longrightarrow X$ in Mod_{A_0}. The rest is routine. $\square$

In the sequel we shall denote by $\mathcal{LG}^*_A$ the full subcategory of $Gr_{A[X]}$ consisting of those pairs $(I^{\cdot}, d^{\cdot})$ such that $I^{\cdot} \in \prod_{k \in \mathbf{Z}} \mathrm{Inj}^{\,k}_A$, i.e., such that I^k is an almost injective graded right A-module cogenerated in degree $-k$, for every $k \in \mathbf{Z}$. Also, we denote by $\mathcal{LC}^*_A$ the full subcategory of $\mathcal{LG}^*_A$ consisting of those $(I^{\cdot}, d^{\cdot})$ which are cochain complexes. The objects of $\mathcal{LC}^*_A$ are called **linear complexes of almost injectives**. Within $\mathcal{LC}^*_A$ we shall also consider the full subcategory $\mathcal{L}c^*_A$ consisting of those $(I^{\cdot}, d^{\cdot}) \in \mathcal{LC}^*_A$ such that I^k is finitely cogenerated, for every $k \in \mathbf{Z}$.

Theorem 2.10. *Let $A = \oplus_{n \geq 0} A_n$ a locally finite positively graded algebra with quiver Q. There is a fully faithful covariant exact functor $\upsilon = \upsilon_A : {}_{KQ}Gr \longrightarrow Gr_{A[X]}$ which induces, by restriction, equivalences of categories ${}_{!A}Gr = Gr_{A^!} \xrightarrow{\cong} \mathcal{LC}^*_A$ and ${}_{!A}lfgr = lfgr_{A^!} \xrightarrow{\cong} \mathcal{L}c^*_A$.*

Proof. The first part of the proof is parallel to the corresponding one in the proof of Theorem 2.4, using the functor H of Lemma 2.8 instead of the functor T and Lemma 2.9 instead of Lemma 2.3. Indeed, the functor H induces a fully faithful exact functor $\hat{H} :_{KQ_0} \mathrm{Mod}^{\mathbf{Z}} = \mathrm{Mod}^{\mathbf{Z}}_{A_0} \longrightarrow Gr^{\mathbf{Z}}_A$ with essential image $\prod_{k \in \mathbf{Z}} \mathrm{Inj}^{\,k}_A$. Notice that, due to Lemma 2.7, we have K-linear isomorphisms

$$
\begin{aligned}
(*) \qquad & \mathrm{Hom}_{Gr_A}(\hat{H}(N)^k, \hat{H}(N)^{k+1}) \\
& \qquad = \mathrm{Hom}_{Gr_A}(\mathcal{H}om(A, N_k)[k], \mathcal{H}om(A, N_{k+1})[k+1]) \\
& \qquad \cong \mathrm{Hom}_{A_0}(\mathcal{H}om_{A_0}(A_1, N_k), N_{k+1}),
\end{aligned}
$$

for all $k \in \mathbf{Z}$. We now define the desired functor υ. On objects, it will take an object $(N, \mu) \in_{KQ_0} Gr$ onto the object $(I^{\cdot}, d^{\cdot})$, where $I^{\cdot} = \hat{H}(N)$ and $d^{\cdot} : \hat{H}(N) \xrightarrow{+1} \hat{H}(N)$ is the morphism of degree $+1$ in $Gr^{\mathbf{Z}}_A$ induced by the family of maps $(\eta(\mu_k) : \mathrm{Hom}_{A_0}(A_1, N_k) \longrightarrow N_{k+1})_{k \in \mathbf{Z}}$ (see Lemma 2.9) and the above isomorphisms $(*)$. On the other hand, if (N, μ) and (N', μ') are objects of ${}_{KQ}Gr$ and $g : \hat{H}(N) \longrightarrow \hat{H}(N')$ is a morphism in $Gr^{\mathbf{Z}}_A$, then there is a uniquely determined morphism $f : N \longrightarrow N'$ in ${}_{KQ_0}\mathrm{Mod}^{\mathbf{Z}} = \mathrm{Mod}^{\mathbf{Z}}_{A_0}$ such that $\hat{H}(f) = g$. Using the last part of Lemma 2.9, one sees that f is a morphism $(N, \mu) \longrightarrow (N', \mu')$ in ${}_{KQ}Gr$ iff g is a morphism $(\hat{H}(N), \eta(\mu)) \longrightarrow (\hat{H}(N'), \eta(\mu'))$ in $Gr_{A[X]}$. That proves that the assignment $f \longrightarrow \upsilon(f) =: \hat{H}(f)$ defines a fully faithful exact functor $\upsilon : {}_{KQ}Gr \longrightarrow Gr_{A[X]}$ with essential image $\mathcal{LG}^*_A$.

For the final part, as in the proof of Theorem 2.4, we view the objects of ${}_{KQ}Gr$ as graded right KQ^{op}-modules. We want to prove that $\upsilon(N)$ is a cochain complex iff $N_k \cdot I_2^{\perp} = 0$ for all $k \in \mathbf{Z}$, where $I = \mathrm{Ker}(\pi_A)$. From that it will follow that υ induces an equivalence of categories ${}_{!A}Gr = Gr_{A^!} \xrightarrow{\cong} \mathcal{LC}^*_A$. We have that $\upsilon(N)$ is a cochain complex iff the composition $\mathcal{H}om_{A_0}(A, N_k)[k] \xrightarrow{d^k} \mathcal{H}om_{A_0}(A, N_{k+1})[k+1] \xrightarrow{d^{k+1}} \mathcal{H}om_{A_0}(A, N_{k+2})[k+2]$ is zero, for all $k \in \mathbf{Z}$. According to Lemma 2.7, that is equivalent to say that its $-(k+2)$-component $\mathrm{Hom}_{A_0}(A_2, N_k) \xrightarrow{d^k} \mathrm{Hom}_{A_0}(A_1, N_{k+1}) \xrightarrow{d^{k+1}} \mathcal{H}om_{A_0}(A_0, N_{k+2}) \cong N_{k+2}$ is zero, for all $k \in \mathbf{Z}$. Using the explicit definition of $d^{\cdot}$, one sees that $(d^{k+1} \circ d^k)(f) = \sum_{p \in Q_2} f(\bar{p})p^o$, for all $f \in \mathrm{Hom}_{A_0}(A_2, N_k)$. Since $A_1^2 = A_1 \cdot A_1$ is a direct summand of A_2 in Mod_{A_0} and $\bar{p} \in A_1^2$, for all $p \in Q_2$, we get that $\upsilon(N)$ is a cochain complex iff $(d^{k+1} \circ d^k)(f) = 0$, for all $f \in \mathrm{Hom}_{A_0}(A_1^2, N_k)$. Now we argue as in the corresponding part of the proof of Theorem 2.4. We have $A_1^2 \cong D(I_2^{\perp})$ and the composition of the canonical isomorphism $N_k \otimes I_2^{\perp} \cong N_k \otimes DD(I_2^{\perp}) \cong \mathrm{Hom}_{A_0}(D(I_2^{\perp}), N_k) \cong \mathrm{Hom}_{A_0}(A_1^2, N_k)$ followed by $d^{k+1} \circ d^k : \mathrm{Hom}_{A_0}(A_1^2, N_k) \longrightarrow N_{k+2}$ is just the canonical multiplication map $N_k \otimes I_2^{\perp} \longrightarrow N_{k+2}$ coming from the right KQ-module structure of N. Therefore $d^{k+1} \circ d^k$ vanishes iff $N_k \cdot I_2^{\perp} = 0$ as desired.

It only remains to see that υ also induces an equivalence ${}_{!A}lfgr = lfgr_{A^!} \xrightarrow{\cong} \mathcal{L}c^*_A$. That reduces to prove that if $X \in \mathrm{Mod}_{A_0}$ then X is finite dimensional iff $H(X) = \mathcal{H}om_{A_0}(A, X)$ is a finitely cogenerated graded right A-module. By Lemma 2.8, that follows immediately from the fact that an object of Inj^0_A is finitely cogenerated in Gr_A iff it is a direct summand of a finite direct sum of copies of $D(A)$. $\square$

The following is dual to Corollary 2.6 and we leave the proof as an exercise:

Corollary 2.11. *Let $\Lambda = KQ/I$ be a graded factor of a path algebra. For $M \in Gr^+_\Lambda$ and $j \in \mathbf{Z}$, the following assertions are equivalent:*

1) *M is generated in degree j.*
2) *For every locally finite positively graded algebra $A = \oplus_{n \geq 0} A_n$ orthogonal to Λ, the cohomology graded A-module $H^k(\upsilon_A(M))$ is cogenerated in degrees $< -k$, for all $k \neq j$.*
3) *There is a positively graded algebra A orthogonal to Λ satisfying 2).*

3. Koszul dualities and the Yoneda algebra.

All throughout this section, $\Lambda = KQ/I$ will be a graded factor of a path algebra and Γ will be its Yoneda algebra. The main goal of this section is to see that $\widetilde{\Lambda}$ and Γ are orthogonal. A first ingredient for that is the next straightforward consequence of Theorem 2.4, which is valid for an arbitrary positively graded algebra.

Corollary 3.1. *Let $A = \oplus_{n\geq 0} A_n$ be a positively graded algebra with quiver Q, let $D : lfgr_{KQ} \longrightarrow {}_{KQ}lfgr$ be the canonical duality and $\iota : {}_{KQ}lfgr \hookrightarrow {}_{KQ}Gr$ be the inclusion functor. The composition $\phi = \phi_A : lfgr_{KQ} \xrightarrow{D} {}_{KQ}lfgr \xhookrightarrow{\iota} {}_{KQ}Gr \xrightarrow{\psi_A} Gr_{A[X]}$ is a fully faithful contravariant exact functor which induces by restriction a duality of categories ${}_{A^!}lfgr = lfgr_{{}^!A} \xrightarrow{\cong} \mathcal{L}c_A$.*

Remark 3.2. When restricted to the full subcategories of lower or upper bounded graded modules, the above duality 'changes signs'. For instance, it induces a duality of categories ${}_{A^!}lfgr^+ \cong \mathcal{L}c_A^-$.

The following is the main result of this section:

Theorem 3.3. *Let $\Lambda = KQ/I$ be a graded factor of a path algebra and let $\Gamma = \oplus_{n\geq 0}\Gamma_n$ be its Yoneda algebra. Then Λ and Γ are orthogonal graded algebras. In particular, $\phi = \phi_\Gamma : {}_{KQ}lfgr = lfgr_{KQ^{\mathrm{op}}} \longrightarrow Gr_{\Gamma[X]}$ induces a duality of categories ${}_{\widetilde{\Lambda}}lfgr \xrightarrow{\cong} \mathcal{L}c_\Gamma$.*

Proof. Let $\pi_\Lambda : KQ \longrightarrow \Lambda$ and $\pi_\Gamma : KQ^{\mathrm{op}} \longrightarrow \Gamma$ the canonical homomorphisms, with kernels I and J. We want to prove that $I_2^\perp = J_2$. Notice that $\pi_\Gamma(\alpha^o) = \widetilde{\alpha}$ can be identified, via the isomorphism $\Gamma_1 = \mathrm{Ext}^1_\Lambda(\Lambda_0, \Lambda_0) \xrightarrow{\cong} \mathrm{Hom}_\Lambda(\Lambda_{\geq 1}, \Lambda_0)$, with the unique Λ-homomorphism $\widetilde{\alpha} : \Lambda_{\geq 1} \longrightarrow \Lambda_0$ mapping an arrow γ onto $\delta_{\alpha\gamma} e_{o(\alpha)}$, where $\delta_{\alpha\gamma}$ is the Kronecker symbol. Our goal is to interpret the Yoneda product $\widetilde{\alpha} \cdot \widetilde{\beta}$ as a Λ-homomorphism $\Omega^2(\Lambda_0) \longrightarrow \Lambda_0$, bearing in mind that $\Gamma_2 = \mathrm{Ext}^2_\Lambda(\Lambda_0, \Lambda_0) \cong \mathrm{Hom}_\Lambda(\Omega^2(\Lambda_0), \Lambda_0)$. First, for the convenience of the reader, we shall adapt to our terminology a known explicit description of $\Omega^2(\Lambda_0)$. Recall that $\Omega^2(\Lambda_0) = \Omega^1(\Lambda_{\geq 1})$ is the kernel of the canonical multiplication map $\mu : \Lambda \otimes \Lambda_1 \longrightarrow \Lambda_{\geq 1}$. Suppose ρ is a homogeneous generating set of the ideal I of relations. We write every $r \in \rho$ as a linear combination $\sum_{\gamma \in Q_1} r_\gamma \gamma$, where $r_\gamma \in KQe_{o(\gamma)}$ is uniquely determined for every $\gamma \in Q_1$. We claim that $\Omega^2(\Lambda_0)$ is the Λ-submodule of $\Lambda \otimes \Lambda_1$ generated by the set $\{\sum_{\gamma \in Q_1} \bar{r}_\gamma \otimes \gamma : r \in \rho\}$, where the bar on top of an element of KQ always means its image by π_Λ. Indeed, let $\{a_\gamma : \gamma \in Q_1\}$ be a family of elements of KQ such that $a_\gamma \in KQe_{o(\gamma)}$ for all $\gamma \in Q_1$ and $\mu(\sum_{\gamma\in Q_1} \bar{a}_\gamma \otimes \gamma) = \sum_{\gamma \in Q_1} \bar{a}_\gamma \gamma = 0$ in Λ. Then $\sum_{\gamma \in Q_1} a_\gamma \gamma \in I$. This means that we have an equality

$$(*)\qquad \sum_{\gamma\in Q_1} a_\gamma\gamma = \sum_{r\in\rho} f_r r g_r + \sum_{r\in\rho} h_r r$$

in KQ, where $f_r, h_r \in KQ$ are all zero but finitely many and where $g_r \in KQ_{\geq 1}$, for all $r \in \rho$. We write $g_r = \sum_{\gamma\in Q_1} g_{r,\gamma}\gamma$ and $r = \sum_{\gamma\in Q_1} r_\gamma\gamma$, with $g_{r,\gamma} \in KQe_{o(\gamma)}$ for every $\gamma \in Q_1$. By substituting in the equality $(*)$, we get that $\sum_{\gamma\in Q_1} a_\gamma\gamma = \sum_{\gamma\in Q_1}(\sum_{r\in\rho} f_r r g_{r,\gamma} + \sum_{r\in\rho} h_r r_\gamma)\gamma$ in KQ. From that it follows that $a_\gamma = \sum_{r\in\rho} f_r r g_{r,\gamma} + \sum_{r\in\rho} h_r r_\gamma$ in KQ, which implies that $\bar{a}_\gamma = \sum_{r\in\rho} \bar{h}_r \bar{r}_\gamma$ in Λ. We then get an equality $\sum_{\gamma\in Q_1} \bar{a}_\gamma \otimes \gamma = \sum_{r\in\rho} \bar{h}_r(\sum_{\gamma\in Q_1} \bar{r}_\gamma \otimes \gamma)$ in $\Lambda \otimes \Lambda_1$ which proves the claim.

Once we have an explicit description of $\Omega^2(\Lambda_0)$, we are ready for an identification of $\widetilde{\alpha} \cdot \widetilde{\beta}$. We consider the morphism $v : \Lambda \otimes \Lambda_1 \longrightarrow \Lambda$ in ${}_\Lambda\mathrm{Mod}$ mapping $\sum_{\gamma\in Q_1} \bar{a}_\gamma \otimes \gamma$ onto $\bar{a}_\alpha e_{o(\alpha)}$. Clearly, $v(\Omega^2(\Lambda_0)) \subseteq \Lambda_{\geq 1}$, so that we get by restriction a Λ-homomorphism $u : \Omega^2(\Lambda_0) \longrightarrow \Lambda_{\geq 1}$ making commute the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \Omega^2(\Lambda_0) \arrow[r] \arrow[d, "u"] & \Lambda\otimes\Lambda_1 \arrow[r] \arrow[d, "v"] & \Lambda_{\geq 1} \arrow[r] \arrow[d, "\widetilde{\alpha}"] & 0 \\
0 \arrow[r] & \Lambda_{\geq 1} \arrow[r] & \Lambda \arrow[r] & \Lambda_0 \arrow[r] & 0
\end{tikzcd}$$

where the rows are the obvious exact sequences. Then the Yoneda product $\widetilde{\alpha} \cdot \widetilde{\beta}$ is represented by the composition $\Omega^2(\Lambda_0) \xrightarrow{u} \Lambda_{\geq 1} \xrightarrow{\widetilde{\beta}} \Lambda_0$. Take a generator $x_r = \sum_{\gamma\in Q_1} \bar{r}_\gamma \otimes \gamma$ of $\Omega^2(\Lambda_0)$. When the path $\beta\alpha$ does not appear in r, that element is mapped onto zero by $\widetilde{\alpha} \cdot \widetilde{\beta} = \widetilde{\beta} \circ u$. When $\beta\alpha$ does appear in r, it is mapped onto $e_{o(\beta)}$. That implies that $\widetilde{\alpha} \cdot \widetilde{\beta}$ vanishes on $\{x_r : r \in \rho \text{ and } \mathrm{length}(r) > 2\}$. That is, the action of $\widetilde{\alpha} \cdot \widetilde{\beta}$ on $\Omega^2(\Lambda_0)$ is completely identified by its action on the K-subspace generated by $\{x_r : r \in \rho \text{ and } \mathrm{length}(r) = 2\}$, which is a Λ_0-submodule of ${}_{\Lambda_0}\Omega^2(\Lambda_0)$ isomorphic to I_2. We then have a restriction map $\varphi : \Gamma_2 \cong \mathrm{Hom}_\Lambda(\Omega^2(\Lambda_0), \Lambda_0) \longrightarrow \mathrm{Hom}_{\Lambda_0}(I_2, \Lambda_0)$ and our argument says that the kernel J_2 of the canonical map $\pi : KQ_2^{\mathrm{op}} \longrightarrow \Gamma_2$ coincides with the kernel of $\varphi \circ \pi$. But the action of $(\varphi \circ \pi)(\alpha^o\beta^o) = \widetilde{\alpha} \cdot \widetilde{\beta}$ on I_2 is the restriction to I_2 of the action of the morphism $(\beta\alpha)^* : KQ_2 \longrightarrow KQ_0$ in ${}_{\Lambda_0}\mathrm{Mod}$, which maps $\beta\alpha$ onto $e_{o(\beta)}$, and the remaining paths of length 2 to zero. Hence, via the isomorphism $KQ_2^{\mathrm{op}} \cong D(KQ_2)$, we have that J_2 is identified with the kernel of the restriction map $D(KQ_2) \longrightarrow \mathrm{Hom}_{\Lambda_0}(I_2, \Lambda_0)$, $f \longrightarrow f_{|I_2}$. Therefore $J_2 = I_2^{\perp}$ as desired.

The last assertion of the theorem follows now directly from Corollary 3.1. $\square$

We now get the following categorical characterization of quadratic algebras:

Corollary 3.4. *Let $\Lambda = KQ/I$ be a graded factor of a path algebra and Γ be its Yoneda algebra. The following assertions are equivalent:*

1) *Λ is quadratic;*
2) *$\psi_A : Gr_{KQ} = {}_{KQ^{\mathrm{op}}}Gr \longrightarrow Gr_{A[X]}$ induces by restriction an equivalence of categories $Gr_\Lambda \cong \mathcal{LC}_A$ (resp. $lfgr_\Lambda \cong \mathcal{L}c_A$), for every (or some) positively graded algebra A which is orthogonal to Λ;*
3) *$\psi_\Gamma : Gr_{KQ} = {}_{KQ^{\mathrm{op}}}Gr \longrightarrow Gr_{\Gamma[X]}$ induces by restriction an equivalence of categories $Gr_\Lambda \cong \mathcal{LC}_\Gamma$ (resp. $lfgr_\Lambda \cong \mathcal{L}c_\Gamma$);*
4) *$\phi_\Gamma : {}_{KQ}lfgr = lfgr_{KQ^{\mathrm{op}}} \longrightarrow Gr_{\Gamma[X]}$ induces by restriction a duality of categories ${}_\Lambda lfgr \cong \mathcal{L}c_\Gamma$;*
5) *$\upsilon_A : Gr_{KQ} = {}_{KQ^{\mathrm{op}}}Gr \longrightarrow Gr_{A[X]}$ induces by restriction an equivalence of categories $Gr_\Lambda \cong \mathcal{LC}^*_A$ (resp. $lfgr_\Lambda \cong \mathcal{L}c^*_A$), for every (or some) locally finite positively graded algebra A which is orthogonal to Λ.*

Proof. Λ is quadratic iff $\Lambda = \widetilde{\Lambda}$ and this is equivalent to say that one (or all) of the categories Gr_Λ, $lfgr_\Lambda$ or ${}_\Lambda lfgr$ coincides with the corresponding category of graded modules over $\widetilde{\Lambda}$, viewed as full subcategories of Gr_{KQ} or ${}_{KQ}Gr$ according to the case. Since $\widetilde{\Lambda} \cong A^!$, for every positively graded algebra A which is orthogonal to Λ, the result follows directly from Theorems 2.4, 2.10 and 3.3. □

We end this section with an interesting consequence of our theorems.

Corollary 3.5. *Let A and B be two quadratically equivalent positively graded algebras. The following assertions hold:*

1) *There is an equivalence of categories $\mathcal{LC}_A \cong \mathcal{LC}_B$ (resp. $\mathcal{L}c_A \cong \mathcal{L}c_B$).*
2) *When A and B are locally finite, there is an equivalence of categories $\mathcal{LC}^*_A \cong \mathcal{LC}^*_B$ (resp. $\mathcal{L}c^*_A \cong \mathcal{L}c^*_B$).*
3) *When A, B are graded factors of path algebras, there is an equivalence of categories $\mathcal{L}c_\Gamma \cong \mathcal{L}c_{\Gamma'}$, where Γ and Γ' are the Yoneda algebras of A and B, respectively.*

Proof. Since $\widetilde{A} \cong \widetilde{B}$ and $A^! \cong B^!$, the result is a direct consequence of Theorems 2.4, 2.10 and 3.3. □

4. Some equivalences of derived categories.

Throughout this section, for every abelian category $\mathcal{A}$, $\mathcal{D}^b(\mathcal{A})$ will be the full subcategory of its derived category $\mathcal{D}(\mathcal{A})$ with objects those isomorphic (in $\mathcal{D}(\mathcal{A})$) to bounded complexes of objects in $\mathcal{A}$. We follow [**17**] for the terminology concerning derived categories. In the sequel, Λ will be a Koszul algebra with Yoneda algebra $\Gamma \cong \Lambda^!$. Recall from [**1**] that we have mutually inverse equivalences of triangulated categories $F : \mathcal{D}^{\downarrow}(\Lambda) \rightleftarrows \mathcal{D}^{\uparrow}(\Gamma) : G$, where we adopt the same terminology of [**1**], but working with right instead of left graded modules. With that in mind, if $M^{\cdot}$ is an object in $\mathcal{D}^{\downarrow}(\Lambda)$ then $F(M^{\cdot})$ is a complex of graded Γ-modules with $F(M^{\cdot})^p = \oplus_{i+j=p} M^i_j \otimes \Gamma[j] = \oplus_{i+j=p}\psi(M^i)^j$, where $\psi = \psi_\Gamma$. Conversely if $N^{\cdot}$ is an object in $\mathcal{D}^{\uparrow}(\Gamma)$, then one has $G(N^{\cdot})^p = \oplus_{i+j=p}\upsilon(N^i)^j$, with $\upsilon = \upsilon_\Lambda$. Moreover, in the particular case when Λ is finite dimensional and Γ is Noetherian, those equivalences induce mutually inverse equivalences $\mathcal{D}^b(gr - \Lambda) \cong \mathcal{D}^b(gr - \Gamma)$ ([**1**, Theor. 2.12.6]). Our next result is a slight extension of this. Recall that a positively graded algebra $A = \oplus_{n \geq 0} A_n$ is graded right coherent in case every finitely generated graded right ideal is finitely presented. In that case, $fpgr_A$ is an abelian category with exact inclusion functor $fpgr_A \longrightarrow Gr_A$.

Proposition 4.1. *Let Λ be a Koszul finite dimensional algebra with graded right coherent Yoneda algebra. The equivalences of categories*

$$F : \mathcal{D}^{\downarrow}(\Lambda) \rightleftarrows \mathcal{D}^{\uparrow}(\Gamma) : G$$

induce by restriction mutually inverse equivalences of triangulated categories $F : \mathcal{D}^b(gr_\Lambda) \rightleftarrows \mathcal{D}^b(fpgr_\Gamma) : G$.

Proof. The restriction of F to Gr^-_Λ, viewed as the full subcategory of $\mathcal{D}^{\downarrow}(\Lambda)$ consisting of stalk complexes at the 0-position, is just ψ_Γ. Then F takes the simple objects of Gr^-_Λ onto indecomposable projective objects of Gr_Γ. But then F and G induce by restriction mutually inverse equivalences between the triangulated subcategories of $\mathcal{D}^{\downarrow}(\Lambda)$ and $\mathcal{D}^{\uparrow}(\Gamma)$ generated by the simple objects of Gr_Λ and the indecomposable projective objects of Gr_Γ, respectively. Those subcategories are $\mathcal{D}^b(gr_\Lambda)$ and $\mathcal{D}^b(fpgr_\Gamma)$. The latter is due to the fact that Γ has finite graded global dimension, because its Yoneda algebra Λ, is finite-dimensional (cf. [**15**, Cor., p. 424]). $\square$

We shall consider the full subcategory $\mathcal{I}_\Lambda$ of $\mathcal{D}^b(gr_\Lambda)$ whose objects are the complexes isomorphic (in $\mathcal{D}^{\downarrow}(\Lambda)$) to bounded complexes of injective graded Λ-modules. On the other hand, when Γ is right coherent, every finite-dimensional graded Γ-module is finitely presented, because so are the simples. We denote by $\mathcal{F}_\Gamma$ the full subcategory of $\mathcal{D}^b(fpgr_\Gamma)$ consisisting (up to isomorphism in $D^b(fpgr_\Gamma)$) of bounded complexes of finite dimensional graded Γ-modules.

Corollary 4.2. *Let Λ be a finite dimensional Koszul algebra such that its Yoneda algebra Γ is graded right coherent. The equivalences $F : \mathcal{D}^b(gr_\Lambda) \rightleftarrows \mathcal{D}^b(fpgr_\Gamma) : G$ induce mutually inverse equivalences of triangulated categories $\mathcal{I}_\Lambda \cong \mathcal{F}_\Gamma$ and $\mathcal{D}^b(gr_\Lambda)/\mathcal{I}_\Lambda \cong \mathcal{D}^b(fpgr_\Gamma)/\mathcal{F}_\Gamma$.*

Proof. The restriction of G to Gr^+_Γ is υ_Λ, so that $G(\Gamma_0)$ is the stalk complex $D(\Lambda)$ at the 0-position. Then F and G induce mutually inverse equivalences between the triangulated subcategories of $\mathcal{D}^b(gr_\Lambda)$ and $\mathcal{D}^b(fpgr_\Gamma)$ generated by $D(\Lambda)$ and Γ_0, respectively, and their corresponding Verdier quotients. Those subcategories are precisely $\mathcal{I}_\Lambda$ and $\mathcal{F}_\Gamma$. □

Let $\mathcal{A}$ be an abelian category. Recall that a Serre subcategory T of $\mathcal{A}$ is a full subcategory satisfying the property that in every short exact sequence of $\mathcal{A}$, say $0 \to A \longrightarrow B \longrightarrow C \to 0$, the central object B belongs to T iff so do A, C. By [**7**], we have a quotient abelian category $\mathcal{A}/T$ and an exact quotient functor $\pi : \mathcal{A} \longrightarrow \mathcal{A}/T$. In case of existence of the respective derived categories, we have an induced exact functor $\mathcal{D}\pi : \mathcal{D}(\mathcal{A}) \longrightarrow \mathcal{D}(\mathcal{A})/T$. The following is well-known (cf. [**13**, Theorem 3.2]):

Lemma 4.3. *The kernel K of $\mathcal{D}\pi$ is the full subcategory with objects the complexes having cohomology in T and $D\pi$ induces an equivalence of triangulated categories $\mathcal{D}^*(\mathcal{A})/K \cong \mathcal{D}^*(\mathcal{A}/T)$, for $* = +, -, b$.*

For every positively graded algebra $A = \oplus_{n\geq 0}A_n$, we shall denote by L_A the full subcategory of Gr_A consisting of finite dimensional graded A-modules. Within the derived category $\mathcal{D}(Gr_A)$ we shall consider the full subcategory $\mathcal{F}_A$ consisting of bounded complexes of objects of L_A. We close $\mathcal{F}_A$ under isomorphism in any full subcategory of $\mathcal{D}(Gr_A)$ containing it that we use in the sequel.

Theorem 4.4. *Let $A = \oplus_{n\geq 0}A_n$ be a positively graded right coherent algebra such that each ideal $A_{\geq s} = \oplus_{n\geq s}A_n$ is finitely generated on the right, for every $s \geq 0$ (e.g., when $A = KQ/I$ is a right coherent generalized graded factor of a path algebra). Then all graded right A-modules of finite length are finitely presented and the canonical quotient functor $\pi : fpgr_A \longrightarrow fpgr_A/L_A$ induces an exact functor of triangulated categories $\mathcal{D}\pi : \mathcal{D}^b(fpgr_A) \longrightarrow \mathcal{D}^b(fpgr_A/L_A)$ with kernel $\mathcal{F}_A$. In particular, it also induces an equivalence of triangulated categories*

$$\mathcal{D}^b(fpgr_A)/\mathcal{F}_A \cong \mathcal{D}^b(fpgr_A/L_A).$$

Proof. If $S = \oplus_{n\in\mathbf{Z}}S_n$ is a simple object of Gr_A, then it is generated by a homogeneous element, so that $S = S_{\geq m}$, with $S_m \neq 0$, for some $m \in \mathbf{Z}$. But the chain $S \cdot A_{\geq 1} \subseteq S_{\geq m+1} \subset S_{\geq m} = S$ of inclusions in Gr_A and the simplicity of S imply that $S \cdot A_{\geq 1} = S_{\geq m+1} = 0$. Then $S = S_m$ is a necessarily simple A_0-module. This argument also proves that $A_{\geq 1}$ is the graded Jacobson radical of A. In particular, S is a direct summand of $A/A_{\geq 1} \cong A_0$ in Gr_A. Since $A/A_{\geq 1}$ is a finitely presented graded right A-module by hypothesis, we conclude that every simple object of Gr_A, and hence every object of finite length, belongs to $fpgr_A$. Clearly, L_A is then a Serre subcategory of $fpgr_A$ and the quotient functor $\pi : fpgr_A \longrightarrow fpgr_A/L_A$ makes sense. Let K be the full subcategory of $D^b(fpgr_A)$ with objects the complexes $X^\cdot$ having cohomology in L_A. Bearing in mind the previous lemma, in order to prove the assertions concerning derived categories, we only need to prove the equality $K = \mathcal{F}_A$. Since we clearly have $\mathcal{F}_A \subseteq K$, we just need to check the converse inclusion. Let $X^\cdot \in K$, then $H^*(X^\cdot) = \oplus_{i \in \mathbf{Z}} H^i(X^\cdot)$ is a (finitely presented) graded A-module of finite length. Then it is finitely graded, i.e., $H^*(X^\cdot)_n = 0$ for all but finitely many $n \in \mathbf{Z}$. We pick up an interval of integers $[k, m]$ such that $H^*(X^\cdot)_n = 0$, for all $n \notin [k, m]$. Then the complexes $X^\cdot_{>m}$ and $X^\cdot / X^\cdot_{\geq k}$ having in their i-th position $X^i_{>m} = \oplus_{n>m} X^i_n$ and $X^i / X^i_{\geq k}$, respectively, are acyclic. Notice that our hypothesis on each $A_{\geq s}$ guarantees that all these complexes are complexes of finitely presented graded modules. Now, from the exact sequences of complexes $0 \to X^\cdot_{>m} \longrightarrow X^\cdot \longrightarrow X^\cdot / X^\cdot_{>m} \to 0$ and $0 \to X^\cdot_{\geq k} / X^\cdot_{>m} \longrightarrow X^\cdot / X^\cdot_{>m} \longrightarrow X^\cdot / X^\cdot_{\geq k} \to 0$, we immediately deduce that $X^\cdot$ is isomorphic in $D^b(fpgr_R)$ to $X^\cdot_{\geq k} / X^\cdot_{>m}$. But the fact that $A_{\geq s}$ is finitely generated as a right ideal, for every $s \geq 0$, easily implies that each A_s has finite length as an A_0-module. From that it follows that, for every finitely presented graded right A-module M, its homogeneous components M_s are all A_0-modules of finite length. In particular, $X^\cdot_{\geq k} / X^\cdot_{>m}$ is a complex in $\mathcal{F}_A$. Hence $K \subseteq \mathcal{F}_A$, and the converse inclusion is clear.

Finally, we clarify the assertion between brackets. If $A = KQ/I$ is a right coherent generalized graded factor of a path algebra, take $d = \max\{\deg(\alpha) : \alpha \in Q_1\}$. If p is a path in Q such that $\bar{p} = p + I \in A_n$, with $n > s + d$, then the decompostion $p = q\alpha$, with $\alpha \in Q_1$, yields that $\deg(q) \geq s$. By iteration of this argument, we conclude that $A_{\geq s}$ is generated by $A_s \oplus \cdots \oplus A_{s+d}$, which is finite dimensional over K. Therefore $A_{\geq s}$ is a finitely generated graded right ideal of A, for every $s \geq 0$. □

If A is a positively graded finite dimensional algebra, we shall denote by $\overline{gr}_A$ the stable category (module injectives) of gr_A. Its objects are those of gr_A and $\mathrm{Hom}_{\overline{gr}_A}(M, N) = \mathrm{Hom}_{gr_A}(M, N)/I(M, N)$, for all $M, N \in gr_A$, where $I(M, N)$ is the vector subspace of $\mathrm{Hom}_{gr_A}(M, N)$ given by the morphisms which factor through an injective object of gr_A. The stable category (modulo projectives) of gr_A, denoted $\underline{gr}_A$ is defined dually. It is well-known that $\overline{gr}_A$ has a structure of suspended category (in the terminology of [**12**]) or right triangulated category (in the terminology of [**3**]) whose stabilization $S(\overline{gr}_A)$ is the triangulated category $\mathcal{D}^b(gr_A)/\mathcal{I}_A$ (cf. [**2**][dual of Theorem 3.8]). We now have:

Corollary 4.5. *Let $\Lambda = KQ/I$ be a finite dimensional Koszul algebra with graded right coherent Yoneda algebra Γ. There is an equivalence of triangulated categories $S(\overline{gr}_\Lambda) \cong \mathcal{D}^b(fpgr_\Gamma/L_\Gamma)$. In particular, when Λ is self-injective, there is an equivalence of triangulated categories $\underline{gr}_\Lambda = \overline{gr}_\Lambda \cong \mathcal{D}^b(fpgr_\Gamma/L_\Gamma)$.*

Proof. Just apply Corollary 4.2 and Theorem 4.4 and the fact that, when Λ is selfinjective, $\underline{gr}_\Lambda = \overline{gr}_\Lambda$ is already a triangulated category coincident with $S(\overline{gr}_\Lambda)$. $\square$

Remark 4.6. When $V \subseteq \mathbf{P}^n$ is a projective irreducible variety with coordinate algebra $K[V]$, Serre's theorem (see [**16**], Chap. III) says that the category coh(V) of coherent sheaves on V is equivalent to the category $gr_{K[V]}/L_{K[V]}$. Hence, last corollary extends and reproves in a different way the well-known result of Bernstein, Gelfand and Gelfand (see, e.g., [**4**] or [**9**, Ch. IV, Section 3]) stating that $\mathcal{D}^b(\mathrm{coh}\,(\mathbf{P}^n))$ is equivalent to $\underline{gr}_\Lambda$, where Λ is the exterior algebra of a $(n+1)$-dimensional vector space.

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