GENERALIZED FOCK SPACES AND WEYL COMMUTATION RELATIONS FOR THE DUNKL KERNEL

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In this paper we study a class of generalized Fock spaces associated with the Dunkl operators. Next we introduce the commutator relations between the Dunkl operators and multiplication operators which lead to a generalized class of Weyl commutation relations for the Dunkl kernel.

1. Introduction.

Fock space (called also Segal-Bargmann space [5]) is the Hilbert space of entire functions of \( \mathbb{C}^d \) with inner product given by

\[
(f, g) := \frac{1}{\pi^d} \int_{\mathbb{C}^d} f(z) \overline{g(z)} e^{-|z|^2} \, dx \, dy, \quad z = x + iy,
\]

where

\[
|z|^2 = \sum_{i=1}^{d} x_i^2 + y_i^2, \quad dx \, dy = \prod_{i=1}^{d} dx_i \, dy_i.
\]

This space which associated with Fock's [12] realization of the creation and annihilation operators of Bose particles is studied by Bargmann [4]. Next, the ordinary Fock space \( \mathcal{A} \) is the subject of many works ([5, 7] and [8]).

In 2001, M. Sifi and F. Soltani [21] introduced a Hilbert space \( \mathcal{A}_\gamma \) of entire functions where the inner product is weighted by a generalized Gaussian distribution. On \( \mathcal{A}_\gamma \) the Dunkl operator on the real line:

\[
T_\gamma(f)(z) := \frac{d}{dz} f(z) + \frac{2\gamma}{z} \left[ \frac{f(z) - f(-z)}{2} \right], \quad \gamma > 0,
\]

and the multiplication by \( z \) are adjoints and satisfy the commutation rule

\[
[T_\gamma, z] = I + 2\gamma B, \quad \text{where} \quad Bf(x) = f(-x).
\]

This commutator rule leads to a generalized class of Weyl commutation relations for the Dunkl kernel in the one dimensional.

In this paper we consider the differential-difference operators \( T_j, \ j = 1, \ldots, d \), on \( \mathbb{R}^d \) introduced by C.F. Dunkl in [9] and called Dunkl operators in the literature. These operators are very important in pure Mathematics and in Physics. They provide a useful tool in the study of special functions associated with root systems [10]. They are closely related to certain
representations of degenerated affine Hecke algebras ([6] and [16]). Moreover the commutative algebra generated by these operators has been used in the study of certain exactly solvable models of quantum Mechanics, namely the Calogero-Sutherland-Moser models, which deal with systems of identical particles in the one dimensional space ([2, 3] and [14]).

The Dunkl kernel $E_k(x, y)$ is the unique solution of the initial problem

$$T_j^x u(x, y) = y_j u(x, y); \quad u(0, y) = 1; \quad j = 1, \ldots, d,$$

see [10, 17] and [18]. This kernel has a unique holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d$. Furthermore, the Dunkl kernel $E_k(z, w); \quad z, w \in \mathbb{C}^d$ can be expanded in a power series in the form

$$E_k(z, w) = \sum_{\nu \in \mathbb{N}^d} \varphi_{\nu}(z)\varphi_{\nu}(w),$$

with some homogeneous orthonormal basis $\{\varphi_{\nu}\}_{\nu \in \mathbb{N}^d}$ of polynomials ([17] and [19]).

We introduce in this paper the generalized Fock space $A_k$ associated with the Dunkl operators. This is a Hilbert space of functions $f$ on $\mathbb{C}^d$ which can be written

$$f(z) = \sum_{\nu \in \mathbb{N}^d} a_{\nu}\varphi_{\nu}(z)$$

with

$$\|f\|_k^2 = (f, f)_k := \sum_{\nu \in \mathbb{N}^d} |a_{\nu}|^2 < \infty.$$

If $f, g \in A_k$, having series expansions $f(z) = \sum_{\nu \in \mathbb{N}^d} a_{\nu}\varphi_{\nu}(z)$ and $g(z) = \sum_{\nu \in \mathbb{N}^d} b_{\nu}\varphi_{\nu}(z)$. Then the inner product is given by the generalized spherical harmonics

$$(f, g)_k = \left(f(T)\tilde{g}\right)(0),$$

where $f(T) = f(T_1, \ldots, T_d)$ and $\tilde{g}(z) = \sum_{\nu \in \mathbb{N}^d} b_{\nu}\varphi_{\nu}(z)$.

The generalized Fock space $A_k$, has also a reproducing kernel $K$ given by

$$K(z, w) = E_k(z, w); \quad z, w \in \mathbb{C}^d.$$ If $f \in A_k$, then we have

$$f(w) = (f, E_k(\cdot, w))_k, \quad w \in \mathbb{C}^d.$$ Thus the Dunkl kernel serves as the generalized Dirac delta function in $A_k$.

The associated operators for the generalized Fock space $A_k$ are $T_j$ and the multiplication operator by $z_j$. They are adjoints in $A_k$ and satisfy a commutation rule:

$$[T_i, z_j] = \delta_{i,j}I + \sum_{\alpha \in R_+} k(\alpha)\alpha_i\alpha_j B_\alpha; \quad i, j = 1, \ldots, d,$$

where $B_\alpha$ a reflection operator, $k(\alpha)$ a multiplicity function and $R_+$ is a positive root system.

These commutators rule lead to a generalized class of Weyl commutation relations for the Dunkl kernel.

These relations are studied in the classical case ($k = 0$) in [13].
Throughout this paper we shall use the standard multi-index notations. For multi-indices $\nu, s \in \mathbb{N}^d$, we write $|\nu| = \sum_{i=1}^{d} \nu_i$, $\nu! = \prod_{i=1}^{d} \nu_i!$, $(\nu,s) = \prod_{i=1}^{d} (\nu_i s_i)$ as well as $z^\nu = \prod_{i=1}^{d} z_{\nu_i}^i$, $D^\nu = \prod_{i=1}^{d} D_{\nu_i}^i$, for $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$ and any family $D = (D_1, \ldots, D_d)$ of commuting operators. Finally, we will need the partial ordering $\leq$ on $\mathbb{N}^d$, which is defined by $s \leq \nu \iff s_i \leq \nu_i$, $i = 1, \ldots, d$.

2. Preliminaries.

In this section we collect some notations and results on Dunkl operators and Dunkl kernel that will be important later on. General references here are [9, 17, 18, 19] and [20].

We consider $\mathbb{R}^d$ with the Euclidean scalar $\langle ., . \rangle$ and $|x| = \sqrt{\langle x, x \rangle}$. On $\mathbb{C}^d$, $|.|$ denotes also the standard Hermitian norm, while $\langle z, w \rangle = \sum_{j=1}^{d} z_j w_j$ and $\ell(z) = \langle z, z \rangle$.

For $\alpha \in \mathbb{R}^d \setminus \{0\}$, let $\sigma_\alpha$ be the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^d$ orthogonal to $\alpha$,

$$
\sigma_\alpha x := x - \frac{2\langle \alpha, x \rangle}{|\alpha|^2} \alpha.
$$

A finite set $R \subset \mathbb{R}^d \setminus \{0\}$ is called a root system if $R \cap \mathbb{R} \alpha = \{-\alpha, \alpha\}$ and $\sigma_\alpha R = R$ for all $\alpha \in R$. We assume that it is normalized by $|\alpha|^2 = 2$ for all $\alpha \in R$. For a given root system $R$ the reflections $\sigma_\alpha, \alpha \in R$ generated a finite group $G \subset O(d)$, the reflection group associated with $R$. All reflections in $G$ correspond to suitable pairs of roots. For a given root system $R$ the reflections $\sigma_\alpha, \alpha \in R$ generated a finite group $G \subset O(d)$, the reflection group associated with $R$. All reflections in $G$ correspond to suitable pairs of roots. For a given $\beta \in H = \mathbb{R}^d \setminus \bigcup_{\alpha \in R} H_\alpha$, we fix the positive subsystem $R_+ = \{\alpha \in R / \langle \alpha, \beta \rangle > 0\}$, then for each $\alpha \in R$ either $\alpha \in R_+$ or $-\alpha \in R_+$. The connected components of $H$ are called the Weyl chambers of $G$.

A function $k : R \to \mathbb{C}$ on a root system $R$ is called a multiplicity function if it is invariant under the action of the associated reflection group $G$. If one regards $k$ as a function on the corresponding reflections, this means that $k$ is constant on the conjugacy classes of reflections in $G$. For abbreviation, we introduce the index

$$
\gamma = \gamma(k) := \sum_{\alpha \in R_+} k(\alpha).
$$

Moreover, let $w_k$ denotes the weight function:

$$
w_k(x) := \prod_{\alpha \in R_+} |\langle \alpha, x \rangle|^{2k(\alpha)}, \quad x \in \mathbb{R}^d,
$$

which is $G$-invariant and homogeneous of degree $2\gamma$. 
For $d = 1$ and $G = \mathbb{Z}_2$, the multiplicity function $k$ is a simple parameter denoted $\gamma > 0$ and
\[ w_k(x) = |x|^{2\gamma}, \quad x \in \mathbb{R}. \]

The Dunkl operators $T_j; j = 1, \ldots, d$, on $\mathbb{R}^d$ associated with the finite reflection group $G$ and multiplicity function $k$ are given for a function $f$ of class $C^1$ on $\mathbb{R}^d$, by
\[ T_j f(x) := \frac{\partial}{\partial x_j} f(x) + \sum_{\alpha \in \mathbb{R}_+} k(\alpha)\alpha_j \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle}. \]

In the case $k = 0$, the $T_j; j = 1, \ldots, d$, reduce to the corresponding partial derivatives. In this paper we will assume throughout that $k \geq 0$.

For $y \in \mathbb{R}^d$, the initial problem
\[ \begin{cases} T_j x u(x, y) = y_j u(x, y); & j = 1, \ldots, d, \\ u(0, y) = 1, \end{cases} \]
admits a unique analytic solution on $\mathbb{R}^d$, which will be denoted $E_k(x, y)$ and called the Dunkl kernel ([17, 18, 19] and [20]). This kernel has a unique holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d$.

**Examples.**

1) If $k = 0$, then $E_k(z, w) = e^{\langle z, w \rangle}$ for $z, w \in \mathbb{C}^d$. (Recall that $\langle ., . \rangle$ was defined to be bilinear on $\mathbb{C}^d \times \mathbb{C}^d$.)

2) If $d = 1$ and $G = \mathbb{Z}_2$, the Dunkl kernel is given by
\[ E_\gamma(z, w) = \Im\frac{1}{z^{\gamma+\frac{1}{2}}} \left( \Im\frac{1}{z^{\gamma-\frac{1}{2}}} + \Im\frac{1}{z^{\gamma+\frac{1}{2}}} \right), \]
where
\[ \Im\frac{1}{z^{\gamma+\frac{1}{2}}} = \Gamma\left( \gamma + \frac{1}{2} \right) \sum_{n=0}^{\infty} \frac{1}{n! \Gamma(n + \gamma + \frac{1}{2})} \left( \frac{z w}{2} \right)^{2n}, \]
is the modified Bessel function of order $\gamma - \frac{1}{2}$ [21].

Let $\mathcal{P} = \mathbb{C}[\mathbb{R}^d]$ denotes the $\mathbb{C}$- Algebra of polynomial functions on $\mathbb{R}^d$ and $\mathcal{P}_n, n \in \mathbb{N}$, the subspace of homogeneous polynomials of degree $n$. In the context of generalized spherical harmonics, C.F. Dunkl in [9] introduced on $\mathcal{P}$ the following bilinear form:
\[ (p, q)_k := \left( p(T)q \right)(0); \quad p, q \in \mathcal{P}. \]

Here $p(T)$ is the operator derived from $p(x)$ by replacing $x_i$ by $T_i$. A useful collection of its properties can be found in [9] and [17]. We recall that $(., .)_k$ is symmetric, positive-definite and $(p, q)_k = 0$, for $p \in \mathcal{P}_n, q \in \mathcal{P}_m$ with $n \neq m$. Moreover, for all $j = 1, \ldots, d$ and $p, q \in \mathcal{P},$
\[ (x_j p, q)_k = (p, T_j q)_k. \]
Let \( \{ \varphi_\nu \}_{\nu \in \mathbb{N}^d} \) be an orthonormal basis of \( P \) with respect to the scalar product \((.,.)_k\) such that \( \varphi_\nu \in P_{|\nu|} \) and the coefficients of the \( \varphi_\nu \) are real. As \( P = \bigoplus_{n \in \mathbb{N}} P_n \) and \( P_n \perp P_m \) for \( n \neq m \), the \( \varphi_\nu \) with \( |\nu| = n \) can for example be constructed by Gram-Schmidt orthogonalization within \( P_n \) from an arbitrary ordered real-coefficients basis of \( P_n \). If \( k = 0 \) the canonical choice of the homogeneous orthonormal basis \( \varphi_\nu \) is just \( \varphi_\nu(x) = \frac{x^\nu}{\sqrt{\nu!}} \).

As in the classical case, M. Rösler obtained in [17, p. 524] the following connection of the basis \( \varphi_\nu \) with the Dunkl kernel:

\[
E_k(z, w) = \sum_{\nu \in \mathbb{N}^d} \varphi_\nu(z) \varphi_\nu(w); \quad z, w \in \mathbb{C}^d,
\]

where the convergence is normal on \( \mathbb{C}^d \times \mathbb{C}^d \).

**Example.** If \( d = 1 \) and \( G = \mathbb{Z}_2 \) every homogeneous orthonormal basis is of the form

\[
\varphi_n(z) = \frac{z^n}{\sqrt{b_n(\gamma)}}, \quad b_n(\gamma) = \frac{2^n ([n/2]!) \Gamma \left( \frac{n + 1}{2} \right) \Gamma \left( \frac{n + 1 + \gamma}{2} + \frac{1}{2} \right)}{\Gamma(\gamma + 1)}.
\]

Here \([n/2]\) is the integer part of \( n/2 \).

From (2), the Dunkl kernel \( E_k \) possesses the following properties ([17, 19] and [20]): For all \( z, w \in \mathbb{C}^d \) and \( \lambda \in \mathbb{C} \),

\[
E_k(z, w) = E_k(w, z), \quad E_k(\lambda z, w) = E_k(z, \lambda w),
\]

\[
E_k(z, w) = E_k(z, \overline{w}), \quad E_k(z, \overline{z}) = \sum_{\nu \in \mathbb{N}^d} |\varphi_\nu(z)|^2,
\]

\[
|E_k(z, w)| \leq e^{||z||_2}.
\]

In [18], M. Rösler establish the Bochner-type representation of the Dunkl kernel

\[
E_k(x, z) = \int_{\mathbb{R}^d} e^{\langle \xi, z \rangle} d\mu_x(\xi); \quad x \in \mathbb{R}^d, \quad z \in \mathbb{C}^d,
\]

where \( \mu_x \) is a probability measure on \( \mathbb{R}^d \) with support in \( \{ \xi \in \mathbb{R}^d / |\xi| \leq |x| \} \).

The Dunkl kernel \( E_k \) is analytic on \( \mathbb{C}^d \times \mathbb{C}^d \). Therefore, there exist unique analytic functions \( m_\nu, \nu \in \mathbb{N}^d \), on \( \mathbb{C}^d \) with

\[
E_k(z, w) = \sum_{\nu \in \mathbb{N}^d} \frac{m_\nu(z)}{\nu!} w^\nu; \quad z, w \in \mathbb{C}^d.
\]

The restriction of \( m_\nu \) to \( \mathbb{R}^d \) are called the \( \nu \)-th moment functions ([18, 19] and [20]). It is given explicitly by

\[
m_\nu(x) = \int_{\mathbb{R}^d} \xi^\nu d\mu_x(\xi), \quad x \in \mathbb{R}^d,
\]

where \( \mu_x \) is the measure given by (7).
The functions $m_\nu$ are homogeneous polynomials of degree $|\nu|$. Among the applications of these moments, we mention the construction of martingales from Dunkl-type Markov processes [19].


In this section we define and study the generalized Fock space for the Dunkl kernel in $d$-dimensions.

**Definition 1.** The generalized Fock space $A_k$ associated with the Dunkl operators is the space of holomorphic functions $f$ on $\mathbb{C}^d$ which can be written

$$f(z) = \sum_{\nu \in \mathbb{N}_d} a_\nu \varphi_\nu(z)$$

with

$$\|f\|_k^2 := \sum_{\nu \in \mathbb{N}_d} |a_\nu|^2 < \infty.$$ 

Hence the inner product in $A_k$ is given for $f(z) = \sum_{\nu \in \mathbb{N}_d} a_\nu \varphi_\nu(z)$ in $A_k$ and $g(z) = \sum_{\nu \in \mathbb{N}_d} b_\nu \varphi_\nu(z) \in A_k$, by

$$(f,g)_k := \sum_{\nu \in \mathbb{N}_d} a_\nu \overline{b_\nu}.$$ 

**Remark.** If $k = 0$, $A_0$ is the ordinary Fock space $A$ [4].

**Proposition 1.**

i) If $f, g \in A_k$ with $f(z) = \sum_{\nu \in \mathbb{N}_d} a_\nu \varphi_\nu(z)$ and $g(z) = \sum_{\nu \in \mathbb{N}_d} b_\nu \varphi_\nu(z)$, we have

$$(f,g)_k = \left( f(T) \bar{g} \right)(0),$$

where $\bar{g}(z) = \sum_{\nu \in \mathbb{N}_d} \overline{b_\nu} \varphi_\nu(z)$.

ii) If $f \in A_k$ with $f(z) = \sum_{\nu \in \mathbb{N}_d} a_\nu \varphi_\nu(z)$, we have

$$|f(z)| \leq e^{||z||^2/2} \|f\|_k.$$ 

**Proof.** i) From [17, p. 529], we have

$$\left( \varphi_\nu(T) \bar{\varphi}_s \right)(0) = \delta_{\nu,s},$$

where $\delta_{\nu,s}$ is the Kronecker symbol.

Thus

$$(f,g)_k = \sum_{\nu,s \in \mathbb{N}_d} a_\nu \overline{b_s} \left( \varphi_\nu(T) \varphi_s \right)(0).$$

Using the continuously of the inner product, we obtain the result.

ii) Using Cauchy-Schwarz’s inequality, then

$$|f(z)|^2 \leq \left[ \sum_{\nu \in \mathbb{N}_d} |a_\nu|^2 \right] \left[ \sum_{\nu \in \mathbb{N}_d} |\varphi_\nu(z)|^2 \right] = \|f\|^2_k E_k(z, \bar{z}).$$
Thus
\[ |f(z)| \leq [E_k(z, z)]^{1/2} \|f\|_k. \]
The result follows from the inequality (6).

From Proposition 1 ii), the map \( f \to f(z), \ z \in \mathbb{C}^d \), is a continuous linear functional on \( A_k \). Thus from Riesz theorem [1], \( A_k \) has a reproducing kernel.

**Proposition 2.** The function \( K \) given for \( w, z \in \mathbb{C}^d \), by
\[ K(z, w) = E_k(z, w), \]
is a reproducing kernel for the generalized Fock spaces \( A_k \), that is:

i) For every \( w \in \mathbb{C}^d \), the function \( z \to K(z, w) \) belongs to \( A_k \).

ii) The reproducing property: For every \( w \in \mathbb{C}^d \) and \( f \in A_k \), we have
\[ (f, K(\cdot, w))_k = f(w). \]

**Proof.** i) Using (5) and (6), we deduce for \( w \in \mathbb{C}^d \),
\[ \|E_k(\cdot, w)\|_k^2 = \sum_{\nu \in \mathbb{N}^d} |\varphi_\nu(w)|^2 = E_k(w, w) \leq e^{|w|^2}, \]
which proves i).

ii) If \( f(z) = \sum_{\nu \in \mathbb{N}^d} a_\nu \varphi_\nu(z) \in A_k \), it follows from (9) that
\[ (f, E_k(\cdot, w))_k = \sum_{\nu \in \mathbb{N}^d} a_\nu \varphi_\nu(w) = f(w). \]

\[ \Box \]

**Corollary 1.**

i) The set \( \{E_k(\cdot, w), w \in \mathbb{C}^d\} \) is complete in \( A_k \).

ii) For all \( z, w \in \mathbb{C}^d \), we have
\[ E_k(z, w) = (E_k(\cdot, z), E_k(\cdot, w))_k. \]

iii) Let \( m \in \mathbb{N}\setminus\{0\} \) and \( z_1, z_2, \ldots, z_m \in \mathbb{C}^d \), with \( z_i \neq z_j \), then
\[ \det \left[ E_k(z_i, z_j) \right]_{i,j=1}^m > 0. \]

**Notation.** We denote by \( L^2(\mu_k) \) the Hilbert space of measurable functions on \( \mathbb{R}^d \), for which
\[ \|f\|_{2, k} := \left[ \int_{\mathbb{R}^d} |f(x)|^2 d\mu_k(x) \right]^{1/2} < \infty. \]

Here \( \mu_k \) is the measure defined on \( \mathbb{R}^d \), by
\[ d\mu_k(x) := c_k w_k(x) dx, \]
with \( c_k = \left( \int_{\mathbb{R}^d} e^{-|x|^2} d\mu_k(x) \right)^{-1} \),
is the Mehta-type constant.
In the next part of this section we establish the unitary equivalence of $L^2(\mu_k)$ and $A_k$. First we recall some properties of the generalized Hermite functions ([17] and [19]):

**Definition 2.** The generalized Hermite polynomials $\{H_\nu\}_{\nu \in \mathbb{N}^d}$ associated with the basis $\{\varphi_\nu\}_{\nu \in \mathbb{N}^d}$ on $\mathbb{C}^d$, are given by

$$H_\nu(z) := 2^{\nu} e^{-\frac{1}{4} \Delta_k} \varphi_\nu(z) = 2^{\nu} \sum_{n=0}^{[\nu/2]} (-1)^n \frac{n^n \Delta^n_k \varphi_\nu(z)}{2^{2n} n!},$$

where $\Delta_k = \sum_{i=1}^d T_i^2$ is the Dunkl Laplacian [17].

Moreover, we define the generalized Hermite functions on $\mathbb{C}^d$, by

$$h_\nu(z) := 2^{-\nu/2} e^{-\frac{1}{2} \ell(z)} H_\nu(z).$$

**Examples.**

1) If $k = 0$, we obtain

$$H_\nu(x) = \frac{2^{\nu} \prod_{i=1}^d e^{-\frac{1}{4} \partial_i^2} (x_i^{\nu_i})}{\sqrt{\nu!}} \prod_{i=1}^d \tilde{H}_{\nu_i}(x_i), \quad x \in \mathbb{R}^d,$$

where

$$\tilde{H}_{\nu_i} = (-1)^{\nu_i} e^{x_i^2} \frac{\partial^{\nu_i}}{\partial x_i^{\nu_i}} (e^{-x_i^2}).$$

2) If $d = 1$ and $G = \mathbb{Z}_2$, we obtain

$$H_n(z) = \sum_{i=0}^{[n/2]} (-1)^i b_{n-2i}(\gamma) \frac{(2x)^{n-2i}}{i!}, \quad x \in \mathbb{R},$$

where $b_n(\gamma)$ are the constants given by (3).

The following lemma is shown in [17, p. 525-529]:

**Lemma 1.**

i) The set $\{h_\nu\}_{\nu \in \mathbb{N}^d}$ is an orthonormal basis of $L^2(\mu_k)$.

ii) For all $z, w \in \mathbb{C}^d$, there is a generating function for the generalized Hermite polynomials,

$$e^{-\ell(w)} E_k(2z, w) = \sum_{\nu \in \mathbb{N}^d} h_\nu(z) \varphi_\nu(w).$$

**Notation.** We denote by $U_k$ the kernel given for $z, w \in \mathbb{C}^d$, by

$$U_k(z, w) := e^{-(\ell(z)+\ell(w))/2} E_k(\sqrt{2}z, w).$$

(10)

**Lemma 2.** For $w, z \in \mathbb{C}^d$, we have

$$U_k(z, w) = \sum_{\nu \in \mathbb{N}^d} h_\nu(z) \varphi_\nu(w).$$
Proof. From Definition 2, we have
\[ \sum_{\nu \in \mathbb{N}^d} h_{\nu}(z)\varphi_{\nu}(w) = e^{-\ell(z)/2} \sum_{\nu \in \mathbb{N}^d} 2^{-|\nu|/2} H_{\nu}(z)\varphi_{\nu}(w). \]
As \( \varphi_{\nu} \) is homogeneous of degree \( |\nu| \), then
\[ \varphi_{\nu}\left(\frac{w}{\sqrt{2}}\right) = 2^{-|\nu|/2} \varphi_{\nu}(w). \]
Thus
\[ \sum_{\nu \in \mathbb{N}^d} h_{\nu}(z)\varphi_{\nu}(w) = e^{-\ell(z)/2} \sum_{\nu \in \mathbb{N}^d} H_{\nu}(z)\varphi_{\nu}\left(\frac{w}{\sqrt{2}}\right). \]
Applying Lemma 1 ii) and (4), we obtain
\[ \sum_{\nu \in \mathbb{N}^d} h_{\nu}(z)\varphi_{\nu}(w) = e^{-\ell(z) + \ell(w)/2} E_k(2z, \frac{w}{\sqrt{2}}) = U_k(z, w). \]
\[ \square \]

Lemma 3.

i) For all \( z, w \in \mathbb{C}^d \), we have
\[ E_k(z, w) = \int_{\mathbb{R}^d} U_k(z, x)U_k(w, x)d\mu_k(x). \]

ii) For all \( z \in \mathbb{C}^d \), the function \( x \to U_k(z, x) \) belongs to \( L^2(\mu_k) \), and we have
\[ \|U_k(z, \cdot)\|_{2,k}^2 = E_k(z, z). \]

iii) For all \( x \in \mathbb{R}^d \), the function \( z \to U_k(z, x) \) belongs to \( A_k \), and we have
\[ \|U_k(\cdot, x)\|_{2,k}^2 = e^{-3|x|^2} E_k(2x, x). \]

Proof. i) We put
\[ I = \int_{\mathbb{R}^d} U_k(z, x)U_k(w, x)d\mu_k(x). \]
From (10), we have
\[ I = e^{-(\ell(z) + \ell(w))/2} \int_{\mathbb{R}^d} e^{-|x|^2/2} E_k(\sqrt{2}z, x)E_k(\sqrt{2}w, x)d\mu_k(x). \]
So from [17, p. 523] and (4), we get
\[ \int_{\mathbb{R}^d} e^{-|x|^2/2} E_k(\sqrt{2}z, x)E_k(\sqrt{2}w, x)d\mu_k(x) = e^{(\ell(z) + \ell(w))/2} E_k(z, w), \]
which proves i).

ii) This assertion follows from i) and (5).

iii) For \( z \in \mathbb{C}^d \), we put
\[ \phi(z) := e^{-\ell(z)/2}. \]
Let \( x \in \mathbb{R}^d \), then from Proposition 2 ii), (10) and (4), we have
\[
\|U_k(\cdot, x)\|_k^2 = e^{-|x|^2} (\phi(\cdot)E_k(\cdot, \sqrt{2}x), E_k(\cdot, \sqrt{2}x))_k = e^{-3|x|^2} E_k(2x, x).
\]

**Definition 3.** The chaotic transform \( C_k \) (also called \( S \)-transform in the stochastic calculus [15]) is the transformation defined on \( L^2(\mu_k) \), by
\[
C_k(f)(z) := \int_{\mathbb{R}^d} U_k(z, x)f(x)d\mu_k(x), \quad z \in \mathbb{C}^d.
\]

**Remark.** The basis elements of \( L^2(\mu_k) \) and \( \mathcal{A}_k \) are called chaos. In the following theorem we shall prove that the transformation \( C_k \) maps the chaos of \( L^2(\mu_k) \) to these of \( \mathcal{A}_k \).

**Theorem 1.** The chaotic transform \( C_k \) is a unitary mapping of \( L^2(\mu_k) \) on \( \mathcal{A}_k \). Moreover, the basis elements are related by
\[
C_k(h_\nu) = \varphi_\nu.
\]

**Proof.** It follows directly from Lemma 1 i) and Lemma 2, that for \( \nu \in \mathbb{N}^d \),
\[
C(h_\nu)(z) = \int_{\mathbb{R}^d} U_k(z, x)h_\nu(x)d\mu_k(x) = \varphi_\nu(z), \quad z \in \mathbb{C}^d.
\]

Consequently \( C_k \) maps the subspace generated by the family \( \{h_\nu\}_{\nu \in \mathbb{N}^d} \) into the polynomials in \( \mathcal{A}_k \). Thus \( C_k \) maps a dense set in \( L^2(\mu_k) \) into a dense set in \( \mathcal{A}_k \). Further, if \( f \in L^2(\mu_k) \), then \( f(z) = \sum_{\nu \in \mathbb{N}^d} a_\nu h_\nu(x) \). For \( \nu \in \mathbb{N}^d \), let \( f_N(x) = \sum_{j=0}^{N} |\nu|_j a_\nu h_\nu(x), \quad x \in \mathbb{R} \). Then
\[
C_k(f_N)(z) = \sum_{j=0}^{N} \sum_{|\nu|_j = j} a_\nu \varphi_\nu(z); \quad \lim_{N \to \infty} \|f - f_N\|_{2,k} = 0.
\]

On the other hand, from Hölder’s inequality and Lemma 3 ii), we have
\[
|C_k(f - f_N)(z)| \leq |E_k(z, z)|^{1/2}\|f - f_N\|_{2,k}.
\]

Thus we obtain
\[
C_k(f)(z) = \sum_{\nu \in \mathbb{N}^d} a_\nu \varphi_\nu(z).
\]

Hence
\[
\|C_k(f)\|_k^2 = \sum_{\nu \in \mathbb{N}^d} |a_\nu|^2 = \|f\|_{2,k}^2.
\]

It follows that \( C_k \) is a unitary transformation from \( L^2(\mu_k) \) into \( \mathcal{A}_k \).

Clearly, if \( g(z) = \sum_{\nu \in \mathbb{N}^d} a_\nu \varphi_\nu(z) \in \mathcal{A}_k \), we have
\[
(11) \quad C_k^{-1}(g)(x) = \sum_{\nu \in \mathbb{N}^d} a_\nu h_\nu(x), \quad x \in \mathbb{R}^d.
\]

Which completes the proof. \( \Box \)
Proposition 3. If \( g \in \mathcal{A}_k \), we have
\[
\mathcal{C}^{-1}_k(g)(x) = (g, U_k(\cdot, x))_k, \quad x \in \mathbb{R}^d.
\]

Proof. Let \( g \in \mathcal{A}_k \). We put for \( x \in \mathbb{R}^d \),
\[
\Psi_k(g)(x) = (g, U_k(\cdot, x))_k.
\]
Using Lemma 2, Lemma 3 iii) and the same method as in the proof of Theorem 1 we obtain
\[
\Psi_k(g)(x) = \sum_{\nu \in \mathbb{N}^d} a_\nu h_\nu(x) = \mathcal{C}^{-1}_k(g)(x), \quad x \in \mathbb{R}^d.
\]
\[\square\]


We define the multiplication operators \( Q_i; i = 1, \ldots, d \) on \( \mathcal{A}_k \) by
\[
Q_i f(z) := z_i f(z), \quad z \in \mathbb{C}^d.
\]
We denote also by \( T_i; i = 1, \ldots, d \) the operators defined on \( \mathcal{A}_k \).
Let
\[
\mathcal{D}(Q_i) = \{ f \in \mathcal{A}_k / Q_i(f) \in \mathcal{A}_k \},
\]
\[
\mathcal{D}(T_i) = \{ f \in \mathcal{A}_k / T_i f \in \mathcal{A}_k \}
\]
denote the domains of \( Q_i \) and \( T_i \) respectively.

We denote by \([,]\) the commutator product \([A, B] = AB - BA\). As in [11], we have the following relations:

Lemma 4. The operators \( Q_i \) and \( T_i; i = 1, \ldots, d \) satisfy on \( \mathcal{A}_k \) the commutation relations:
\[
[T_i, T_j] = [Q_i, Q_j] = 0; \quad i, j = 1, \ldots, d,
\]
\[
[T_i, Q_j] = \delta_{i,j} I + \sum_{\alpha \in R_+} k(\alpha) \alpha_i \alpha_j B_\alpha; \quad i, j = 1, \ldots, d,
\]
where \( I \) the identity operator and \( B_\alpha \) is the reflection operator \((B_\alpha^2 = I)\) given by
\[
B_\alpha f(z) = f(\sigma_\alpha z).
\]

Proof. Using the fact that \( \sigma_\alpha^2 = id \) and \( \langle \alpha, \sigma_\alpha z \rangle = -\langle \alpha, z \rangle \), we obtain
\[
T_i T_j f(z) = T_i \left( \frac{\partial}{\partial z_j} f \right)(z) + \sum_{\alpha \in R_+} k(\alpha) \alpha_j \frac{\partial}{\partial z_i} \left( \frac{f(z) - f(\sigma_\alpha z)}{\langle \alpha, z \rangle} \right).
\]
Since
\[
\frac{\partial}{\partial z_i} (f(\sigma_\alpha z)) = \frac{\partial}{\partial z_i} f(\sigma_\alpha z) - \sum_{\ell=1}^d \alpha_i \alpha_\ell \frac{\partial}{\partial z_\ell} f(\sigma_\alpha z),
\]

we have
\[ T_i T_j f(z) = -\frac{\partial^2}{\partial z_i \partial z_j} f(z) + T_i \left( \frac{\partial}{\partial z_j} f \right) (z) + T_j \left( \frac{\partial}{\partial z_i} f \right) (z) \]
\[ - \sum_{\alpha \in R_+} k(\alpha) \alpha_i \alpha_j \left[ \frac{f(z) - f(\sigma_\alpha z)}{(\langle \alpha, z \rangle)^2} - \sum_{\ell=1}^d \alpha_\ell \frac{\partial}{\partial z_\ell} f(\sigma_\alpha z) \right]. \]
Thus
\[ [T_i, T_j] f(z) = 0. \]
The other relations are evident. □

**Proposition 4.** Let
\[ f(z) = \sum_{\nu \in \mathbb{N}^d} a_\nu \varphi_\nu(z) \in D(Q_i) \quad \text{and} \quad g(z) = \sum_{\nu \in \mathbb{N}^d} a_\nu \varphi_\nu(z) \in D(T_i), \]
then
\[ (Q_i f, g)_k = (f, T_i g)_k. \]

**Proof.** Applying Proposition 1 i), we get
\[ (Q_i f, g)_k = (Q_i f(T)g)(0) = \sum_{\nu, s \in \mathbb{N}^d} a_\nu b_\nu T_i \varphi_\nu(T) \varphi_s(0). \]
Then from (12) we obtain
\[ (Q_i f, g)_k = \sum_{\nu, s \in \mathbb{N}^d} a_\nu b_\nu T_i \varphi_\nu(T) \varphi_s(0) = (f, T_i g)_k. \]
□

**Lemma 5.** If \( f \in A_k \), then \( B_\alpha f \in A_k \), and we have
\[ \| Q_i f \|_k^2 = \| T_i f \|_k^2 + \| f \|_k^2 + \sum_{\alpha \in R_+} k(\alpha) \alpha_i^2 (f, B_\alpha f)_k, \]
where \( B_\alpha \) is the operator given by (14).

**Proof.** Let \( f \in A_k \). Applying the chaotic transform, in view of Theorem 1, it suffices to show that \( C_k^{-1}(B_\alpha f) \in L^2(\mu_k) \). From (11), we have
\[ C_k^{-1}(B_\alpha f)(x) = C_k^{-1}(f)(\sigma_\alpha x), \quad x \in \mathbb{R}^d. \]
Putting \( u = \sigma_\alpha x \), we get
\[ d\mu_k(x) = |J_\alpha| d\mu_k(u) \quad \text{where} \quad J_\alpha = \det \left[ \delta_{i,j} - \alpha_i \alpha_j \right]_{i,j=1}^d. \]
Since \( J_\alpha = -1 \), we obtain
\[ \| C_k^{-1}(B_\alpha f) \|_{2,k}^2 = \int_{\mathbb{R}^d} |C_k^{-1}(f)(u)|^2 d\mu_k(u). \]
Which proves that $B_{\alpha} f \in \mathcal{A}_k$.

On the other hand, from Proposition 4, we deduce

$$\|Q_i f\|^2_k = (f, T_i Q_i f)_k.$$ 

But from (13), we have

$$T_i Q_i f = Q_i T_i f + f + \sum_{\alpha \in \mathbb{R}_+} k(\alpha) \alpha_i^2 B_{\alpha} f.$$ 

Thus

$$\|Q_i f\|^2_k = (f, T_i Q_i f)_k + \|f\|^2_k + \sum_{\alpha \in \mathbb{R}_+} k(\alpha) \alpha_i^2 (f, B_{\alpha} f)_k.$$ 

Using another time Proposition 4, we obtain the result. □

**Proposition 5.** The operators $Q_i$ and $T_i$ are closed densely defined operators on $\mathcal{A}_k$, and we have

$$\mathcal{D}(Q_i) = \mathcal{D}(T_i); \quad Q_i^* = T_i; \quad T_i^* = Q_i,$$

where $Q_i^*$ and $T_i^*$ are the adjoints operators of $Q_i$ and $T_i$, respectively.

**Proof.** These results follow from [4, Theorem 1.2], Lemma 5 and Proposition 4 by using the same method as [21, Proposition 6]. □

**Lemma 6.** For $\nu \in \mathbb{N}_d \setminus \{0\}$, we have the following relations:

i) 

$$[T^\nu, Q_j] = \nu_j T_1^{\nu_1} \cdots T_{i-1}^{\nu_{i-1}} T_i^{\nu_i} T_{i+1}^{\nu_{i+1}} \cdots T_d^{\nu_d}$$

$$+ B_{\alpha} \sum_{i=1}^{d} \sum_{\ell=0}^{\nu_i} \sum_{\alpha \in \mathbb{R}_+} k(\alpha) \alpha_i \alpha_j H_1^{\nu_1} \cdots H_{i-1}^{\nu_{i-1}} H_i^{\nu_i} T_{i+1}^{\nu_{i+1}} \cdots T_d^{\nu_d},$$

where $H_i; \ i = 1, \ldots, d$, are given by the differential-difference operators

$$H_i = -T_i + 2 \frac{\partial}{\partial x_i} - \sum_{\ell=1}^{d} \alpha_i \alpha_\ell \frac{\partial}{\partial x_\ell}.$$ 

ii) 

$$[T_j, Q^\nu] = \nu_j Q_1^{\nu_1} \cdots Q_{i-1}^{\nu_{i-1}} Q_i^{\nu_i} Q_{i+1}^{\nu_{i+1}} \cdots Q_d^{\nu_d}$$

$$+ B_{\alpha} \sum_{i=1}^{d} \sum_{\ell=0}^{\nu_i} \sum_{\alpha \in \mathbb{R}_+} k(\alpha) \alpha_i \alpha_j Z_1^{\nu_1} \cdots Z_{i-1}^{\nu_{i-1}} Z_i^{\nu_i} Z_{i+1}^{\nu_{i+1}} Q_{i+1}^{\nu_{i+1}} \cdots Q_d^{\nu_d},$$

where $Z_i; \ i = 1, \ldots, d$, are the multiplication operators given by

$$Z_i = Q_i - \sum_{\ell=1}^{d} \alpha_i \alpha_\ell Q_\ell.$$
Proof. From (13), we have
\[
[T^\nu_i, Q_j] = \sum_{\ell=0}^{\nu_i-1} T^\ell_i [T_i, Q_j] T^{\nu_i-\ell-1}_i
\]
\[
= \nu_i \delta_{ij} T^\nu_i \quad + \sum_{\ell=0}^{\nu_i-1} k(\alpha) \alpha_i \alpha_j T^\ell_i B_\alpha T^{\nu_i-\ell-1}_i.
\]
From this equality, we get
\[
[T^\nu, Q_j] = \sum_{i=1}^{d} T^\nu_i \ldots T^\nu_{i-1} [T^\nu_i, Q_j] T^\nu_{i+1} \ldots T^\nu_d
\]
\[
= \nu_j T^\nu_1 \ldots T^\nu_{i-1} T^\nu_i \ldots T^\nu_{i+1} T^\nu_{i+1} \ldots T^\nu_d
\]
\[
+ \sum_{\ell=0}^{\nu_i-1} k(\alpha) \alpha_i \alpha_j T^\nu_i \ldots T^\nu_{i-1} T^\nu_i B_\alpha T^{\nu_i-\ell-1}_i T^{\nu_i+1}_{i+1} \ldots T^\nu_d.
\]
But
\[
T^\nu_i B_\alpha = B_\alpha H^\nu_i,
\]
where
\[
H_i = -T_i + 2\frac{\partial}{\partial x_i} - \sum_{\ell=1}^{d} \alpha_i \alpha_\ell \frac{\partial}{\partial x_\ell}.
\]
Thus we obtain Assertion i). And similarly we get ii). \(\square\)

Notation. For \(x \in \mathbb{R}^d\) and \(z \in \mathbb{C}^d\), we denote by
\[
I_k(z, x) := \frac{E_k(z, x) - E_k(z, \sigma_\alpha x)}{\langle \alpha, x \rangle}.
\]
From [17, p. 533], we can write the function \(I_k(z, x)\) in the form
\[
I_k(z, x) = \langle \nabla_x E_k(z, x), \alpha \rangle + \frac{1}{2} \langle \alpha, x \rangle \alpha^t D^2_x E_k(z, \xi) \alpha,
\]
with some \(\xi\) on the line segment between \(x\) and \(\sigma_\alpha x\).
(Here \(\nabla\) and \(D^2 f(x)\) denote the usual gradient and Hessian of \(f\) in \(x\).)

Lemma 7. For \(a, b \in \mathbb{C}^d\), we have the following commutation relations:

i) \([E_k(a, T), Q_j] = a_j E_k(a, T) - R_{k, j}(a, T),\) where
\[
R_{k, j}(a, T) = \sum_{\alpha \in \mathbb{R}^+} k(\alpha) \alpha_j I_k(a, T)
\]
\[
- B_\alpha \sum_{\nu_i \in \mathbb{N}^d} \sum_{i=1}^{d} \sum_{\ell=0}^{\nu_i-1} k(\alpha) \alpha_i \alpha_j \frac{m_\alpha(a)}{\nu!} H^\nu_1 \ldots H^\nu_{i-1}_i H^\nu_i T^{\nu_i-\ell-1}_i T^{\nu_i+1}_{i+1} \ldots T^\nu_d.
\]
ii) \( [T_j, E_k(b, Q)] = b_j E_k(b, Q) - S_{k,j}(b, Q) \), where
\[
S_{k,j}(b, Q) = \sum_{\alpha \in R^+} k(\alpha) \alpha_j I_k(b, Q)
- B_\alpha \sum_{\nu \in \mathbb{N}^d} \sum_{i=1}^d \sum_{\ell=0}^{\nu_i-1} k(\alpha) \alpha_i \alpha_j \frac{m_\nu(b)}{\nu!} Z_1^{\nu_1} \cdots Z_{i-1}^{\nu_{i-1}} Z_i^{\nu_i-\ell-1} Q_i^{\nu_{i+1}} \cdots Q_d^{\nu_d}.
\]

Proof. Using (8) and Lemma 6 i), we obtain
\[
[E_k(a, T), Q_j] = \sum_{\nu \in \mathbb{N}^d} \frac{m_\nu(a)}{\nu!} [T^\nu, Q_j]
= \sum_{\nu \in \mathbb{N}^d} \frac{m_\nu(a)}{\nu!} \nu_j T_1^{\nu_1} \cdots T_d^{\nu_d}
+ B_\alpha \sum_{\nu \in \mathbb{N}^d} \sum_{i=1}^d \sum_{\ell=0}^{\nu_i-1} k(\alpha) \alpha_i \alpha_j \frac{m_\nu(a)}{\nu!} H_1^{\nu_1} \cdots H_{i-1}^{\nu_{i-1}} H_i^\ell T_i^{\nu_i-\ell-1} T_{i+1}^{\nu_{i+1}} \cdots T_d^{\nu_d}.
\]

Applying the relation
\[
\frac{\partial}{\partial w_j} E_k(z, w) = z_j E_k(z, w) - \sum_{\alpha \in R^+} k(\alpha) \alpha_j I_k(z, w); \quad z, w \in \mathbb{C}^d,
\]
we obtain
\[
[E_k(a, T), Q_j] = a_j E_k(a, T) - R_{k,j}(a, T).
\]
This proves i). Similarly, we can prove ii). \( \square \)

Remark. If \( d = 1 \) and \( G = \mathbb{Z}_2 \) \([21]\), we have
\[
R_\gamma(a, T_\gamma) = \frac{2\gamma}{2\gamma + 1} a(T_\gamma - I) \mathfrak{Z}_{\gamma+\frac{1}{2}}(aT_\gamma),
\]
\[
S_\gamma(b, Q) = \frac{2\gamma}{2\gamma + 1} b(T_\gamma - I) \mathfrak{Z}_{\gamma+\frac{1}{2}}(bQ),
\]
where \( B f(x) = f(-x) \).

Since \( E_k(a, 0) = 1 \), the Dunkl kernel \( E_k(a, z); a, z \in \mathbb{C}^d \), is a unit in the integral domain formal power series over \( \mathbb{C}^d \). We define
\[
E_k^{-1}(a, z) := \sum_{\nu \in \mathbb{N}^d} \frac{t_\nu(a)}{\nu!} z^\nu.
\]

Writing
\[
E_k(a, z) E_k^{-1}(a, z) = E_k^{-1}(a, z) E_k(a, z) = 1,
\]
we obtain
\[
t_0(a) = 1, \sum_{\nu \in \mathbb{N}^d} \left\{ \sum_{s \leq \nu} \binom{\nu}{s} m_{\nu-s}(a) t_s(a) \right\} \frac{z^{\nu}}{\nu!} = 1.
\]
Thus \(\{t_\nu(a)\}_{\nu \in \mathbb{N}^d}\) is a sequence of moment functions in \(a\) determined by
\[
t_0(a) = 1, \ t_\nu(a) = -\sum_{s \leq \nu-1} \binom{\nu}{s} m_{\nu-s}(a) t_s(a).
\]

The function \(E_k^{-1}(a, z)\) occurs in the generalized Weyl commutation relations for the Dunkl kernel.

**Theorem 2.** Let \(a, b \in \mathbb{C}^d\), then:

i) \(E_k(b, Q) E_k(a, T) = E_k(a, T) E_k(b, P_a), \ P_a = (P_{a,1}, \ldots, P_{a,d}), \) where
\[
P_{a,j} = Q_j - a_j I + E_k^{-1}(a, T) R_{k,j}(a, T).
\]

ii) \(E_k(a, T) E_k(b, Q) = E_k(b, Q) E_k(a, L_b), \ L_b = (L_{b,1}, \ldots, L_{b,d}), \) where
\[
L_{b,j} = T_j + b_j I - E_k^{-1}(b, Q) S_{k,j}(b, Q).
\]

iii) \(E_k(a, Q) E_k(b, Q) = E_k(a\#b, Q), \ E_k(a, T) E_k(b, T) = E_k(a\#b, T), \)

where \(a\#b\) is the convolution of \(a\) and \(b\) given by
\[
m_\nu(a\#b) = \sum_{s \leq \nu} \binom{\nu}{s} m_s(a)m_{\nu-s}(b).
\]

**Proof.** We shall prove i), ii) follows in the same way. For \(j = 1, 2, \ldots, d\), we have
\[
E_k^{-1}(a, T) Q_j E_k(a, T) = E_k^{-1}(a, T) \left\{ E_k(a, T) Q_j - [E_k(a, T), Q_j] \right\}.
\]
Using Lemma 7 i), we obtain
\[
E_k^{-1}(a, T) Q_j E_k(a, T) = Q_j - a_j I + E_k^{-1}(a, T) R_{k,j}(a, T).
\]
Thus implies that for \(\nu \in \mathbb{N}^d:\)
\[
E_k^{-1}(a, T) Q^\nu E_k(a, T) = P^\nu_{a,1}, \ P_a = (P_{a,1}, \ldots, P_{a,d}), \)
where
\[
P_{a,j} = Q_j - a_j I + E_k^{-1}(a, T) R_{k,j}(a, T).
\]
Multiplying by \(\frac{m_\nu(b)}{\nu!}\) and summing, we get
\[
E_k^{-1}(a, T) E_k(b, Q) E_k(a, T) = E_k(b, P_a).
\]
Then i) follows upon multiplication by \(E_k(a, T)\).

iii) It suffices to prove the first relation.
Using (8) and (12), we can write
\[
E_k(a, Q)E_k(b, Q) = \sum_{\nu, s \in \mathbb{N}^d} \frac{m_{\nu}(a)m_s(b)}{\nu! s!} Q^{\nu+s}
\]
\[
= \sum_{\nu \in \mathbb{N}^d} \left\{ \sum_{s \leq \nu} \left( \begin{array}{c} \nu \\ s \end{array} \right) m_s(a)m_{\nu-s}(b) \right\} \frac{Q^\nu}{\nu!}
\]
\[
= \sum_{\nu \in \mathbb{N}^d} \frac{m_{\nu}(a\#b)}{\nu!} Q^\nu.
\]
Thus we obtain
\[
E_k(a, Q)E_k(b, Q) = E_k(a\#b, Q).
\]
\[\square\]

**Remarks.**

1) In the classical case \((k = 0)\) [13, p. 223], the Weyl commutation relations are given by
\[
e^{(a,P)}e^{(b,Q)} = e^{(a,b)}e^{(b,Q)}e^{(a,P)},
\]
\[
e^{(a,P)}e^{(b,P)} = e^{(a+b,P)},
\]
\[
e^{(a,Q)}e^{(b,Q)} = e^{(a+b,Q)},
\]
where \(P = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d})\) and \(Q = (Q_1, \ldots, Q_d)\).

2) If \(d = 1\) and \(G = \mathbb{Z}_2\) [21], the Weyl commutation relations are given by
\[
E_\gamma(bQ)E_\gamma(aT_\gamma) = E_\gamma(aT_\gamma)E_\gamma(bP_a);
\]
\[
E_\gamma(aT_\gamma)E_\gamma(bQ) = E_\gamma(bQ)E_\gamma(aL_b),
\]
where
\[
P_a = Q - aI + \frac{2\gamma}{2\gamma + 1} aE_\gamma^{-1}(aT_\gamma)(I - B)\mathbb{S}_{\alpha+1}(aT_\gamma),
\]
and
\[
L_b = T_\gamma + bI - \frac{2\gamma}{2\gamma + 1} bE_\gamma^{-1}(bQ)(I - B)\mathbb{S}_{\alpha+1}(bQ).
\]

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References


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