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ON CERTAIN DIRICHLET SERIES BUILT FROM THE FOURIER COEFFICIENTS OF MODULAR FUNCTIONS

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In this paper we give a new construction of Dirichlet series — built from the q-product expansion of certain modular functions — which have the line $\Re(s) = 0$ as natural border of analyticity.

1. Introduction

Dirichlet series of the form

$$\sum_{m\geq 1} a(m)m^{-s} \qquad (\sigma := \Re(s) \gg 0)$$

where $a(m)$ is a multiplicative function and $a(\ell^n)$ (ℓ a prime, $n \geq 0$) depends only on the exponent n, are quite common in number theory.

Classical examples — with $a(m)$ being an elementary number-theoretic function — (after the Riemann zeta function $\zeta(s)$ or its inverse) include

$$\sum_{m\geq 1} \sigma_0(m)m^{-s} = \zeta(s)^2 \qquad (\sigma > 1)$$

and

$$\sum_{m\geq 1} 2^{\omega(m)}m^{-s} = \frac{\zeta(s)^2}{\zeta(2s)} \qquad (\sigma > 1),$$

where $\sigma_0(m)$ resp. $\omega(m)$ is the number of positive divisors resp. the number of distinct prime divisors of m.

A somewhat more intrinsic example is the series

$$D(s) = \sum_{m\geq 1} g(m)m^{-s} \qquad (\sigma > 1)$$

where $g(m)$ is the number of non-isomorphic abelian groups of order m. Elementary considerations show that $g(\ell^n) = p(n)$ (ℓ a prime, $n \geq 0$) where $p(n)$ is the number of partitions of n, with the convention $p(0) = 1$. Therefore — using the well-known generating series

$$\sum_{n\geq 0} p(n)q^n = \prod_{n\geq 1}(1 - q^n)^{-1}$$

— one finds that

$$D(s) = \prod_{n\geq 1} \zeta(ns) \qquad (\sigma > 1). \tag{1}$$

As is easy to see, the product on the right-hand side of (1) has a meromorphic continuation to $\sigma > 0$ but cannot be meromorphically continued into a neighborhood of $s_0 = 0$.

We shall show that the above example generalizes to a larger class of Dirichlet series whose coefficients are built from the Fourier coefficients of certain modular functions. Note that

$$\prod_{n\geq 1} (1-q^n)^{-1} = q^{1/24} \frac{1}{\eta(z)}$$

where $\eta(z)$ is the classical Dedekind eta function of weight $\frac{1}{2}$; here $z \in \mathcal{H}$, the complex upper half-plane and $q = e^{2\pi i z}$.

To formulate our result, let f be a nonconstant meromorphic modular function of rational weight k with multiplier system v of finite order on a subgroup Γ of finite index in $\Gamma_1 := SL_2(\mathbf{Z})$. Write

$$f(z) = \sum_{n\geq h} a(n) q_M^n \qquad (z \in \mathcal{H})$$

where $q_M = e^{2\pi i z/M}$, M is the least positive integer with $\begin{pmatrix} 1 & M \\ 0 & 1 \end{pmatrix} \in \Gamma$ and h is the order of f at infinity. Suppose that f is normalized, i.e., $a(h) = 1$.

For a prime power ℓ^n $(n \geq 0)$ put

$$A(\ell^n) := a(n+h)$$

and then extend the function A multiplicatively to the whole of $\mathbf{N}$. Define a formal Dirichlet series by

$$D_f(s) := \sum_{m\geq 1} A(m) m^{-s}.$$

Theorem. *Let f be as above and suppose in addition that $a(n)$ is an integer for all n, $a(h+1) > 0$ and f has no zeros or poles on $\mathcal{H}$. Then one has:*

i) *$D_f(s)$ has abscissa of convergence equal to* 1.

ii) *$D_f(s)$ has a meromorphic continuation to $\sigma > 0$, but cannot be meromorphically continued into a neighborhood of $s_0 = 0$.*

Note that in [**5**, **6**] and [**7**] Kurokawa using his theory of generalized Euler products gave several examples of Dirichlet series (mainly attached to Galois representations) that have the line $\sigma = 0$ as a natural boundary of meromorphicity. His examples are different from the above. The proof of ii), however is similar to the proof of the corresponding statement in [**5**] and is based on the fact that the relevant Dirichlet series can be represented as an infinite weighted product of elementary zeta functions (in our case just the Riemann zeta function $\zeta(s)$).

2. Proof of Theorem

We shall first prove that $D_f(s)$ is (absolutely) convergent for $\sigma > 1$.

The usual discriminant function

$$\Delta = \eta^{24} = q \prod_{n\geq 1}(1-q^n)^{24}$$

has weight 12 and is on Γ_1. Since Γ has only finitely many cusps and f has no poles on $\mathcal{H}$ by our assumption, one can choose a large positive integer N such that $g := \Delta^N f$ is a holomorphic cusp form of weight say $k_1 \geq 2$ with multiplier system v on Γ. By Hecke's classical estimate, the n-th Fourier coefficient of g is $\ll_f n^{k_1/2}$.

The coefficients $p(n)$ of $\prod_{n\geq 1}(1-q^n)^{-1}$ satisfy $p(n) \leq e^{C\sqrt{n}}$ where $C > 0$ is an absolute constant (in fact, as is well-known one has the more precise result $p(n) \sim \frac{e^{K\sqrt{n}}}{4n\sqrt{3}}$ as $n \to \infty$ where $K = \pi\sqrt{2/3}$, cf. e.g., [**1**, Chap. 5.1]).

Therefore we easily see that the Fourier coefficients $a(n)$ of $f = g\Delta^{-N}$ satisfy $|a(n)| \leq e^{C_1\sqrt{n-h}}$, where $C_1 > 0$ is a constant depending on f.

Let $\epsilon > 0$. If $n \in \mathbf{N}$ and we denote by $n = \ell_1^{n_1} \dots \ell_r^{n_r}$ the canonical prime factorization of n, then it follows for almost all n that

$$\begin{aligned} |A(n)| &= |a(n_1+h)| \cdot \dots \cdot |a(n_r+h)| \\ &\leq e^{C_1(\sum_{\nu=1}^r \sqrt{n_\nu})} \\ &\leq e^{C_1(\sum_{\nu=1}^r \log \ell_\nu^{\epsilon n_\nu})} \\ &= n^{C_1\epsilon}. \end{aligned}$$

Hence $D_f(s)$ is absolutely convergent for $\sigma > 1$. Like any nonzero, normalized, M-periodic and meromorphic function on $\mathcal{H} \cup \{\infty\}$, f has a product expansion

$$f(z) = q_M^h \prod_{n\geq 1}(1-q_M^n)^{c(n)} \tag{2}$$

where the right-hand side of (2) is absolutely uniformly convergent in a small ϵ-neighborhood of $q_M = 0$ and the $c(n)$ are uniquely determined complex numbers ([**2**] and [**3**]). Here we understand that the complex powers are defined by the principal branch of the complex logarithm log. Furthermore, since by assumption $a(n) \in \mathbf{Z}$ for all n, it follows that $c(n) \in \mathbf{Z}$ for all n [**3**].

Since f by assumption has no zeros or poles on $\mathcal{H}$, we infer from [**4**] that in fact identity (2) holds in $0 < |q_M| < 1$ and

$$c(n) \ll_{\epsilon,f} n^{\epsilon} \tag{3}$$

for all $\epsilon > 0$. In fact, in [**4**] this was proved under the additional assumption that k is integral and $v = 1$, but the general case follows by taking appropriate powers.

We now claim that

$$D_f(s) = \prod_{n\geq 1} \zeta(ns)^{-c(n)} \qquad (\sigma > 1) \tag{4}$$

and that the product on the right-hand side of (4) has a meromorphic continuation to $\sigma > 0$. We shall in fact prove that given $\delta > 0$, then there exists $N = N(\delta) \in \mathbf{N}$ such that the product

$$\prod_{\ell \text{ prime}, n\geq N} (1 - \ell^{-ns})^{c(n)}$$

is absolutely uniformly convergent in $\sigma \geq \delta$. In particular then (4) follows by formally putting $q_M = \ell^{-s}$ ($\sigma > 1$) for each prime ℓ in the infinite product on the right-hand side of (2) and then multiplying these absolutely convergent products for all ℓ. To prove the desired convergence, we must show that

$$\sum_{\ell \text{ prime}, n\geq N} c(n) \log(1 - \ell^{-ns}) \tag{5}$$

is absolutely uniformly convergent for $\sigma \geq \delta$. Now

$$|\log(1 - \ell^{-ns})| \ll \ell^{-n\sigma} \qquad (\sigma \geq \delta, n \gg_{\delta} 0)$$

and hence from (3) we find that the sum of the absolute terms of the sum in (5) for $\sigma \geq \delta$ and N large is bounded by

$$\begin{aligned}
&\ll_{\epsilon,f} \sum_{\ell \text{ prime}, n\geq N} n^{\epsilon} \ell^{-n\sigma} \\
&\ll_{\epsilon,f} \sum_{\ell \text{ prime}, n\geq N} \ell^{-n\delta/2} \\
&\ll_{\epsilon,f} \sum_{\ell \text{ prime}} \ell^{-N\delta/2} (1 - \ell^{-\delta/2})^{-1} \\
&\ll_{\epsilon,\delta,f} \sum_{\ell \text{ prime}} \ell^{-N\delta/2} \\
&\ll_{\epsilon,\delta,f} 1.
\end{aligned}$$

This proves our claim.

We still must prove that $D_f(s)$ has abscissa of convergence equal to 1 and cannot be meromorphically continued to $s_0 = 0$.

The first claim follows from the assumption that $a(h+1) > 0$. Indeed, from (2) we find that $c(1) = -a(h+1)$, hence by (4), $D_f(s)$ has a pole at $s = 1$. The second claim is clear. Indeed, first note that there are infinitely many n with $c(n) \neq 0$, since otherwise f would reduce to a rational function in q_M, hence would be constant which contradicts our assumption. If there are infinitely many n with $c(n) < 0$, then by (4), $D_f(s)$ has a pole at $s = \frac{1}{n}$ for those n and $\frac{1}{n} \to 0$ $(n \to \infty)$. Suppose that there are infinitely many n with $c(n) > 0$. Choose a zero $s_1 = \frac{1}{2} + it_1$ on the critical line of $\zeta(s)$ (as is well-known there are infinitely many such zeros). For those n then by (4), $D_f(s)$ has a pole at $s = \frac{1}{2n} + \frac{it_1}{n} \to 0$ $(n \to \infty)$, which proves the second claim.

Our Theorem is therefore proved.

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