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**To each Levi subgroup of a general linear group there corresponds a set of general linear groups of smaller order. One may therefore construct an irreducible representation of such a Levi subgroup by taking the tensor product of irreducible representations of the smaller general linear groups. We generalize this construction to the context of metaplectic coverings over a $\mathfrak{p}$-adic field.**

## 1. Introduction

The representation theory of general linear groups is simpler than that of most reductive groups. This is due in part to the structural similarity between general linear groups and their Levi subgroups. Let us consider this matter over a $\mathfrak{p}$-adic field $F$. Suppose $P \subsetneq \mathrm{GL}(r, F)$ is a parabolic subgroup with Levi subgroup $M$. The Levi subgroup $M$ may be decomposed as a direct product

$$M = M_1 \cdots M_\ell \cong M_1 \times \cdots \times M_\ell \tag{1}$$

in which each subgroup $M_i \subset M$ is isomorphic to a general linear group of rank less than $r$. Suppose $\rho_i$ is an irreducible representation of $M_i$ for $1 \leq i \leq \ell$. Then the tensor product $\otimes\rho_i$ is an irreducible representation of $M$ and the induced representation $\mathrm{ind}_P^{\mathrm{GL}(r,F)}(\otimes\rho_i)$ is a representation of $\mathrm{GL}(r, F)$. This method of induction is essential in understanding the representations of $\mathrm{GL}(r, F)$, for it provides a bridge between a representation of $\mathrm{GL}(r, F)$ and representations of lower rank general linear groups.

A metaplectic covering of a general linear group is a covering group which shares many of its structural properties with its underlying group. However, the differences in their Levi subgroups do not allow us to take tensor products in the straightforward manner as above. Suppose $\widetilde{M}$ is a metaplectic covering of the Levi subgroup $M$. We cannot decompose $\widetilde{M}$ as in (1) because the analogous subgroups $\widetilde{M_i}$ and $\widetilde{M_j}$ do not commute when $i \neq j$.

In an attempt to avoid this difficulty, a method of parabolic induction is proposed in §26.2 [**4**], in which $\widetilde{M}$ is replaced by a subgroup $\widetilde{M}^B$ which is isomorphic to (an obvious quotient of) $\widetilde{M}_1^B \times \cdots \times \widetilde{M}_\ell^B$. Despite this nice decomposition, H. Sun (§4 [**13**]) has pointed out that the centralizer of $\widetilde{M}^B$ in $\widetilde{M}$ is not necessarily contained in the product of $\widetilde{M}^B$ and the centre of $\widetilde{G}$. As a result, the representation of $\widetilde{M}$ constructed in §26.2 [**4**] is not necessarily irreducible, as is claimed there. In other words, this metaplectic analogue for the usual tensor product representation may be *reducible*.

We shall modify the ideas of §26.2 [**4**] to produce an *irreducible* metaplectic tensor product. This allows us to form the unique parabolically induced representation $\mathrm{ind}_{\widetilde{P}}^{\widetilde{G}}(\otimes\rho_i)$ as in Proposition 26.2 [**4**]. This has consequences for several metaplectic correspondences: The local correspondence of §27.3 [**4**] holds for all coverings (Appendix [**3**]); the assumption on the local correspondence of p. 99 [**10**] may be removed; and the global correspondence of [**11**] holds in more generality. (To be precise, Assumption 1 [**11**] may be weakened to the assumption that $n$ is relatively prime to $r!$ and $r+2mr-1$.)

In the final section we determine the extent to which metaplectic tensor products are unique. As we shall see, this also determines the extent to which the process of taking metaplectic tensor products is reversible.

Ideally, one would like to have a useful definition of metaplectic tensor products available for the wider class of indecomposable representations. As of yet, this has only been completed for two-fold metaplectic coverings of $\mathrm{GL}(2,F)$ ([**7**]).

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## 2. Metaplectic coverings

Let $n$ be a positive integer and $F$ be a $\mathfrak{p}$-adic field containing the $n$th roots of unity $\mu_n$. The $n$th Hilbert symbol is a familiar map $(\cdot,\cdot)_F : F^\times \times F^\times \to \mu_n$ in local class field theory and underlies the definition of a metaplectic covering. It satisfies $(x_1, x_2) = 1$ whenever $x_1$ or $x_2$ belongs to the $n$th power subgroup $F^{\times n}$.

An $n$-fold metaplectic covering is a particular example of an abelian extension of a group $H$ by $\mu_n$. A group $\widetilde{H}$ is such an abelian extension if it is equal to $H \times \mu_n$ as a set, and there is an exact sequence of groups

$$1 \to \mu_n \xrightarrow{\mathbf{i}} \widetilde{H} \xrightarrow{\mathbf{p}} H \to 1,$$

where the maps are defined by

$$\mathbf{i}(\zeta) = (1, \zeta),\ \zeta \in \mu_n,$$

$$\mathbf{p}(x, \zeta) = x,\ (x, \zeta) \in \widetilde{H}.$$

Suppose $r \geq 2$ is an integer and $M_0$ is the diagonal subgroup of $\mathrm{GL}(r, F)$. According to Corollary 1 [**14**], to any pair of abelian extensions

$$1 \to \mu_n \to \widetilde{\mathrm{SL}}(r, F) \to \mathrm{SL}(r, F) \to 1$$

$$1 \to \mu_n \to \widetilde{M_0} \to M_0 \to 1$$

there exists (up to isomorphism) a unique abelian extension

$$1 \to \mu_n \xrightarrow{\mathbf{i}} \widetilde{\mathrm{GL}}(r,F) \xrightarrow{\mathbf{p}} \mathrm{GL}(r,F) \to 1$$

such that $\widetilde{\mathrm{GL}}(r,F) = \widetilde{\mathrm{SL}}(r,F)\widetilde{M_0}$. According to Theorem 8.1 [**12**], the abelian extension $\widetilde{\mathrm{SL}}(r,F)$ is determined uniquely by its diagonal subgroup. We may therefore specify a particular extension $\widetilde{\mathrm{GL}}(r,F)$ by defining the product of two diagonal elements in $\widetilde{M_0}$ to be

$$(2) \qquad (\mathrm{diag}(t_1,\ldots,t_r),\zeta)\,(\mathrm{diag}(t_1',\ldots,t_r'),\zeta') \\ = \left(\mathrm{diag}(t_1t_1',\ldots,t_rt_r'),\zeta\zeta'(\det(t),\det(t'))_F^c \prod_{1\le i<j\le r}(t_i,t_j')_F\right).$$

In this equation, and for the duration of this paper, $0 \le c \le n-1$ is a fixed integer. This equation determines (an isomorphism class of) an $n$-fold metaplectic covering group $\widetilde{\mathrm{GL}}(r,F)$ as defined in any of [**8**], [**4**] or [**10**].

In the case of $\mathrm{GL}(1,F)$ an $n$-fold metaplectic covering $\widetilde{GL}(1,F)$ is defined by the multiplication

$$(t,\zeta)(t',\zeta') = (tt',\zeta\zeta'(t,t')_F^c),\ t,t' \in F^\times, \zeta,\zeta' \in \mu_n,$$

which is determined by the fixed integer $c$.

## 3. Subgroups and representations

The reader is assumed to be familiar with the basic structure theory and representation theory of reductive groups. We shall merely establish some notation. Let $P_0$ be the subgroup of upper-triangular matrices in $\mathrm{GL}(r,F)$. For us, a standard parabolic subgroup is a parabolic subgroup containing $P_0$. Given any subgroup $H$ of $\mathrm{GL}(r,F)$ we set $\widetilde{H} = \mathbf{p}^{-1}(H)$. In keeping with [**8**] and [**4**], we denote the centre of $\widetilde{\mathrm{GL}}(r,F)$ by $\widetilde{Z}$, even though $\widetilde{Z}$ is not necessarily equal to the pre-image of the centre of $\mathrm{GL}(r,F)$ under $\mathbf{p}$. Let $G$ be equal to $\mathrm{GL}(r,F)$ so that $\widetilde{G} = \widetilde{\mathrm{GL}}(r,F)$.

Henceforth, we fix an embedding of $\mu_n$ into the complex numbers. We shall primarily consider *genuine* representations of a subgroup $\widetilde{H}$, namely representations $\sigma$ of a complex vector space which satisfy

$$\sigma(x,\zeta) = \zeta\sigma(x,1),\ x \in H,\ \zeta \in \mu_n,$$

under the above embedding. Given an element $g \in \widetilde{G}$, the conjugate representation $\sigma^g$ of $g\widetilde{H}g^{-1}$ is defined by

$$\sigma^g(x) = \sigma(g^{-1}xg),\ x \in g\widetilde{H}g^{-1}.$$

The group $\widetilde{G}$ has a standard topology which makes it a topological extension. This topology is Hausdorff and possesses a neighbourhood base of open compact subgroups at the identity (Proposition 0.1.2 [**8**]). As such, $\widetilde{G}$ is an $l$-group in the sense of §1 [**1**]. One therefore has the notion of an admissible representation of $\widetilde{G}$. We denote the set of (equivalence classes of) genuine irreducible admissible representations of $\widetilde{G}$ by $\Pi(\widetilde{G})$. More generally, we denote the set of (equivalence classes of) genuine irreducible admissible representations of any $l$-group $H$ by $\Pi(H)$. All of the groups we shall consider are $l$-groups.

## 4. Metaplectic tensor products

Suppose $P$ is a standard parabolic subgroup of $G$ with Levi subgroup $M$. Our assumption that $P$ be standard is made purely for the sake of simplicity and what follows may be carried out for any parabolic subgroup.

Recall decomposition (1) and observe that any element $m \in \widetilde{M}$ may be written as a product $m_1 \cdots m_\ell$, in which $\mathbf{p}(m_i) \in M_i$ for $1 \leq i \leq \ell$. Using this notation, set

$$\widetilde{M}^n = \{m \in \widetilde{M} : \det(\mathbf{p}(m_1)), \ldots, \det(\mathbf{p}(m_\ell)) \in F^{\times n}\}$$

and $\widetilde{M}_i^n = \widetilde{M}^n \cap \widetilde{M}_i$. One can show that the elements of $\widetilde{M}_i^n$ commute with those of $\widetilde{M}_j$ if $i \neq j$ using Proposition 0.1.5 [**8**]. However $\widetilde{M}^n$ is not isomorphic to $\widetilde{M}_1^n \times \cdots \times \widetilde{M}_\ell^n$ as there are too many copies of $\mathbf{i}(\mu_n)$ in the direct product. This obstruction is removed by taking a quotient with the group

$$\Xi = \{(\mathbf{i}(\zeta_1), \ldots, \mathbf{i}(\zeta_\ell)) : \zeta_1, \ldots, \zeta_\ell \in \mu_n,\ \zeta_1 \cdots \zeta_\ell = 1\}.$$

We have the decomposition

$$\widetilde{M}^n \cong (\widetilde{M}_1^n \times \cdots \times \widetilde{M}_\ell^n)/\Xi. \tag{3}$$

Fix a genuine character $\omega$ of the centre $\widetilde{Z}$. Suppose we are given representations $\rho_i \in \Pi(\widetilde{M}_i)$, $1 \leq i \leq \ell$, which satisfy

$$\rho_1(m_1) \cdots \rho_\ell(m_\ell) = \omega(m),\ m \in \widetilde{Z} \cap \widetilde{M}^n. \tag{4}$$

We shall define the metaplectic tensor product of these representations in four steps. After each step we will provide an example in the case $\widetilde{G} = \widetilde{\mathrm{GL}}(2, F)$. Although we wish to define metaplectic tensor products only up to equivalence, we shall often consider $\rho_1, \ldots, \rho_\ell$ as actual representations (as opposed to equivalence classes) to make our construction more precise.

*Step* 1: We begin by restricting to the level of the subgroup $\widetilde{M}^n$ where we have the direct product decomposition (3). According to Lemma 2.1 [**6**], the restriction of $\rho_i$ to $\widetilde{M}_i^n$ has a decomposition

$$\rho_i|_{\widetilde{M}_i^n} \cong \sum_m k_i(\rho_i')^m, \tag{5}$$

where $m$ runs over a finite subset of $\widetilde{M}_i$, $k_i$ is a positive multiplicity and the conjugate representations of the sum are mutually inequivalent. We may identify the tensor product $\otimes_{i=1}^{\ell}\rho_i'$ with a unique irreducible representation of $\widetilde{M}^n$ after taking the quotient by $\Xi$.

**Example** $(r = 2)$**:** According to (2), multiplication in $\widetilde{M}_1$ is given by

$$(\mathrm{diag}(t_1, 1), \zeta)(\mathrm{diag}(t_1', 1), \zeta') = \mathrm{diag}(t_1t_1', \zeta\zeta'(t_1, t_1')_F^c).$$

This means that $\widetilde{M}_1$ is isomorphic to $\widetilde{\mathrm{GL}}(1, F)$. The same is also true for $\widetilde{M}_2$. The subgroups $\widetilde{M}_1^n$ and $\widetilde{M}_2^n$ are central, and are both isomorphic to the abelian group $F^{\times n} \times \mu_n$. Decomposition (5) therefore reduces to $\rho_i|_{\widetilde{M}_i^n} \cong k_i\rho_i'$, where $k_i$ is the dimension of $\rho_i$, and $\rho_i'$ is a genuine one-dimensional representation of $\widetilde{M}_i^n$. The representation $\rho_1' \otimes \rho_2'$ is trivial on $\Xi = \{(\mathbf{i}(\zeta), \mathbf{i}(\zeta^{-1})) : \zeta \in \mu_n\}$ and consequently passes to a one-dimensional genuine representation of $\widetilde{M}^n \cong (\widetilde{M}_1 \times \widetilde{M}_2)/\Xi$.

*Step* 2: The next step in our procedure is to extend $\otimes_{i=1}^{\ell}\rho_i'$ using our choice of central character $\omega$. The representation $\otimes_{i=1}^{\ell}\rho_i'$ agrees with $\omega$ on the subgroup $\widetilde{Z} \cap \widetilde{M}^n$ by assumption (4). We may therefore set $\rho'$ to be the irreducible representation of $\widetilde{Z}\widetilde{M}^n$ given by taking the tensor product of $\omega$ with $\otimes_{i=1}^{\ell}\rho_i'$ over the group ring of $\widetilde{Z} \cap \widetilde{M}^n$.

**Example** $(r = 2)$**:** By Proposition 0.1.1 [**8**], the centre $\widetilde{Z}$ is equal to

$$\{(\mathrm{diag}(t, t), \zeta) : t \in F^{\times n/\gcd(n, r+2rc-1)}, \zeta \in \mu_n\}.$$

The intersection $\widetilde{Z} \cap \widetilde{M}^n$ is equal to $\{(\mathrm{diag}(t, t), \zeta) : t \in F^{\times n}, \zeta \in \mu_n\}$ and the representation $\rho'$ satisfies

$$\rho'(\mathrm{diag}(t, t), \zeta) = \zeta\, \omega(\mathrm{diag}(t, t), 1) = \zeta\, \rho_1'(\mathrm{diag}(t, 1), 1)\, \rho_2'(\mathrm{diag}(1, t), 1)$$

for all $t \in F^{\times n}$. If $\gcd(n, r + 2rc - 1) = 1$ then $\widetilde{Z} \subset \widetilde{M}^n$ and $\rho'$ is actually equal to $\rho_1' \otimes \rho_2'$. Otherwise $\rho'$ is a veritable extension of $\rho_1' \otimes \rho_2'$.

*Step* 3: Here we extend $\rho'$ to a subgroup $\widetilde{M}_{\rho''} \supset \widetilde{Z}\widetilde{M}^n$ so that the resulting extension $\rho''$ satisfies Mackey's irreducibility criterion. By this we mean that if $\rho'' \cong (\rho'')^m$ for some $m \in \widetilde{M}$ then $m$ must lie in $\widetilde{M}_{\rho''}$. We shall define this extension inductively by considering the quotient group $\widetilde{M}/\widetilde{Z}\widetilde{M}^n$.

Since $F^\times/F^{\times n}$ is finite, the normal subgroup $\widetilde{M}^n$ is of finite index in $\widetilde{M}$. It is not difficult to see that representatives for the elements of $\widetilde{M}/\widetilde{M}^n$ can be chosen to be diagonal elements of $\widetilde{M}$. It therefore follows that $\widetilde{M}/\widetilde{Z}\widetilde{M}^n$

is a finite abelian group. Suppose that $m \in \widetilde{M}$ does not lie in $\widetilde{Z}\widetilde{M}^n$ and satisfies $(\rho')^m \cong \rho'$. (If this does not happen for any element outside of $\widetilde{Z}\widetilde{M}^n$ then $\rho'$ already satisfies Mackey's criterion and we may take $\rho'' = \rho'$.) According to Lemma 14.22 [**9**], the representation $\rho'$ may be extended to the normal subgroup generated by $m$ and $\widetilde{Z}\widetilde{M}^n$. The quotient of $\widetilde{M}$ with respect to this new subgroup is now a finite abelian group whose order is less than the order of the previous quotient. It is clear that we may repeat the above procedure a finite number of times until we obtain a representation $\rho''$ with the desired properties.

**Example** $(r = 2)$**:** Using Equation (2) it is simple to show that $\widetilde{Z}\widetilde{M}^n$ is contained in the centre of $\widetilde{M}$. Suppose $m \in \widetilde{M}$ does not lie in $\widetilde{Z}\widetilde{M}^n$. Let $k$ be the order of $m\widetilde{Z}\widetilde{M}^n$ in $\widetilde{M}/\widetilde{Z}\widetilde{M}^n$ and $T_m$ be an $k$th root of $\rho'(m^k)$. Since $(\rho')^m = \rho'$, we are able to extend $\rho'$ to the subgroup $\widetilde{M}_\sigma \subset \widetilde{M}$ generated by $m$ and $\widetilde{Z}\widetilde{M}^n$ by defining

$$\sigma(m^i x) = T_m^i\, \rho'(x),\ x \in \widetilde{Z}\widetilde{M}^n,\ i \in \mathbf{Z}.$$

Indeed, $\sigma$ is easily verified to be a homomorphism and $\sigma(m^k) = \rho'(m^k)$.

Now suppose $m' \in \widetilde{M}$ does not lie in $\widetilde{M}_\sigma$, $k'$ is the order of $m'\widetilde{M}_\sigma$ in $\widetilde{M}/\widetilde{M}_\sigma$, and $T_{m'}$ is an nonzero intertwining operator between $\sigma$ and $\sigma^{m'}$. Then $T_{m'}^k$ intertwines $\sigma$ with $\sigma^{(m')^k}$ and so does the operator $\sigma((m')^{-k})$. Since these representations are irreducible, Schur's lemma implies that $T_{m'}^k$ is a nonzero multiple of $\sigma((m')^{-k})$. We may assume that the multiple is equal to 1 and then define an extension $\sigma'$ of $\sigma$ to the subgroup $\widetilde{M}_{\sigma'}$ generated by $m'$ and $\widetilde{M}_\sigma$ by setting

$$\sigma'((m')^i x) = T_{m'}^i\, \sigma(x),\ x \in \widetilde{M}_\sigma,\ i \in \mathbf{Z}.$$

It is once again easily verified that $\sigma'$ is a homomorphism and an extension of $\sigma$. Clearly this process may be continued until we arrive at an extension $\rho''$ which is inequivalent to any of its nontrivial conjugates in $\widetilde{M}$.

The observant reader might have noticed that our treatment of the extension to $\widetilde{M}_{\sigma'}$ was more abstract than necessary. After all, $\sigma$ is one-dimensional so $T_{m'}$ is really just a scalar. Furthermore, since $M$ is abelian and $\sigma$ is genuine we have

$$\mathbf{i}^{-1}((m')^{-1} x m' x^{-1}) = \sigma^{m'}(x)\sigma(x^{-1}) = T_{m'},\ x \in \widetilde{M}_\sigma.$$

The function on the left defines a character of $\widetilde{M}_\sigma$ into $\mu_n$. Therefore $T_{m'}$ is equal to one, that is $M_{\sigma'}$ is abelian and $\sigma^{m'}$ is actually *equal* to $\sigma$. This was the circumstance of the earlier extension of $\rho'$ to $\sigma$ and may therefore be handled in exactly the same way. This argument also shows that the extension of $\rho'$ to any maximal abelian subgroup satisfies Mackey's criterion. This fact has been exploited by several authors in the case that $M = M_0$

is the diagonal subgroup (§5.1 [**5**], p. 59 [**8**], §1 [**2**]). We have included the slightly more general perspective taken in the extension to $\widetilde{M}_{\sigma'}$ because it is necessary in the case that $M$ is *not* abelian.

Before moving to the fourth and final step in our definition of metaplectic tensor products some additional remarks are in order. First note that our definition of $\rho''$ is far from canonical. This can be seen in the arbitrary choice of the elements $m$ and $m'$ in the example above. Moreover, even after we fix a subgroup, say $\widetilde{M}_{\sigma'}$ as in the example, we could replace $T_{m'}$ by its multiple with a $k'$th root of unity to obtain an inequivalent extension to $M_{\sigma'}$. The following lemma, which is paraphrased from Lemma 2 [**13**], determines how this second type of discrepancy affects the resulting extensions:

**Lemma 4.1.** *Suppose $\widetilde{H} \supset \widetilde{Z}\widetilde{M}^n$ and $\pi$ is an extension of $\rho'$ to $\widetilde{H}$. Then:*

(i)
$$\operatorname{ind}_{\widetilde{Z}\widetilde{M}^n}^{\widetilde{H}} \rho' \cong \bigoplus_{\chi} \chi\pi,$$
*where $\chi$ runs over the finite set of characters of $\widetilde{H}$ which are trivial on $\widetilde{Z}\widetilde{M}^n$.*

(ii) *The representation $\chi\pi$ is inequivalent to $\pi$ for any such $\chi$, unless $\chi$ is trivial.*

(iii) *Any extension of $\rho'$ to $\widetilde{H}$ is of the form $\chi\pi$ as above.*

*Proof.* Suppose $\chi$ is a character as above, $\pi$ is equivalent to $\chi\pi$, and $T$ is a nonzero intertwining operator between them. Both of these representations restrict to $\rho'$ on $\widetilde{Z}\widetilde{M}^n$, and $T$ remains an intertwining operator. By Schur's lemma, $T$ is a scalar. This implies that $\chi\pi = \pi$ and so $\chi$ is trivial. We have just proved the second assertion. To prove the first, suppose again that $\chi$ is arbitrary. We apply Frobenius reciprocity to obtain

$$\operatorname{Hom}_{\widetilde{H}}\left(\chi\pi, \operatorname{ind}_{\widetilde{Z}\widetilde{M}^n}^{\widetilde{H}} \rho'\right) = \operatorname{Hom}_{\widetilde{Z}\widetilde{M}^n}\left((\chi\pi)|_{\widetilde{Z}\widetilde{M}^n}, \rho'\right) = \operatorname{Hom}_{\widetilde{Z}\widetilde{M}^n}(\rho', \rho') \cong \mathbf{C}.$$

Consequently, each $\chi\pi$ occurs as a subrepresentation of $\operatorname{ind}_{\widetilde{Z}\widetilde{M}^n}^{\widetilde{H}} \rho'$. Since there are exactly $|\widetilde{H}/\widetilde{Z}\widetilde{M}^n|$ characters of the form $\chi$, this induced representation contains $|\widetilde{H}/\widetilde{Z}\widetilde{M}^n|$ inequivalent subrepresentations. It contains exactly this number of irreducible subrepresentations, as Mackey's decomposition theorem implies that

$$\operatorname{Hom}_{\widetilde{H}}\left(\operatorname{ind}_{\widetilde{Z}\widetilde{M}^n}^{\widetilde{H}} \rho', \operatorname{ind}_{\widetilde{Z}\widetilde{M}^n}^{\widetilde{H}} \rho'\right) \cong \operatorname{Hom}_{\widetilde{Z}\widetilde{M}^n}\left(\rho', \oplus_{h \in \widetilde{H}/\widetilde{Z}\widetilde{M}^n} (\rho')^h\right),$$

and the space on the right is of dimension no more than $|\widetilde{H}/\widetilde{Z}\widetilde{M}^n|$. A similar computation (on contragredients) shows that it contains exactly $|\widetilde{H}/\widetilde{Z}\widetilde{M}^n|$ irreducible quotients. The first assertion is therefore proven. For the third assertion, suppose that $\pi'$ is an extension of $\rho'$ to $\widetilde{H}$. It must be of the desired form by the first assertion and the isomorphism

$$\mathrm{Hom}_{\widetilde{H}}\left(\pi', \mathrm{ind}_{\widetilde{Z}\widetilde{M}^n}^{\widetilde{H}} \rho'\right) \cong \mathrm{Hom}_{\widetilde{Z}\widetilde{M}^n}(\rho', \rho').$$

□

*Step* 4: Mackey's criterion ensures that the induced representation $\mathrm{ind}_{\widetilde{M}_{\rho''}}^{\widetilde{M}} \rho''$ is irreducible. Following §26.2 [**4**], we denote this induced representation by $\otimes_{i=1}^{\ell} \rho_i$ (or simply $\otimes \rho_i$) and refer to it as a metaplectic tensor product. Although the central character $\omega$ is suppressed from the notation, it is important to remember that different choices of central character will lead to metaplectic tensor products which differ in their central characters.

**Example** ($r = 2$)**:** Suppose $U$ is the group of units of $F$ and $\widetilde{M}_{\rho''}$ is the maximal abelian subgroup

$$\left\{(\mathrm{diag}(xt_1, xt_2), \zeta) : x \in UF^{\times n/\gcd(n, r+2rc-1)},\ t_1, t_2 \in UF^{\times n}\right\}$$

defined on p. 59 [**8**]. Since $\rho''$ is one-dimensional, the dimension of the metaplectic tensor product $\rho_1 \otimes \rho_2 = \mathrm{ind}_{\widetilde{M}_{\rho''}}^{\widetilde{M}} \rho''$ is equal to the order of $\widetilde{M}/\widetilde{M}_{\rho''}$, which may be computed to be $n^2/\gcd(n, r+2rc-1)$.

To be well-defined, we must show that $\otimes \rho_i$ is independent of the particular extension of $\rho'$ to $\widetilde{M}_{\rho''}$, the choice of subgroup $\widetilde{M}_{\rho''}$, and the choice of the irreducible representation in (5). We prove each of these in sequence keeping the notation of the previous four steps.

**Lemma 4.2.** *Suppose $\rho^1$ and $\rho^2$ are extensions of $\rho'$ to $\widetilde{M}_{\rho''}$. Then $\mathrm{ind}_{\widetilde{M}_{\rho''}}^{\widetilde{M}} \rho^1$ is equivalent to $\mathrm{ind}_{\widetilde{M}_{\rho''}}^{\widetilde{M}} \rho^2$.*

*Proof.* Suppose $i = 1, 2$. The distribution character of $\mathrm{ind}_{\widetilde{M}_{\rho''}}^{\widetilde{M}} \rho^i$ is an invariant function on $\widetilde{M}$ under conjugation. It is also a $\mathbf{i}(\mu_n)$-equivariant (i.e., genuine) function. Combining these facts with Proposition 0.1.4 [**8**] leads to the conclusion that this character is supported in a subset of $\widetilde{Z}\widetilde{M}^n$. We can expand the character of $\mathrm{ind}_{\widetilde{M}_{\rho''}}^{\widetilde{M}} \rho^i$ in terms of the character of $\rho^i$. Since its support lies in $\widetilde{Z}\widetilde{M}^n$, the expansion is actually in terms of the character of $\rho^i|_{\widetilde{Z}\widetilde{M}^n} = \rho'$. Therefore the characters of $\mathrm{ind}_{\widetilde{M}_{\rho''}}^{\widetilde{M}} \rho^1$ and $\mathrm{ind}_{\widetilde{M}_{\rho''}}^{\widetilde{M}} \rho^2$ are equal. By the linear independence of characters, the representations are equivalent. □

**Lemma 4.3.** *Suppose $\widetilde{H} \supset \widetilde{Z}\widetilde{M}^n$ is a subgroup of $\widetilde{M}_{\rho''}$ and $\sigma$ is an extension of $\rho'$ to $\widetilde{H}$. Then $\sigma$ has an extension to $\widetilde{M}_{\rho''}$ which satisfies Mackey's criterion.*

*Proof.* By Lemma 4.1 (iii), there is a character $\chi$ of $\widetilde{H}$, trivial on $\widetilde{Z}\widetilde{M}^n$, such that $\sigma$ is equal to $\chi\rho''|_{\widetilde{H}}$. Since $\widetilde{M}/\widetilde{Z}\widetilde{M}^n$ is finite and abelian, we can easily extend $\chi$ to a character $\chi'$ of $\widetilde{M}_{\rho''}$. Obviously, $\chi'\rho''$ is an extension of $\sigma$ to $\widetilde{M}_{\rho''}$. We may extend $\chi'$ further to a character $\chi''$ of $\widetilde{M}$. According to a well-known property of induction, we have

$$\mathrm{ind}_{\widetilde{M}_{\rho''}}^{\widetilde{M}} \chi'\rho'' = \chi''\mathrm{ind}_{\widetilde{M}_{\rho''}}^{\widetilde{M}} \rho''.$$

The representation on the right is irreducible by the definition of $\rho''$. As a result, $\chi'\rho''$ satisfies Mackey's criterion. □

**Lemma 4.4.** *Suppose $\tau$ is an another extension of $\rho'$ to a subgroup $\widetilde{H}$ and $\tau$ satisfies Mackey's criterion. Then $\mathrm{ind}_{\widetilde{H}}^{\widetilde{M}}\tau$ is equivalent to $\mathrm{ind}_{M_{\rho''}}^{\widetilde{M}}\rho''$.*

*Proof.* Since $\tau$ satisfies Mackey's criterion $\mathrm{ind}_{\widetilde{H}}^{\widetilde{M}}\tau$ is irreducible. Frobenius reciprocity and Mackey's decomposition theorem lead to

$$\begin{aligned}
1 &\geq \dim \mathrm{Hom}_{\widetilde{M}}\left(\mathrm{ind}_{\widetilde{H}}^{\widetilde{M}}\tau, \mathrm{ind}_{M_{\rho''}}^{\widetilde{M}}\rho''\right) \\
&= \dim \mathrm{Hom}_{\widetilde{H}}\left(\tau, \sum_{m\in\widetilde{H}\backslash\widetilde{M}/\widetilde{M}_{\rho''}} \mathrm{ind}_{\widetilde{H}\cap m\widetilde{M}_{\rho''}m^{-1}}^{\widetilde{H}} (\rho'')^m|_{\widetilde{H}\cap m\widetilde{M}_{\rho''}m^{-1}}\right) \\
&\geq \dim \mathrm{Hom}_{\widetilde{H}}\left(\tau, \mathrm{ind}_{\widetilde{H}\cap\widetilde{M}_{\rho''}}^{\widetilde{H}}\rho''|_{\widetilde{H}\cap\widetilde{M}_{\rho''}}\right) \\
&= \dim \mathrm{Hom}_{\widetilde{H}\cap\widetilde{M}_{\rho''}}\left(\tau|_{\widetilde{H}\cap\widetilde{M}_{\rho''}}, \rho''|_{\widetilde{H}\cap\widetilde{M}_{\rho''}}\right).
\end{aligned}$$

According to Lemma 4.2, the equivalence class of $\mathrm{ind}_{M_{\rho''}}^{\widetilde{M}}\rho''$ does not depend on the choice of extension of $\rho'$ to $\widetilde{M}_{\rho''}$. By Lemma 4.3, we may assume that the irreducible representation $\rho''|_{\widetilde{H}\cap\widetilde{M}_{\rho''}}$ is equal to $\tau|_{\widetilde{H}\cap\widetilde{M}_{\rho''}}$, which means the above inequalities may be replaced with equalities. □

**Corollary 4.5.** *Suppose $m \in \widetilde{M}$ and $\sigma$ is an extension of $(\rho')^m$ to a subgroup $\widetilde{M}_\sigma$ satisfying Mackey's criterion. Then $\mathrm{ind}_{\widetilde{M}_\sigma}^{\widetilde{M}}\sigma$ is equivalent to $\mathrm{ind}_{M_{\rho''}}^{\widetilde{M}}\rho''$.*

*Proof.* The representation $\mathrm{ind}_{\widetilde{M}_\sigma}^{\widetilde{M}}\sigma$ is equivalent to its conjugate by $m^{-1}$, which is itself equivalent to $\mathrm{ind}_{m^{-1}\widetilde{M}_\sigma m}^{\widetilde{M}}(\sigma^{m^{-1}})$. Clearly, $\sigma^{m^{-1}}$ is an extension of $\rho'$, so we may apply Lemma 4.4 with $\widetilde{H} = m^{-1}\widetilde{M}_\sigma m$ and $\tau = \sigma^{m^{-1}}$ to obtain the corollary. $\square$

## 5. Uniqueness

Because our definition of metaplectic tensor products involves restriction to subgroups, it is possible for inequivalent representations to have equivalent metaplectic tensor products. The following lemma gives an exact description of this failure of uniqueness. We preserve the notation of the previous section.

**Lemma 5.1.** *Suppose $\rho_i, \tau_i \in \Pi(\widetilde{M}_i)$ for $i = 1, \ldots, \ell$. Then the following are equivalent:*

(a) *The metaplectic tensor products $\otimes\rho_i$ and $\otimes\tau_i$ are equal.*
(b) *For each $i = 1, \ldots, \ell$, there exists a character $\omega_i$ of $\widetilde{M}_i$ such that $\omega_i$ is trivial on $\widetilde{M}_i^n$ and $\tau_i = \omega \otimes \rho_i$.*

*Proof.* It is an immediate consequence of the definitions that (b) implies (a). Let us therefore assume that (a) holds. This implies that the restrictions of $\otimes\rho_i$ and $\otimes\tau_i$ to $\widetilde{M}^n$ are equal. According to Lemma 2.1 [**6**], these restrictions are sums of conjugate representations

$$\sum_{m \in \widetilde{M}/\widetilde{M}_{\rho''}} \left(\otimes_{i=1}^{\ell} \rho_i'\right)^m = (\otimes\rho_i)|_{\widetilde{M}^n} = (\otimes\tau_i)|_{\widetilde{M}^n} = \sum_{m \in \widetilde{M}/\widetilde{M}_{\tau''}} \left(\otimes_{i=1}^{\ell} \tau_i'\right)^m.$$

Consequently there exists $m \in \widetilde{M}$ such that $\otimes\tau_i' \cong (\otimes\rho_i')^m$. The element $m$ decomposes as a product $m_1 \cdots m_\ell$, where $\mathbf{p}(m_i) \in M_i$. Proposition 0.1.5 [**8**] ensures that $m_i$ commutes with the elements of $\widetilde{M}_j^n$ for $i \neq j$. This implies that

$$\tau_i' \cong (\rho_i')^{m_i}, \ 1 \leq i \leq \ell. \tag{6}$$

Let $Z_{\widetilde{M}_i}$ be the centre of $\widetilde{M}_i$, and let $\omega_{\rho_i}$ and $\omega_{\tau_i}$ be the central characters of $\rho_i$ and $\tau_i$ respectively. Equivalence (6) implies that the restrictions of these central characters to $Z_{\widetilde{M}_i} \cap \widetilde{M}_i^n$ are equal. Therefore the character $\omega_{\tau_i}\omega_{\rho_i}^{-1}$ may be extended trivially to $Z_{\widetilde{M}_i}\widetilde{M}_i^n$. Since this extension is trivial on $\widetilde{M}_i^n$ and $\widetilde{M}_i/Z_{\widetilde{M}_i}\widetilde{M}_i^n$ is a finite abelian group, we may extend it further to a character $\omega_i$ of $\widetilde{M}_i$. By Lemma 2.1 [**6**], we have decompositions

$$\omega_i\rho_i|_{Z_{\widetilde{M}_i}\widetilde{M}_i^n} \cong \sum_{m' \in \widetilde{M}_i/\widetilde{M}_{i,\rho_i''}} \omega_{\tau_i}\omega_{\rho_i}^{-1}(\omega_{\rho_i}\rho_i')^{m'} = \sum_{m' \in \widetilde{M}_i/\widetilde{M}_{i,\rho_i''}} (\omega_{\tau_i}\rho_i')^{m'}$$

and

$$\begin{aligned}\tau_i|_{Z_{\widetilde{M}_i}\widetilde{M}_i^n} &\cong \sum_{m'\in\widetilde{M}_i/\widetilde{M}_{i,\tau_i''}} (\omega_{\tau_i}\tau_i')^{m'} \\ &\cong \sum_{m'\in\widetilde{M}_i/\widetilde{M}_{i,\tau_i''}} (\omega_{\tau_i}(\rho_i')^{m_i})^{m'} \\ &\cong \sum_{m'\in\widetilde{M}_i/\widetilde{M}_{i,\rho_i''}} (\omega_{\tau_i}\rho_i')^{m'}.\end{aligned}$$

Here, the representations $\rho_i''$ and $\tau_i''$ are the obvious analogues of the representations in Step 3 of our construction of metaplectic tensor products and the products in parentheses are to be regarded as tensor products over $Z_{\widetilde{M}_i} \cap \widetilde{M}_i^n$. Comparing these decompositions, we see that the restrictions of $\omega_i\rho_i$ and $\tau_i$ to $Z_{\widetilde{M}_i}\widetilde{M}_i^n$ are equivalent. However, as already pointed out in the proof of Lemma 4.2, the distribution characters of $\omega_i\rho_i$ and $\tau_i$ are supported in $Z_{\widetilde{M}_i}\widetilde{M}_i^n$. Consequently, the distribution characters are equal. This implies the equivalence of $\omega_i\rho_i$ and $\tau_i$. $\square$

It is obvious that the conditions of (b) Lemma 5.1 form an equivalence relation on $\ell$-tuples of representations in $\Pi(\widetilde{M}_1) \times \cdots \times \Pi(\widetilde{M}_\ell)$. A apparent corollary of this lemma is that to each representation in $\Pi(\widetilde{M})$ there is a unique equivalence class of such $\ell$-tuples.

**Example:** Let us consider the case $\widetilde{G} = \widetilde{\mathrm{GL}}(3, F)$ and $M = M_1 \times M_2$ such that $M_1 \cong \mathrm{GL}(2, F)$ and $M_2 \cong \mathrm{GL}(1, F)$. Suppose $\rho \in \Pi(\widetilde{M})$. Then $\rho|_{\widetilde{M}^n}$ is equal to $\sum_m (\rho_1' \otimes \rho_2')^m$, for some irreducible representations $\rho_i'$ of $\widetilde{M}_i^n$ and $m$ running over finite subset of $\widetilde{M}$. Following steps one through four for each of $\rho_1'$ and $\rho_2'$ we obtain $\rho_1 \in \Pi(\widetilde{M}_1)$ and $\rho_2 \in \Pi(\widetilde{M}_2)$. Let $(F^\times/F^{\times n})^\wedge$ be the set of characters of the finite group $F^\times/F^{\times n}$. To each character in $(F^\times/F^{\times n})^\wedge$ we may identify a unique character of $\widetilde{M}_i$ which is trivial on $\widetilde{M}^n$ by composing with $\mathbf{p}$ and the determinant map. With this identification, the equivalence class of representations whose metaplectic tensor products are equivalent to $\rho$ is equal to

$$\{(\omega_1\rho_1, \omega_2\rho_2) : \omega_1, \omega_2 \in (F^\times/F^{\times n})^\wedge\}.$$

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