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Let F be a non-archimedean local field with residual field of odd characteristic. Given a reductive group G defined over F, equipped with an involution denoted $g \mapsto g^*$, let K be a maximal compact of G. G acts on the space $\{x \in G \mid x = x^*\}$ by $g \cdot x = g\,x\,g^*$. Let $s_0 \in G$ be fixed by the involution and let $S = G \cdot s_0$ and $H = \mathrm{Stab}_G(s_0)$. A relative spherical function on S is a K-invariant function on S, which is an eigenfunction of the Hecke algebra of G relative to K. The problem at hand is to classify all such functions, compute them explicitly in terms of Macdonald polynomials and obtain an explicit Plancherel measure. We obtain a complete solution in three cases relevant to the theory of Automorphic Forms. Namely:
***Case* 1: $G = GL(2n, F)$, $H = GL(n, F) \times GL(n, F)$.**
***Case* 2: $G = GL(m, E)$, $H = GL(m, F)$.**
***Case* 3: $G = GL(2n, F)$, $H = GL(n, E)$.**
E is an unramified quadratic extension of F.

1. Introduction

Let F be a non-archimedean local field, $\mathcal{O}_\mathcal{F}$ the ring of integers of F, $\wp_F$ the maximal ideal of $\mathcal{O}_\mathcal{F}$ and ϖ a uniformizer in $\wp_F$. Let

$$q = \#\left(\mathcal{O}_\mathcal{F}/\wp_F\right).$$

We assume q is odd. The problem at hand may be roughly described as follows: Let G be a reductive group defined over F, equipped with an involution - an anti-automorphism of order two - denoted $g \mapsto g^*$. The group G acts on the space of all $x \in G$ for which there is $a \in F^\times$ such that $x^* = a\,x$, by

$$g \cdot x = g\,x\,g^*.$$

Let $s_0 \in G$ be fixed, up to a scalar factor, by the involution and let H be the stabilizer of s_0 in G. We wish to study the spherical functions on G relative to H. We consider three different cases:

Case 1 and *Case* 3: $G = GL(2n, F)$.
Case 2: $G = GL(m, E)$.

E is an unramified quadratic extention of F. We denote by $a \mapsto \bar{a}$ the nontrivial automorphism of E over F. Let $\iota \in \mathcal{O}_E^\times$ be such that $E = F[\iota]$ and $\bar{\iota} = -\iota$ and let $\tau = \iota^2$ a non-square in F. For $X = (X_{ij}) \in M_r(E)$ denote $\overline{X} = \left(\overline{X}_{ij}\right)$. Let

$$q_1 = \begin{cases} q & \text{Case 1 and Case 3} \\ q^2 & \text{Case 2} \end{cases}$$

and denote by $|\,|$, the normalized absolute value on F in Case 1 and in Case 3, respectively on E in Case 2, so that $|\varpi^{-1}| = q_1$.

Let $g \mapsto g^*$ denote the involution on G defined by:

Case 1: $g^* = \epsilon\, g^{-1}\, \epsilon$, where

$$\epsilon = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} \in G.$$

Case 2: $g^* = \bar{g}^{-1}$.
Case 3: $g^* = g^{-1}$.

For the sake of a more uniform notation, we let $m = 2n$ in Case 1 and in Case 3, and $n = [\frac{m}{2}]$ in Case 2, where $[x]$ is the integral part of x.

Denote by w_j the element of $GL(j, F)$ with ones in the anti-diagonal entries and zeroes elsewhere. Let

$$s_0 = \begin{cases} I_m & \text{Case 1 and Case 2} \\ \begin{pmatrix} 0 & w_n \\ \tau\, w_n & 0 \end{pmatrix} & \text{Case 3} \end{cases}$$

and define

$$S = G \cdot s_0.$$

Note that in Case 3 $s_0^* = \tau^{-1}\, s_0$ so s_0 is only fixed, up to a scalar factor, by the involution. In fact we could reduce ourselves to the case where s_0 is fixed by the involution. We observe that $S\, s_0^{-1}$ is the orbit of the identity element in the space of elements $x \in G$ fixed by the involution $g \mapsto s_0\, g^{-1}\, s_0^{-1}$. We chose the translated S as above since it helps unify notations with the other cases.

Let H be the stabilizer of s_0 in G.
In Case 1:

$$H = \left\{ \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \mid g_i \in GL(n, F),\, i = 1, 2 \right\}.$$

In Case 2:

$$H = GL(m, F).$$

In Case 3:

$$H = \left\{ \begin{pmatrix} a & b \\ \tau\, w_n\, b\, w_n & w_n\, a\, w_n \end{pmatrix} \in G \mid a, b \in M_n(F) \right\} \simeq GL(n, E).$$

Define the map $\theta : G \to S$,

$$\theta(g) = g\, s_0\, g* = g \cdot s_0. \tag{1}$$

It induces a bijection

$$G/H \overset{\theta}{\cong} S. \tag{2}$$

In Case 1 and in Case 3: Let

$$K = GL(m, \mathcal{O}_F).$$

In Case 2: Let

$$K = GL(m, \mathcal{O}_E).$$

Denote by $\mathcal{H}(G,K)$ the Hecke algebra of G with respect to K. It is the convolution algebra of compactly supported, K-bi-invariant, complex valued functions on G. Let $C^\infty(K\backslash S)$ be the space of K-invariant complex valued functions on S. We define an $\mathcal{H}(G,K)$-module structure on $C^\infty(K\backslash S)$ by the convolution:

$$f * \varphi(s) = \int_G f(g)\, \varphi\left(g^{-1} \cdot s\right) dg \tag{3}$$

where $f \in \mathcal{H}(G,K)$, $\varphi \in C^\infty(K\backslash S)$ and dg is the Haar measure on G normalized such that $\int_K dg = 1$. $\mathcal{H}(G,K)$ is then an algebra of convolution operators on $C^\infty(K\backslash S)$. Let

$$\Lambda_n^+ = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \cdots \geq \lambda_n \geq 0\}.$$

For $j > n$ we may and will view Λ_n^+ as a subset of Λ_j^+ through the embedding $(\lambda_1, \ldots, \lambda_n) \mapsto (\lambda_1, \ldots, \lambda_n, 0, \ldots, 0)$. For $j = (j_1, \ldots, j_n) \in \mathbb{Z}^n$ let

$$\varpi^j = \begin{pmatrix} & & & \varpi^{j_1} \\ & & \cdot & \\ & \cdot & & \\ \varpi^{j_n} & & & \end{pmatrix}$$

and let $j^* = (-j_n, \ldots, -j_1)$. Note that $\left(\varpi^j\right)^{-1} = \varpi^{j^*}$. For $\lambda \in \Lambda_n^+$, define in Case 1:

$$d_\lambda = \begin{pmatrix} 0 & \varpi^\lambda \\ -\varpi^{\lambda^*} & 0 \end{pmatrix}.$$

In Case 2:

$$d_\lambda = \begin{pmatrix} & \varpi^\lambda \\ \varpi^{\lambda^*} & \end{pmatrix}$$

if m is even, and

$$d_\lambda = \begin{pmatrix} & & \varpi^\lambda \\ & 1 & \\ \varpi^{\lambda^*} & & \end{pmatrix}$$

if m is odd.
In Case 3:

$$d_\lambda = \begin{pmatrix} & \varpi^\lambda \\ \tau\,\varpi^{\lambda^*} & \end{pmatrix}.$$

S is the disjoint union of the K-orbits $K \cdot d_\lambda,\ \lambda \in \Lambda_n^+$ (Proposition 3.1).

Definition 1.1. A **relative spherical function** on S, is an eigenfunction $\Omega \in C^\infty(K\backslash S)$ of the Hecke algebra $\mathcal{H}(G,K)$, normalized so that $\Omega(d_0) = 1$.

We remark that in Case 1, if Y is the symmetric space of all $y \in G$, such that $y^2 = I_m$, then G acts on Y by conjugation, $S\,\epsilon$ is the orbit of ϵ in Y, and H is the centralizer of ϵ. Therefore, in Case 1 we essentially study the relative spherical functions on an orbit of the symmetric space defined by the equation $s^2 = I_m$, whereas in Case 2 we study the relative spherical functions on the symmetric space defined by the equation $s\,\bar{s} = I_m$ and in Case 3 by the equation $s^2 = \tau\, I_m$.

The Macdonald polynomials, defined in [**18**] (10.1), are associated to an 'admissible pair' (R, Σ) of root systems, in the sense of [**18**] Introduction. Let Σ be the reduced root system of type B_n. Let R be the root system of type BC_n. (R, Σ) is an admissible pair. The root systems R and Σ may be realized in the same vector space $\mathbb{C}^n$. Let ϵ_i, $i = 1, \dots, n$ be the standard basis of $\mathbb{C}^n$, and let Σ^+ (respectively R^+) be the set of positive roots in Σ (respectively R) then:

$$\Sigma^+ = \{\epsilon_i \pm \epsilon_j \,|\, 1 \le i < j \le n\} \cup \{\epsilon_i \,|\, 1 \le i \le n\} \tag{4}$$

and

$$R^+ = \{\epsilon_i \pm \epsilon_j \,|\, 1 \le i < j \le n\} \cup \{\epsilon_i, 2\,\epsilon_i \,|\, 1 \le i \le n\}. \tag{5}$$

We remark that our choice of positive roots for Σ amounts to fixing the basis $\Delta_\Sigma = \{\epsilon_1 - \epsilon_2, \dots, \epsilon_{n-1} - \epsilon_n, \epsilon_n\}$ of simple roots in Σ. The root systems R and Σ have the same Weyl group Γ which is the Weyl group of Sp_n. There is then a natural action of Γ on $\mathbb{C}^n$. The Macdonald polynomials associated to the pair (R, Σ) are:

$$P_\lambda^{BC}(e^{\epsilon_i}) = V_\lambda^{-1} \sum_{\sigma \in \Gamma} \sigma \left(e^\lambda \prod_{\alpha \in R^+} \frac{1 - t_\alpha\, t_{2\alpha}^{\frac{1}{2}}\, e^{-\alpha}}{1 - t_{2\alpha}^{\frac{1}{2}}\, e^{-\alpha}} \right) \tag{6}$$

where $\lambda \in \Lambda_n^+$ is identified with dominant weights of R, and $\{e^{\epsilon_i} \,|\, 1 \le i \le n\}$ are the independent variables of the polynomial. For $x \in \mathbb{C}^n$, $\sigma\, e^x = e^{\sigma\, x}$. V_λ is given in [**18**], and is independent of the e^{ϵ_i}'s. The $\{t_\alpha \,|\, \alpha \in R\}$ are parameters. We assign them values as follows:

In Case 1: If α is a short root of Σ, let $t_\alpha = -1$ and $t_{2\alpha}^{\frac{1}{2}} = -q^{-\frac{1}{2}}$, if α is a long root of Σ, let $t_\alpha = q^{-1}$.

In Case 2: If α is a short root of Σ, let $t_\alpha = -q^2$ and $t_{2\alpha}^{\frac{1}{2}} = -q^{-1}$, if α is a long root of Σ, let $t_\alpha = q^{-2}$.
In Case 3: If α is a short root of Σ, let $t_\alpha = 1$ and $t_{2\alpha}^{\frac{1}{2}} = -q^{-\frac{1}{2}}$, if α is a long root of Σ, let $t_\alpha = q^{-1}$.

If α is not a root in R we set $t_\alpha^{\frac{1}{2}} = 1$. For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, let

$$P_z(\lambda) = P_\lambda^{BC}(e^{\epsilon_i})_{|e^{\epsilon_i} := q_1^{z_i}}$$

be the value of $P_\lambda^{BC}(e^{\epsilon_i})$ after assigning for all $i = 1, \ldots, n$

$$e^{\epsilon_i} = e^{\epsilon_i}(z) = q_1^{z_i}.$$

It is clear from the definitions that

$$P_{\sigma z}(\lambda) = P_z(\lambda), \ \ \sigma \in \Gamma.$$

For $x = (x_1, \ldots, x_n), \ y = (y_1, \ldots, y_n) \in \mathbb{C}^n$ let $x \cdot y = \sum_{i=1}^n x_i y_i$ and let

$$\rho = \left(n - \frac{1}{2}, n - \frac{3}{2}, \ldots, \frac{1}{2}\right) \in \mathbb{C}^n.$$

The first main result of this work is:

Theorem 1.2. *Let $\Omega(s)$ be a relative spherical function on S, then $\exists\, z \in \mathbb{C}^n$ such that $\forall \lambda \in \Lambda_n^+$*

$$\Omega(d_\lambda) = q_1^{-(\lambda\cdot\rho)} \frac{V_\lambda}{V_0} P_z(\lambda). \tag{7}$$

We then have $\Omega = \Omega_z$, where Ω_z is defined in (40). Let $\mathcal{S}(K\backslash S)$ be the $\mathcal{H}(G, K)$-submodule of K-invariant functions on S, which are compactly supported. For $\phi \in \mathcal{S}(K\backslash S)$ we define its spherical Fourier transform:

$$\hat{\phi}(z) = \int_S \phi(s)\, \Omega_z(s)\, ds \tag{8}$$

where ds is the G-invariant measure on S normalized so that $\int_{K\cdot d_0} ds = 1$. To describe the support of the Plancherel measure we introduce the following notation: We let X_0 be the direct product of n copies of $\sqrt{-1}\left(\mathbb{R}/\frac{2\pi}{\log q_1}\mathbb{Z}\right)$. In Case 2 we also let

$$X^{(1)} = \left\{ z = (z_1, \ldots, z_n) \,\middle|\, z_1 = \frac{1}{2},\, z_i \in \sqrt{-1}\left(\mathbb{R}/\frac{2\pi}{\log q_1}\mathbb{Z}\right),\, i > 1 \right\}$$

and

$$X_1 = \Gamma\, X^{(1)}.$$

Thus X_1 is the set of all n-tuples, with one co-ordinate being equal to $\pm\frac{1}{2}$ and the other co-ordinates in the circles $\sqrt{-1}\left(\mathbb{R}/\frac{2\pi}{\log q_1}\mathbb{Z}\right)$. Define

$$\Delta(z) = \prod_{\alpha\in R} \frac{1 - t_{2\alpha}^{\frac{1}{2}}\, e^{\alpha}}{1 - t_{2\alpha}^{\frac{1}{2}}\, t_\alpha\, e^{\alpha}},$$

and in Case 2 let

$$\Delta^{(1)}\left(z^{(1)}\right) = \lim_{z_1\to\frac{1}{2}} \Delta(z)\left(1 + t_{2\epsilon_1}^{\frac{1}{2}} e^{\epsilon_1}\right),$$

here $z^{(1)} = (z_2, \ldots, z_n)$ is the $(n-1)$-tuple with no 1-st coordinate. In fact we will view $\Delta^{(1)}$ as a function on $X^{(1)}$ and as in [**16**] Chapter V we define the Γ-invariant function Δ_1 on X_1 by

$$\Delta_1(\sigma\, z) = \Delta^{(1)}(z)$$

for $z \in X^{(1)}$, $\sigma \in \Gamma$. Let

$$\Gamma_1 = \{\sigma \in \Gamma \,|\, \sigma\, X^{(1)} = X^{(1)}\},$$

then $|\Gamma| = 2^n\, n!$ and $|\Gamma_1| = 2^{n-1}\,(n-1)!$.

Theorem 1.3. *There is a measure $d_\mu(z)$ such that for $\phi \in \mathcal{S}(K\backslash S)$:*

$$\phi(s) = \int \hat{\phi}(z)\, \Omega_z(s)\, d_\mu(z)\,. \tag{9}$$

In Case 1 and in Case 3 the measure $d_\mu(z)$ is supported on X_0, and is given by:

$$d_\mu(z) = \frac{1}{|\Gamma|}\, V_0\, \Delta(z)\, dz. \tag{10}$$

In Case 2 the measure $d_\mu(z) = d_{\mu_0}(z) + d_{\mu_1}(z)$ where $d_{\mu_0}(z)$ is supported on X_0 and is given by:

$$d_{\mu_0}(z) = \frac{1}{|\Gamma|}\, V_0\, \Delta(z)\, dz \tag{11}$$

and $d_{\mu_1}(z)$ is supported on X_1 and is given by:

$$d_{\mu_1}(z) = \frac{1}{|\Gamma_1|}\, V_0\, \Delta_1(z)\, dz. \tag{12}$$

In all cases dz is the Haar-Lebesgue measure of volume one.

The remainder of this work is structured as follows: Chapter 2 is a collection of generalities to be used in what follows. In Chapter 3 the decomposition of the symmetric spaces into K-orbits is proved. Chapter 4 is a qualitative classification of the relative spherical functions. It is an adaptation to the relevant cases of the method used in [**13**]. In Chapter 5 a formula for the relative spherical functions is computed and the main results are proved.

Here the method is that used in [3] and in [4]. A new component is the need to show the vanishing of some 'irrelevant' intertwining operators. In [19], Z. Mao and S. Rallis solved a similar problem where $G = Sp_{2n}(F)$ and $H = Sp_n(F) \times Sp_n(F)$. Proposition 5.15 is a straightforward application of their work. Chapter 6 is an application of the classification of the relative spherical functions on S. It classifies the H-distinguished, irreducible, admissible, spherical representations of G.

One hopes that the results of this work will contribute to the study of the automorphic spectrum in the sense of [14], of the three cases of symmetric spaces. The study of distinguished representations has its origins in [11]. Amongst the papers relevant to the three cases discussed in this work, are: [8], where S. Friedberg and H. Jacquet obtain a characterization of distinguished representations relevant to Case 1 in terms of poles of certain L-functions, a result suggested by [2]. [7] and [6], where Y. Flicker studies $GL(m, F)$-distinguished representations on $GL(m, E)$ relevant to Case 2, and compares them with representations on the unitary group. In [9], Guo proves a fundamental lemma for the Hecke unit element, comparing between orbital integrals associated to Case 1 and to Case 3. Motivated by the success of Z. Mao and S. Rallis [19] in a different case of a fundamental lemma, now that the relevant Plancherel measures are available, one hopes to generalize Guo's fundamental lemma to a general Hecke element.

This work was given to me as a thesis problem by my advisor Hervé Jacquet, it is with great pleasure that I thank him for making it possible. Many thanks to Z. Mao and S. Rallis for their helpful advice. I also thank the referee for filling up a gap in the definition of the relative spherical functions.

2. Preliminaries

2.1. Root systems and Macdonald polynomials. Let Φ be the reduced root system of type A_m. Let $\{e_i \mid i = 1, \dots, m\}$ be the standard basis of $\mathbb{C}^m$. We fix a choice of positive roots Φ^+ in Φ:

$$\Phi^+ = \{e_i - e_j \mid 1 \le i < j \le m\}. \tag{13}$$

The natural action of the Weyl group W of Φ on $\mathbb{C}^m$ identifies W with the symmetric group of m variables. As in (6), we recall here the definition of the Macdonald polynomials attached to the admissible pair (Φ, Φ) of root systems ([18] (10.1)):

$$P_\lambda^A(E^{e_i}) = \left(V_\lambda^A\right)^{-1} \sum_{w\in W} w\left(E^\lambda \prod_{a\in\Phi^+} \frac{1 - t_a\, t_{2a}^{\frac{1}{2}}\, E^{-a}}{1 - t_{2a}^{\frac{1}{2}}\, E^{-a}}\right) \tag{14}$$

where $\lambda \in \Lambda_m^+$ is identified with dominant weights of Φ, and $\{E^{e_i} \mid i = 1, \dots, m\}$ are the independent variables of the polynomial. The parameters

t_a are assigned the values $t_a = q^{-1}, a \in \Phi$ and $t_a^{\frac{1}{2}} = 1$ if a is not a root in Φ. V_λ^A is given in [**18**] and is independent of the E^{e_i}'s. For $\nu = (\nu_1, \ldots, \nu_m) \in \mathbb{C}^m$ let $Q_\nu^A(\lambda)$ be the value of $P_\lambda^A(E^{e_i})$ after assigning $E^{e_i} = q_1^{-\nu_i}$, $i = 1, \ldots, m$. The polynomials $Q_\nu^A(\lambda)$ are also known as the Hall-Littlewood polynomials ([**17**] (2.1)). For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ let

(15) $$\nu(z) = (z_1, \ldots, z_n, -z_n, \ldots, -z_1)$$

if m is even, and

(16) $$\nu(z) = (z_1, \ldots, z_n, 0, -z_n, \ldots, -z_1)$$

if m is odd. We will be interested in $Q_{\nu(z)}^A(\lambda)$, $\lambda \in \Lambda_n^+$, where Λ_n^+ is viewed as a subset of Λ_m^+. For the root systems R and Σ defined in (4) and (5), the natural action of the Weyl group Γ on $\mathbb{C}^n$ identifies Γ with the signed permutation group in n variables. We may also view Γ as a subgroup of W through the action:

(17) $$\sigma\,\nu(z) = \nu(\sigma\,z).$$

Given any root system Σ with Weyl group W_Σ and a fixed choice of positive roots Σ^+, for any $w \in W_\Sigma$ we denote $\Sigma_w^+ = \{\alpha \in \Sigma^+ \mid w\,\alpha \notin \Sigma^+\}$. Let $\nu = (\nu_1, \ldots, \nu_m) \in \mathbb{C}^m$. For $a = e_i - e_j \in \Phi$ define

$$c_a(\nu) = \frac{1 - q_1^{-1}\, q_1^{\nu_j - \nu_i}}{1 - q_1^{\nu_j - \nu_i}}.$$

For $w \in W$ let

(18) $$c_w(\nu) = \prod_{a \in \Phi_w^+} c_a(\nu).$$

We list here results on the Macdonald polynomials $P_z(\lambda)$. For proofs we refer to [**18**]. We should remark, that all definitions and results in [**18**] are in terms of the P_λ^{BC}'s, our translation to the $P_z(\lambda)$'s, should be thought of as applying the specialization, defined in Chapter 1, in terms of the complex variable $z \in \mathbb{C}^n$, after performing the algebraic operations in terms of the P_λ^{BC}'s. We denote

$$\mathbb{C}[q_1^z]^\Gamma = \mathbb{C}[q_1^{z_1}, \ldots, q_1^{z_n}, q_1^{-z_n}, \ldots, q_1^{-z_1}]^\Gamma.$$

Let

$$m_\lambda = \sum_{\mu \in \Gamma \cdot \lambda} e^\mu.$$

The set $\{m_\lambda \mid \lambda \in \Lambda_n^+\}$, is the standard basis of $\mathbb{C}[q_1^z]^\Gamma$. Define a partial order in Λ_n^+ by $\lambda > \mu$ if and only if $\lambda \neq \mu$ and $\lambda - \mu \in \mathbb{N}^n$. It is proved in

[18] that $\forall\,\lambda \in \Lambda_n^+$, there are constants $u_{\mu\,\lambda} \in \mathbb{C}$ such that:

$$P_z\,(\lambda) = m_\lambda + \sum_{\mu<\lambda} u_{\mu\,\lambda}\, m_\mu. \tag{19}$$

Let

$$\Delta = \prod_{\alpha\in R} \frac{1 - t_{2\alpha}^{\frac{1}{2}}\, e^\alpha}{1 - t_{2\alpha}^{\frac{1}{2}}\, t_\alpha\, e^\alpha}$$

where the parameters t_α are assigned values as in Chapter 1. In [18] (3.4) a scalar product on $\mathbb{C}[q_1^z]^\Gamma$ is defined by:

$$\langle f, g\rangle = |\Gamma|^{-1}[f\bar{g}\Delta]_1. \tag{20}$$

Notations are the same as in [18] Section 3. The following is proved in [18]: If $\lambda \neq \mu$, in Λ_n^+ then:

$$\langle P_z\,(\lambda)\,, P_z\,(\mu)\rangle = 0 \tag{21}$$

and

$$\langle P_z\,(\lambda)\,, P_z\,(\lambda)\rangle = V_\lambda^{-1}. \tag{22}$$

2.2. Intertwining operators. Let $P = A\,N$ be the standard Borel subgroup of G, N is its unipotent radical and A is the diagonal subgroup. For $\nu = (\nu_1, \dots, \nu_m) \in \mathbb{C}^m$, let χ_ν be the character on P defined by,

$$\chi_\nu\,(a\,n_1) = \prod_{i=1}^m |a_i|^{\nu_i}$$

where $a = \mathrm{diag}[a_1, \dots, a_m] \in A$, $n_1 \in N$. We will also denote then $\chi_\nu = (|\,|^{\nu_1}, \dots, |\,|^{\nu_m})$. Let $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ be such that $\mathrm{Re}\, z_i > z_{i+1} + 1$, $i = 1, \dots, n-1$ and $\mathrm{Re}\, z_n > 1$, and let $\chi = \chi_{\nu(z)}$. χ is then regular in the sense that if ${}^w\chi = \chi$ for $w \in W$, then $w = 1$. Let $I\,(\chi)$ denote the space of the principal series, unramified representation of G induced from χ. It is the action $R(g)$ of G, by right translations, on the space $I(\chi)$ of functions $\varphi : G \to \mathbb{C}$ which are right invariant by some open subgroup of G and satisfy

$$\varphi(p\,g) = \chi\,\delta^{\frac{1}{2}}(p)\,\varphi(g)$$

for all $p \in P$ and $g \in G$. Here δ is the topological module of P, defined by

$$\delta(a\,n_1) = \prod_{i=1}^m |a_i|^{m+1-2i}$$

whenever $a = \mathrm{diag}[a_1, \dots, a_m] \in A$ and $n_1 \in N$. Under our assumptions on z, $I(\chi)$ is irreducible. Defined in [3], there is a projection $\mathcal{P}_\chi : C_c^\infty\,(G) \to I\,(\chi)$. For $f \in C_c^\infty\,(G)$ it is given by:

$$\mathcal{P}_\chi\,(f)\,(g) = \int_P \chi^{-1}\,\delta^{1/2}\,(p)\;f\,(pg)\;d_L p \tag{23}$$

where $d_L p$ is the left Haar measure on P such that $\int\limits_{P\cap K} d_L p = 1$. $\mathcal{P}_\chi$ is G-equivariant under right translations, i.e., for all $g, g' \in G$,

$$\mathcal{P}_\chi\left(R\left(g'\right) f\right)(g) = \mathcal{P}_\chi(f)\left(gg'\right). \tag{24}$$

For a compact open set $X \subset G$, let

$$\varphi_{X,\chi} = \mathcal{P}_\chi(\mathrm{ch}_X)$$

be the image of the characteristic function of X under the projection $\mathcal{P}_\chi$. Let

$$\mathcal{D}(G) = C_c^\infty(G)^*$$

be the space of distributions on G. For $T \in \mathcal{D}(G)$, $f \in C_c^\infty(G)$ denote by $\langle T, f\rangle$ the value of f applied to T. By [**12**], the map dual to $\mathcal{P}_\chi$ defines an isomorphism

$$\mathcal{P}_\chi^* : I(\chi)^* \xrightarrow{\sim} \mathcal{D}(G)_{\chi^{-1}} \tag{25}$$

where

$$\mathcal{D}(G)_{\chi^{-1}} = \Big\{T \in \mathcal{D}(G) \mid \langle T, f^{p^{-1}}\rangle = \chi^{-1}\,\delta^{1/2}(p)\,\langle T, f\rangle, \\ f \in C_c^\infty(G),\ p \in P\Big\}$$

and $f^p(g) = f(p\,g)$. For $\nu \in \mathbb{C}^m$ we denote

$$c_w(\chi_\nu) = c_w(\nu),$$

for the constants $c_w(\nu)$ defined in (18). In what follows we define certain intertwining operators between spaces of unramified principal series representations and we list their properties relevant to this work. For a more complete treatment, one may refer to [**3**]. For $a \in \Phi^+$, let N_a be the subgroup of N associated to the root a, notations being as in [**3**]. For $w \in W$, let $N_w = \prod\limits_{a\in\Phi_w^+} N_a$, then $N_w \simeq \left(wNw^{-1} \cap N\right)\backslash N$. Whenever $\mathrm{Re}\,\nu_1 > \cdots > \mathrm{Re}\,\nu_m$, the intertwining operator $T_w = T_{w,\chi_\nu} : I(\chi_\nu) \to I({}^w\chi_\nu)$ is defined by the convergent integral:

$$(T_w\varphi)(g) = \int\limits_{N_w} \varphi\left(w^{-1}\,n\,g\right)\,dn \tag{26}$$

for all $\varphi \in I(\chi_\nu)$, $g \in G$. The Haar measure on N_w is normalized through the isomorphism with $\left(wNw^{-1} \cap N\right)\backslash N$ so that the orbit of I_m under $N \cap K$ has measure 1 in the N-invariant measure on $\left(wNw^{-1} \cap N\right)\backslash N$. For a general $\nu \in \mathbb{C}^m$ the intertwining operator T_{w,χ_ν} is defined by analytic continuation. It satisfies

$$T_w(\varphi_{K,\chi}) = c_w(\chi)\ \varphi_{K,{}^w\chi}. \tag{27}$$

In [**12**] it is shown that T_w extends to an intertwining operator $\widetilde{T}_w : I\left(\chi^{-1}\right)^* \to I\left({}^w\chi^{-1}\right)^*$, which is a constant multiple of the operator $T^*_{w^{-1}}$ dual to $T_{w^{-1}} : I\left({}^w\chi^{-1}\right) \to I\left(\chi^{-1}\right)$. The constant is given by:

$$\widetilde{T}_w = \frac{c_w\left(\chi\right)}{c_{w^{-1}}\left({}^w\chi^{-1}\right)} T^*_{w^{-1}}. \tag{28}$$

3. *K*-orbit decomposition of *S*

For $g \in G$, $1 \le i \le m$ let

$$\|g\|_i = \max\left\{|\det X| \mid X \text{ is an } i\times i \text{ minor of } g\right\}.$$

Proposition 3.1. *The K-orbits of S are given by the disjoint union*

$$S = \coprod_{\lambda\in\Lambda_n^+} K\cdot d_\lambda. \tag{29}$$

3.1. Case 1.

Proof. For $\lambda \in \Lambda_n^+$ let $g_\lambda = \begin{pmatrix} -I_n & I_n \\ \varpi^{\lambda^*} & \varpi^{\lambda^*} \end{pmatrix}$, then

$$\theta\left(g_\lambda\right) = d_\lambda. \tag{30}$$

Since for $\mu \ne \lambda$ in Λ_n^+, $\exists\, i \le n$ such that

$$q^{\lambda_1+\cdots+\lambda_i} = \|d_\lambda\|_i \ne \|d_\mu\|_i = q^{\mu_1+\cdots+\mu_i},$$

we get that $\underset{\lambda\in\Lambda_n^+}{\cup} K\cdot d_\lambda$ is indeed a disjoint union in S. To prove the equality it is enough to show that

$$G = \underset{\lambda\in\Lambda_n^+}{\cup} K\, g_\lambda\, H.$$

Let $g \in G$, by the Iwasawa decomposition $\exists\, k \in K$, $h \in H$, $X \in M_n\left(F\right)$ such that

$$g = k\begin{pmatrix} I_n & X \\ & I_n \end{pmatrix} h.$$

Since $\forall\, k_1,\, k_2 \in GL\left(n, \mathcal{O}_{\mathcal{F}}\right)$,

$$g = k\begin{pmatrix} k_1^{-1} & 0 \\ 0 & k_2 \end{pmatrix}\begin{pmatrix} I_n & k_1\, X\, k_2 \\ 0 & I_n \end{pmatrix}\begin{pmatrix} k_1 & 0 \\ 0 & k_2^{-1} \end{pmatrix} h,$$

using the Cartan decomposition of X, $\exists\, k \in K$, $h \in H$, $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ satisfying $m_1 \ge \cdots \ge m_n$, such that

$$g = k\begin{pmatrix} I_n & \varpi^m \\ & I_n \end{pmatrix} h.$$

Note that for all $Y \in M_n\left(\mathcal{O}_F\right)$,

$$g = k\begin{pmatrix} I_n & -Y \\ & I_n \end{pmatrix}\begin{pmatrix} I_n & Y + \varpi^m \\ & I_n \end{pmatrix} h.$$

By choosing Y to be the anti-diagonal matrix with y_i in the $(n+1-i,i)$-entry, where

$$y_i = \begin{cases} 1-\varpi^{m_i} & m_i > 0 \\ 0 & \text{else,} \end{cases}$$

we may assume $m_1 \le 0$. Thus $\exists\, \lambda \in \Lambda_n^+$, $k \in K$, $h \in H$ such that

$$g = k \begin{pmatrix} I_n & \varpi^{\lambda^*} \\ & I_n \end{pmatrix} h.$$

It is now enough to show that

$$\begin{pmatrix} I_n & \varpi^{\lambda^*} \\ & I_n \end{pmatrix} \in K\, g_\lambda\, H.$$

Let

$$k = \begin{pmatrix} \varpi^{\lambda} & \varpi\,\varpi^{\lambda} - 2I_n \\ -I_n & \varpi\, I_n \end{pmatrix} \in K,\; h = \begin{pmatrix} -\varpi^{\lambda^*} & \\ & \left(\varpi\,\varpi^{\lambda} - I_n\right)^{-1} \end{pmatrix} \in H$$

then

$$k \begin{pmatrix} I_n & \varpi^{\lambda^*} \\ & I_n \end{pmatrix} h = g_\lambda.$$

□

3.2. Case 2 and Case 3.

Proof. We start with the following two lemmas:

Lemma 3.2.

$$S = \begin{cases} \{g \in G \,|\, g\bar{g} = I_m\} & \textit{Case } 2 \\ \{g \in G \,|\, g^2 = \tau\, I_m\} & \textit{Case } 3. \end{cases} \tag{31}$$

Proof. In Case 3 this is proved in [10]. For Case 2 clearly, $s\bar{s} = I_m$ for all $s \in S$. By [1] Lemma 1.1, if $x\,\bar{x}$ and $y\,\bar{y}$ are H- conjugate then x and y are G-twisted conjugate, for all $x, y \in G$, i.e., $\exists\, g \in G$ such that $g\,x\,\bar{g}^{-1} = y$. Thus for any $s \in G$ such that $s\,\bar{s} = I_m$, s is twisted conjugate to I_m, and hence $\exists\, g \in G$ such that $\theta\,(g) = s$. □

Lemma 3.3. *In Case* 2:

$$S \cap K = K \cdot d_0. \tag{32}$$

Proof. Since $K \cdot d_0 \subset S \cap K$, it is enough to show that $S \cap K$ is a unique K-orbit. We will show that $S \cap K = K \cdot I_m$. Since $\theta\,(K\,H) = K \cdot I_m$, to show $S \cap K = K \cdot I_m$, it is enough to show that if $g \in G$ is such that $\theta\,(g) \in K$ then $g \in K\,H$. Thus given $g \in G$ such that $\theta\,(g) \in K$, we are free to conclude the result on $k\,g\,h$ for any $k \in K$, $h \in H$. Multiplying by some $k \in K$ from the left we may assume $g \in P$. If the diagonal entries of g are $u_i\,\varpi^{n_i}$, $u_i \in \mathcal{O}_E^\times$, $n_i \in \mathbb{Z}$, $i = 1, \ldots, m$, then multiplying by $\mathrm{diag}[u_1^{-1}, \ldots, u_m^{-1}] \in K$ from the left, and by $\mathrm{diag}[\varpi^{-n_1}, \ldots, \varpi^{-n_m}] \in H$

from the right, we may assume $g \in N$. Thus $g = h_1 + \iota\, h_2$ where $h_1 \in N\,(F)$ and $h_2 \in M_m\,(F)$ is an upper triangular nilpotent matrix. So multiplying by h_1^{-1} from the right, we may assume $g \in N$ is such that its entries above the diagonal are all in $\iota\, F$. Let $x_i \in \iota\, F$ be the $i, i+1$ entry of g, $i = 1, \ldots, m-1$. Since $\theta\,(g) = g\,\bar{g}^{-1} \in S \cap K$ and since $\left(g\,\bar{g}^{-1}\right)_{i\,i+1} = x_i - \bar{x}_i = 2\,x_i$ we see that $x_i \in \iota\,\mathcal{O}_F$, $i = 1, \ldots, m-1$. So the matrix

$$k = \begin{pmatrix} 1 & -x_1 & & \\ & \ddots & \ddots & \\ & & & -x_{m-1} \\ & & & 1 \end{pmatrix}$$

is in K. Replacing g by $k\,g$ we may assume $g \in N$ is such that $(g)_{i\,i+1} = 0$, $i = 1, \ldots, m-1$. We now proceed by induction. If $g \in N$ is such that $\theta\,(g) \in K$ and $(g)_{i\,i+j} = 0$, $1 \leq j < j_0$, $i \leq m - j$, then multiplying g from the right by the inverse of its 'real' part, as before, we may assume in addition that all entries of g above the diagonal are in $\iota\, F$. This combined with the fact that $\theta\,(g) \in K$ implies that $(g)_{i\,i+j_0} \in \iota\,\mathcal{O}_F$ for all $i \leq j_0$, and therefore, $\exists\, k \in K$ such that $(k\,g)_{i\,j} = 0$, $1 \leq j \leq j_0$, $i \leq m - j$. So we showed that $\exists\, k \in K$, $h \in H$ such that $k\,g\,h = I_m$. $\square$

As in Case 1, the right-hand side of (29) is a disjoint union in S. Note that in Case 2, for each $s \in S$, since $s\bar{s} = I_m$, we have $|\det s| = 1$. So $S \cap K = S \cap M_m\,(\mathcal{O}_F)$ and for $s \in S$ we get, $s \in K$ if and only if $\|s\|_1 = 1$. Let $s = (s_{i\,j}) \in S$. If $\|s\|_1 \leq 1$ then by the above remark $\|s\|_1 = 1$ and by Lemma 3.3, $s \in K \cdot d_0$. So in Case 2 we may assume $\|s\|_1 > 1$. We first show that $\exists\, i, j$, $1 \leq i \neq j \leq m$, such that $\|s\|_1 = |s_{i\,j}|$. In Case 3 if $\|s\|_1 \leq 1$ then since $s^2 = \tau I_m$ we have $\|s\|_1 = 1$ and if $1 = |s_{i\,i}| > |s_{i\,j}|$ for all $j \neq i$ comparing the (i,i)-entries of $s^2 = \tau I_m$ we see that $|\tau - s_{i\,i}^2| < 1$ which means the residual fields associated to E and F are the same. This contradicts our assumption that E/F is unramified. If $\|s\|_1 > 1$ is not obtained in an entry off the diagonal, then for some i,

$$\|s\|_1 = |s_{i\,i}| > |s_{i\,j}|,\, |s_{j\,i}|,$$

for all $j \neq i$. Since $s\,\bar{s} = I_m$, we have

$$1 = \left|\sum_{j=1}^{m} s_{i\,j}\bar{s}_{j\,i}\right| = |s_{i\,i}\bar{s}_{i\,i}|$$

in Case 2, and since $s^2 = \tau I_m$ we have

$$1 = \left|\sum_{j=1}^{m} s_{i\,j}\, s_{j\,i}\right| = |s_{i\,i}^2|$$

in Case 3, in contradiction to our assumption. Thus if $i \neq j$ are such that $\|s\|_1 = |s_{i\,j}|$, let $w \in G$ be the permutation matrix associated to the permutation that interchanges between i and m and between j and 1. Since in Case 2 $w \in H$, in both cases it acts on S by standard conjugation, so $(w \cdot s)_{m\,1} = s_{i\,j}$. Replacing s by $w \cdot s$ we may assume $\|s\|_1 = |s_{m\,1}|$. So the matrix

$$k = \begin{pmatrix} 1 & & & -\frac{s_{1\,1}}{s_{m\,1}} \\ & \ddots & & \vdots \\ & & 1 & -\frac{s_{m-1\,1}}{s_{m\,1}} \\ & & & 1 \end{pmatrix}$$

is in K, and the first column of $k \cdot s$ is

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ s_{m\,1} \end{pmatrix}.$$

Imposing the condition $k \cdot s\left(\overline{k \cdot s}\right) = I_m$ in Case 2 and $(k \cdot s)^2 = \tau\, I_m$ in Case 3 we get that

$$k \cdot s = \begin{pmatrix} 0 & & \widetilde{s}_{m\,1} \\ \vdots & & 0 \\ & * & \\ 0 & & \vdots \\ s_{m\,1} & & 0 \end{pmatrix} \tag{33}$$

where

$$\widetilde{s}_{m\,1} = \begin{cases} \bar{s}_{m\,1}^{-1} & \text{Case 2} \\ \tau\, s_{m\,1}^{-1} & \text{Case 3.} \end{cases}$$

Replacing s by $k \cdot s$ we may assume that s has the form (33). The matrix

$$k_1 = \begin{pmatrix} 1 & -\frac{s_{m\,2}}{s_{m\,1}} & \cdots & -\frac{s_{m\,m-1}}{s_{m\,1}} & 0 \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$$

is again in K. We have, $\bar{k}_1^{-1} \cdot s$ in Case 2 and $k_1^{-1} \cdot s$ in Case 3 has the form

$$\begin{pmatrix} 0 & & & \\ \vdots & & * & \\ 0 & & & \\ s_{m\,1} & 0 & \dots & 0 \end{pmatrix},$$

and a matrix of that form in S must have the form

$$\begin{pmatrix} & & \widetilde{s}_{m\,1} \\ & s' & \\ s_{m\,1} & & \end{pmatrix}$$

where $s' \in GL\,(m-2, E)$ is such that $s'\,\overline{s'} = I_{m-2}$ in Case 2 and ${s'}^2 = \tau\, I_{m-2}$ in Case 3. We assume then that s is of that form. If

$$s_{m\,1} = \begin{cases} u\,\varpi^{-\lambda} & \text{Case 2} \\ u\,\tau\,\varpi^{-\lambda} & \text{Case 3,} \end{cases}$$

where $\lambda > 0$ and $|u| = 1$, then $k_2 = \mathrm{diag}[1, \ldots, 1, u^{-1}] \in K$.

In Case 2:

$$k_2 \cdot s = \begin{pmatrix} & & \varpi^{\lambda} \\ & s' & \\ \varpi^{-\lambda} & & \end{pmatrix}. \tag{34}$$

In Case 3:

$$k_2 \cdot s = \begin{pmatrix} & & \varpi^{\lambda} \\ & s' & \\ \tau\,\varpi^{-\lambda} & & \end{pmatrix}. \tag{35}$$

Using Lemma 3.2, the proposition now follows by induction on m. For the sake of completeness we must remark that the base of induction is the cases $m = 0$ where there is nothing to prove, and in Case 2 $m = 1$ where the proposition follows from Hilbert 90. □

4. The relative spherical functions

For $\nu = (\nu_1, \ldots, \nu_m) \in \mathbb{C}^m$ let Φ_ν be the function on G defined by

$$\Phi_\nu\,(g) = \prod_{i=1}^{m} |a_i|^{\nu_i - \frac{1}{2}(m+1-2i)} \tag{36}$$

where $g = n_1\,a\,k$, is the Iwasawa decomposition of g, $a = \mathrm{diag}[a_1, \ldots, a_m] \in A$, $n_1 \in N$, $k \in K$. The Satake transform of a function $f \in \mathcal{H}\,(G, K)$ is defined by:

$$\hat{f}\,(\nu) = \int_G f\,(g)\,\Phi_\nu\,(g)\,dg. \tag{37}$$

By [20], it defines an isomorphism of the algebras:

$$\mathcal{H}\,(G, K) \simeq \mathbb{C}[q_1^{\pm\nu_1}, \ldots, q_1^{\pm\nu_m}]^W. \tag{38}$$

For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ define

$$\widetilde{f}\,(z) = \hat{f}\,(\nu\,(z)).$$

By abuse of notation, denote:

$$\mathbb{C}[q_1^{-z}, q_1^{z}]^W = \Big\{P(q_1^{-z_1}, \dots, q_1^{-z_n}, q_1^{z_n}, \dots, q_1^{z_1}) \,|\, P\,(X_1, \dots, X_m) \in \mathbb{C}[X_1, \dots, X_m]^W\Big\},$$

whenever m is even and,

$$\mathbb{C}[q_1^{-z}, q_1^{z}]^W = \Big\{P(q_1^{-z_1}, \dots, q_1^{-z_n}, 1, q_1^{z_n}, \dots, q_1^{z_1}) \,|\, P\,(X_1, \dots, X_m) \in \mathbb{C}[X_1, \dots, X_m]^W\Big\},$$

whenever m is odd. It is then clear from (38), that the transform $f \mapsto \widetilde{f}\,(z)$ is a surjective homomorphism of algebras:

$$\mathcal{H}\,(G, K) \to \mathbb{C}[q_1^{-z}, q_1^{z}]^W. \tag{39}$$

4.1. Definition of the relative spherical functions. For $s \in S$, let $d_i\,(s)$ be the determinant of the lower left $i \times i$ block of s, $i = 1, \dots, n$. Let

$$S' = \left\{ s \in S \;\Big|\; \prod_{i=1}^{n} d_i(s) \neq 0 \right\},$$

and let $\mathrm{ch}_{S'}$ be the characteristic function of S'. We define the functions

$$d_t\,(s) = \mathrm{ch}_{S'}\,(s) \prod_{i=1}^{n} |d_i\,(s)\,|^{t_i},$$

for $t = (t_1, \dots, t_n) \in \mathbb{C}^n$, $s \in S$. Let

$$\omega_t\,(s) = \int_K d_t\,(k \cdot s)\,dk$$

and define

$$\Omega_z\,(s) = \frac{\omega_t\,(s)}{\omega_t\,(d_0)} \tag{40}$$

where $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ is related to t through the linear translations:

$$\begin{cases} t_i = z_i - z_{i+1} - 1 & 1 \leq i \leq n-1 \\ t_n = z_n - \frac{1}{2} \end{cases} \tag{41}$$

$$z_i = t_i + \dots + t_n + n - i + \frac{1}{2},\ i = 1, \dots, n.$$

To justify our definitions we state the following result of B. Deshommes [5]:

Theorem 4.1. *Let*

$$f=(f_1,\dots,f_m):F^k\to F^m$$

be a polynomial function, and let

$$D=\left\{x\in F^k \,\middle|\, \prod_{i=1}^{m} f_i\left(x\right)=0\right\}.$$

For $t=(t_1,\dots,t_m)\in\mathbb{C}^m$ *define* $|f\left(x\right)|^t=\prod_{i=1}^{m}|f_i\left(x\right)|^{t_i}$. *Let* Φ *be a smooth function of compact support on* F^k *and let* $w=(w_1,\dots,w_m)$ *with* $w_i=q^{-t_i}$. *Define*

$$Z_\Phi\left(w\right)=\int_{F^k-D}|f\left(x\right)|^t\,\Phi\left(x\right)\,dx.$$

The integral defining $Z_\Phi\left(w\right)$ *is convergent to a holomorphic function on* $0<|w_i|<1$. *Furthermore,* $Z_\Phi\left(w\right)$ *extends to a rational function of* w.

For each $k\in K$ we write $k^*=k'\det k^{-1}$. Since $|\det k|=1$ we have

$$\int d_t(k\cdot s)\,dk=\int d_t(k\,s\,k')\,dk. \tag{42}$$

In Case 1 and in Case 3 the entries of $k\,s\,k'$ are polynomials in the entries of k. In Case 2 only the norm $d_i(k\,s\,k')\,\bar{d}_i(k\,s\,k')$ is a polynomial in the entries of k, viewed over F, but

$$|d_i(k\,s\,k')|=|d_i(k\,s\,k')\,\bar{d}_i(k\,s\,k')|_F.$$

Hence in all three cases, over the ground field F, for all $i=1,\dots,n$ the functions

$$k\mapsto|d_i(k\,s\,k')|^{t_i}$$

are complex powers of polynomials in the entries of k. Therefore the right-hand side of (42) is indeed the integral of a product of complex powers of polynomials, taken over an open set. Thus by Theorem 4.1, the integral (40) converges for $Re\,t_1,\dots,Re\,t_n\geq 0$ and $\omega_t(s)$ extends to a rational function of $q^{z_1},\dots,q^{z_n}$. In particular this is true for $\omega_t(d_0)$. Note that $\omega_t\left(d_0\right)\neq 0$, because when all $t_i>0$ then $d_t\left(k\cdot d_0\right)=1$, for all k in the open subgroup of K, of matrices that project to diagonal matrices over the residual field. This justifies the definition of $\Omega_z(s)$ in (40). We deduce that $\Omega_z(s)$ is a rational function in $q^{z_1},\dots,q^{z_n}$ that satisfies

$$\Omega_z(d_0)=1.$$

The following shows that $\{\Omega_z\,|\,z\in\mathbb{C}^n\}$, is a family of relative spherical functions on S:

Lemma 4.2. *Let* $z\in\mathbb{C}^n$, *for all* $f\in\mathcal{H}\left(G,K\right)$*:*

$$f*\Omega_z\left(s\right)=\widetilde{f}\left(z\right)\Omega_z\left(s\right). \tag{43}$$

Proof. We compute

$$\begin{aligned}\omega_t(d_0)(f*\Omega_z)(s) &= \omega_t(d_0)\int\limits_G f(g)\,\Omega_z\left(g^{-1}\cdot s\right)dg\\ &= \int\limits_G f(g)\int_K d_t\left(k\,g^{-1}\cdot s\right)dk\,dg\\ &\underset{g\mapsto gk}{=}\int\limits_G f(g)\,d_t\left(g^{-1}\cdot s\right)dg\\ &= \int\limits_P f(p)\int\limits_K d_t\left(p^{-1}k^{-1}\cdot s\right)dk\,d_Rp,\end{aligned}$$

where d_Rp is the right Haar measure on P, such that $d_Rp = \delta(p)\,d_Lp$. If

$$s = \begin{pmatrix} * & * \\ C & * \end{pmatrix} \in S,$$

with C the bottom left $n\times n$ block of s, and if $p\in P$ has p_1 as the top left $n\times n$ block and p_2 as the bottom right $n\times n$ block, then

$$p^{-1}\cdot s = \begin{pmatrix} * & * \\ X & * \end{pmatrix}$$

where in Case 1 and in Case 3: $X = p_2^{-1}\,C\,p_1$ and in Case 2: $X = p_2^{-1}\,C\,\bar{p}_1$.

Thus if

$$p_1 = \begin{pmatrix} a_1 & & * \\ & \ddots & \\ & & a_n \end{pmatrix},\ p_2 = \begin{pmatrix} a_{n+1} & & * \\ & \ddots & \\ & & a_{2n} \end{pmatrix},$$

then for $i = 1,\dots,n$

$$\left|d_i\left(p^{-1}\cdot s\right)\right| = \prod_{j=1}^{i}\left|\frac{a_j}{a_{m+1-j}}\right| d_i(s),$$

and for all $s\in S$ we get:

$$d_t\left(p^{-1}\cdot s\right) = \prod_{i=1}^{n}\left|\prod_{j=1}^{i}\frac{a_j}{a_{m+1-j}}\right|^{t_i} d_t(s). \tag{44}$$

Using the linear relation (41) between t and z and replacing d_Rp with $\delta(p)\,d_Lp$ the integral above becomes:

$$= \left\{\int\limits_P f(p)\,\Phi_{\nu(z)}(p)\,d_Lp\right\}\omega_t(s) = \left\{\int\limits_G f(g)\,\Phi_{\nu(z)}(g)\,dg\right\}\omega_t(s),$$

and the lemma follows. ∎

4.2. The functional equations. The space $\mathcal{S}(K\backslash S)$ defined in the introduction, is spanned by the functions $\{\mathrm{ch}_\lambda \mid \lambda \in \Lambda_n^+\}$, where ch_λ is the characteristic function of the K-orbit $K \cdot d_\lambda$. It is a $\mathcal{H}(G,K)$- submodule of $C^\infty(K\backslash S)$. The spherical Fourier transform $\hat{}$ on $\mathcal{S}(K\backslash S)$, is defined in (8).

Proposition 4.3. *For all $s \in S$, $z \mapsto \Omega_z(s)$ is an entire function of $z \in \mathbb{C}^n$. Moreover it lies in $\mathbb{C}[q_1^{-z}, q_1^{z}]^W$. Equivalently, the image of $\mathcal{S}(K\backslash S)$ under the spherical Fourier transform $\hat{}$ is contained in $\mathbb{C}[q_1^{-z}, q_1^{z}]^W$.*

Proof. For all $\lambda \in \Lambda_n^+$, we have

$$\widehat{\mathrm{ch}_\lambda}(z) = \int\limits_{K\cdot d_\lambda} \Omega_z(s)\, ds = \left\{ \int\limits_{K\cdot d_\lambda} ds \right\} \Omega_z(d_\lambda). \tag{45}$$

Thus showing that for all $s \in S, \Omega_z(s) \in \mathbb{C}[q_1^{-z}, q_1^{z}]^W$ is indeed equivalent to showing that the image of the spherical Fourier transform lies in $\mathbb{C}[q_1^{-z}, q_1^{z}]^W$. Once this is proved, $\Omega_z(s)$ is entire. Thus it is enough to prove that $\widehat{\mathrm{ch}_\lambda}(z) \in \mathbb{C}[q_1^{-z}, q_1^{z}]^W$ for all $\lambda \in \Lambda_n^+$. To prove Proposition 4.3, we follow Hironaka-Sato [**13**]. Lemma 4.6 proves the difference equations relevant to the symmetric space S.

Lemma 4.4. *For all $f \in \mathcal{H}(G,K), \varphi \in \mathcal{S}(K\backslash S)$ the spherical Fourier transform satisfies*

$$(f * \varphi)\hat{}\,(z) = \widetilde{f}(z)\, \hat{\varphi}(z). \tag{46}$$

Proof. For $f \in \mathcal{H}(G,K)$ let $\check{f}(g) = f\left(g^{-1}\right), g \in G$. Then

$$\begin{aligned}
(f * \varphi)\hat{}\,(z) &= \int\limits_S \int\limits_G f(g)\, \varphi\left(g^{-1} \cdot s\right)\, dg\; \Omega_z(s)\, ds \\
&\underset{s\mapsto g\cdot s}{=} \int\limits_G \int\limits_S \Omega_z(g\cdot s)\ \varphi(s)\ ds\, f(g)\ dg \\
&= \int\limits_S \varphi(s) \int\limits_G \check{f}(g)\, \Omega_z\left(g^{-1}\cdot s\right) dg\, ds \\
&= \int\limits_S \varphi(s) \left(\check{f} * \Omega_z\right)(s)\ ds \underset{\text{Lemma 4.2}}{=} \widetilde{\check{f}}(z)\, \hat{\varphi}(z).
\end{aligned}$$

Since the Satake transform satisfies $\widehat{\check{f}}(\nu) = \hat{f}(-\nu)$ and since elements in $\mathbb{C}[q_1^{-z}, q_1^{z}]^W$ are invariant under $z_i \mapsto -z_i$ we get that $\widetilde{\check{f}}(z) = \widetilde{f}(z)$. □

For $\lambda \in \Lambda_n^+$, denote

$$|\lambda| = \sum_{i=1}^{n} \lambda_i,\, n(\lambda) = \sum_{i=1}^{n} (i-1)\,\lambda_i.$$

The length $l(\lambda)$ of a partition $\lambda \in \Lambda_n^+$ is defined to be the number of nonzero λ_i's. We define the order $\prec$ on Λ_n^+ by: $\mu \prec \lambda$ if and only if $\mu \neq \lambda$ and $\mu_{j_0} < \lambda_{j_0}$, where $j_0 = \max_{\{j \mid \mu_j \neq \lambda_j\}} j$. For $\lambda \in \Lambda_m^+$, let c_λ be the characteristic function of the double coset $K\,h_\lambda\,K$, where

$$h_\lambda = \begin{pmatrix} \varpi^{\lambda_1} & & \\ & \ddots & \\ & & \varpi^{\lambda_m} \end{pmatrix}.$$

By our convention, for $\lambda \in \Lambda_n^+$,

$$h_\lambda = \begin{pmatrix} \varpi^{\lambda_1} & & & \\ & \ddots & & \\ & & \varpi^{\lambda_n} & \\ & & & I_{m-n} \end{pmatrix}.$$

For a positive integer r let $h_r = h_{(r,0,\dots,0)}$, and denote $c_r = c_{(r,0,\dots,0)}$. $\forall\, \mu, \lambda \in \Lambda_n^+$ define,

$$N_\mu^\lambda(r) = \#\{K\,x \subset K\,h_r\,K \mid x \cdot d_\mu \in K \cdot d_\lambda\}.$$

Lemma 4.5. $\check{c}_r * \mathrm{ch}_\lambda = \sum_{\mu \in \Lambda_n^+} N_\mu^\lambda(r)\,\mathrm{ch}_\mu$.

Proof. Let $\varphi = \check{c}_r * \mathrm{ch}_\lambda$, then as a function in $\mathcal{S}(K\backslash S)$ we have:

$$\varphi = \sum_{\mu \in \Lambda_n^+} \varphi(d_\mu)\,\mathrm{ch}_\mu.$$

On the other hand:

$$\varphi(s) = \int_{K\,h_r\,K} \mathrm{ch}_\lambda(g \cdot s)\,dg = \sum_{Kx \subset K h_r K} \int_{Kx} \mathrm{ch}_\lambda(g \cdot s)\,dg$$
$$= \sum_{Kx \subset K h_r K} \mathrm{ch}_\lambda(x \cdot s).$$

Thus,

$$\varphi(d_\mu) = \sum_{Kx \subset K h_r K} \mathrm{ch}_\lambda(x \cdot d_\mu) = N_\mu^\lambda(r).$$

$\square$

Lemma 4.6. *Let* $\lambda = (\lambda_1, \dots, \lambda_l, 0, \dots, 0) \in \Lambda_n^+$ *with* $l(\lambda) = l$ *and denote* $r = \lambda_l$ *and* $\lambda' = (\lambda_1, \dots, \lambda_{l-1}, 0, \dots, 0)$*, then*

$$\check{c}_r * \mathrm{ch}_{\lambda'} = \alpha_\lambda \, \mathrm{ch}_\lambda + \sum_{\substack{\mu \prec \lambda \\ |\mu| \leq |\lambda|}} \beta_\mu \, \mathrm{ch}_\mu$$

where $\alpha_\lambda > 0,\ \forall \mu,\ \beta_\mu \geq 0$ *and* $\alpha_\lambda,\ \beta_\mu$ *are all integers.*

Proof. By Lemma 4.5 we have:

$$\check{c}_r * \mathrm{ch}_{\lambda'} = \sum_{\mu \in \Lambda_n^+} N_\mu^{\lambda'}(r) \, \mathrm{ch}_\mu .$$

Since for $D = \mathrm{diag}[1, \dots, 1, \underbrace{\varpi^r}_{(n-l+1)\text{-place}}, 1, \dots, 1] \in GL(n, F)$, we have

$$\begin{pmatrix} I_{m-n} & 0 \\ 0 & D \end{pmatrix} \in K \, h_r \, K$$

and

$$\begin{pmatrix} I_{m-n} & 0 \\ 0 & D \end{pmatrix} \cdot d_\lambda = d_{\lambda'}$$

we get that $N_\lambda^{\lambda'}(r) > 0$. Hence it is enough to show that if $N_\mu^{\lambda'}(r) \neq 0$ then $\mu \preceq \lambda$ and $|\mu| \leq |\lambda|$. We proceed by the following steps:

Step 1: For all $\upsilon \in \Lambda_n^+$, if $K \, h_\upsilon \, K \subset K \, h_{\lambda'} \, K \, h_r \, K$ then $\upsilon \preceq \lambda$ and $|\upsilon| = |\lambda|$.

Step 2: If $\exists \, y \in K \cdot d_\mu$ such that $h_\upsilon \cdot y \in K \cdot d_0$ then $\mu_i \leq \upsilon_i$, $i = 1, \dots, n$.

Step 3: If $N_\mu^{\lambda'}(r) \neq 0$ then $\exists \, \upsilon \in \Lambda_n^+$ such that $K \, h_\upsilon \, K \subset K \, h_{\lambda'} \, K \, h_r \, K$ and $\exists \, y \in K \cdot d_\mu$ such that $h_\upsilon \cdot y \in K \cdot d_0$.
Assuming the 3 steps: $N_\mu^{\lambda'}(r) \neq 0 \;\Rightarrow\; \exists \, \upsilon$ as in Step 3, by Step 1 we get $\upsilon \preceq \lambda$ and $|\upsilon| = |\lambda|$, and by Step 2 we get $\mu_i \leq \upsilon_i$, $i = 1, \dots, n$, hence $\mu \preceq \lambda$ and $|\mu| \leq |\upsilon| = |\lambda|$. So the 3 steps prove the lemma.

Proof of Step 1: Let $x \in K \, h_\upsilon \, K$ such that $x = h_{\lambda'} \, k \, h_r$ for some $k \in K$. Since $|\det h_\mu| = q_1^{-|\mu|}$ for all $\mu \in \Lambda_n^+$, by comparing determinants we get $|\upsilon| = |\lambda'| + r = |\lambda|$. By comparing rank in the residual field, since $\mathrm{rank}\,(h_{\lambda'} \, k \, h_r) \geq \mathrm{rank}\, h_{\lambda'} - 1$, we get

$$l(\upsilon) = m - \mathrm{rank}\, x \leq m - \mathrm{rank}\, h_{\lambda'} + 1 = l\left(\lambda'\right) + 1 = l(\lambda).$$

For $y \in K \, h_\mu \, K$, $\|y\|_{m-i} = {q_1}^{-(\mu_n + \cdots + \mu_{i+1})}$ for all $\mu \in \Lambda_n^+$, $1 \leq i \leq n$. Denote $h_{\lambda'} \, k = (a_{i\,j})$, and note that $|a_{i\,j}| \leq 1$ for all $i,\ j$. Thus

$$x = h_{\lambda'} \, k \, h_r = \begin{pmatrix} \varpi^r a_{1\,1} & a_{1\,2} & \dots & a_{1\,m} \\ \vdots & \vdots & & \vdots \\ \varpi^r a_{m\,1} & a_{m\,2} & \dots & a_{m\,m} \end{pmatrix}.$$

Since $l(\lambda') = l-1$, we have $\|h_{\lambda'}\, k\|_{m-l+1} = 1$, hence $q_1^{-\upsilon_l} = \|x\|_{m-l+1} \geq q_1^{-r}$ and therfore $\upsilon_l \leq r$. To prove that $\upsilon \preceq \lambda$ we now show by induction on i that if $\upsilon_{l-j} = \lambda_{l-j}$ for all $j < i$, then $\upsilon_{l-i} \leq \lambda_{l-i}$. Since $\|h_{\lambda'}\, k\|_{m-l+i+1} = q_1^{-(\lambda_{l-1}+\cdots+\lambda_{l-i})}$, from the presentation of x in terms of the entries of $h_{\lambda'}\, k$ we get,

$$\begin{aligned} q_1^{-(\lambda_l+\cdots+\lambda_{l-i+1}+\upsilon_{l-i})} &= q_1^{-(\upsilon_l+\cdots+\upsilon_{l-i})} \\ &= \|x\|_{m-l+i+1} \geq q_1^{-r}\|h_{\lambda'}\, k\|_{m-l+i+1} \\ &= q_1^{-(r+\lambda_{l-1}+\cdots+\lambda_{l-i})} \\ &= q_1^{-(\lambda_l+\lambda_{l-1}+\cdots+\lambda_{l-i})}. \end{aligned}$$

Therefore $\upsilon_{l-i} \leq \lambda_{l-i}$.

Proof of Step 2: By assumption, $\exists\, y \in K \cdot d_0$ such that $h_\upsilon{}^{-1} \cdot y \in K \cdot d_\mu$. Denote $y = (b_{i\,j})$, then since in Case 1 and in Case 2 $h_\upsilon \in H$, in all cases it act by conjugation on S and, $h_\upsilon{}^{-1} \cdot y = (\varpi^{\upsilon_j - \upsilon_i}\, b_{i\,j})$ (by our convention $\upsilon_i = 0$ for $i > n$). Note that $\forall\, x \in K \cdot d_\mu$, $i = 1, \dots, n$, $\|x\|_i = q_1^{(\mu_1+\cdots+\mu_i)}$. Since $\forall\, i, j$ $|b_{i\,j}| \leq 1$, the entries in the $\geq i^{th}$ rows of $h_\upsilon{}^{-1} \cdot y$ all have absolute value $\leq q_1^{\upsilon_i}$. As every determinant of an $i \times i$ minor of $h_\upsilon{}^{-1} \cdot y$ is a linear combination of $(i-1) \times (i-1)$ minors with coefficients in some row $\geq i^{th}$ row, we get

$$q_1^{\mu_1+\cdots+\mu_i} = \|h_\upsilon{}^{-1} \cdot y\|_i \leq q_1^{\upsilon_i}\, \|h_\upsilon{}^{-1} \cdot y\|_{i-1} = q_1^{\mu_1+\cdots+\mu_{i-1}+\upsilon_i}$$

therefore $\mu_i \leq \upsilon_i$.

Proof of Step 3: If $N_\mu^{\lambda'}(r) \neq 0$ then $\exists\, x = k_1\, h_r\, k_2 \in K\, h_r\, K$, $k_1, k_2 \in K$, such that $x \cdot d_\mu = d_{\lambda'}$. Note that $\alpha\, h_{\lambda'}\, w_m \cdot d_{\lambda'} = d_0$, where

$$\alpha = \begin{cases} I_m & \text{Case 1 and Case 2} \\ w_m & \text{Case 3} \end{cases}$$

so $\alpha\, h_{\lambda'}\, w_m\, x \cdot d_\mu = d_0$. Since $\alpha\, h_{\lambda'}\, w_m\, x \in K\, h_{\lambda'}\, K\, h_r\, K$, there is $\upsilon \in \Lambda_m^+$ such that $K\, h_\upsilon\, K \subset K\, h_{\lambda'}\, K\, h_r\, K$ and $\alpha\, h_{\lambda'}\, w_m\, x \in K\, h_\upsilon\, K$. By Step 1, $\upsilon \in \Lambda_n^+$. So $\exists\, k \in K$, such that $h_\upsilon\, k \in K\, \alpha\, h_{\lambda'}\, w_m\, x$. Let $y = k \cdot d_\mu$, then $y \in K \cdot d_\mu$ and $h_\upsilon \cdot y \in K \cdot d_0$. □

We are now ready for the last step in proving Proposition 4.3. For $z \in \mathbb{C}^n$, $\lambda \in \Lambda_n^+$, clearly $Q^A_{\nu(z)}(\lambda) \in \mathbb{C}[q_1^{-z}, q_1^{z}]^W$. In order to complete the proof of Proposition 4.3 it is enough to show that $\widehat{\mathrm{ch}_\lambda}(z)$ is a linear combination of $Q^A_{\nu(z)}(\mu)$'s.

Lemma 4.7. $\forall\, \lambda \in \Lambda_n^+$,

$$\widehat{\mathrm{ch}_\lambda}(z) = \alpha_\lambda\, Q^A_{\nu(z)}(\lambda) + \sum_{\mu \prec \lambda,\, |\mu| \leq |\lambda|} \beta_\mu\, Q^A_{\nu(z)}(\mu) \tag{47}$$

where $\alpha_\lambda > 0$, $\forall \mu\, \beta_\mu,\, \alpha_\lambda \in \mathbb{Q}$.

Proof. We will prove the lemma by induction on λ with respect to the order $\mu \prec \lambda$ and $|\mu| \leq |\lambda|$. For $\lambda = 0$ the lemma is clear. Indeed

$$\widehat{\mathrm{ch}_0}(z) = 1 = Q^A_{\nu(z)}(0).$$

Applying Lemma 4.4 to the equality obtained in Lemma 4.6 we get

$$\widehat{\mathrm{ch}_\lambda}(z) = {\alpha_\lambda}^{-1}\,\widetilde{c_r}(z)\,\widehat{\mathrm{ch}_{\lambda'}}(z) - {\alpha_\lambda}^{-1} \sum_{\mu\prec\lambda,\,|\mu|\leq|\lambda|} \alpha_\mu\,\widehat{\mathrm{ch}_\mu}(z)$$

for some integers α_μ, $\mu \preceq \lambda$ where $\alpha_\lambda > 0$. Collecting relevant results on Hall-Littlewood polynomials we have:

- $\widetilde{c_r}(z) = q_1^{\frac{1}{2}(m-1)\,r} Q^A_{\nu(z)}((r))$, where $Q^A_{\nu(z)}((r)) = Q^A_{\nu(z)}(r, 0, \ldots, 0)$, [17] Ch.V, §3.3, p. 299.
- $Q^A_{\nu(z)}((r))\, Q^A_{\nu(z)}(\upsilon) = \sum \varphi^\upsilon_\mu\, Q^A_{\nu(z)}(\mu)$, where φ^υ_μ satisfies the following properties:
  1. $\varphi^\upsilon_\mu \in \mathbb{Q}$.
  2. If $\upsilon \preceq \lambda'$ then $\varphi^\upsilon_\mu = 0$ unless $\mu \preceq \lambda$ and $|\mu| = |\upsilon| + r$, and $\varphi^\upsilon_\lambda \neq 0$ if and only if $\upsilon = \lambda'$, and then $\varphi^\upsilon_\lambda > 0$, [17] Ch.V, §2.6, p. 295.

Since [13] supplies us with the relevant facts in: 1. Preliminaries, I omit all details. Thus applying the above and the induction hypothesis we get:

$$\begin{aligned}
&\widehat{\mathrm{ch}_\lambda}(z)\\
&= \alpha_\lambda^{-1}\, q_1^{\frac{1}{2}(m-1)\,r}\, Q^A_{\nu(z)}((r)) \cdot \left\{ \beta_{\lambda'}\, Q^A_{\nu(z)}(\lambda') + \sum_{\upsilon\prec\lambda',\,|\upsilon|\leq|\lambda'|} \beta_\mu\, Q^A_{\nu(z)}(\upsilon) \right\}\\
&\quad - \alpha_\lambda^{-1} \left\{ \sum_{\mu\prec\lambda,\,|\mu|\leq|\lambda|} \gamma_\mu\, Q^A_{\nu(z)}(\mu) \right\}\\
&= \alpha_\lambda^{-1}\, \beta_{\lambda'}\, q_1^{\frac{1}{2}(m-1)\,r}\, \varphi^{\lambda'}_\lambda\, Q^A_{\nu(z)}(\lambda) + \sum_{\mu\prec\lambda,\,|\mu|\leq|\lambda|} \delta_\mu\, Q^A_{\nu(z)}(\mu).
\end{aligned}$$

□

This completes the proof of Proposition 4.3. □

4.3. Parametrization of all relative spherical functions on *S*.

Lemma 4.8. $\left\{ Q^A_{\nu(z)}(\lambda) \,\middle|\, \lambda \in \Lambda^+_n \right\}$ *is a basis for* $\mathbb{C}[q_1^{-z}, q_1^z]^W$, *over* $\mathbb{C}$.

Proof. For $k \in \mathbb{N}$, denote by S_k the group of permutations in k variables. S_k has a natural action on $\mathbb{C}^k$. For $\lambda \in \Lambda^+_k$, let

$$m_\lambda\left((E^{e_i})_{i=1}^k \right) = \sum_{\mu\,\in\, S_k\cdot\lambda} E^\mu.$$

Let $\lambda \in \Lambda_n^+$ and define,

$$\widetilde{m}_\lambda \left((E^{e_i})_{i=1}^n \right) = m_{(\lambda,0,\ldots,0)} \left(\left(E^{e_i}; E^{-e_i} \right)_{i=1}^n \right)$$

if m is even, and

$$\widetilde{m}_\lambda \left((E^{e_i})_{i=1}^n \right) = m_{(\lambda,0,\ldots,0)} \left((E^{e_i})_{i=1}^n ; 1; \left(E^{-e_i} \right)_{i=1}^n \right)$$

if m is odd. λ is viewed as an element of Λ_m^+, in the right-hand side of both equations. It is clear that $\{\widetilde{m}_\lambda \left((q^{z_i})_{i=1}^n \right) | \lambda \in \Lambda_n^+\}$, forms a $\mathbb{C}$-basis for $\mathbb{C}[q_1^{-z}, q_1^z]^W$. By [18],

$$Q_{\nu(z)}^A (\lambda) = \widetilde{m}_\lambda \left((q^{z_i})_{i=1}^n \right) + \sum_{\mu \prec \lambda} u_{\mu\,\lambda} \widetilde{m}_\mu \left((q^{z_i})_{i=1}^n \right)$$

for some constants $u_{\mu\,\lambda} \in \mathbb{C}$, indeed the triangularization of $Q_{\nu(z)}^A (\lambda)$ with respect to $\{\widetilde{m}_\mu \left((q^{z_i})_{i=1}^n \right) | \mu \in \Lambda_n^+\}$, is proved there with respect to a partial order on Λ_n^+, which is contained in the order $\prec$. □

Motivated by Lemma 4.4, we define an $\mathcal{H}(G,K)$-module structure on $\mathbb{C}[q_1^{-z}, q_1^z]^W$, natural to our setting:

$$f \cdot P = \widetilde{f}(z)\ P \tag{48}$$

where $f \in \mathcal{H}(G,K)\,,\ \ P \in \mathbb{C}[q_1^{-z}, q_1^z]^W$.

Proposition 4.9. *The spherical Fourier transform defines an isomorphism of $\mathcal{H}(G,K)$-modules*

$$\mathcal{S}(K \backslash S) \simeq \mathbb{C}[q_1^{-z}, q_1^z]^W.$$

Proof. It is into $\mathbb{C}[q_1^{-z}, q_1^z]^W$ by Proposition 4.3. By Lemma 4.4 and (48), it is indeed an $\mathcal{H}(G,K)$-morphism, and since $\widehat{\mathrm{ch}_0}(z) = 1$, the surjectivity in (39), together with Lemma 4.4 implies the surjectivity of $\widehat{\ }$. It is injective by Lemma 4.7 and Lemma 4.8. □

Proposition 4.10. *Any eigenfunction in $C^\infty(K \backslash S)$ of the Hecke algebra $\mathcal{H}(G,K)$ is a constant multiple of Ω_z for some $z \in \mathbb{C}^n$.*

Proof. We follow [13], Theorem 2. Consider the bilinear form $\langle\, , \rangle$ on $\mathcal{S}(K \backslash S) \times C^\infty(K \backslash S)$ defined by:

$$\langle \varphi\,,\, \psi \rangle = \int_S \varphi(s)\ \psi(s)\ ds.$$

The following properties of $\langle\, , \rangle$ are easy to verify:

- $\langle \mathrm{ch}_0\,,\, \psi \rangle = \psi(d_0)\,,\ \psi \in C^\infty(K \backslash S)\,.$
- $\langle f * \varphi\,,\, \psi \rangle = \langle \varphi\,,\, \check{f} * \psi \rangle$ for all $f \in \mathcal{H}(G,K)\,,\ \varphi \in \mathcal{S}(K \backslash S)\,,\ \psi \in C^\infty(K \backslash S)$.
- If for all $\varphi \in \mathcal{S}(K \backslash S)\,,\ \langle \varphi\,, \psi \rangle = 0$ then $\psi = 0$.

We will use the above three properties freely throughout the proof of the proposition. Let $\Omega \in C^{\infty}(K\backslash S)$, $\Omega \neq 0$ an eigenfunction of the Hecke algebra. Denote by $\omega : \mathcal{H}(G,K) \to \mathbb{C}$ the eigenvalue of Ω. Let $f \in \mathcal{H}(G,K)$ be such that $\widetilde{f}(z) = 0$ for all z, then for all $\varphi \in \mathcal{S}(K\backslash S)$ we have

$$\left(\widehat{\check{f} * \varphi}\right)(z) = \widetilde{f}(z)\,\hat{\varphi}(z) = 0,$$

hence by Proposition 4.9,

$$\check{f} * \varphi = 0.$$

There exists $\varphi \in \mathcal{S}(K\backslash S)$ such that $\langle \varphi\,,\,\Omega\rangle \neq 0$. But

$$\omega(f)\langle \varphi\,,\,\Omega\rangle = \langle \varphi\,,\,f*\Omega\rangle = \langle \check{f}*\varphi\,,\,\Omega\rangle = \langle 0\,,\,\Omega\rangle = 0$$

therefore $\omega(f) = 0$. Since

$$\mathcal{H}(G,K)/\{\widetilde{f}(z) = 0\} \simeq \mathbb{C}[q_1^{-z}, q_1^{z}]^W,$$

ω defines an algebra homomorphism $\omega_1 : \mathbb{C}[q_1^{-z}, q_1^{z}]^W \to \mathbb{C}$ such that

$$\omega(f) = \omega_1(\widetilde{f}(z)).$$

In turn, since $\mathbb{C}[q_1^{-z}, q_1^{z}]$ is integral over $\mathbb{C}[q_1^{-z}, q_1^{z}]^W$, ω_1 extends to an algebra homomorphism from $\mathbb{C}[q_1^{-z}, q_1^{z}]$ to $\mathbb{C}$. Hence there is $z_0 \in \mathbb{C}^n$, such that $\omega_1(P) = P(z_0)$, $P \in \mathbb{C}[q_1^{-z}, q_1^{z}]$. We therefore have $\omega(f) = \widetilde{f}(z_0)$ for all $f \in \mathcal{H}(G,K)$. To complete the proof we now show that $\Omega = \Omega(d_0)\,\Omega_{z_0}$. Let $\varphi \in \mathcal{S}(K\backslash S)$ and let $f \in \mathcal{H}(G,K)$ such that $\widetilde{f}(z) = \hat{\varphi}(z)$, then by Proposition 4.9, Lemma 4.4 and the fact that $\widehat{\mathrm{ch}_0}(z) = 1$ we have, $\varphi = f * \mathrm{ch}_0$. Therefore

$$\begin{aligned}
\langle \varphi\,,\,\Omega - \Omega(d_0)\,\Omega_{z_0}\rangle &= \langle f*\mathrm{ch}_0\,,\,\Omega - \Omega(d_0)\,\Omega_{z_0}\rangle \\
&= \langle \mathrm{ch}_0\,,\,\check{f}*(\Omega - \Omega(d_0)\,\Omega_{z_0})\rangle \\
&= \langle \mathrm{ch}_0\,,\,\omega(\check{f})\,\Omega - \widetilde{f}(z_0)\,\Omega(d_0)\,\Omega_{z_0}\rangle \\
&\underset{\omega(\check{f})=\widetilde{f}(z_0)}{=} \omega(\check{f})\,\langle \mathrm{ch}_0\,,\,\Omega - \Omega(d_0)\,\Omega_{z_0}\rangle \\
&= \omega(\check{f})\,(\Omega(d_0) - \Omega(d_0)\,\Omega_{z_0}(d_0)) = 0.
\end{aligned}$$

Hence indeed $\Omega = \Omega(d_0)\,\Omega_{z_0}$. □

5. Computation of $\Omega_z(d_\lambda)$

In order to prove Theorem 1.2, we only need to verify now that Ω_z satisfies (7). We let $z \in \mathbb{C}^n$ and unless otherwise stated, we will assume that

(49) $$\operatorname{Re} z_i > \operatorname{Re} z_{i+1} + 1,\ i = 1, \ldots, n-1,\ \operatorname{Re} z_n > 1.$$

We will use the Casselman-Shalika method to show that the spherical functions Ω_z satisfy (7), for all z in the open set defined by (49). Theorem 1.2

will then follow by analytic continuation of Ω_z. Only then we will remove the restriction (49) on z. Throughout the chapter z and t are related by (41). We let

$$\chi = \chi_{\nu(z)}$$

and denote $\chi_i = |\,|^{z_i}$, $i = 1, \dots, n$. We remark that as long as z satisfies (49) the representation $I(\chi)$ is irreducible.

5.1. Convergence of the period integral. We choose an element $\xi \in G$, such that $\theta(\xi) = d_0$ as follows:

Case 1: $\xi = \begin{pmatrix} I_n & w_n \\ -w_n & I_n \end{pmatrix}$.

Case 2: $\xi = \begin{pmatrix} \iota\, I_n & w_n \\ -\iota\, w_n & I_n \end{pmatrix}$ if m is even, and $\xi = \begin{pmatrix} \iota\, I_n & & w_n \\ & 1 & \\ -\iota\, w_n & & I_n \end{pmatrix}$ if m is odd.

Case 3: $\xi = I_m$.

Let $H_\xi = H \cap \xi^{-1} P \xi$.

Proposition 5.1. *The integral,*

$$\int_{H_\xi \backslash H} \varphi(\xi\, h)\, dh \tag{50}$$

is convergent whenever $\varphi \in I(\chi)$ *and* $\operatorname{Re} z_1 > \cdots > \operatorname{Re} z_n > \frac{1}{2}$.

Proof. It is enough to prove the convergence of the integral for $\varphi_{K,\chi}$. We fix some notation and then prove each case separately. Let

$$\xi' = \begin{cases} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} & \text{Case 1} \\ \begin{pmatrix} \iota & 1 \\ -\iota & 1 \end{pmatrix} & \text{Case 2} \\ \begin{pmatrix} 1 & 1 \\ \iota & -\iota \end{pmatrix} & \text{Case 3,} \end{cases}$$

and in Case 3, let $\xi_1 = \begin{pmatrix} I_n & w_n \\ \iota\, w_n & -\iota\, I_n \end{pmatrix}$.

Define

$$\xi_0 = \begin{pmatrix} \xi' & & \\ & \ddots & \\ & & \xi' \end{pmatrix}$$

and $w_0 \in W$, the Weyl element such that ${}^{w_0}\chi = \left(\chi_1, \chi_1^{-1}, \ldots, \chi_n, \chi_n^{-1}\right)$ if m is even, and

$$\xi_0 = \begin{pmatrix} \xi' & & & \\ & \ddots & & \\ & & \xi' & \\ & & & 1 \end{pmatrix}$$

and $w_0 \in W$ the Weyl element such that ${}^{w_0}\chi = \left(\chi_1, \chi_1^{-1}, \ldots, \chi_n, \chi_n^{-1}, 1\right)$ if m is odd. Then $\xi = w_0^{-1}\,\xi_0\,w_0$ in Case 1 and in Case 2 and $\xi_1 = w_0^{-1}\,\xi_0\,w_0$ in Case 3. Note also that in Case 3,

$$\xi_1^{-1} H\,\xi_1 = \left\{ \begin{pmatrix} \alpha & \\ & w_n\,\overline{\alpha}\,w_n \end{pmatrix} \;\middle|\; \alpha \in GL\,(n, E) \right\}.$$

Define also $K_0 = K \cap H$, then

in Case 1: $K_0 = \left\{ \begin{pmatrix} k_1 & \\ & k_2 \end{pmatrix} \;\middle|\; k_1, k_2 \in GL\,(n, \mathcal{O}_F) \right\}$;

in Case 2: $K_0 = GL\,(m, \mathcal{O}_F)$;

in Case 3: $K_0 = \left\{ \xi_1 \begin{pmatrix} \alpha & \\ & w_n\,\overline{\alpha}\,w_n \end{pmatrix} \xi_1^{-1} \;\middle|\; \alpha \in GL\,(n, \mathcal{O}_E) \right\} \simeq GL\,(n, \mathcal{O}_E)$.

For $g_1, \ldots, g_n \in GL\,(2, F)$ in Case 1 and in Case 3, and $g_1, \ldots, g_n \in GL\,(2, E)$ in Case 2, let

$$\Delta\,(g_1, \ldots, g_n) = \prod_{i=1}^{n} |\det\, g_i|^{2i-(n+1)}$$

if m is even, and

$$\Delta\,(g_1, \ldots, g_n) = \prod_{i=1}^{n} |\det\, g_i|^{2i-\left(n+\frac{3}{2}\right)}$$

if m is odd. Let $\Pi : I\left({}^{w_0}\chi\right) \to \bigotimes_{i=1}^{n} I\left(\chi_i, \chi_i^{-1}\right)$ be the map, in Case 1 and in Case 2, defined by:

(51)

$$\left(\Pi\varphi'\right)(g_1, \ldots, g_n) = \Delta\,(g_1, \ldots, g_n) \int_{K_0} \varphi' \left[\begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_n \end{pmatrix} w_0\,k_0 \right] dk_0$$

and in Case 3, defined by:

(52)

$$\left(\Pi\varphi'\right)(g_1, \ldots, g_n) = \Delta\,(g_1, \ldots, g_n) \int_{K_0} \varphi' \left[\begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_n \end{pmatrix} \xi_0\,w_0\,\xi_1^{-1}\,k_0 \right] dk_0$$

whenever $\varphi' \in I\left({}^{w_0}\chi\right)$. We will reduce the proposition to the case $m = 2$, but first let us compute the period integral explicitly in that case.

Lemma 5.2 (Case 1). *Let* $\chi = \left(\chi_1, \chi_1^{-1}\right)$, *where* $\chi_1 = |\,|^z$. *If* $\operatorname{Re} z > -\frac{1}{2}$, *then the integral* $\int_{F^\times} \varphi\left[\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right] d^\times a$ *is convergent for all* $\varphi \in I(\chi)$. *Normalizing the Haar measure on* $F^\times$ *so that* $\int_{\mathcal{O}_F^\times} d^\times a = 1$, *we have:*

$$\int_{F^\times} \varphi_{K_2,\chi}\left[\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right] d^\times a = \frac{1 + q^{-\frac{1}{2}} q^{-z}}{1 - q^{-\frac{1}{2}} q^{-z}}. \tag{53}$$

Proof. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, F)$, we have

$$\varphi_{K_2,\chi}(g) = \frac{|\det g|^{z+\frac{1}{2}}}{\max(|c|, |d|)^{2z+1}}.$$

Thus,

$$\begin{aligned}
&\int_{F^\times} \varphi_{K_2,\chi}\left[\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right] d^\times a \\
&= \int_{F^\times} \frac{|a|^{z+\frac{1}{2}}}{\max(|a|, 1)^{2z+1}}\, d^\times a \\
&= \int_{|a|\le 1} |a|^{z+\frac{1}{2}}\, d^\times a + \int_{|a|>1} |a|^{-\left(z+\frac{1}{2}\right)}\, d^\times a \\
&= \int_{|a|=1} d^\times a + 2\int_{|a|<1} |a|^{z+\frac{1}{2}}\, d^\times a = 1 + 2\sum_{n=1}^{\infty} q^{-\left(z+\frac{1}{2}\right)n}.
\end{aligned}$$

The right-hand side is convergent whenever $\operatorname{Re} z > -\frac{1}{2}$, and equals (53). ☐

Lemma 5.3 (Case 2). *Let* $\chi = \left(\chi_1, \chi_1^{-1}\right)$, *where* $\chi_1 = |\,|^z$. *If* $\operatorname{Re} z > \frac{1}{2}$ *then the integral* $\int_{F\times F^\times} \varphi\left[\begin{pmatrix} \iota & 1 \\ -\iota & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\right] d^\times a\, db$ *is convergent for all* $\varphi \in I(\chi)$. *Normalizing the Haar measure on* $F^\times$ *so that* $\int_{\mathcal{O}_F^\times} d^\times a = 1$ *and on* F *so that* $\int_{\mathcal{O}_F} db = 1$, *we have:*

$$\int_{F\times F^\times} \varphi_{K_2,\chi}\left[\begin{pmatrix} \iota & 1 \\ -\iota & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\right] d^\times a\, db = \frac{1 + q^{-1} q^{-2z}}{1 - q\, q^{-2z}}. \tag{54}$$

Proof. We have

$$\int\limits_{F\times F^\times} \varphi_{K_2,\chi}\left[\begin{pmatrix} \iota & 1 \\ -\iota & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\right] d^\times a\, db$$

$$= \int\limits_{F\times F^\times} \frac{|a|^{z+\frac{1}{2}}}{\max\{|a|, |1-\iota\, b|\}^{2z+1}}\, d^\times a\, db.$$

Since $1-\iota\, b \in \mathcal{O}_E^\times$ for all $b\in\mathcal{O}_F$, the period integral becomes:

$$= \int\limits_{|b|\leq 1} \frac{|a|^{z+\frac{1}{2}}}{\max\{|a|, 1\}^{2z+1}}\, d^\times a\, db + \int\limits_{|b|>1} \frac{|a|^{z+\frac{1}{2}}}{\max\{|a|, |b|\}^{2z+1}}\, d^\times a\, db = I_1 + I_2,$$

where I_j is the j-th summand, $j=1,2$. I_1 is computed in Lemma 5.2, we have:

$$I_1 = \frac{1+q^{-1}q^{-2z}}{1-q^{-1}q^{-2z}}.$$

We compute I_2:

$$\int\limits_{|b|>1} \frac{|a|^{z+\frac{1}{2}}}{\max\{|a|, |b|\}^{2z+1}}\, d^\times a\, db$$

$$= \int\limits_{|b|>\max\{1,|a|\}} \frac{|a|^{z+\frac{1}{2}}}{|b|^{2z+1}}\, d^\times a\, db + \int\limits_{|a|\geq|b|>1} |a|^{-z-\frac{1}{2}}\, d^\times a\, db$$

$$= \sum_{n=1}^{\infty}\sum_{m=1-n}^{\infty} q^{-(2z+1)m}\, q^{-(4z+2)n}\left(q^{2n}-q^{2n-2}\right) + \sum_{m=1}^{\infty} q^{-(2z+1)m}\left(q^{2m}-1\right).$$

The right-hand side is convergent whenever $\operatorname{Re} z > \frac{1}{2}$, and I_1+I_2 equals (54). □

Lemma 5.4 (Case 3)**.** *Let $\chi = \left(\chi_1, \chi_1^{-1}\right)$, where $\chi_1 = |\,|^z$. Let*

$$H_2 = \left\{\begin{pmatrix} a & b \\ \tau b & a \end{pmatrix} \in GL\,(2,F)\ \middle|\ a,b\in F\right\}$$

and $(H_2)_\xi = H_2 \cap P_2$, then $(H_2)_\xi\backslash H_2$ is compact and the integral $\int\limits_{(H_2)_\xi\backslash H_2} \varphi\,(h)\, dh$ is convergent for all $\varphi\in I\,(\chi)$. Normalizing the Haar measure on $(H_2)_\xi\backslash H_2$ so that $\int\limits_{(H_2)_\xi\backslash H_2} dh = 1$ we have:

$$\int\limits_{(H_2)_\xi\backslash H_2} \varphi_{K_2,\chi}\,(h)\, dh = 1. \tag{55}$$

Proof. The isomorphism $H_2 \simeq E^\times$ defined by $h \mapsto {\xi'}^{-1} h\, \xi'$, $h \in H_2$ induces an isomorphism $(H_2)_\xi \backslash H_2 \simeq F^\times \backslash E^\times$, hence $(H_2)_\xi \backslash H_2$ is indeed compact and the convergence of the period integral is clear. Since $\varphi_{K_2, \chi|H_2} \equiv 1$, (55) follows. ☐

Whenever $\mathrm{Re}\, z_1 > \cdots > \mathrm{Re}\, z_n > \frac{1}{2}$, we may now define the linear form $\lambda = \bigotimes_{i=1}^{n} \lambda_i$ on $\bigotimes_{i=1}^{n} I\left(\chi_i, \chi_i^{-1}\right)$, where λ_i is the linear form on $I\left(\chi_i, \chi_i^{-1}\right)$ given by, Lemma 5.2 in Case 1, by Lemma 5.3 in Case 2 and by Lemma 5.4 in Case 3. We rewrite the integral over $H_\xi \backslash H$ using an Iwasawa decomposition of H.

Case 1: For $h = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \in H$, $g_1, g_2 \in GL(n, F)$,

$$h = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_{2n} \end{pmatrix} k_0,$$

where $m_1 \in N_n$, $m_2 \in {}^tN_n$-the group of lower triangular unipotent matrices, $a_i \in F^\times$, $i = 1, \ldots, 2n$ and $k_0 \in K_0$. The integral becomes:

$$\int \varphi \left[\xi \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_{2n} \end{pmatrix} k_0 \right] dm_1\, dm_2 \cdot$$

$$\prod_{i=1}^{n} |a_i\, a_{2n+1-i}|^{2i-(n+1)} \prod_{i=1}^{2n} d^\times a_i\, dk_0 \tag{56}$$

where the integral over the a_i's is taken modulo the relations $a_i = a_{2n+1-i}$, $i = 1, \ldots, n$. Denote the entries of m_1 by $(m_1)_{ij} = x_{ij}$, $1 \le i < j \le n$, and $(m_2)_{ij} = y_{ij}$, $1 \le j < i \le n$. Then

$$w_0 \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} w_0^{-1} = \begin{pmatrix} I_2 & & \alpha_{ij} \\ & \ddots & \\ & & I_2 \end{pmatrix},$$

where for $i < j$, $\alpha_{ij} = \begin{pmatrix} x_{ij} & 0 \\ 0 & y_{n+1-i\, n+1-j} \end{pmatrix}$. Thus

$$\xi \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} = w_0^{-1}\, \xi_0 \begin{pmatrix} I_2 & & \alpha_{ij} \\ & \ddots & \\ & & I_2 \end{pmatrix} w_0 = w_0^{-1} \begin{pmatrix} I_2 & & \beta_{ij} \\ & \ddots & \\ & & I_2 \end{pmatrix} \xi_0\, w_0,$$

where $\beta_{ij} = \xi' \alpha_{ij} \xi'^{-1} = \begin{pmatrix} \frac{1}{2}\left(y^{ij} + x_{ij}\right) & \frac{1}{2}\left(y^{ij} - x_{ij}\right) \\ \frac{1}{2}\left(y^{ij} - x_{ij}\right) & \frac{1}{2}\left(y^{ij} + x_{ij}\right) \end{pmatrix}$, the notation being: $y^{ij} = y_{n+1-i\, n+1-j}$. So the period integral takes the form:

$$\int \varphi \left[w_0^{-1} \begin{pmatrix} I_2 & & \beta_{ij} \\ & \ddots & \\ & & I_2 \end{pmatrix} \xi_0 \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix} w_0\, k_0 \right] \prod_{1\le i<j\le n} dx_{ij} \prod_{1\le j<i\le n} dy_{ij} \cdot$$

$$\Delta\left(\alpha_1, \ldots, \alpha_n\right) \prod_{i=1}^{2n} d^\times a_i\, dk_0 \tag{57}$$

where $\alpha_i = \begin{pmatrix} a_i & 0 \\ 0 & a_{2n+1-i} \end{pmatrix}$. Define the change of variables $u_{ij} = \frac{1}{2}(y^{ij} + x_{ij})$, $v_{ij} = \frac{1}{2}(y^{ij} - x_{ij})$. Let

$$m = \begin{pmatrix} I_2 & & \beta_{ij} \\ & \ddots & \\ & & I_2 \end{pmatrix} = n_1\, n_2,\; n_2 = \begin{pmatrix} I_2 & & \gamma_{ij} \\ & \ddots & \\ & & I_2 \end{pmatrix},\; \gamma_{ij} = \begin{pmatrix} 0 & 0 \\ v_{ij} & u_{ij} \end{pmatrix}$$

and $n_1 = m\, n_2^{-1}$. Then $w_0^{-1} n_1 w_0 \in N$ and therefore for $g \in G$

$$\varphi\left(w_0^{-1} m\, g\right) = \varphi\left(w_0^{-1} n_2\, g\right).$$

Note that n_2 varies over N_{w_0} as the u_{ij}, v_{ij}'s vary in F, thus the integral becomes:

$$\int \varphi \left[w_0^{-1} \eta\, \xi_0 \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix} w_0\, k_0 \right] d\eta\, \Delta\left(\alpha_1, \ldots, \alpha_n\right) \prod_{i=1}^{2n} d^\times a_i\, dk_0 \tag{58}$$

where $\eta \in N_{w_0}$. Let $T_{w_0} = T_{w_0,\chi}$ and $\varphi' = T_{w_0}\varphi$ then by (26), (58) becomes:

$$\int \varphi' \left[\begin{pmatrix} \xi' & & \\ & \ddots & \\ & & \xi' \end{pmatrix} \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix} w_0\, k_0 \right] \Delta\left(\alpha_1, \ldots, \alpha_n\right) \prod_{i=1}^{2n} d^\times a_i\, dk_0. \tag{59}$$

Let $\varphi'' = \Pi\varphi'$. From (51) and (59), we see that the period integral is equal to:

$$\int \varphi''\left(\xi'\, \alpha_1, \ldots, \xi'\, \alpha_n\right) \prod_{i=1}^{2n} d^\times a_i = \lambda\left(\varphi''\right). \tag{60}$$

The integral (26), defining the intertwining operator T_{w_0}, is convergent for $\operatorname{Re} z_1 > \cdots > \operatorname{Re} z_n > 0$ and by Lemma 5.2, λ is well-defined for $\operatorname{Re} z_i > \frac{1}{2}$. Case 1 of the proposition is now proved.

Case 2: Let N^1 be the unipotent radical of the standard parabolic subgroup of H of type $(2,\dots,2)$ if m is even, and of type $(2,\dots,2,1)$ if m is odd. Let M be the corresponding Levi subgroup of G, i.e.,

$$M=\left\{\left.\begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_n\end{pmatrix}\right| g_i\in GL\left(2,F\right)\right\}$$

if m is even and

$$M=\left\{\left.\begin{pmatrix} g_1 & & & \\ & \ddots & & \\ & & g_n & \\ & & & a\end{pmatrix}\right| g_i\in GL\left(2,F\right),a\in F^\times\right\}$$

if m is odd. We use the Iwasawa decomposition

$$H=\left(w_0^{-1}N^1w_0\right)\left(w_0^{-1}Mw_0\right)K_0$$

to rewrite the period integral as:

$$\int\varphi\left[\xi\, w_0^{-1}\,n^{(1)}\begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_n\end{pmatrix}k_0\right]\,dn^{(1)}\,\Delta\left(g_1,\dots,g_n\right)\prod_{i=1}^n dg_i\,dk_0 \tag{61}$$

if m is even and,

$$\int\varphi\left[\xi\, w_0^{-1}\,n^{(1)}\begin{pmatrix} g_1 & & & \\ & \ddots & & \\ & & g_n & \\ & & & 1\end{pmatrix}k_0\right]\,dn^{(1)}\,\Delta\left(g_1,\dots,g_n\right)\prod_{i=1}^n dg_i\,dk_0 \tag{62}$$

if m is odd. The integral over $g_i\in GL\left(2,F\right)$ is taken modulo

$$H_{\xi'}=\left\{\left.\begin{pmatrix} a & b\\ \iota^2\,b & a\end{pmatrix}\right|\,(a,b)\neq(0,0)\ \text{ in } F^2\right\},$$

i.e., g_i is integrated over $\left\{\left.\begin{pmatrix}\alpha & \beta\\ 0 & 1\end{pmatrix}\right|\,\alpha\in F^\times,\,\beta\in F\right\}$. Denote

$$n^{(1)}=\begin{pmatrix} I_2 & & \alpha_{i\,j}\\ & \ddots & \\ & & I_2\end{pmatrix}\ \text{if } m \text{ is even, and}$$

$$n^{(1)}=\begin{pmatrix} I_2 & & \alpha_{i\,j} & a_i\\ & \ddots & & \\ & & I_2 & \\ & & & 1\end{pmatrix}\ \text{if } m \text{ is odd,}$$

where $\alpha_{ij} \in M_2(F)$, $1 \le i < j \le n$, $a_i \in M_{2\times 1}(F)$, $1 \le i \le n$. Then

$$\xi\, w_0^{-1}\, n^{(1)} = w_0^{-1} \begin{pmatrix} I_2 & & \beta_{ij} \\ & \ddots & \\ & & I_2 \end{pmatrix} \xi_0$$

if m is even, and

$$\xi\, w_0^{-1}\, n^{(1)} = w_0^{-1} \begin{pmatrix} I_2 & & \beta_{ij} & b_i \\ & \ddots & & \\ & & I_2 & \\ & & & 1 \end{pmatrix} \xi_0$$

if m is odd, where $\beta_{ij} = \xi'\,\alpha_{ij}\,\xi'^{-1}$ and $b_i = \xi'\,a_i$. Let $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(F)$. Let $x = \frac{1}{2}[(d+a) - \iota\left(b + \frac{c}{\iota^2}\right)]$ and $y = \frac{1}{2}[(d-a) - \iota\left(b - \frac{c}{\iota^2}\right)]$, then $\xi'\,\alpha\,\xi'^{-1} = \begin{pmatrix} \bar{x} & \bar{y} \\ y & x \end{pmatrix}$. Similarly let $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and let $z = a_2 - \iota\,a_1$, then $\xi'\,a = \begin{pmatrix} \bar{z} \\ z \end{pmatrix}$. Thus after the appropriate change of variables we may write $\beta_{ij} = \begin{pmatrix} \overline{x_{ij}} & \overline{y_{ij}} \\ y_{ij} & x_{ij} \end{pmatrix}$, and $b_i = \begin{pmatrix} \overline{z_i} \\ z_i \end{pmatrix}$. The variables x_{ij}, y_{ij}, z_i all range over E as the α_{ij}'s range over $M_2(F)$ and the a_i's range over $M_{2\times 1}(F)$. Similar to Case 1, we let

$$\begin{pmatrix} I_2 & & \beta_{ij} \\ & \ddots & \\ & & I_2 \end{pmatrix} = n_1 \begin{pmatrix} I_2 & & \gamma_{ij} \\ & \ddots & \\ & & I_2 \end{pmatrix}$$

if m is even, and

$$\begin{pmatrix} I_2 & & \beta_{ij} & b_i \\ & \ddots & & \\ & & I_2 & \\ & & & 1 \end{pmatrix} = n_1 \begin{pmatrix} I_2 & & \gamma_{ij} & c_i \\ & \ddots & & \\ & & I_2 & \\ & & & 1 \end{pmatrix}$$

if m is odd, where $\gamma_{ij} = \begin{pmatrix} 0 & 0 \\ y_{ij} & x_{ij} \end{pmatrix}$ and $c_i = \begin{pmatrix} 0 \\ z_i \end{pmatrix}$. Then $n_1 \in N \cap w_0\, N\, w_0^{-1}$. Also

$$\begin{pmatrix} I_2 & & \gamma_{ij} \\ & \ddots & \\ & & I_2 \end{pmatrix} \text{ if } m \text{ is even } \left(\text{resp. } \begin{pmatrix} I_2 & & \gamma_{ij} & c_i \\ & \ddots & & \\ & & I_2 & \\ & & & 1 \end{pmatrix} \text{ if } m \text{ is odd} \right)$$

range over N_{w_0} as the $x_{i\,j}, y_{i\,j}$ (respectively $x_{i\,j}$, $y_{i\,j}$, z_i) range over E. Thus (61) becomes:

$$\int \varphi\left[w_0^{-1}\,\eta\,\xi_0\begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_n\end{pmatrix} w_0\,k_0\right]\Delta\,(g_1,\ldots,g_n)\prod_{i=1}^n dg_i\,dk_0 \tag{63}$$

and (62) becomes:

$$\int \varphi\left[w_0^{-1}\,\eta\,\xi_0\begin{pmatrix} g_1 & & & \\ & \ddots & & \\ & & g_n & \\ & & & 1\end{pmatrix} w_0\,k_0\right]\Delta\,(g_1,\ldots,g_n)\prod_{i=1}^n dg_i\,dk_0 \tag{64}$$

where $\eta \in N_{w_0}$. Similar to Case 1, (63) for m even and (64) for m odd combined with Lemma 5.3 show that for $\operatorname{Re} z_1 > \cdots \operatorname{Re} z_n > \frac{1}{2}$ the period integral converges and equals:

$$(\lambda \circ \Pi \circ T_{w_0})\;\varphi. \tag{65}$$

Case 3: We apply the standard Iwasawa decomposition of $GL\,(n,E)$ to decompose H, through the isomorphism $H \simeq GL\,(n,E)$. Thus for $h \in H$ we write

$$h = \xi_1\begin{pmatrix} n^{(1)}\,a & \\ & w_n\,\overline{n}^{(1)}\,\bar{a}\,w_n\end{pmatrix}\xi_1^{-1}\,k_0,$$

where $n^{(1)}$ is upper triangular unipotent, $a = \operatorname{diag}[a_1,\ldots,a_n]$ is diagonal in $GL\,(n,E)$, and $k_0 \in K_0$. We rewrite the period integral as:

(66)

$$\int \varphi\left[\xi_1\begin{pmatrix} n^{(1)}\,a & \\ & w_n\,\overline{n}^{(1)}\,\bar{a}\,w_n\end{pmatrix}\xi_1^{-1}\,k_0\right]dn^{(1)}\prod_{i=1}^n |a_i|^{2(2i-(n+1))}\prod_{i=1}^n d^\times a_i\,dk_0.$$

The integral over each of the a_i's is taken modulo $F^\times$. Denote $n^{(1)} = (x_{i\,j})$ and $\alpha_i = \begin{pmatrix} a_i & \\ & \bar{a}_i\end{pmatrix}$, $i = 1,\ldots,n$, then

$$\begin{aligned}
&\xi_1\begin{pmatrix} n^{(1)}\,\alpha & \\ & w_n\,\overline{n}^{(1)}\,\overline{\alpha}\,w_n\end{pmatrix}\\
&= w_0^{-1}\,\xi_0\begin{pmatrix} I_2 & & \alpha_{i\,j} \\ & \ddots & \\ & & I_2\end{pmatrix}\begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n\end{pmatrix} w_0\\
&= w_0^{-1}\begin{pmatrix} I_2 & & \beta_{i\,j} \\ & \ddots & \\ & & I_2\end{pmatrix}\xi_0\begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n\end{pmatrix} w_0,
\end{aligned}$$

where $\alpha_{i\,j} = \begin{pmatrix} x_{i\,j} & \\ & \bar{x}_{i\,j} \end{pmatrix}$ and

$$\beta_{i\,j} = \xi'\,\alpha_{i\,j}\,\xi'^{-1} = \begin{pmatrix} x_{i\,j} + \bar{x}_{i\,j} & \frac{\iota}{\tau}\,(x_{i\,j} - \bar{x}_{i\,j}) \\ \iota\,(x_{i\,j} - \bar{x}_{i\,j}) & x_{i\,j} + \bar{x}_{i\,j} \end{pmatrix}.$$

Using the change of variables $u_{i\,j} = \iota\,(x_{i\,j} - \bar{x}_{i\,j})$ and $v_{i\,j} = x_{i\,j} + \bar{x}_{i\,j}$, let $\gamma_{i\,j} = \begin{pmatrix} 0 & 0 \\ u_{i\,j} & v_{i\,j} \end{pmatrix}$, then

$$\begin{pmatrix} I_2 & & \beta_{i\,j} \\ & \ddots & \\ & & I_2 \end{pmatrix} = n_1 \begin{pmatrix} I_2 & & \gamma_{i\,j} \\ & \ddots & \\ & & I_2 \end{pmatrix},$$

where $w_0^{-1}\,n_1\,w_0 \in N$ and

$$N_{w_0} = \left\{ \begin{pmatrix} I_2 & & \gamma_{i\,j} \\ & \ddots & \\ & & I_2 \end{pmatrix} \middle|\, u_{i\,j}, v_{i\,j} \in F \right\}.$$

As in the previous cases we may now write (66) as:

(67)

$$\int \varphi \left[w_0^{-1}\,\eta \begin{pmatrix} \xi'\,\alpha_1 & & \\ & \ddots & \\ & & \xi'\,\alpha_n \end{pmatrix} w_0\,\xi_1^{-1}\,k_0 \right] d\eta\,\Delta\,(\alpha_1, \ldots, \alpha_n) \prod_{i=1}^{n} d^{\times} a_i\,dk_0$$

where $\eta \in N_{w_0}$. With analogy to the previous cases we now observe that (67) is in fact equal to:

$$(\lambda \circ \Pi \circ T_{w_0})\;\varphi. \tag{68}$$

The convergence of the integral follows from Lemma 5.4 and (26). □

From now on we fix an H-invariant measure on $H_\xi \backslash H$. For $\varphi \in I\,(\chi)$ define,

$$\Lambda_\chi\,(\varphi) = \int_{H_\xi \backslash H} \varphi\,(\xi\,h)\;dh. \tag{69}$$

5.2. Redefining Ω_z.

Lemma 5.5. *Let (π, V) be an irreducible, admissible, unramified, representation of G. The space of H-invariant linear forms on V is at most one dimensional.*

Proof. For Case 1 this is true even if π is not unramified by the uniqueness of linear periods [**15**]. In fact the lemma follows, for all cases, from Proposition 4.10. Let $\Lambda \in V^*$ be an H-invariant form. Define $\Omega\,(\theta\,(g)) =$

$\Lambda\left(\pi\left(g^{-1}\right) v_K\right)$, where v_K is a nonzero K-invariant vector in V. For $\phi \in \mathcal{H}(G,K)$, let $\hat{\phi}(\pi) \in \mathbb{C}$ be defined by $\pi(\phi) v_K = \hat{\phi}(\pi) v_K$, we have

$$\begin{aligned}(\phi * \Omega)(\theta(g_0)) &= \int_G \phi(g) \Lambda\left(\pi\left(g_0^{-1} g\right) v_K\right) dg \\ &= \Lambda\left(\pi\left(g_0^{-1}\right) \int_G \phi(g) \pi(g) v_k \, dg\right) \\ &= \Lambda\left(\pi\left(g_0^{-1}\right) \pi(\phi) v_K\right) = \hat{\phi}(\pi) \Omega(\theta(g_0)).\end{aligned}$$

Therefore $\Omega \in C^\infty(K\backslash S)$, is an eigenfunction of $\mathcal{H}(G,K)$ and by Proposition 4.10, $\exists\, z \in \mathbb{C}^n$, such that $\Omega = \Omega(d_0)\, \Omega_z$. Choose g_0 such that $\theta(g_0) = d_0$ then

$$\Lambda(\pi(g^{-1})v_K) = \Lambda(\pi(g_0^{-1})v_K)\, \Omega_z(\theta(g)).$$

Since z depends only on π and since π is irreducible, this shows that Λ is determined by its value on $\pi(g_0^{-1})v_K$ which proves the lemma. □

We now give a different definition to ω_t in a way that will enable us to apply the Casselman-Shalika method and proceed with the computation. For $s \in S$, let

$$D_z^s(g) = d_t(g \cdot s),$$

then $\omega_t(s) = \int_K D_z^s(k)\, dk$. Note that by (44), for $p \in P$ and $g \in G$

$$D_z^s(p\, g) = \chi^{-1} \delta^{\frac{1}{2}}(p)\ D_z^s(g). \tag{70}$$

Using again the theory of complex powers of polynomial functions [**5**], the distribution defined for $\operatorname{Re} t_i \geq 0$ by:

$$\phi \mapsto \int_G \phi(g)\ D_z^s(g)\ dg,$$

has a meromorphic continuation to a distribution on G. By (70), $D_z^s \in \mathcal{D}(G)_{\chi^{-1}}$. Note also that $D_z^{I_m}(g\, h) = D_z^{I_m}(g)$, $g \in G$, $h \in H$. Let $\Lambda_{0,\chi} \in I(\chi)^*$ be such that

$$\mathcal{P}_\chi^*(\Lambda_{0,\chi}) = D_z^{I_m}.$$

$\Lambda_{0,\chi}$ is uniquely defined this way through the isomorphism (25). $\Lambda_{0,\chi}$ is then an H-invariant linear form on $I(\chi)$. The action R of G on $I\left(\chi^{-1}\right)$ extends to an action on $I(\chi)^*$,

$$(R(g)\Lambda)(\varphi) = \Lambda\left(R\left(g^{-1}\right)\varphi\right),$$

where $\Lambda \in I(\chi)^*$, $\varphi \in I(\chi)$.

Lemma 5.6. *Let $s \in S$ and $g_s \in G$ such that $\theta(g_s) = s$ then,*

$$\omega_t(s) = (R(g_s)\Lambda_{0,\chi})(\varphi_{K,\chi}).$$

Proof. Using the equivariance of $\mathcal{P}_\chi$ (24), and the definition of $\Lambda_{0,\chi}$, we have

$$\begin{aligned}(R(g_s)\Lambda_{0,\chi})(\varphi_{K,\chi}) &= \Lambda_{0,\chi}\left(R\left(g_s^{-1}\right)\mathcal{P}_\chi(\mathrm{ch}_K)\right)\\ &= \Lambda_{0,\chi}\circ\mathcal{P}_\chi(\mathrm{ch}_{K\,g_s})\\ &= \langle D_z^{I_m}, \mathrm{ch}_{K\,g_s}\rangle\\ &= \int\limits_G d_t(\theta(g))\,\mathrm{ch}_{K\,g_s}(g)\,dg\\ &= \int\limits_K d_t(\theta(k\,g_s))\,dk\\ &= \omega_t(s).\end{aligned}$$

□

5.3. Expansion in the Casselman basis. Let B be the standard Iwahori subgroup of G. It is the pullback of the standard Borel subgroup of GL_m over the residual field. In [3], Casselman introduced a basis $\{f_{w,\chi^{-1}}\,|\,w\in W\}$ of $I\left(\chi^{-1}\right)^B$, the space of B-invariant vectors in $I\left(\chi^{-1}\right)$, that satisfies for $w, w'\in W$

$$\left(T_w\,f_{w',\chi^{-1}}\right)(1) = \delta_{w,w'}.$$

Here $T_w = T_{w,\chi^{-1}}$. For $\Lambda\in I(\chi)^*$, let

$$(R(B)\ \Lambda)(\varphi) = \int\limits_B \Lambda(R(b)\ \varphi)\ db$$

be the projection of $I(\chi)^*$ onto $I\left(\chi^{-1}\right)^B$, where the measure is normalized so that $\int\limits_B db = 1$. Let $g_s\in G$ be such that $\theta(g_s) = s$. Since $R(B)\ R(g_s)\ \Lambda_{0,\chi}\in I\left(\chi^{-1}\right)^B$, there exist constants $\alpha_w(\chi, s)$ such that

$$R(B)\ R(g_s)\ \Lambda_{0,\chi} = \sum_{w\in W}\alpha_w(\chi,s)\ f_{w,\chi^{-1}}. \tag{71}$$

Applying $T_w(\cdot)(1)$ to both sides we get

$$\alpha_w(\chi,s) = (T_w(R(B)\ R(g_s)\ \Lambda_{0,\chi}))(1),$$

hence

$$\begin{aligned}\omega_t(s) &= (R(g_s)\ \Lambda_{0,\chi})(\varphi_{K,\chi})\\ &= \langle R(B)\,R(g_s)\ \Lambda_{0,\chi},\ \varphi_{K,\chi}\rangle_K\\ &= \sum_{w\in W}(T_w(R(B)\ R(g_s)\,\Lambda_{0,\chi}))(1)\ \langle f_{w,\chi^{-1}},\ \varphi_{K,\chi}\rangle_K.\end{aligned}$$

In [3], Casselman computed:

$$\langle f_{w,\chi^{-1}}, \varphi_{K,\chi}\rangle_K = \int_K f_{w,\chi^{-1}}(k)\, dk = Q^{-1} \frac{c_{\sigma_l}({}^w\chi)}{c_w(\chi^{-1})} \tag{72}$$

where Q is a constant independent of χ, and σ_l is the longest element of W, which is also the longest element of Γ. Since $\widetilde{T}_w$ is an intertwining operator, using (28),

$$\begin{aligned} T_w(R(B)\, R(g_s)\, \Lambda_{0,\chi}) &= R(B)\, R(g_s)\, \widetilde{T}_w\, \Lambda_{0,\chi} \\ &= \frac{c_w(\chi^{-1})}{c_{w^{-1}}({}^w\chi)} R(B)\, R(g_s)\, T^*_{w^{-1}}\, \Lambda_{0,\chi}. \end{aligned}$$

So we get

$$\omega_t(s) = Q^{-1} \sum_{w\in W} \frac{c_{\sigma_l}({}^w\chi)}{c_{w^{-1}}({}^w\chi)} \left(R(B)\, R(g_s)\, T^*_{w^{-1}}\, \Lambda_{0,\chi}\right)(1).$$

Denote

$$a_{w,\chi}(s) = \left(R(B)\, R(g_s)\, T^*_{w^{-1}}\, \Lambda_{0,\chi}\right)(1) \tag{73}$$

then,

$$\omega_t(s) = Q^{-1} \sum_{w\in W} \frac{c_{\sigma_l}({}^w\chi)}{c_{w^{-1}}({}^w\chi)}\, a_{w,\chi}(s). \tag{74}$$

If $w \notin \Gamma$ we call $a_{w,\chi}$ irrelevant.

5.4. Vanishing of the irrelevant terms. We show here that $a_{w,\chi}(d_\lambda) = 0$ whenever $\lambda \in \Lambda_n^+$ and $w \notin \Gamma$. So when evaluated at d_λ, the expression in (74) is actually a sum over Γ. Recall, $S' = \left\{ s \in S \;\middle|\; \prod_{i=1}^n d_i(s) \neq 0 \right\}$ is open in S.

Lemma 5.7. $S' = P \cdot d_0 = \theta(P\,\xi\,H)$.

Proof. Since $\theta(\xi) = d_0$ the second equality is clear. For

$$s = \begin{pmatrix} * & * \\ X & * \end{pmatrix} \in S, \tag{75}$$

with X an $n \times n$ matrix and $p \in P$, such that p_1 is its top left $n \times n$ block and p_2 is its bottom right $n \times n$ block

in Case 1 and in Case 3: $p \cdot s = \begin{pmatrix} * & * \\ p_2\, X\, p_1^{-1} & * \end{pmatrix}$;

in Case 2: $p \cdot s = \begin{pmatrix} * & * \\ p_2\, X\, \bar{p}_1^{-1} & * \end{pmatrix}$.

Hence S' is P-stable and clearly $P \cdot d_0 \subset S'$. If $s \in S'$ has the form (75), with X as above, then X must be in the Bruhat cell $P\, w_n\, P$. So $\exists\, p_1, p_2$, $n \times n$ upper triangular matrices such that in Case 1: $p_2\, X\, p_1^{-1} = -w_n$, in Case 2: $p_2\, X\, \bar{p}_1^{-1} = w_n$ and in Case 3: $p_2\, X\, p_1^{-1} = \tau\, w_n$. We may assume then, that the bottom left $n \times n$ block of s is $-w_n$ in Case 1, w_n in Case 2 and $\tau\, w_n$ in Case 3. If m is odd, s has the form

$$\begin{pmatrix} * & \alpha & * \\ \beta & c & \gamma \\ w_n & \delta & * \end{pmatrix},$$

where $\gamma,\, \beta \in M_{1\times n}(E)$, $\alpha,\, \delta \in M_{n\times 1}(E)$, $c \in E$. So

$$\begin{pmatrix} I_n & & \\ & 1 & -\beta\, w_n \\ & & I_n \end{pmatrix} \cdot s$$

has the form

$$\begin{pmatrix} * & \alpha & * \\ 0 & c & \gamma \\ w_n & \delta & * \end{pmatrix},$$

and a matrix of that form in S must also satisfy $\gamma = 0$. We may assume s is of this form, thus

$$\begin{pmatrix} I_n & w_n\, \bar{\delta} & \\ & 1 & \\ & & I_n \end{pmatrix} \cdot s = \begin{pmatrix} * & 0 & * \\ 0 & c & 0 \\ w_n & 0 & * \end{pmatrix}$$

for some $c \in E$. We can once more assume s is of this form. Since $s \in S$, $c\,\bar{c} = 1$, so by Hilbert 90, $c = \frac{u}{\bar{u}}$ for some $u \in \mathcal{O}_E^{\times}$. Thus

$$\begin{pmatrix} I_n & & \\ & u^{-1} & \\ & & I_n \end{pmatrix} \cdot s = \begin{pmatrix} * & 0 & * \\ 0 & 1 & 0 \\ w_n & 0 & * \end{pmatrix}$$

and we may therefore assume that $c = 1$. Back to a general m, imposing the condition $s \in S$, we find that there is an $n \times n$ matrix A such that:

In Case 1:

$$s = \begin{pmatrix} A & \left(I_n - A^2\right) w_n \\ -w_n & w_n\, A\, w_n \end{pmatrix}.$$

In Case 2:

$$s = \begin{pmatrix} -A & \left(I_n - A\,\overline{A}\right) w_n \\ w_n & w_n\, \overline{A}\, w_n \end{pmatrix}$$

if m is even, and

$$s = \begin{pmatrix} -A & & \left(I_n - A\,\overline{A}\right) w_n \\ & 1 & \\ w_n & & w_n\, \overline{A}\, w_n \end{pmatrix}$$

if m is odd.

In Case 3:

$$s = \begin{pmatrix} -A & \left(I_n - \tau^{-1} A^2\right) w_n \\ \tau\, w_n & w_n\, A\, w_n \end{pmatrix}.$$

In Case 1 and in Case 2 let

$$p = \begin{pmatrix} I_n & A\, w_n \\ & I_n \end{pmatrix} \text{ if } m \text{ is even, and } p = \begin{pmatrix} I_n & & A\, w_n \\ & 1 & \\ & & I_n \end{pmatrix} \text{ if } m \text{ is odd,}$$

and in Case 3 let

$$p = \begin{pmatrix} I_n & \tau^{-1} A\, w_n \\ & I_n \end{pmatrix},$$

then $p \cdot s = d_0$. □

Note that from Lemma 5.7 we get that $P\,\xi\,H$ is the pre-image of the open set S' under θ and therefore $P\,\xi\,H$ is open in G. We also get that

$$d_t\left(\theta\left(g\right)\right) = D_z^{I_m}\left(g\right) = \mathrm{ch}_{P\,\xi H}\left(g\right) \prod_{i=1}^{n} \left|d_i\left(\theta\left(g\right)\right)\right|^{t_i}. \tag{76}$$

Lemma 5.8. *For all $\lambda \in \Lambda_n^+$ we have*

$$PB \cdot d_\lambda = S'.$$

Moreover for all $b \in B$, $\lambda \in \Lambda_n^+$, $i = 1, \ldots, n$, we have:

$$\left|d_i\left(b \cdot d_\lambda\right)\right| = \left|d_i\left(d_\lambda\right)\right|. \tag{77}$$

Proof. It is easy to see that $P \cdot d_0 = P \cdot d_\lambda$ therefore the inclusion $S' \subset PB \cdot d_\lambda$ follows from Lemma 5.7. The other inclusion will follow once we prove (77). From (44), it is clear that $\left|d_i\left(b \cdot s\right)\right| = \left|d_i\left(s\right)\right|$, $\forall\, b \in P \cap B$, $s \in S$. Let $N_0 = N_0\left(m\right)$ be the subgroup of lower triangular unipotent matrices in K projecting to the identity matrix over the residual field. By the Iwahori decomposition, $B = \left(B \cap P\right)\, N_0$, it is enough to prove the lemma for $\eta \in N_0$. Denote by $-n_1^{-1}$ in Case 1, by $\overline{n_1}^{-1}$ in Case 2 and by $\tau\, n_1^{-1}$ in Case 3 the top left $n \times n$ block of η, and let n_2 be the bottom right $n \times n$ block of η. If X is the bottom left $n \times n$ block of $\eta \cdot d_\lambda$ then,

$$\left\| X - n_2\, \varpi^{\lambda^*}\, n_1 \right\|_1 < 1. \tag{78}$$

Let $\gamma = \left(\gamma_{i\,j}\right)$ be an $n \times n$ matrix, satisfying the following property:

$$\begin{array}{ll} \left|\gamma_{i\,j}\right| < \left|\varpi^{-\lambda_{n+1-i}}\right| & i + j < n + 1 \\ \left|\gamma_{i\,j}\right| < \left|\varpi^{-\lambda_j}\right| & i + j > n + 1 \\ \left|\gamma_{i\,j}\right| = \left|\varpi^{-\lambda_j}\right| & i + j = n + 1, \end{array} \tag{79}$$

i.e., the absolute value of each anti-diagonal entry is strictly greater then the absolute values of the entries below it in the same column, and then the entries to its left in the same row. For any permutation σ of $1, \ldots, n$, we have $|\gamma_{i\,\sigma(i)}| \leq |\varpi^{-\lambda_{\sigma(i)}}|$ and if equality holds then $i+\sigma(i)=n+1$. This is clear from (79) if $i+\sigma(i)\geq n+1$ and from (79) combined with the fact that $\lambda\in\Lambda_n^+$, if $i+\sigma(i)<n+1$. So

$$\prod_{i=1}^{n}|\gamma_{i\,\sigma(i)}|\leq|\varpi^{-|\lambda|}|,$$

and equality holds if and only if σ is the permutation associated to the permutation matrix w_n. Hence $|\det\gamma|=|\varpi^{-|\lambda|}|$. Note that if γ satisfies the property (79) with respect to $\lambda=(\lambda_1,\dots,\lambda_n)\in\Lambda_n^+$, then the $i\times i$ bottom left block of γ satisfies the property (79) with respect to $(\lambda_1,\dots,\lambda_i)$. Since $d_i\,(d_\lambda)=\varpi^{-(\lambda_1+\cdots+\lambda_i)}$, it is now enough to show that X has the property (79) with respect to λ. It is also clear from (78) that X has the property (79) with respect to λ if and only if $n_2\,\varpi^{\lambda^*}\,n_1$ has the property (79) with respect to λ. Note that $-n_1\in N_0\,(n)$ in Case 1, $n_1\in N_0\,(n)$ in Case 2 and $\tau\,n_1\in N_0\,(n)$ in Case 3. So multiplying by a unit, we may now assume $n_1,\,n_2\in N_0\,(n)$. Denote $n_2=(\alpha_{i,j})$ and $n_1=(\beta_{i,j})$, then

$$\left(n_2\,\varpi^{\lambda^*}\,n_1\right)_{i,j}=\sum_{k=1}^{\min\{n+1-j,\,i\}}\alpha_{i,k}\,\varpi^{-\lambda_{(n+1-k)}}\,\beta_{n+1-k,j}.$$

If $i+j<n+1$ then the sum is taken over $1\leq k\leq i$, and since $n+1-k\geq n+1-i>j$ we have $|\beta_{n+1-k,j}|\leq|\varpi|$, so

$$|\alpha_{i,k}\,\varpi^{-\lambda_{(n+1-k)}}\,\beta_{n+1-k,j}|\leq|\varpi^{1-\lambda_{n+1-k}}|\leq|\varpi^{1-\lambda_{n+1-i}}|.$$

Therefore $|\left(n_2\,\varpi^{\lambda^*}\,n_1\right)_{i,j}|<|\varpi^{-\lambda_{n+1-i}}|$. Similarly if $i+j>n+1$, then the sum is taken over $1\leq k\leq n+1-j$, and $|\alpha_{i,k}|\leq|\varpi|$, so

$$|\alpha_{i,k}\,\varpi^{-\lambda_{(n+1-k)}}\,\beta_{n+1-k,j}|\leq|\varpi^{1-\lambda_{n+1-k}}|\leq|\varpi^{1-\lambda_j}|.$$

Therefore $|\left(n_2\,\varpi^{\lambda^*}\,n_1\right)_{i,j}|<|\varpi^{-\lambda_j}|$. If $i+j=n+1$ then $\alpha_{i,i}=1=\beta_{n+1-i,j}$ and for $k<n+1-j$,

$$|\alpha_{i,k}\,\varpi^{-\lambda_{(n+1-k)}}\,\beta_{n+1-k,j}|<|\varpi^{-\lambda_{n+1-k}}|<|\varpi^{-\lambda_j}|,$$

hence

$$\left|\left(n_2\,\varpi^{\lambda^*}\,n_1\right)_{i,j}\right|=|\varpi^{-\lambda_j}|.$$

□

In Definition 5.9 and Proposition 5.10 we remove the restriction (49) and assume z is any element of $\mathbb{C}^n$. We still denote $\chi=\chi_{\nu(z)}$.

Definition 5.9. We will say that a linear form $\Lambda\in I\,({}^w\chi)^*$ is **supported away from** $P\,\xi\,H$ if the restriction of the linear form to the open set $P\,\xi\,H$ (i.e., to the space of functions supported on $P\,\xi\,H$) is 0.

Proposition 5.10. *Let $\lambda \in \Lambda_n^+$, and $g_\lambda \in G$, such that $\theta(g_\lambda) = d_\lambda$. If a linear form $\Lambda \in I(\chi)^*$ is supported away from $P\,\xi\,H$, then $(R(B)\ R(g_\lambda)\ \Lambda)(1) = 0$.*

Proof. For $\varphi \in I\left(\chi^{-1}\right)^B$, since $\varphi_{B,\chi}$ restricted to K is equal to the characteristic function of B, we have

$$\varphi(1) = (R(B)\,\varphi)(1) = \int_B \varphi(b)\ db = \int_K \varphi(k)\ \varphi_{B,\chi}(k)\ dk = \langle \varphi, \varphi_{B,\chi}\rangle_K.$$

So

$$(R(B)\ R(g_\lambda)\ \Lambda)(1) = \langle R(B)\ R(g_\lambda)\ \Lambda, \varphi_{B,\chi}\rangle_K = \Lambda\left(R\left(g_\lambda^{-1}\right)\ \varphi_{B,\chi}\right).$$

Let $g \in G$ be such that $R\left(g_\lambda^{-1}\right)\ \varphi_{B,\chi}(g) \neq 0$, then since the support of $\varphi_{B,\chi}$ is $P\,B$, we get that $g\,g_\lambda^{-1} \in P\,B$, hence by Lemma 5.7 and Lemma 5.8 we have $\theta(g) \in P\,B \cdot d_\lambda = P \cdot d_0$ and therefore $g \in P\,\xi H$. □

Proposition 5.11. *$T^*_{w^{-1}}\,\Lambda_{0,\chi}$ is supported away from $P\,\xi\,H$ unless $w \in \Gamma$.*

Proof. Let r be the restriction of ${}^w\chi\delta^{\frac{1}{2}}$ to $\xi\,H_\xi\,\xi^{-1}$.

In Case 1 and in Case 3:

$$\xi\,H_\xi\,\xi^{-1} = \left\{ \left(\begin{array}{cc} a & 0 \\ 0 & w_n\,a\,w_n \end{array} \right) \middle|\ a = \mathrm{diag}[a_1, \ldots, a_n],\ a_i \in F^\times \right\}.$$

In Case 2:

$$\xi\,H_\xi\,\xi^{-1} = \left\{ \left(\begin{array}{cc} a & 0 \\ 0 & w_n\,\bar{a}\,w_n \end{array} \right) \middle|\ a = \mathrm{diag}[a_1, \ldots, a_n],\ a_i \in E^\times \right\}$$

if m is even, and

$$\xi\,H_\xi\,\xi^{-1} = \left\{ \left(\begin{array}{ccc} a & & \\ & b & \\ & & w_n\,\bar{a}\,w_n \end{array} \right) \middle|\ a = \mathrm{diag}[a_1, \ldots, a_n],\ a_i \in E^\times,\ b \in F^\times \right\}$$

if m is odd.

Thus $r \equiv 1$ if and only if $w \in \Gamma$. The subspace of $I({}^w\chi)$ of all functions supported in the open double coset $P\,\xi\,H$, is naturally isomorphic to the space $\mathcal{S}(H_\xi\backslash H, r)$, of all complex valued functions f on H of compact support modulo H_ξ, that are right invariant under some open subgroup of H and satisfy for $h_0 \in H_\xi$, $h \in H$:

$$f(h_0\,h) = r\left(\xi\,h_0\,\xi^{-1}\right) f(h).$$

For $\varphi \in I({}^w\chi)$ with support in $P\,\xi\,H$, we denote by f^φ its image in $\mathcal{S}(H_\xi\backslash H, r)$, then

$$f^\varphi(h) = \varphi(\xi\,h).$$

The isomorphism is clearly H-equivariant. If $T^*_{w^{-1}}\,\Lambda_{0,\chi}$ is not supported away from $P\,\xi\,H$, then there is a nonzero, H-invariant form Λ on $\mathcal{S}(H_\xi\backslash H, r)$.

There is an H-equivariant projection of $C_c^\infty(H)$ onto $\mathcal{S}(H_\xi\backslash H, r)$, where H is acting by the right action on $C_c^\infty(H)$, defined for $F \in C_c^\infty(H)$ by

$$f_F(h) = \int_{H_\xi} r\left(\xi\, h_0^{-1}\, \xi^{-1}\right) F(h_0\, h)\, dh_0,$$

where dh_0 is a Haar measure on H_ξ. Let $T \in \mathcal{D}(H)$ be defined by:

$$\langle T, F\rangle = \Lambda(f_F),$$

then T is a nonzero, H-invariant distribution on H and hence upto a complex scalar it is a right Haar measure. Since H is unimodular, T is also left invariant by H. For $F \in C_c^\infty(H)$ we denote by F^{h_1} the function defined by $F^{h_1}(h) = F(h_1\, h),\ \ h, h_1 \in H$. Note that for all $F \in C_c^\infty(H)$ and $h_0 \in H_\xi$ we have

$$f_{F^{h_0}} = r\left(\xi\, h_0\, \xi^{-1}\right) f_F.$$

So

$$\langle T, F\rangle = \langle T, F^{h_0}\rangle = \Lambda(f_{F^{h_0}}) = r\left(\xi\, h_0\, \xi^{-1}\right) \Lambda(f_F) = r\left(\xi\, h_0\, \xi^{-1}\right) \langle T, F\rangle.$$

Therefore $r = {}^w\chi\delta^{\frac{1}{2}}_{|\xi\, H_\xi\, \xi^{-1}} \equiv 1$, which implies that $w \in \Gamma$. □

Combining Proposition 5.10, Proposition 5.11 and (73), indeed, for $w \in W$, $w \notin \Gamma$ and $\lambda \in \Lambda_n^+$ we have

$$a_{w,\chi}(d_\lambda) = 0. \tag{80}$$

5.5. The explicit functional equations. For $\lambda \in \Lambda_n^+$, (74) now takes the form

$$\omega_t(d_\lambda) = Q^{-1} \sum_{\sigma\in\Gamma} \frac{c_{\sigma_l}({}^\sigma\chi)}{c_{\sigma^{-1}}({}^\sigma\chi)}\, a_{\sigma,\chi}(d_\lambda). \tag{81}$$

For $\sigma \in \Gamma$, and $T_{\sigma^{-1}} = T_{\sigma^{-1},{}^\sigma\chi},\ \ T^*_{\sigma^{-1}}\, \Lambda_{0,\chi}$ is an H-invariant linear form on $I({}^\sigma\chi)$. Lemma 5.5 implies that there is a constant $A_\sigma(\chi)$ such that

$$T^*_{\sigma^{-1}}\, \Lambda_{0,\chi} = A_\sigma(\chi)\, \Lambda_{0,{}^\sigma\chi}. \tag{82}$$

Computing as in the proof of Proposition 5.10,

$$\begin{aligned} a_{\sigma,\chi}(d_\lambda) &= \left(R(B)\ R(g_\lambda)\ T^*_{\sigma^{-1}}\, \Lambda_{0,\chi}\right)(1) \\ &= A_\sigma(\chi)\ \left(R(B)\ R(g_\lambda)\ \Lambda_{0,{}^\sigma\chi}\right)(1) \\ &= A_\sigma(\chi)\ \Lambda_{0,{}^\sigma\chi}\left(R\left(g_\lambda^{-1}\right)\ \varphi_{B,\chi}\right) \\ &= A_\sigma(\chi)\ \Lambda_{0,{}^\sigma\chi} \circ \mathcal{P}_{{}^\sigma\chi}\left(\mathrm{ch}_{B\, g_\lambda}\right) \\ &= A_\sigma(\chi)\ \langle D^{I_m}_{\sigma\, z}, \mathrm{ch}_{B\, g_\lambda}\rangle \\ &= A_\sigma(\chi) \int_B d_{\sigma\, t}(b\cdot d_\lambda)\ dg, \end{aligned}$$

where $\sigma\, t$ is related to $\sigma\, z$ by (41). By Lemma 5.8 we obtain

$$\begin{aligned} a_{\sigma,\chi}\left(d_\lambda\right) &= A_\sigma\left(\chi\right)\left\{\int\limits_B dg\right\} d_{\sigma\, t}\left(d_\lambda\right) \\ &= A_\sigma\left(\chi\right)\left\{\int\limits_B dg\right\} q_1^{\sum_{i=1}^{n}\lambda_i\left(\sum_{j=i}^{n}(\sigma\, t)_i\right)} \\ &= A_\sigma\left(\chi\right)\left\{\int\limits_B dg\right\} q_1^{\sum_{i=1}^{n}\lambda_i\left((\sigma\, z)_i-\left(n-i+\frac{1}{2}\right)\right)} \\ &= A_\sigma\left(\chi\right)\left\{\int\limits_B dg\right\} q_1^{\lambda\cdot(\sigma\, z-\rho)}. \end{aligned}$$

Recall that for $z \in \mathbb{C}^n$, we assigned $e^{\epsilon_i} = q_1^{z_i}$, thus,

$$a_{\sigma,\chi}\left(d_\lambda\right) = A_\sigma\left(\chi\right)\left\{\int\limits_B dg\right\} q_1^{-(\lambda\cdot\rho)}\, e^{\sigma\,\lambda}. \tag{83}$$

Combining all this we obtain:

Lemma 5.12.

$$\omega_t\left(d_\lambda\right) = \left\{\int\limits_B dg\right\} Q^{-1}\, q_1^{-(\lambda\cdot\rho)} \sum_{\sigma\in\Gamma} \frac{c_{\sigma_l}\left({}^{\sigma}\chi\right)}{c_{\sigma^{-1}}\left({}^{\sigma}\chi\right)}\, A_\sigma\left(\chi\right)\, e^{\sigma\,\lambda}. \tag{84}$$

Let Σ^{+L} (respectively Σ^{+S}) be the subset of long (respectively short) roots in Σ^+.

In Case 1 let

$$\zeta\left(\chi\right) = \prod_{\alpha\in\Sigma^{+L}} \frac{1-q^{-1}\, e^{-\alpha}}{1-e^{-\alpha}} \prod_{\alpha\in\Sigma^{+S}} \frac{1+q^{-\frac{1}{2}}e^{-\alpha}}{1-q^{-\frac{1}{2}}e^{-\alpha}}.$$

In Case 2 let

$$\zeta\left(\chi\right) = \prod_{\alpha\in\Sigma^{+L}} \frac{1-q^{-2}\, e^{-\alpha}}{1-e^{-\alpha}} \prod_{\alpha\in\Sigma^{+S}} \frac{1+q^{-1}e^{-\alpha}}{1-q\, e^{-\alpha}}$$

if m is even and

$$\zeta\left(\chi\right) = \prod_{\alpha\in\Sigma^{+L}} \frac{1-q^{-2}\, e^{-\alpha}}{1-e^{-\alpha}} \prod_{\alpha\in\Sigma^{+S}} \frac{1+q^{-1}e^{-\alpha}}{1-q\, e^{-\alpha}}\, \frac{1-q^{-2}e^{-\alpha}}{1-e^{-\alpha}}$$

if m is odd.

In Case 3 let

$$\zeta(\chi) = \prod_{\alpha\in\Sigma^{+L}} \frac{1-q^{-1}\,e^{-\alpha}}{1-e^{-\alpha}}.$$

We remind the reader that we assume z satisfies (49). We will use the following lemma for the computation of the spherical functions:

Lemma 5.13. *There is a positive constant c, independent of χ, such that*

$$\Lambda_{0,\chi} = c\,\Lambda_\chi. \tag{85}$$

Proof. By Lemma 5.5, the equality (85) holds with a constant $c = c_\chi$. In what follows we show that c is independent of χ. By definition of $\Lambda_{0,\chi}$,

$$\Lambda_{0,\chi}\left(\mathcal{P}_\chi(\phi)\right) = \langle D_z^{I_m}, \phi\rangle$$

for all $\phi \in C_c^\infty(G)$. Since we assume (49), by (41) $\operatorname{Re} t_i > 0,\ i = 1,\dots,n$, and by [**5**], the integral defining the distribution $D_z^{I_m}$ is convergent. Hence

$$\Lambda_{0,\chi}\left(\mathcal{P}_\chi(\phi)\right) = \int_G \phi(g)\; d_t(\theta(g))\; dg = \int_{P\,\xi\,H} \phi(g)\; d_t(\theta(g))\; dg.$$

From (44) we get that

$$d_t(\theta(g)) = \chi^{-1}\,\delta^{1/2}(p(g))$$

for all $g \in P\,\xi\,H$, where $g = p(g)\,\xi\,h$ independent of the choice of $p(g) \in P$ and $h \in H$. So

$$\Lambda_{0,\chi}\left(\mathcal{P}_\chi(\phi)\right) = \int_{P\,\xi\,H} \chi^{-1}\,\delta^{1/2}(p(g))\;\phi(g)\;dg.$$

We let $P \times H$ act on G through the right action $g^{(p,h)} = p^{-1}gh$. Then

$$P\,\xi\,H \simeq \operatorname{Stab}_\xi\backslash(P\times H)$$

and

$$\operatorname{Stab}_\xi = \widetilde{\Delta}_{H_\xi} = \{\left(\xi\,h\,\xi^{-1}, h\right) \mid h \in H_\xi\}.$$

So

$$\Lambda_{0,\chi}\left(\mathcal{P}_\chi(\phi)\right) = \int_{\widetilde{\Delta}_{H_\xi}\backslash(P\times H)} \chi^{-1}\,\delta^{1/2}\left(p\left(\xi^\beta\right)\right)\;\phi\left(\xi^\beta\right)\;d\beta.$$

Computing formally first, we get

$$\Lambda_{0,\chi}\left(\mathcal{P}_\chi(\phi)\right) = \int_{\left(P\times H_\xi\right)\backslash(P\times H)}\;\int_{\widetilde{\Delta}_{H_\xi}\backslash\left(P\times H_\xi\right)} \chi^{-1}\,\delta^{1/2}\left(p\left(\xi^{\alpha\,\beta}\right)\right)\;\phi\left(\xi^{\alpha\,\beta}\right)\;d\alpha\,d\beta.$$

Clearly $(P\times H_\xi)\backslash(P\times H) \simeq H_\xi\backslash H$. The $(P\times H)$-invariant measure on $(P\times H_\xi)\backslash(P\times H)$ transforms to a positive multiple $c_1\,dh$ of dh. Also

$\widetilde{\Delta}_{H_\xi}\backslash(P\times H_\xi)\simeq P$, through the isomorphism $\widetilde{\Delta}_{H_\xi}(p,1)\mapsto p$. This isomorphism transforms the $(P\times H_\xi)$-invariant measure on $\widetilde{\Delta}_{H_\xi}\backslash(P\times H_\xi)$ to a right Haar measure $d_R p$ on P. Since $d_R\left(p^{-1}\right)$ is a left Haar measure on P, there is a positive constant c_2 such that $d_R\left(p^{-1}\right)=c_2\,d_L p$. Hence we obtain,

$$\begin{aligned}\Lambda_{0,\chi}\left(\mathcal{P}_\chi(\phi)\right)&=c_1\int\limits_{H_\xi\backslash H}\int\limits_P\phi\left(p^{-1}\xi h\right)\chi^{-1}\delta^{\frac12}\left(p^{-1}\right)\,d_R p\,dh\\&=c_1\,c_2\int\limits_{H_\xi\backslash H}\int\limits_P\phi\left(p\,\xi h\right)\chi^{-1}\delta^{\frac12}\left(p\right)\,d_L p\,dh\\&=c_1\,c_2\int\limits_{H_\xi\backslash H}\left(\mathcal{P}_\chi(\phi)\right)(\xi\,h)\,dh\\&=c_1\,c_2\,\Lambda_\chi\left(\mathcal{P}_\chi(\phi)\right).\end{aligned}$$

The convergences of the integrals are justified by Proposition 5.1 and (23). □

Proposition 5.14. *There is a positive constant c, independent of χ, such that*

$$\Omega_z=c\,\frac{\omega_t}{\zeta(\chi)}.\tag{86}$$

Proof. From the definition of Ω_z in (40), we need to show that the ratio between $\omega_t(d_0)$ and $\zeta(\chi)$ is independent of χ. From Lemma 5.6, Lemma 5.13 and the fact that for our choice of ξ, $\xi\in K$ we get that

$$\omega_t(d_0)=c\left(R(\xi)\,\Lambda_\chi\right)(\varphi_{K,\chi})=c\,\Lambda_\chi(\varphi_{K,\chi})=c\int\limits_{H_\xi\backslash H}\varphi_{K,\chi}(\xi\,h)\,dh,$$

for some constant c independent of χ. In Proposition 5.1 we showed that

$$\int\limits_{H_\xi\backslash H}\varphi_{K,\chi}(\xi\,h)\,dh=\left(\lambda\circ\Pi\circ T_{w_0}\right)\varphi_{K,\chi},$$

where λ and Π are defined in the proof of the proposition. By (28) we have

$$T_{w_0}\,\varphi_{K,\chi}=c_{w_0}(\chi)\,\varphi_{K,{}^{w_0}\chi}.$$

So in Case 1 and in Case 2:

$$\begin{aligned}&(\Pi \circ T_{w_0})\, \varphi_{K,\chi}\,(g_1,\dots,g_n)\\ &= c_{w_0}(\chi)\ \Delta(g_1,\dots,g_n) \int\limits_{K_0} \varphi_{K,{}^{w_0}\chi} \left[\begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_n \end{pmatrix} w_0\, k_0 \right] dk_0 \\ &= c_{w_0}(\chi)\ \Delta(g_1,\dots,g_n)\ \varphi_{K,{}^{w_0}\chi} \left[\begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_n \end{pmatrix} \right],\end{aligned}$$

where $g_1,\dots,g_n \in GL(2,F)$ in Case 1, and $g_1,\dots,g_n \in GL(2,E)$ in Case 2 and an equality between the left and right-hand sides similarly holds in Case 3 for $g_1,\dots,g_n \in GL(2,F)$. It is therefore easy to verify that

$$(\Pi \circ T_{w_0})\, \varphi_{K,\chi} = c_{w_0}(\chi) \left(\bigotimes_{i=1}^{n} \varphi_{K_2,(\chi_i,\chi_i^{-1})} \right).$$

Using Lemma 5.2 for Case 1, Lemma 5.3 for Case 2 and Lemma 5.4 for Case 3 we then see that in Case 1:

$$\omega_t(d_0) = c\, c_{w_0}(\chi) \prod_{i=1}^{n} \frac{1+q^{-\frac{1}{2}} q^{-z_i}}{1-q^{-\frac{1}{2}} q^{-z_i}} = c\, c_{w_0}(\chi) \prod_{\alpha\in\Sigma^{+S}} \frac{1+q^{-\frac{1}{2}} e^{-\alpha}}{1-q^{-\frac{1}{2}} e^{-\alpha}}; \tag{87}$$

in Case 2:

$$\omega_t(d_0) = c\, c_{w_0}(\chi) \prod_{i=1}^{n} \frac{1+q^{-1} q^{-2z_i}}{1-q\, q^{-2z_i}} = c\, c_{w_0}(\chi) \prod_{\alpha\in\Sigma^{+S}} \frac{1+q^{-1} e^{-\alpha}}{1-q\, e^{-\alpha}} \tag{88}$$

and in Case 3:

$$\omega_t(d_0) = c\, c_{w_0}(\chi) \tag{89}$$

for some constant c, independent of χ. To compute $c_{w_0}(\chi)$ explicitly, we note that

$$\Phi^{+}_{w_0} = \{e_i - e_j \,|\, n < i < j \le m \text{ or } 1 \le m+1-j < i \le n\}$$

is in bijection with Σ^{+L} through

$$e_i - e_j \mapsto \begin{cases} \epsilon_{m+1-j} - \epsilon_{m+1-i} & n < i < j \le m \\ \epsilon_{m+1-j} + \epsilon_i & 1 \le m+1-j < i \le n \end{cases}$$

if m is even, and

$$\Phi^{+}_{w_0} = \{e_i - e_j \,|\, n < i < j \le m \text{ or } 1 \le m+1-j < i \le n\}$$

is in bijection with Σ^+ through

$$e_i - e_j \mapsto \begin{cases} \epsilon_{m+1-j} & i = n+1 \\ \epsilon_{m+1-j} - \epsilon_{m+1-i} & n < i < j \leq m \\ \epsilon_{m+1-j} + \epsilon_i & 1 \leq m+1-j < i \leq n \end{cases}$$

if m is odd. If $a \in \Phi^+_{w_0}$ is associated to $\alpha \in \Sigma^+$, then

$$c_a(\chi) = \frac{1 - q_1^{-1} e^{-\alpha}}{1 - e^{-\alpha}}.$$

Thus by the definition of $c_{w_0}(\chi)$ (18), we get

$$c_{w_0}(\chi) = \prod_{\alpha \in \Sigma^{+L}} \frac{1 - q_1^{-1} e^{-\alpha}}{1 - e^{-\alpha}}$$

if m is even, and

$$c_{w_0}(\chi) = \prod_{\alpha \in \Sigma^{+}} \frac{1 - q_1^{-1} e^{-\alpha}}{1 - e^{-\alpha}}$$

if m is odd. This combined with (87) in Case 1, with (88) in Case 2 and with (89) in Case 3, indeed implies that $\omega_t(d_0)$ is a constant multiple of $\zeta(\chi)$, the constant being independent of χ. $\square$

5.6. Proof of the main theorems. By Lemma 5.12 and Proposition 5.14, there is a constant c independent of z, such that for all $\lambda \in \Lambda_n^+$

$$\Omega_z(d_\lambda) = c\, Q^{-1} q^{-(\lambda \cdot \rho)} \sum_{\sigma \in \Gamma} c(\sigma, \chi)\, e^{\sigma \lambda} \tag{90}$$

where

$$c(\sigma, \chi) = \frac{c_{\sigma_l}({}^{\sigma}\chi)}{c_{\sigma^{-1}}({}^{\sigma}\chi)} \frac{A_\sigma(\chi)}{\zeta(\chi)}.$$

Note that $z \mapsto e^{\sigma \lambda} = e^{\sigma \lambda}(z)$, $\sigma \in \Gamma$, are linearly independent additive characters in z. Let

$$\epsilon(\chi) = c(1, \chi) = \frac{c_{\sigma_l}(\chi)}{\zeta(\chi)}.$$

Then for $\tau \in \Gamma$, comparing the coefficient of $e^{\tau \lambda}(z) = e^{\lambda}(\tau z)$, in (90) applied to the equality $\Omega_z(d_\lambda) = \Omega_{\tau z}(d_\lambda)$, given by the functional equation in Proposition 4.3, we obtain

$$c(\tau, \chi) = c(1, {}^{\tau}\chi) = \epsilon({}^{\tau}\chi),$$

so

$$\Omega_z(d_\lambda) = c\, Q^{-1} q^{-(\lambda \cdot \rho)} \sum_{\sigma \in \Gamma} \sigma\left(\epsilon(\chi)\, e^{\lambda}\right). \tag{91}$$

By the definition of $c_{\sigma_l}(\chi)$ in (18),

$$c_{\sigma_l}(\chi) = \prod_{\alpha\in\Sigma^{+L}} \left(\frac{1-q_1^{-1}\,e^{-\alpha}}{1-e^{-\alpha}}\right)^2 \prod_{\alpha\in\Sigma^{+S}} \left(\frac{1-q_1^{-1}e^{-2\,\alpha}}{1-e^{-2\,\alpha}}\right)$$

if m is even, and

$$c_{\sigma_l}(\chi) = \prod_{\alpha\in\Sigma^{+L}} \left(\frac{1-q_1^{-1}\,e^{-\alpha}}{1-e^{-\alpha}}\right)^2 \prod_{\alpha\in\Sigma^{+S}} \left(\frac{1-q_1^{-1}e^{-2\,\alpha}}{1-e^{-2\,\alpha}}\right)\left(\frac{1-q^{-2}e^{-\alpha}}{1-e^{-\alpha}}\right)$$

if m is odd.

We then have

in Case 1:

$$\epsilon(\chi) = \prod_{\alpha\in\Sigma^{+L}} \left(\frac{1-q^{-1}\,e^{-\alpha}}{1-e^{-\alpha}}\right) \prod_{\alpha\in\Sigma^{+S}} \left(\frac{1-q^{-\frac{1}{2}}e^{-\alpha}}{1+q^{-\frac{1}{2}}e^{-\alpha}}\right)\left(\frac{1-q^{-1}\,e^{-2\,\alpha}}{1-e^{-2\,\alpha}}\right);$$

in Case 2:

$$\epsilon(\chi) = \prod_{\alpha\in\Sigma^{+L}} \left(\frac{1-q^{-2}\,e^{-\alpha}}{1-e^{-\alpha}}\right) \prod_{\alpha\in\Sigma^{+S}} \left(\frac{1-q\,e^{-\alpha}}{1+q^{-1}e^{-\alpha}}\right)\left(\frac{1-q^{-2}\,e^{-2\,\alpha}}{1-e^{-2\,\alpha}}\right);$$

in Case 3:

$$\epsilon(\chi) = \prod_{\alpha\in\Sigma^{+L}} \left(\frac{1-q^{-1}\,e^{-\alpha}}{1-e^{-\alpha}}\right) \prod_{\alpha\in\Sigma^{+S}} \left(\frac{1-q^{-1}\,e^{-2\,\alpha}}{1-e^{-2\,\alpha}}\right).$$

Comparing (91) with (6) and the definition of $P_z(\lambda)$ we obtain:

$$\Omega_z(d_\lambda) = c\,Q^{-1}\,q^{-(\lambda\cdot\rho)}\,V_\lambda\,P_z(\lambda).$$

Since $P_z(0) = 1 = \Omega_z(d_0)$ we see that $c = \frac{Q}{V_0}$, hence

$$\Omega_z(d_\lambda) = q_1^{-(\lambda\cdot\rho)}\,\frac{V_\lambda}{V_0}\,P_z(\lambda). \tag{92}$$

Theorem 1.2 now follows from Proposition 4.10 by the analytic continuation of Ω_z to $\mathbb{C}^n$.

We pass to the proof of Theorem 1.3. We first need to compute the volumes of the K-orbits in S. The computation is a straight forward application of the work of Mao and Rallis [**19**]. For the rest of this work z is any element in $\mathbb{C}^n$.

Proposition 5.15.

$$\int\limits_{K\cdot d_\lambda} ds = q_1^{2\,(\lambda\cdot\rho_0)}\,\frac{V_0}{V_\lambda}.$$

Proof. The proof is that of Z. Mao and S. Rallis, we repeat it here for the reader's convenience. As in [**19**], we start with the following:

Lemma 5.16. *For $\lambda \in \Lambda_n^+$,*

$$\langle (P_z(\lambda))^2, 1\rangle = V_\lambda^{-1} \tag{93}$$

where the scalar product on $\mathbb{C}[q_1^z]^\Gamma$ is defined in (20).

Proof. By the definition of the scalar product $\langle (P_z(\lambda))^2, 1\rangle$ is the value of the constant term of $|\Gamma|^{-1}\left(P_\lambda^{BC}\right)^2 \Delta$, after the specialization in terms of z defined in Chapter 1. Denote by Γ_λ the subgroup of Γ that fixes λ. It follows from the proof of (10.1) in [**18**], that:

$$|\Gamma_\lambda|^{-1} V_\lambda P_\lambda^{BC} \Delta = m_\lambda + \sum_{\mu>\lambda} u_{\mu\,\lambda} m_\mu \tag{94}$$

for some constants $u_{\mu\,\lambda}$. In fact the argument in [**18**] shows that for $\lambda \in \Lambda_n^+$,

$$|\Gamma_\lambda|^{-1} V_\lambda P_\lambda^{BC} \Delta = m_\lambda + \sum_{\mu>\lambda} u_{\mu\,\lambda} m_\mu + \sum_{\mu>\lambda} v_{\mu\,\lambda} |\Gamma_\mu|^{-1} P_\mu^{BC} \Delta \tag{95}$$

for some constants $u_{\mu\,\lambda}$, $v_{\mu\,\lambda}$. We can then proceed using (95), for each of the (finitely many) summands $P_\mu^{BC}\Delta$. Since there exist r, and $\mu_1, \ldots, \mu_r \in \Lambda_n^+$, maximal such that m_{μ_i} appears with a nonzero coefficient in the sum representing $|\Gamma_\lambda|^{-1} V_\lambda P_\lambda^{BC}\Delta$ in term of the basis $\{m_\mu \mid \mu \in \Lambda_n^+\}$, after finitely many steps the sum (95) will become of the form (94). Since $P_\lambda^{BC} = m_\lambda + \sum_{\mu<\lambda} u_{\mu\,\lambda} m_\mu$, and since for $\mu_1 > \mu_2$, $m_{\mu_1} m_{\mu_2}$ has no constant term, the constant term of $|\Gamma|^{-1}\left(P_\lambda^{BC}\right)^2 \Delta = \left(V_\lambda^{-1} |\Gamma|^{-1} |\Gamma_\lambda|\right) |\Gamma_\lambda|^{-1} V_\lambda P_\lambda^{BC} \Delta \cdot P_\lambda^{BC}$ is the constant term of $\left(V_\lambda^{-1} |\Gamma|^{-1} |\Gamma_\lambda|\right) m_\lambda^2$, which is computed in [**18**] and equals V_λ^{-1}. □

Since $\mathcal{S}(K\backslash S)$ is an $\mathcal{H}(G,K)$-module, for every $f \in \mathcal{H}(G,K)$ there are constants c_μ, $\mu \in \Lambda_n^+$, all but finitely many equal zero, such that:

$$f * \mathrm{ch}_0 = \sum_{\mu \in \Lambda_n^+} c_\mu \, \mathrm{ch}_\mu. \tag{96}$$

We compute $(f * \Omega_z)(d_\lambda)$ in two different ways. On the one hand using (96),

$$\begin{aligned}(f * \Omega_z)(d_\lambda) &= \sum_{\mu \in \Lambda_n^+} (f * \mathrm{ch}_\mu)(d_\mu)\, \Omega_z(d_\mu) \\ &= (f * \mathrm{ch}_0)(d_0)\, \Omega_z(d_0) + \sum_{\mu \neq 0} (f * \mathrm{ch}_\mu)(d_\mu)\, \Omega_z(d_\mu).\end{aligned}$$

On the other hand by Lemma 4.2,

$$(f * \Omega_z)(d_\lambda) = \widetilde{f}(z)\, \Omega_z(d_\lambda).$$

Applying Theorem 1.2 to the equality

$$\widetilde{f}(z)\ \Omega_z(d_\lambda) = (f * \mathrm{ch}_0)(d_0)\ \Omega_z(d_0) + \sum_{\mu\neq 0} (f * \mathrm{ch}_\mu)(d_\mu)\ \Omega_z(d_\mu),$$

we get

$$q_1^{-(\lambda\cdot\rho)} \frac{V_\lambda}{V_0} \widetilde{f}(z)\, P_z(\lambda) = c(\lambda) + \sum_{\mu\neq 0} d(\mu)\, P_z(\mu),$$

for some constants $d(\mu)$ independent of z. Taking inner product with $P_z(0) = 1$, and using (21) and (22) we have:

$$q_1^{-(\lambda\cdot\rho)} V_\lambda \langle \widetilde{f}(z)\, P_z(\lambda), 1\rangle = c(\lambda). \tag{97}$$

By Lemma 4.2 and (45),

$$(f * \mathrm{ch}_0)\hat{} = \widetilde{f}(z).$$

On the other hand using (96),

$$(f * \mathrm{ch}_0)\hat{} = \sum_{\mu\in\Lambda_n^+} c(\mu)\, \hat{\mathrm{ch}}_\mu.$$

Therefore using (45) and Theorem 1.2 once more, we get:

$$\widetilde{f}(z) = \sum_{\mu\in\Lambda_n^+} \left\{ \int_{K\cdot d_\mu} ds \right\} q_1^{-(\mu\cdot\rho)} \frac{V_\mu}{V_0}\, c(\mu)\, P_z(\mu).$$

Taking inner product with $P_z(\lambda)$, and using (21) and (22) we get:

$$\langle \widetilde{f}(z), P_z(\lambda)\rangle = \left\{ \int_{K\cdot d_\lambda} ds \right\} q_1^{-(\lambda\cdot\rho)} \frac{c(\lambda)}{V_0}. \tag{98}$$

From (97) and (98) we get:

$$\int_{K\cdot d_\lambda} ds = q_1^{2(\lambda\cdot\rho)} \frac{V_0}{V_\lambda} \frac{\langle \widetilde{f}(z), P_z(\lambda)\rangle}{\langle \widetilde{f}(z)\, P_z(\lambda), 1\rangle}. \tag{99}$$

Since this is true for all $f \in \mathcal{H}(G,K)$, by (39), we may now pick f such that $\widetilde{f}(z) = P_z(\lambda)$. From (93) and (22) we get

$$\frac{\langle P_z(\lambda), P_z(\lambda)\rangle}{\langle (P_z(\lambda))^2, 1\rangle} = 1.$$

□

The spherical Fourier inversion formula (Theorem 1.3), and the computation of the Plancherel measure now follow as in [**16**] Chapter V. In Case 1 and in Case 3 it follows as in Theorem (5.1.2). Case 2 falls into what Mcdonald refers to as the exceptional case, and the Plancherel measure follows as in Theorem (5.2.10).

6. The H-distinguished spherical representations

Definition 6.1. A representation (π, V) of G is called H-distinguished if there is a nonzero, H-invariant, linear form on V.

Proposition 6.2. *Let (π, V) be an irreducible, H-distinguished, spherical representation of G, then there exists $z \in \mathbb{C}^n$ such that π is isomorphic to a sub-quotient of $I\left(\chi_{\nu(z)}\right)$.*

Proof. Let $v_K \in V$ be a nonzero K-invariant vector. The isomorphism class of π is determined by the character $f \mapsto \hat{f}(\pi)$ of $\mathcal{H}(G, K)$, defined by

$$\pi(f)\, v_K = \hat{f}(\pi)\, v_K.$$

For $\nu \in \mathbb{C}^m$, the character of $\mathcal{H}(G, K)$ associated to the irreducible sub-quotient of $I(\chi_\nu)$ is the Satake transform, $f \mapsto \hat{f}(\nu)$ defined in (37). Let Λ be a nonzero H-invariant form on V, from the proof of Lemma 5.5 we have $\Lambda(v_K) \neq 0$, so replacing Λ by a constant multiple we may assume $\Lambda(v_K) = 1$. As in Lemma 5.5, define

$$\Omega(\theta(g)) = \Lambda\left(\pi\left(g^{-1}\right) v_K\right),$$

then Ω is a relative spherical function on S, with eigenvalue $f \mapsto \hat{f}(\pi)$ on the Hecke algebra $\mathcal{H}(G, K)$. By Proposition 4.10, $\exists\, z \in \mathbb{C}^n$, such that

$$\Omega = \Omega_z.$$

By Lemma 4.2, we then have

$$\hat{f}(\pi) = \widetilde{f}(z) = \hat{f}(\nu(z)),$$

hence (π, V) is isomorphic to the irreducible spherical sub-quotient of $I\left(\chi_{\nu(z)}\right)$. ☐

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