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AN APPROACH TO SPECTRAL PROBLEMS ON
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ISAAC PESENSON

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It is shown that eigenvalues of the Laplace–Beltrami operator on a compact Riemannian manifold can be determined as limits of eigenvalues of certain finite-dimensional operators in spaces of polyharmonic functions with singularities. In particular, a bounded set of eigenvalues can be determined using a space of such polyharmonic functions with a fixed set of singularities. It also shown that corresponding eigenfunctions can be reconstructed as uniform limits of the same polyharmonic functions with appropriate fixed set of singularities.

1. Introduction and main results

Given an appropriate set x_1, x_2, \dots, x_N of points “uniformly” distributed over a compact manifold M , $\dim M = d$, and a natural $k > d/2$, we construct a N -dimensional subspace of polyharmonic functions $S^k(x_1, x_2, \dots, x_N)$ with singularities at x_1, x_2, \dots, x_N . In other words, the space $S^k(x_1, x_2, \dots, x_N)$ is the set of solutions of the following equation:

$$\Delta^{2k}u = \sum_{\gamma=1}^N \alpha_\gamma \delta(x_\gamma), k > d/2,$$

where Δ is the Laplace–Beltrami operator of M , $\delta(x_\gamma)$ is the Dirac measure at the point x_γ and

$$\alpha_1 + \alpha_2 + \dots + \alpha_N = 0.$$

The main result (Theorem 1.3) shows that eigenvalues of the Laplace–Beltrami operator on M can be determined in two different ways:

- 1) Eigenvalues on an interval $[0, \omega]$, $\omega > 0$, can be determined as limits of eigenvalues of some finite-dimensional operators in $S^k(x_1, x_2, \dots, x_N)$ when smoothness k goes to infinity, but the set of points x_1, x_2, \dots, x_N is fixed;
- 2) all eigenvalues can be determined as limits of eigenvalues of the same finite-dimensional operators in $S^k(x_1, x_2, \dots, x_N)$ when k is fixed, but the number of points x_1, x_2, \dots, x_N is increasing.

Technically this result is a specific realization of the Rayleigh-Ritz method ([2] and [3]).

The above result is based on the fact (Theorem 1.4) that eigenfunctions of Δ can be reconstructed from their values on appropriate sets x_1, x_2, \dots, x_N as uniform limits of polyharmonic functions with singularities at x_1, x_2, \dots, x_N . A weaker result in terms of homogeneous manifolds and $L_2(M)$ -convergence is contained in [6].

Let Δ be the Laplace–Beltrami operator on a compact, orientable Riemannian manifold $M, \dim M = d$, with metric tensor g . It is known that Δ is a self-adjoint positive definite operator in the corresponding space $L_2(M)$ constructed from g . Domains of the powers $\Delta^{s/2}, s \in \mathbb{R}$, coincide with the Sobolev spaces $H^s(M), s \in \mathbb{R}$. To choose norms on spaces $H^s(M)$, we consider a finite cover of M by balls $B(y_\nu, \sigma)$ where $y_\nu \in M$ is the center of the ball and σ is its radius. For a partition of unity φ_ν subordinate to the family $\{B(y_\nu, \sigma)\}$ we introduce Sobolev space $H^s(M)$ as the completion of $C_0^\infty(M)$ with respect to the norm

$$(1.1) \quad \|f\|_{H^s(M)} = \left(\sum_\nu \|\varphi_\nu f\|_{H^s(B(y_\nu, \sigma))}^2 \right)^{1/2}.$$

The Regularity Theorem for the Laplace–Beltrami operator Δ states that the norm (1.1) is equivalent to the graph norm $\|f\| + \|\Delta^s f\|$.

We assume that the Ricci curvature Ric satisfies (as a form) the inequality

$$(1.2) \quad Ric \geq -kg, k \geq 0.$$

The volume of the ball $B(x, \rho)$ will be denoted by $|B(x, \rho)|$. Our assumptions about the manifold imply that there exists a constant $b > 0$ such that

$$(1.3) \quad b^{-1} \leq \frac{|B(x, \rho)|}{|B(y, \rho)|} \leq b, x, y \in M, \rho < r,$$

where r is the injectivity radius of the manifold. The Bishop-Gromov Comparison Theorem (see [8]) implies that for any $0 < \sigma < \lambda < r/2$ the following inequality holds true:

$$|B(x, \lambda)| \leq (\lambda/\sigma)^d e^{(kr(d-1))^{1/2}} |B(x, \sigma)|.$$

In what follows we use the notation

$$R_0(M) = 12^d b e^{(kr(d-1))^{1/2}},$$

where d is the dimension of the manifold, r is the injectivity radius and constants k, b are from (1.2) and (1.3) respectively.

The following Covering Lemma plays an important role for the paper:

Lemma 1.1. *If M satisfy the above assumptions then for any $0 < \rho < r/6$ there exists a finite set of points $\{x_i\}$ such that:*

- 1) Balls $B(x_i, \rho/4)$ are disjoint,

- 2) balls $B(x_i, \rho/2)$ form a cover of M ,
- 3) multiplicity of the cover by balls $B(x_i, \rho)$ is not greater $R_0(M)$.

We will need the following definition:

Definition 1. For a given $0 < \rho < r/6$ we say that a finite set of points $M_\rho = \{x_i\}$ is ρ -admissible if it satisfies properties 1)-3) from Lemma 1.1.

Given a ρ -admissible set $M_\rho, |M_\rho| = N$, and a sequence of complex numbers $\{v_i\}_1^N$, we consider the following variational problem:

Find a function $f \in H^{2k}(M), k \in \mathbb{N}, k > d - 1$, such that:

- 1) $f(x_i) = v_i, i = 1, \dots, N$,
- 2) f is a minimizer of the functional $u \rightarrow \|\Delta^k f\|$.

We show that this problem does have a unique solution.

For a ρ -admissible set M_ρ and a function $f \in H^k(M), k$ is large enough, the solution of the above variational problem that interpolates f on the set M_ρ will be denoted by $s_k(f)$. In fact, the function $s_k(f)$ depends on the set M_ρ , but we hope our notation will not cause any confusion. The following lemma implies in particular that the set of minimizers is linear:

Lemma 1.2. *The set of solutions of the variational problem is the same as the set $S^k(M_\rho)$ of all solutions of the equation*

$$(1.4) \quad \Delta^{2k}u = \sum_{x_\gamma \in M_\rho} \alpha_\gamma \delta(x_\gamma), k > d/2,$$

where $\delta(x_\gamma)$ is the Dirac measure at the point x_γ and

$$(1.5) \quad \alpha_1 + \alpha_2 + \dots + \alpha_N = 0, |M_\rho| = N.$$

Elements of the set $S^k(M_\rho)$ will be called polyharmonic splines. Since zero is the simple eigenvalue of the scalar Laplace–Beltrami operator on a compact connected manifold and because Equation (1.4) on a compact manifold is solvable only under Assumption (1.5) the dimension of the space $S^k(M_\rho)$ is exactly $|M_\rho| = N$.

It was shown in [6] that there are functions $L_\nu^k \in S^k(M_\rho)$ such that L_ν^k takes value 1 at x_ν and 0 at all other points of M_ρ . Moreover these functions form a basis of $S^k(M_\rho)$.

Let $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_j$ be the sequence of the first j eigenvalues of the operator Δ in $L_2(M)$ counted with their multiplicities and $\varphi_1, \varphi_2, \dots, \varphi_j$ is the corresponding set of orthonormal eigenfunctions. Throughout the paper $\|\cdot\|$ denotes the $L_2(M)$ norm.

According to the min-max principle for a self-adjoint positive definite operator D in a Hilbert space E the j -th eigenvalue can be calculated by the formula

$$\lambda_j = \inf_{F \subset E} \sup_{f \in F} \frac{\|D^{1/2}f\|_E^2}{\|f\|_E^2}, f \neq 0,$$

where inf is taken over all j -dimensional subspaces F of E .

We introduce the numbers $\lambda_j^{(k)}(M_\rho)$ by the formula

$$(1.6) \quad \lambda_j^{(k)}(M_\rho) = \inf_{F \subset S^k(M_\rho)} \sup_{f \in F} \frac{\|\Delta^{1/2} f\|^2}{\|f\|^2}, f \neq 0,$$

where inf is taken over all j -dimensional subspaces of $S^k(M_\rho)$.

As a consequence of the min-max principle we obtain that the numbers $\lambda_j^{(k)}(M_\rho)$ are the eigenvalues of the matrix $D^{(k)} = D^{(k)}(M_\rho)$ with entries

$$(1.7) \quad d_{\gamma, \nu}^{(k)} = \int_M (\Delta L_\gamma^k) L_\nu^k dx,$$

where dx is the Riemannian density.

Now we can formulate our main result which shows that eigenvalues of matrices $D^{(k)}$ approximate eigenvalues of the Laplace–Beltrami operator and the rate of convergence is exponential.

Theorem 1.3. *There exists a $C_0 = C_0(M)$ such that for any given $\omega > 0$ if $0 < \rho < (C_0\omega)^{-1/2}$ then for every ρ -admissible set M_ρ , every eigenvalue $\lambda_j \leq \omega$ and all $k = (2^l + 1)d, l = 0, 1, \dots$,*

$$(1.8) \quad \lambda_j^{(k)}(M_\rho) - \omega^{2d} \gamma^{2(k-d)} \leq \lambda_j \leq \lambda_j^{(k)}(M_\rho),$$

where $\gamma = C_0 \rho^2 \omega < 1$.

The inequality (1.8) shows that there are three different ways to determine eigenvalues λ_j :

- 1) Eigenvalues from the interval $[0, \omega]$ can be determined by keeping a set M_ρ with $0 < \rho < (C_0\omega)^{-1/2}$ fixed and by letting k go to infinity.
- 2) By letting ρ go to zero and keeping k fixed one can determine all of the eigenvalues.
- 3) The convergence will be even faster if ρ goes to zero and at the same time k goes to infinity.

The following Approximation Theorem plays a key role in the proof of Theorem 1.3:

Theorem 1.4. *There exist constants $C(M), \rho(M) > 0$ such that for any $0 < \rho < \rho(M)$, any ρ -admissible set M_ρ , any smooth function f and any $t \leq d$ the following inequality holds true:*

$$(1.9) \quad \|\Delta^t(s_k(f) - f)\| \leq (C(M)\rho^2)^{k-d} \|\Delta^k f\|,$$

for any $k = (2^l + 1)d, l = 0, 1, \dots$. In particular, if f is a linear combination of orthonormal eigenfunctions whose corresponding eigenvalues belong to the interval $[0, \omega]$, then for any $t \leq d$

$$(1.10) \quad \|\Delta^t(s_k(f) - f)\| \leq \omega^d (C(M)\rho^2\omega)^{k-d} \|f\|,$$

where $k = (2^l + 1)d, l = 0, 1, \dots$

Moreover, we have the following estimates in the uniform norm on the manifold:

$$\sup_{x \in M} |(s_k(f)(x) - f(x))| \leq (C(M)\rho^2)^{k-d} \|\Delta^k f\|, k = (2^l + 1)d, l = 0, 1, \dots$$

and respectively,

$$\sup_{x \in M} |(s_k(f)(x) - f(x))| \leq \omega^d (C(M)\rho^2\omega)^{k-d} \|f\|, k = (2^l + 1)d, l = 0, 1, \dots,$$

if f belongs to the span of eigenfunctions whose eigenvalues are not greater than ω .

Proofs of Theorems 1.3 and 1.4 show that the constants $C_0(M), C(M), \rho(M)$ depend on the bounds on the curvature of M .

We obtain our Approximation Theorem as a consequence of the following inequality that is a Poincare-type inequality:

Theorem 1.5. *There exist $C(M), \rho(M) > 0$ such that for any $0 < \rho < \rho(M)$, any ρ -admissible set M_ρ and any $f \in H^{2dm}(M)$ whose restriction to M_ρ is zero the following inequality holds true:*

$$(1.11) \quad \|f\| \leq \left(C(M)\rho^{2d}\right)^m \|\Delta^{dm} f\|,$$

where $m = 2^l, l = 0, 1, \dots$

According to our main Theorem 1.3, if we are going to determine the spectrum on an interval $[0, \omega], \omega > 0$, we have to use an ρ -admissible set of points M_ρ , where $0 < \rho < (C_0\omega)^{-1/2}$. It is clear that the cardinality of M_ρ i.e., $N = |M_\rho|$ cannot be less than the number of eigenvalues on the interval $[0, \omega]$. In fact if $\rho = \epsilon(C_0\omega)^{-1/2}, 0 < \epsilon < 1$, then the number of points in M_ρ is approximately

$$(1.12) \quad N = |M_\rho| \asymp \frac{\text{Vol } M}{\rho^d} = \epsilon^{-d} C_0^{d/2} \text{Vol } M \omega^{d/2}.$$

Note, that according to the Weyl's asymptotic formula the number of eigenvalues on an interval $[0, \omega]$ is asymptotically

$$(1.13) \quad c \text{Vol } M \omega^{d/2}.$$

In other words our method requires an ‘‘almost’’ optimal number of points for admissible sets M_ρ .

It is important to realize an interesting feature of the inequalities (1.9)–(1.11): All the constants and the interval for admissible ρ 's depend solely on the manifold, while the exponents k and m can be made arbitrary large. These inequalities are consequences of the inequality (1.11). To obtain (1.11) we establish it first for $m = 1$ and then apply the following result which allows to ‘‘exponentiate’’ right-hand sides of some inequalities:

Lemma 1.6. *If for some $f \in H^{2s}(M)$, $a, s > 0$,*

$$(1.14) \quad \|f\| \leq a \|\Delta^s f\|,$$

then for the same f, a, s and all $t \geq 0, m = 2^l, l = 0, 1, \dots$,

$$(1.15) \quad \|\Delta^t f\| \leq a^m \|\Delta^{ms+t} f\|,$$

if $f \in H^{2(ms+t)}(M)$.

As an application of our main result we prove that the zeta-function $\zeta(s)$ of the Laplace–Beltrami operator

$$(1.16) \quad \zeta(s) = \sum_{\lambda_i \neq 0} \lambda_i^{-s}$$

is the uniform limit of zeta-functions for finite-dimensional operators D^k . Namely, we choose a sequence η_n that goes to zero and for every η_n construct a set M_{η_n} . For a fixed k that is large enough we consider the space $S^k(M_{\eta_n})$ and the eigenvalues of the corresponding operator $D^{(k)}$ defined by (1.7) we denote as $\lambda_i^{(k)}(\eta_n)$. The ζ -function for a finite-dimensional operator $D^{(k)}$ is denoted by $\zeta_{\eta_n}(s)$.

Theorem 1.7. *The sequence of ζ -functions $\zeta_{\eta_n}(s)$ converges uniformly to $\zeta(s)$ on compact subsets of the set $\{s = u + iv \mid u > d/2\}$.*

2. Proof of Theorem 1.3

Let $P_{M_\rho}^k$ be the projector from $H^{d/2+1}(M)$ onto the space $S^k(M_\rho)$ defined by the formula $P_{M_\rho}^k f = s_k(f)$. Note that the function $s_k(f)$ depends on the set M_ρ .

For a given $\omega > 0$ let $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_{j(\omega)} \leq \omega$ be the set of all eigenvalues counted with their multiplicities which are not greater than ω . If $\varphi_1, \varphi_2, \dots, \varphi_{j(\omega)}$ is the set of corresponding orthonormal eigenfunctions then their *span* is denoted by E_ω . Note, that $\dim E_{\lambda_i} = i$. If $\omega \in [\lambda_{j(\omega)}, \lambda_{j(\omega)+1})$ then $E_\omega = E_{\lambda_{j(\omega)}}$ and $\dim E_\omega = \dim E_{\lambda_{j(\omega)}} = j(\omega)$.

According to Approximation Theorem 1.4, inequality (1.10), for any φ_i such that the corresponding $\lambda_i \leq \omega$ we have

$$\|s_k(\varphi_i) - \varphi_i\| \leq \omega^d (C_0 \rho^2 \omega)^{k-d}, s_k(\varphi_i) \in S^k(M_\rho), k = d(2^l + 1), l = 0, 1, \dots$$

The right-hand side in the last inequality goes to zero for $0 < \rho < (C_0 \omega)^{-1/2}$ and large k . Thus, the dimension of $P_{M_\rho}^k(E_\omega)$ is $j(\omega)$ as long as $0 < \rho < (C_0 \omega)^{-1/2}$ and k is large enough.

Next according to the min-max principle the eigenvalue λ_j of Δ can be defined by the formula

$$\lambda_j = \inf_{F \subset L_2(M)} \sup_{f \in F} \frac{\|\Delta^{1/2} f\|^2}{\|f\|^2}, f \neq 0,$$

where inf is taken over all j -dimensional subspaces of $L_2(M)$.

It is clear that

$$(2.1) \quad \lambda_j \leq \lambda_j^{(k)}(M_\rho) \leq \sup_{f \in P_{M_\rho}^k(E_{\lambda_j})} \frac{\|\Delta^{1/2} f\|^2}{\|f\|^2}, f \neq 0,$$

where $\lambda_j^{(k)}$ is defined by (1.6), $\lambda_j \leq \omega, 0 < \rho < (C_0\omega)^{-1/2}$ and k is large enough.

For any $\psi \in E_{\lambda_j}$, set $h_k = s_k(\psi) - \psi$, and

$$h_k = h_{k,j} + h_{k,j}^\perp,$$

where $h_{k,j} \in E_{\lambda_j}, h_{k,j}^\perp \in E_{\lambda_j}^\perp$.

It gives

$$\Delta^{1/2} h_k = \Delta^{1/2} h_{k,j} + \Delta^{1/2} h_{k,j}^\perp.$$

Since Δ is self adjoint and E_{λ_j} is its invariant subspace the terms on the right are orthogonal and we obtain

$$(2.2) \quad \|\Delta^{1/2} h_{k,j}^\perp\| \leq \|\Delta^{1/2} h_k\|.$$

It is clear that the orthogonal projection of $s_k(\psi)$ onto E_{λ_j} is $\psi + h_{k,j} = \psi_{k,j}$. Since $s_k(\psi) = \psi_{k,j} + h_{k,j}^\perp$, we have

$$\|s_k(\psi)\|^2 \geq \|\psi_{k,j}\|^2$$

and we also have

$$\Delta^{1/2} s_k(\psi) = \Delta^{1/2} \psi_{k,j} + \Delta^{1/2} h_{k,j}^\perp,$$

that implies

$$\|\Delta^{1/2} s_k(\psi)\|^2 = \|\Delta^{1/2} \psi_{k,j}\|^2 + \|\Delta^{1/2} h_{k,j}^\perp\|^2.$$

After all we obtain the following inequality:

$$\frac{\|\Delta^{1/2} s_k(\psi)\|^2}{\|s_k(\psi)\|^2} \leq \frac{\|\Delta^{1/2} \psi_{k,j}\|^2}{\|\psi_{k,j}\|^2} + \frac{\|\Delta^{1/2} h_{k,j}^\perp\|^2}{\|s_k(\psi)\|^2}.$$

The last inequality along with inequalities (2.1) and (2.2) gives

$$(2.3) \quad \frac{\|\Delta^{1/2} s_k(\psi)\|^2}{\|s_k(\psi)\|^2} \leq \lambda_j + \frac{\|\Delta^{1/2} h_k\|^2}{\|s_k(\psi)\|^2}.$$

In what follows we will use the notation

$$h_k^{(i)} = h_k^{(i)}(M_\rho) = s_k(\varphi_i) - \varphi_i,$$

where φ_i is the i -th orthonormal eigenfunction.

According to Approximation Theorem 1.4, $\|h_k^{(i)}(M_\rho)\|$ can be done arbitrary small for large k if corresponding eigenvalue $\lambda_i \leq \omega$ and $0 < \rho < (C_0\omega)^{-1/2}$ because

$$\|h_k^{(i)}(M_\rho)\| \leq \omega^d (C_0 \rho^2 \omega)^{k-d}, k = d(2^l + 1), l = 0, 1, \dots$$

Assume that $0 < \rho < (C_0\omega)^{-1/2}$ and k is so large that

$$(2.4) \quad \sum_{i=1}^{j(\omega)} \|h_k^{(i)}(M_\rho)\|^2 \leq 1/2$$

where $j(\omega)$ is the number of all eigenvalues (counting with their multiplicities) which are not greater than ω .

Using the fact that $\Delta^{1/2}$ is a self adjoint operator one can show that

$$\|\Delta^{1/2}h_k\| \leq \|\psi\| \left(\sum_{i=1}^{j(\omega)} \|\Delta^{1/2}h_k^{(i)}\|^2 \right)^{1/2},$$

where $h_k = s_k(\psi) - \psi$.

The last inequality and the inequality (2.3) imply

$$(2.5) \quad \begin{aligned} \lambda_j^{(k)}(M_\rho) - \lambda_j &\leq \sup_{\psi \in E_{\lambda_j}} \frac{\|\Delta^{1/2}s_k(\psi)\|^2}{\|s_k(\psi)\|^2} - \lambda_j \\ &\leq \sup_{\psi \in E_{\lambda_j}} \frac{\|\Delta^{1/2}h_k\|^2}{\|s_k(\psi)\|^2} \\ &\leq \sup_{\psi \in E_{\lambda_j}} \frac{\|\psi\|^2 \sum_{i=1}^{j(\omega)} \|\Delta^{1/2}h_k^{(i)}\|^2}{\|s_k(\psi)\|^2}. \end{aligned}$$

Since

$$\|\psi\| = \|s_k(\psi) - h_k\| \leq \|s_k(\psi)\| + \|h_k\|$$

and

$$\|h_k\|^2 \leq \|\psi\|^2 \sum_{i=1}^{j(\omega)} \|h_k^{(i)}\|^2 \leq \frac{1}{2} \|\psi\|^2,$$

we have

$$\|s_k(\psi)\|^2 \geq (\|\psi\| - \|h_k\|)^2 \geq \frac{1}{4} \|\psi\|^2.$$

After using (2.5) we obtain

$$\lambda_j^{(k)}(M_\rho) - \lambda_j \leq 4 \sum_{i=1}^{j(\omega)} \|\Delta^{1/2}h_k^{(i)}(M_\rho)\|^2.$$

Because the Sobolev space $H^s(M)$ is continuously embedded into the space $H^t(M)$ if $s > t$, we obtain that according to the estimates (1.9) and (2.4)

$$\begin{aligned} \|\Delta^{1/2}h_k^{(i)}(M_\rho)\|^2 &\leq \omega^{2d} (C(M)\rho^2\omega)^{2(k-d)} \|h_k^{(i)}(M_\rho)\|^2 \\ &\leq \omega^{2d} (C_0\rho^2\omega)^{2(k-d)}. \end{aligned}$$

Finally we have

$$\lambda_j \leq \lambda_j^{(k)}(M_\rho) \leq \lambda_j + \omega^{2d} (C_0 \rho^2 \omega)^{2(k-d)}, k = d(2^l + 1), l = 0, 1, \dots$$

where $\lambda_j \leq \omega, 0 < \rho < (C_0 \omega)^{-1/2}$ and k is large enough.

Theorem 1.3 is proved.

3. Approximation of the zeta-function of the Laplace–Beltrami operator

The zeta-function $\zeta(s)$ of the Laplace–Beltrami operator is defined by formula (1.16). Since the paper of Minakshisundaram and Pleijel [4] it is known that this series converges absolutely for every $s = u + iv$ where $u > d/2$. As a result it converges uniformly on every half-plane whose closure is a proper subset of the set $\{s = u + iv | u > d/2\}$.

Now we choose a sequence η_n that goes to zero and for every η_n construct a set M_{η_n} . For a fixed $k > d/2$ we consider the space $S^k(M_{\eta_n})$ and the eigenvalues of the corresponding operator $D^{(k)}$ we denote as $\lambda_i^{(k)}(\eta_n)$. The ζ -function for a finite-dimensional operator $D^{(k)}$ is denoted by $\zeta_{\eta_n}(s)$.

Theorem 3.1. *The sequence of ζ -functions $\zeta_{\eta_n}(s)$ converges uniformly to $\zeta(s)$ on compact subsets of the set $\{s = u + iv | u > d/2\}$ as η_n goes to zero.*

Proof. Let $\Omega \subset \{s = u + iv | u > d/2\}$ be a compact set. For a fixed $\varepsilon > 0$ let $m \in \mathbb{N}$ a such integer that

$$\sum_{j \geq m} |\lambda_j^{-s}| = \sum_{j \geq m} \lambda_j^{-Re s} \leq \varepsilon/3$$

for all $s \in \Omega$.

Since for every k , we have that $\lambda_j^{(k)}(\eta_n) \geq \lambda_j$,

$$|(\lambda_j^{(k)}(\eta_n))^{-s}| = (\lambda_j^{(k)}(\eta_n))^{-Re s} \leq \lambda_j^{-Re s} = |\lambda_j^{-s}|.$$

Thus

$$(3.1) \quad \left| \sum_{j \geq m} \lambda_j^{-s} - \sum_{j=m}^{N(n)} (\lambda_j^{(k)}(\eta_n))^{-s} \right| \leq \frac{2\varepsilon}{3}.$$

Next, because $\lambda_j^{(k)}(\eta_n)$ goes to λ_j as η_n goes to zero, we can find $n = n(\varepsilon)$ such that for $n > n(\varepsilon), s \in \Omega$

$$(3.2) \quad \left| \sum_{j \leq m} \lambda_j^{-s} - \sum_{j \leq m}^{N(n)} (\lambda_j^{(k)}(\eta_n))^{-s} \right| \leq \varepsilon/3,$$

where we assume that $\lambda_j \neq 0, \lambda_j^{(k)}(\eta_n) \neq 0$.

The last inequalities imply

$$|\zeta(s) - \zeta_{\eta_n}(s)| \leq \varepsilon$$

if $n > n(\varepsilon), s \in \Omega$. The theorem is proved. □

4. A Poincare type inequality and spline approximation on manifolds

We consider a compact orientable Riemannian manifold whose Ricci curvature satisfies (1.2). First, we prove Covering Lemma 1.1 from the Introduction (compare to [1]).

Proof. Let us choose a family of disjoint balls $B(x_i, \rho/4)$ such that there is no ball $B(x, \rho/4), x \in M$, which has empty intersections with all balls from our family. Then the family $B(x_i, \rho/2)$ is a cover of M . Every ball from the family $\{B(x_i, \rho)\}$, that has nonempty intersection with a particular ball $\{B(x_j, \rho)\}$ is contained in the ball $\{B(x_j, 3\rho)\}$. Since any two balls from the family $B(x_i, \rho/4)$ are disjoint, it gives the following estimate for the index of multiplicity R of the cover $B(x_i, \rho)$:

$$(4.1) \quad R \leq \frac{\sup_{y \in M} |B(y, 3\rho)|}{\inf_{x \in M} |B(x, \rho/4)|}.$$

As it was mentioned in the Introduction, the Bishop-Gromov comparison theorem (see [8]) implies that for any $0 < \sigma < \lambda < r/2$

$$(4.2) \quad |B(x, \lambda)| \leq (\lambda/\sigma)^d e^{(kr(d-1))^{1/2}} |B(x, \sigma)|.$$

This property along with (4.1) allows to continue the estimation of R :

$$R \leq 12^d e^{(kr(d-1))^{1/2}} \frac{\sup_{y \in M} |B(y, \rho/4)|}{\inf_{x \in M} |B(x, \rho/4)|} \leq 12^d b e^{(kr(d-1))^{1/2}} = R_0(M).$$

□

We will need the following result which is in fact a global Poincare type inequality:

Lemma 4.1. *For any $k > d - 1$ there exist constants $C(M, k) > 0, \rho(M, k) > 0$ such that for any $\rho < \rho(M, k)$ and any ρ -admissible set $M_\rho = \{x_i\}$ the following inequality holds true:*

$$(4.3) \quad \|f\| \leq C(M, k) \left\{ \rho^{d/2} \left(\sum_{x_i \in M_\rho} |f(x_i)|^2 \right)^{1/2} + \rho^{2k} \|\Delta^k f\| \right\}, k > d - 1.$$

Proof. Let $M_\rho = \{x_i\}$ be a ρ -admissible set and $\{\varphi_\nu\}$ the partition of unity from (1.1). For any $f \in C^\infty(M)$, every fixed $B(x_i, \rho)$ and every $x \in B(x_i, \rho/2)$

$$(4.4) \quad (\varphi_\nu f)(x) = (\varphi_\nu f)(x_i) + \sum_{1 \leq |\alpha| \leq n-1} \frac{1}{\alpha!} \partial^{|\alpha|}(\varphi_\nu f)(x_i)(x - x_i)^\alpha + \sum_{|\alpha|=n} \frac{1}{(n-1)!} \int_0^\tau t^{n-1} \partial^{|\alpha|}(\varphi_\nu f)(x_i + t\vartheta) \vartheta^\alpha dt,$$

where $x = (x_1, \dots, x_d), x_i = (x_1^i, \dots, x_d^i), \alpha = (\alpha_1, \dots, \alpha_d), x - x_i = (x_1 - x_1^i)^{\alpha_1} \dots (x_d - x_d^i)^{\alpha_d}, \tau = \|x - x_i\|, \vartheta = (x - x_i)/\tau$.

We are going to make use of the following inequality:

$$(4.5) \quad |\partial^{|\alpha|}(\varphi_\nu f)(x_i)| \leq C_{d,m} \sum_{|\mu| \leq m} \rho^{|\mu+\alpha|-d/2} \|\partial^{|\mu+\alpha|}(\varphi_\nu f)\|_{L_2(B(x_i, \rho))},$$

where $\mu = (\mu_1, \mu_2, \dots, \mu_d), m > d/2$. To prove this inequality we first recall the following inequality (see [5]):

There exists a constant $c_{d,m}$ such that for every $\psi \in C_0^\infty(B(x_i, \rho/2))$

$$|\psi(x_i)| \leq c_{d,m} \rho^{m-d/2} \|\psi\|_{H^m(B(x_i, \rho))}, m > d/2.$$

We consider the function $\xi(x) = e \exp(1/(\|x\|^2 - 1))$, if $\|x\| < 1$ and $\xi(x) = 0$, if $\|x\| \geq 1$. It is clear that $\xi \in C_0^\infty(U_0) \subset \mathbb{R}^d, \xi(0) = 1$, where U_0 is the unit ball of \mathbb{R}^d . Set $\xi_\rho(x) = \xi(2\rho^{-1}(x - x_i))$. Since for any $\psi \in C^\infty(B(x_i, \rho))$ we have that $\xi_\rho \psi \in C_0^\infty(B(x_i, \rho/2))$ and $\xi_\rho \psi(x_i) = \psi(x_i)$, we can use the last inequality to obtain

$$|\psi(x_i)| \leq c_{d,m} \rho^{m-d/2} \|\xi_\rho \psi\|_{H^m(B(x_i, \rho))}, m > d/2.$$

This inequality shows that there exist constants $C(d, m, n)$ such that for any $\psi \in C^\infty(B(x_i, \rho))$

$$|\psi(x_i)| \leq \sum_{n \leq m} C(d, m, n) \rho^{n-d/2} \|\psi\|_{H^n(B(x_i, \rho))}, m > d/2.$$

This inequality implies the inequality (4.5) when $\psi = \partial^{|\alpha|}(\varphi_\nu f)$.

Now we continue the estimation of the second term in (4.4).

$$\begin{aligned} & \int_{B(x_i, \rho/2)} \left| \sum_{1 \leq |\alpha| \leq n-1} \frac{1}{\alpha!} \partial^{|\alpha|}(\varphi_\nu f)(x_i)(x - x_i)^\alpha \right|^2 dx \\ & \leq \Omega_d \sum_{1 \leq |\alpha| \leq n-1} (1/2)^{2|\alpha|+d} |\partial^{|\alpha|}(\varphi_\nu f)(x_i)|^2 \rho^{2|\alpha|+d} \\ & \leq C_{d,n} \sum_{|\gamma| \leq n+m-1} \rho^{2|\gamma|} \|\partial^{|\gamma|}(\varphi_\nu f)\|_{L_2(B(x_i, \rho))}^2. \end{aligned}$$

Next, to estimate the third term in (4.4) we use the Schwartz inequality and the assumption $n > d/2$

$$\begin{aligned} & \left| \int_0^\tau t^{n-1} \partial^{|\alpha|}(\varphi_\nu f)(x_i + t\vartheta) \vartheta^\alpha dt \right|^2 \\ & \leq \left(\int_0^\tau t^{n-d/2-1/2} |t^{d/2-1/2} \partial^{|\alpha|}(\varphi_\nu f)(x_i + t\vartheta)| dt \right)^2 \\ & \leq C_{d,n} \tau^{2n-d} \int_0^\tau t^{d-1} |\partial^{|\alpha|}(\varphi_\nu f)(x_i + t\vartheta)|^2 dt. \end{aligned}$$

We integrate both sides of this inequality over the ball $B(x_i, \rho/2)$ using the spherical coordinate system (τ, ϑ) .

$$\begin{aligned} & \int_0^{\rho/2} \tau^{d-1} \int_0^{2\pi} \left| \int_0^\tau t^{n-1} \partial^{|\alpha|}(\varphi_\nu f)(x_i + t\vartheta) \vartheta^\alpha dt \right|^2 d\vartheta d\tau \\ & \leq C_{d,n} \int_0^{\rho/2} t^{d-1} \left(\int_0^{2\pi} \int_0^{\rho/2} \tau^{2n-d} |\partial^{|\alpha|}(\varphi_\nu f)(x_i + t\vartheta)|^2 \tau^{d-1} d\tau d\vartheta \right) dt \\ & \leq C_{d,n} \rho^{2n} \|\partial^{|\alpha|}(\varphi_\nu f)\|_{L_2(B(x_i, \rho))}^2, \end{aligned}$$

where $\tau = \|x - x_i\| \leq \rho/2$, $|\alpha| = n$.

Finally, if $n > d/2$ and $k = n + m - 1$,

$$\begin{aligned} & \|\varphi_\nu f\|_{L_2(B(x_i, \rho/2))}^2 \\ & \leq C_{d,k} \left(\rho^d |f(x_i)|^2 + \sum_{j=1}^k \sum_{1 \leq |\alpha| \leq j} \rho^{2|\alpha|} \|\partial^{|\alpha|}(\varphi_\nu f)\|_{L_2(B(x_i, \rho))}^2 \right), \end{aligned}$$

where $k > d - 1$ since $n > d/2$ and $m > d/2$. Since balls $B(x_i, \rho/2)$ cover the manifold and the cover by $B(x_i, \rho)$ has a finite multiplicity $\leq R_0(M)$ the summation over all balls gives

$$\|f\|_{L_2(M)}^2 \leq C(M, k) \left\{ \rho^d \left(\sum_{i=1}^\infty |f(x_i)|^2 \right) + \sum_{j=1}^k \rho^{2j} \|f\|_{H^j(M)}^2 \right\}, k > d - 1.$$

Using this inequality and the regularity theorem for Laplace–Beltrami operator (see [9]) we obtain

$$\begin{aligned} & \|f\|_{L_2(M)} \\ & \leq C(M, k) \left\{ \rho^{d/2} \left(\sum_{i=1}^\infty |f(x_i)|^2 \right)^{1/2} + \sum_{j=1}^k \rho^j \left(\|f\| + \|\Delta^{j/2} f\| \right) \right\}, k > d - 1. \end{aligned}$$

For the self-adjoint Δ for any $a > 0, \rho > 0, 0 \leq j \leq k$ we have the following interpolation inequality:

$$\rho^j \|\Delta^{j/2} f\| \leq a^{2k-j} \rho^{2k} \|\Delta^k f\| + c_k a^{-j} \|f\|.$$

Because in the last inequality we are free to choose any $a > 0$ we are coming to our main claim. \square

The next goal is to extend the last estimate to the Sobolev norm.

Theorem 4.2. *For any $k > d-1$ there exist constants $C(M, k) > 0, \rho(M, k) > 0$, such that for any $0 < \rho < \rho(M, k)$, any admissible set $M_\rho = \{x_i\}$, any $m = 2^l, l = 0, 1, \dots$, any smooth f which is zero on M_ρ and any $t \geq 0$*

$$(4.6) \quad \|\Delta^t f\| \leq \left(C(M, k)\rho^{2k}\right)^m \|\Delta^{km+t} f\|, t \geq 0.$$

We will obtain this estimate as a consequence of the following lemma:

Lemma 4.3. *If for some $f \in H^{2s}(M), a, s > 0$,*

$$(4.7) \quad \|f\| \leq a\|\Delta^s f\|,$$

then for the same f, a, s and all $t \geq 0, m = 2^l, l = 0, 1, \dots$,

$$(4.8) \quad \|\Delta^t f\| \leq a^m \|\Delta^{ms+t} f\|,$$

if $f \in H^{2(ms+t)}(M)$.

Proof. Let us remind that $\{\lambda_j\}$ is the set of eigenvalues of the operator Δ and $\{\varphi_j\}$ is the set of corresponding orthonormal eigenfunctions. Let $\{c_j = \langle f, \varphi_j \rangle\}$ be the set of Fourier coefficients of the function f with respect to the orthonormal basis $\{\varphi_j\}$. Using the Plancherel Theorem we can write Assumption (4.7) in the form

$$\|f\|^2 \leq a^2 \left(\sum_{\lambda_j \leq a^{-1/s}} \lambda_j^{2s} |c_j|^2 + \sum_{\lambda_j > a^{-1/s}} \lambda_j^{2s} |c_j|^2 \right).$$

Since for the first sum $a^2 \lambda_j^{2s} \leq 1$,

$$0 \leq \sum_{\lambda_j \leq a^{-1/s}} (|c_j|^2 - a^2 \lambda_j^{2s} |c_j|^2) \leq \sum_{\lambda_j > a^{-1/s}} (a^2 \lambda_j^{2s} |c_j|^2 - |c_j|^2).$$

Multiplication of this inequality by $a^2 \lambda_j^{2s}$ will only improve the existing inequality and then using the Plancherel Theorem once again we will obtain

$$\|f\| \leq a\|\Delta^s f\| \leq a^2\|\Delta^{2s} f\|.$$

It is now clear that using induction we can prove

$$\|f\| \leq a^m \|\Delta^{ms} f\|, m = 2^l, l \in \mathbb{N}.$$

But then, using the same arguments we have for any $\tau > 0$

$$\begin{aligned} 0 &\leq \sum_{\lambda_j \leq a^{-1/s}} \left(a^{2\tau} \lambda_j^{2\tau s} |c_j|^2 - a^{2(m+\tau)} \lambda_j^{2(m+\tau)s} |c_j|^2 \right) \\ &\leq \sum_{\lambda_j > a^{-1/s}} \left(a^{2(m+\tau)} \lambda_j^{2(m+\tau)s} |c_j|^2 - a^{2\tau} \lambda_j^{2\tau s} |c_j|^2 \right), \end{aligned}$$

that gives the desired inequality (4.8) if $t = s\tau$. □

To prove (4.6) from Theorem 4.2 it is enough to apply the last Lemma 4.3 to Lemma 4.1 with $a = C(M, k)\rho^{2k}$.

Next we are going to construct polyharmonic splines on manifolds. We will need the following lemma that gives an equivalent norm on Sobolev spaces. Recall that the norm in the Sobolev space were introduced in the Introduction.

Lemma 4.4. *For any $k > d - 1$ and any ρ -admissible set $M_\rho = \{x_i\}$, the norm of the Sobolev space $H^{2k}(M)$ is equivalent to the norm*

$$(4.9) \quad \|\Delta^k f\| + \left(\sum_{x_\gamma \in M_\rho} |f(x_\gamma)|^2 \right)^{1/2}.$$

The proof of the lemma can be obtained as a consequence of Theorem 4.2, the Sobolev Embedding Theorem and regularity of the Laplace–Beltrami operator.

Given a ρ -admissible set $M_\rho, |M_\rho| = N$, and a sequence of complex numbers $\{v_\gamma\}_1^N$ we will be interested to find a function $s_k \in H^{2k}(M), k$ is large enough such that:

- a) $s_k(x_\gamma) = v_\gamma, x_\gamma \in M_\rho;$
- b) function s_k minimizes functional $u \rightarrow \|\Delta^k u\|.$

Lemma 4.5. *The minimization problem has a unique solution if $k > d - 1$.*

Proof. According to the last Lemma 4.4 it is enough to minimize the norm (4.9). For the given sequence v_γ consider a function f from $H^{2k}(M)$ such that $f(x_\gamma) = v_\gamma$. We consider $H^{2k}(M)$ as the Hilbert space with the inner product

$$\langle f, g \rangle = \sum_{x_\gamma \in M_\rho} f(x_\gamma)g(x_\gamma) + \langle \Delta^{k/2} f, \Delta^{k/2} g \rangle.$$

Let Pf denote the orthogonal projection of the function f on the subspace $U^{2k}(M_\rho) = \{f \in H^{2k}(M) | f(x_\gamma) = 0\}$ with respect to the new scalar product. Then the function $g = f - Pf$ will be the unique solution to the above minimization problem for the functional $u \rightarrow \|\Delta^k u\|, k > d - 1$. □

We prove Lemma 1.2 from the Introduction, i.e., a function $u \in H^{2k}(M)$ is a solution of the variational problem 1)-2) if and only if it satisfies the following equation in the sense of distributions:

$$(4.10) \quad \Delta^{2k}u = \sum_{\nu=1}^N \alpha_\nu \delta(x_\nu)$$

where $\delta(x_\nu)$ is the Dirac measure at x_ν .

Indeed, we already know that for every solution u of the above variational problem

$$(4.11) \quad 0 = \langle \Delta^k u, \Delta^k h \rangle = \int_M \Delta^k u \overline{\Delta^k h},$$

where h is any function which is zero on M_ρ .

Let $\{\xi_\gamma\}$ be the set of $C_0^\infty(M)$ functions such that their supports are disjoint and $\xi_\nu(x_\mu) = \delta_{\mu\nu}$, where $\delta_{\nu\mu}$ is the Kronecker delta. Then for any $\psi \in C_0^\infty(M)$ the function

$$\psi - \sum_{\nu=1}^N \psi(x_\nu) \xi_\nu$$

is zero on M_ρ and

$$\begin{aligned} 0 &= \int_M \Delta^k u \overline{\Delta^k \left(\psi - \sum_{\nu=1}^N \psi(x_\nu) \xi_\nu \right)} \\ &= \int_M \Delta^{2k} u \overline{\psi} - \sum_{\nu} \langle \Delta^k u, \Delta^k \xi_\nu \rangle \overline{\psi(x_\nu)}. \end{aligned}$$

In other words $\Delta^{2k}u$ is a distribution of the form

$$\Delta^{2k}u = \sum \alpha_\nu \delta(x_\nu),$$

where $\alpha_\nu = \langle \Delta^k u, \Delta^k \xi_\nu \rangle$.

So every solution of the variational problem is a solution of (4.10).

Conversely, if u is a solution of (4.10) then since the Dirac measure belongs to the space $H_{-\varepsilon-d/2}(M)$, $d = \dim M$, $\varepsilon > 0$, the Regularity Theorem for elliptic operator Δ^{2k} of order $4k$ implies that $u \in H_{2k}(M)$ and for any h which is zero on M_ρ we have

$$\langle \Delta^k u, \Delta^k h \rangle = \langle \Delta^{2k} u, h \rangle = 0,$$

that shows that u is a the solution for 1)-2).

Lemma 1.2 is proved.

Now we can prove the Approximation Theorem 1.4, which plays a key role in the proof of Theorem 1.3.

Proof. Using Theorem 4.2 with $k = t = d$ and the continuous embedding $H^d(M) \subset H^s(M)$, $d \geq s$, we obtain for every $s \leq d$

$$\begin{aligned} \|\Delta^s(s_n(f) - f)\| &\leq C(M)\|\Delta^d(s_n(f) - f)\| \\ &\leq \left(C(M)\rho^{2d}\right)^m \|\Delta^{d(m+1)}(s_n(f) - f)\|, \end{aligned}$$

where $n = d(m + 1)$, $m = 2^l$, $l = 0, 1, \dots$. By minimization property we obtain

$$\|\Delta^s(s_n(f) - f)\| \leq (C(M)\rho^2)^{n-d} \|\Delta^n f\|, n = d(2^l + 1).$$

If $f \in E_\omega$, i.e., f is a linear combination of eigenfunctions whose eigenvalues belong to $[0, \omega]$, then $\|\Delta^n f\| \leq \omega^n \|f\|$, and

$$\|\Delta^s(s_n(f) - f)\| \leq \omega^d (C(M)\rho^2\omega)^{n-d} \|f\|, n = d(2^l + 1), s \leq d.$$

To obtain corresponding estimates in the uniform norm it is enough to combine the above inequalities with the Sobolev Embedding Theorem. The Approximation Theorem 1.4 is proved. \square

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DEPARTMENT OF MATHEMATICS
TEMPLE UNIVERSITY
PHILADELPHIA, PA 19122
E-mail address: pesenson@math.temple.edu

