Let $k$ be a field of positive characteristic $p$. One considers some categories, whose objects are given classes of finite $p$-groups, and morphisms are given classes of $k$-virtual bisets, i.e., linear combinations of bisets with coefficients in $k$. The category of $k$-linear functors from such a category to the category of $k$-vector spaces is abelian, and one can try to classify and describe its simple objects, or its projective objects.

By specific subfunctors of the Burnside functor, which have a unique simple quotient $S_{Q,k}$, one will get some estimates on the $k$-dimension of the evaluations of these simple functors. These evaluations are equalities for abelian $p$-groups, and for such groups $P$ the result is even stronger, since it provides some explicit $k$-bases for the evaluations $S_{Q,k}(P)$.

Introduction

Le formalisme des bi-ensembles permet de donner une description uniforme des opérations d’induction, de restriction, d’inflation, de déflation, et de transport par isomorphisme. Cette description permet une interprétation des formules classiques, comme la formule de Mackey, en termes de produits de bi-ensembles.

Etant donné un anneau commutatif $k$, il est alors naturel de considérer des catégories dont les objets sont certaines classes de groupes finis, et les morphismes certaines classes de bi-ensembles $k$-virtuels, i.e., des combinaisons linéaires à coefficients dans $k$ de bi-ensembles. La catégorie des foncteurs $k$-linéaires d’une telle catégorie vers la catégorie des $k$-modules est abélienne, et on peut essayer de classifier et décrire ses objets simples ou ses objets projectifs.

En particulier, S. Bouc a étudié la catégorie dont les objets sont tous les groupes finis, et les morphismes tous les bi-ensembles virtuels, l’anneau $k$ étant un corps de caractéristique zéro (cf. [1]). La classification des facteurs simples du foncteur de Burnside dans ce cas fait apparaître la notion de "b-groupes", et les facteurs en question sont les foncteurs simples $S_{H,k}$, où $H$ est un "b-groupe".

Plus récemment, S. Bouc et J. Thévenaz ont décrit certains foncteurs simples $S_{P,k}$, où $P$ est un $p$-groupe fini et $k$ est un corps de caractéristique
positive $q$ différente de $p$, définis sur la catégorie dont les objets sont les $p$-groupes finis (cf. [2]).

Dans ce travail on essaiera de compléter ce dernier cas, en prenant pour $k$ un corps de caractéristique $p$. Grâce à des sous-foncteurs du foncteur de Burnside, qui ont un unique quotient simple $S_{Q,k}$, on obtiendra des estimations de la dimension sur $k$ des évaluations de ces foncteurs simples. Dans le cas où l’on se restreint aux $p$-groupes abéliens, ces estimations sont des égalités: soient $Q$ et $P$ deux $p$-groupes abéliens finis, tel que $Q$ est d’ordre au moins $p^2$, alors $\dim_k S_{Q,k}(P)$ est égale au nombre de sous-groupes $R$ de $P$ tels que $P/R \simeq Q$; et $\dim_k S_{1,k}(P)$ est égale au nombre de sous-groupes $R$ de $P$ tels que $|P/R| \leq p$. Le cas où $Q$ est d’ordre $p$, est donc laissé complètement ouvert.

1. Notations et rappels

- Un bi-ensemble $U$, est un ensemble sur lequel un groupe $G$ agit à gauche et un autre groupe $H$ agit à droite, de façon que

$$g(xh) = (gx)h$$

pour tous $g \in G$, $x \in U$ et $h \in H$.

Soit $k$ un corps de caractéristique $p$ positive, et $\mathcal{C}_k$ la catégorie dont les objets sont les $p$-groupes finis et les morphismes sont les combinaisons $k$-linéaires de bi-ensembles. Soit $\mathcal{F}_k$ la catégorie abélienne des foncteurs $k$-linéaires de $\mathcal{C}_k$ dans la catégorie des $k$-espaces vectoriels.

Si $P$ est un $p$-groupe, soit $B(P)$ l’anneau de Burnside de $P$, alors la correspondance $P \mapsto k \otimes \mathbb{Z} B(P)$ est naturellement un objet de $\mathcal{F}_k$ (cf. [1], Introduction).

Si $G$ et $H$ sont deux objets de $\mathcal{C}_k$, alors $\text{Hom}_{\mathcal{C}_k}(H, G)$ est le produit tensoriel par $k$ du groupe de Grothendieck de la catégorie des $G$-ensembles-$H$, le produit de deux morphismes étant défini par $k$-linéarité à partir du produit d’ensembles défini comme suit:

Soient $G$ et $H$ deux groupes finis. Si $L$ est un sous-groupe du produit $G \times H$ on notera $(G \times H)/L$ le bi-ensemble formé des classes $(g, h)L$ pour $(g, h) \in G \times H$, considéré comme $G$-ensemble-$H$ pour l’action

$$x \cdot (g, h)L \cdot y = (xg, y^{-1}h)L.$$ 

Soit $G'$ un autre groupe fini, $E$ un $G$-ensemble-$H$ et $F$ un $H$-ensemble-$G'$, on note $E \times_H F$ l’ensemble des orbites de $H$ par son action sur le produit $E \times F$ donnée par $h \cdot (x, y) = (xh^{-1}, hy)$. C’est un $G$-ensemble-$G'$: si $g \in G$ et $g' \in G'$, alors par définition

$$g \cdot (x, y) \cdot g' = (gx, yg')$$

où $(x, y)$ désigne l’image de $(x, y)$ dans $E \times_H F$.

Soit $H$ un sous-groupe de $G$, soit $N$ un sous-groupe normal de $G$. Rappelons la définition des bi-ensembles Ind$_H^G$, Inf$_G^{G/N}$, Def$_G^{G/N}$ et Res$_H^G$: 
on a

\[
\text{Ind}_G^G = (G \times H)/\{(g, g)|g \in H\}
\]
\[
\text{Ind}_{G/N}^G = (G \times (G/N))/\{(g, gN)|g \in G\}
\]
\[
\text{Def}_{G/N}^G = ((G/N) \times G)/\{(gN, g)|g \in G\}
\]
\[
\text{Res}_H^G = (H \times G)/\{(h, h)|h \in H\}.
\]

Soit \(\varphi\) un isomorphisme d’un groupe fini \(G\) dans un autre \(G'\), on définit le bi-ensemble \(\text{Iso}^G_{G'}\) par
\[
\text{Iso}^G_{G'} = (G' \times G)/\{(\varphi(g), g)|g \in G\}.
\]

La notation ne fait pas apparaître \(\varphi\), mais cela ne créera aucune confusion en pratique.

Soient \(G\) et \(G'\) deux groupes finis et \(L\) un sous-groupe du produit \(G \times G'\), alors on note \(p_1(L)\) (resp. \(p_2(L)\)) la projection de \(L\) sur \(G\) (resp. sur \(G'\)).

De même on note
\[
k_1(L) = \{g \in G|(g, 1) \in L\}
\]
\[
k_2(L) = \{h \in G'|(1, h) \in L\}.
\]

Pour tout \(y\) élément de \(p_2(L)\), il existe \(x_y\) élément de \(G\) tel que le couple \((x_y, y)\) soit élément de \(L\). Alors en associant à \(yk_2(L)\) l’élément \(x_yk_1(L)\) on obtient un isomorphisme, dit canonique, entre \(p_2(L)/k_2(L)\) et \(p_1(L)/k_1(L)\).

Si \(G''\) est un autre groupe fini, et \(M\) un sous-groupe de \(G' \times G''\), alors on pose
\[
L \star M = \{(g, g'') \in G \times G''|\exists g' \in G', (g', g) \in L, (g', g'') \in M\}.
\]
C’est un sous-groupe de \(G \times G''\).

Rappelons la formule de Mackey relative aux bi-ensembles (cf. [1], 3.2): soient \(G\) et \(H\) deux groupes finis, soit \(L\) un sous-groupe du produit \(G \times H\) et \(M\) un sous-groupe du produit \(H \times K\). Alors
\[
(G \times H/L) \times_H (H \times K/M) = \sum_{h \in p_2(L)\setminus H/p_1(M)} (G \times K)/(L \star^{(h, 1)}M).
\]
Si \(L\) est un sous-groupe du produit \(G \times H\), alors d’après ([1], Lemme 3, p. 672) le bi-ensemble \(G \times H/L\) se décompose en un produit de morphismes dans \(\mathcal{C}_k\):
\[
G \times H/L = \text{Ind}_k^G \text{Ind}^L_{p_1(L)} \text{Inf}^L_{p_1(L)/k_1(L)} \text{Iso}^L_{p_2(L)/k_2(L)} \text{Def}^L_{p_1(L)/k_1(L)} \text{Res}_H^G
\]
\[
\text{ou } \text{Iso}^L_{p_2(L)/k_2(L)} \text{ est relatif à l’isomorphisme canonique entre } p_2(L)/k_2(L) \text{ et } p_1(L)/k_1(L).
\]

- Les foncteurs simples \(S_{Q,k}\) (cf. [1], Proposition 2).

Les objets simples de \(\mathcal{F}_k\) sont paramétrés par des couples \((Q, V)\), où \(Q\) est un \(p\)-groupe et \(V\) est un \(k\text{Out}(Q)\)-module simple.
Si $S$ est un foncteur simple, le couple $(Q,V)$ correspondant est défini en prenant pour $Q$ un groupe d’ordre minimal tel que $S(Q)$ est différent de 0 et en considérant $V = S(Q)$ comme $k\text{Out}(Q)$-module.

Inversement, étant donné un couple $(Q,V)$ on a un foncteur simple noté $S_{Q,V}$ défini, pour un $p$-groupe $P$, par

$$S_{Q,V}(P) = \left( \text{Hom}_{C_k}(Q,P) \otimes_{\text{End}_{C_k}(Q)} V \right) / \left\{ \sum_i \varphi_i \otimes v_i \mid \forall \psi \in \text{Hom}_{C_k}(P,Q), \sum_i (\psi \varphi_i) v_i = 0 \right\}.$$ 

La notation $S_{Q,k}$ désignera le foncteur simple associé au $k\text{Out}(Q)$-module trivial $k$.

2. Certains foncteurs simples en caractéristique $p$

**Lemme 1.** Soient $Q$ et $P$ deux $p$-groupes finis, soit $A$ un sous-$\text{End}(Q)$-module de $B(Q)$ et $J$ un sous-$\text{End}(Q)$-module maximal de $A$.

Soit $N_{Q,A}$ le sous-foncteur du foncteur de Burnside $B$, défini par

$$N_{Q,A}(P) = \text{Hom}_{C_k}(Q,P) \times_Q A$$

et $N_{Q,A,J}$ le sous-foncteur de $N_{Q,A}$, défini par

$$N_{Q,A,J}(P) = \{ u \in N_{Q,A}(P) \mid \forall \phi \in \text{Hom}_{C_k}(P,Q) : (\phi \times_P u) \in J \}.$$ 

La correspondance qui à $J$ associe $N_{Q,A,J}$, est une bijection entre l’ensemble des sous-$\text{End}(Q)$-modules maximaux de $A$ et l’ensemble des sous-foncteurs maximaux de $N_{Q,A}$.

**Preuve.** Vérifions que $N_{Q,A,J}$ est un sous-foncteur de $N_{Q,A}$:

Soient $P$ et $P'$ deux $p$-groupes finis, soit $\psi \in \text{Hom}_{C_k}(P',P)$, vérifions qu’on a

$$\psi \times_{P'} N_{Q,A,J}(P') \subset N_{Q,A,J}(P).$$

D’après l’associativité du produit des bi-ensembles, on a

$$\forall u \in N_{Q,A,J}(P'), \forall \psi' \in \text{Hom}_{C_k}(P,Q), \ (\psi' \times_P (\psi \times_{P'} u)) \times_{P'} u = (\psi' \times_P \psi) \times_{P'} u.$$ 

Or d’après la définition de $N_{Q,A,J}(P')$:

$$\text{Hom}_{C_k}(P',Q) \times_{P'} u \subset J$$

et $(\psi' \times_{P} \psi) \in \text{Hom}_{C_k}(P',Q)$, donc $(\psi' \times_{P} \psi) \times_{P'} u \in J$, ainsi $\psi' \times_{P} (\psi \times_{P'} u)$ est dans $J$, et $N_{Q,A,J}$ est un sous-foncteur de $N_{Q,A}$. De plus $N_{Q,A,J} \neq N_{Q,A}$, car par exemple $N_{Q,A,J}(Q) = J$ alors que $N_{Q,A}(Q) = A$.

Soit $L$ un sous-foncteur de $N_{Q,A}$, en particulier on a $L(Q) \subset N_{Q,A}(Q) = A$. Alors il y a deux cas à envisager.
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(1) Ou bien $L(Q) \subset J$, dans ce cas
\[ \forall u \in L(P), \forall \varphi \in \text{Hom}_{C_k}(P, Q) : (\varphi \times_P u) \in L(Q) \]
donc $L \subset N_{Q,A,J}$.

(2) Ou bien $L(Q) \not\subset J$, donc $L(Q) + J = A$ et par suite $L + N_{Q,J} = N_{Q,A}$, car
\[
N_{Q,A}(P) = \text{Hom}_{C_k}(Q, P) \times_Q A \\
= \text{Hom}_{C_k}(Q, P) \times_Q (L(Q) + J) \\
= (\text{Hom}_{C_k}(Q, P) \times_Q L(Q)) + N_{Q,J}(P).
\]
Comme $L$ est un sous-foncteur de $N_{Q,A}$, on a
\[
\text{Hom}_{C_k}(Q, P) \times_Q L(Q) \subset L(P)
\]
d'où
\[
N_{Q,A}(P) \subset L(P) + N_{Q,J}(P).
\]
On a alors
\[
L + N_{Q,J} = N_{Q,A}
\]
or $N_{Q,J} \subset N_{Q,A,J}$, alors
\[
L + N_{Q,A,J} = N_{Q,A}.
\]
D'après ce qui précède on a établi que si $L$ est sous-foncteur de $N_{Q,A}$ alors
ou bien $L \subset N_{Q,A,J}$ ou bien $L + N_{Q,A,J} = N_{Q,A}$, donc $N_{Q,A,J}$ est un sous-foncteur maximal de $N_{Q,A}$, et le quotient $N_{Q,A}/N_{Q,A,J}$ est un foncteur simple.

D'autre part, si $L$ est un sous-foncteur maximal (propre) de $N_{Q,A}$, alors $L(Q) \neq A$, car sinon on aurait $\text{Hom}_{C_k}(Q, P) \times_Q L(Q) = \text{Hom}_{C_k}(Q, P) \times_Q A$, i.e., $N_{Q,A} = L$. Il existe un sous-$\text{End}(Q)$-module maximal de $A$, noté $J_L$ tel que $L(Q) \subset J_L$, par le premier cas on déduit que $L \subset N_{Q,A,J_L}$, et comme $L$ est un sous-foncteur maximal de $N_{Q,A}$ on a $L = N_{Q,A,J_L}$.

Finalement, si $J$ et $J'$ sont deux sous-$\text{End}(Q)$-modules maximaux de $A$ tels que $J \neq J'$, alors $N_{Q,A,J} \neq N_{Q,A,J'}$, car par exemple $N_{Q,A,J}(Q) = J$ alors que $N_{Q,A,J'}(Q) = J'$.

Soient $H$ et $P$ deux $p$-groupes finis, dans toute la suite on notera “$H$ sous-groupe de $P$” par $H \vartriangleleft P$.

Soit $Q$ un $p$-groupe fini non-trivial. Considérons dans $k \otimes \mathbb{Z} B(Q)$ le $Q$-ensemble virtuel $\xi_Q$, défini par
\[
\xi_Q = Q/1 - \sum_{|Z|=p, Z \vartriangleleft \mathbb{Z}(Q)} Q/Z
\]
où $\mathbb{Z}(Q)$ désigne le centre du groupe $Q$.

Lemme 2. Soit $Q$ un $p$-groupe non-trivial, alors dans $k \otimes \mathbb{Z} B$ on a:
• Def$_{Q/N}$ $\xi_Q = 0$, si $N \unlhd Q$ et $N \neq 1$.

• Res$_{P}$ $\xi_Q = 0$, si $Q$ est d’ordre au moins $p^2$ et $P$ est un sous-groupe propre de $Q$.

\textbf{Preuve.}

• Soit $N \unlhd Q$ et $N \neq 1$. Rappelons que si $H$ est un sous-groupe de $Q$, alors

$$\text{Def}_{Q/N} Q/H = (Q/N)/(H \cdot N/N).$$

Ainsi

$$\text{Def}_{Q/N} \xi_Q = (Q/N)/(N/N) - \sum_{|Z|=p, Z < Z(Q)} (Q/N)/(Z \cdot N/N)$$

$$= (Q/N)/(N/N) - \sum_{|Z|=p, Z < Z(Q) \cap N} (Q/N)/(N/N)$$

$$- \sum_{|Z|=p, Z \not< Z(Q) \cap N} (Q/N)/(Z \cdot N/N).$$

Or $N$ est un sous-groupe normal non-trivial de $Q$, donc $N \cap Z(Q) \neq 1$, et $|\{Z < (Z(Q) \cap N), |Z| = p\}|$ est le nombre de sous-groupes d’ordre $p$ qui engendrent le $p$-groupe abélien élémentaire non-trivial $\Omega_1(Z(Q) \cap N)$, donc

$$|\{Z < (Z(Q) \cap N), |Z| = p\}| = \frac{|\Omega_1(Z(Q) \cap N)| - 1}{p - 1}$$

par suite

$$|\{Z < (Z(Q) \cap N), |Z| = p\}| \equiv 1 (p).$$

Donc

$$\text{Def}_{Q/N} \xi_Q = - \sum_{|Z|=p, Z \not< Z(Q)} (Q/N)/(Z \cdot N/N)$$

ou encore

$$\text{Def}_{Q/N} \xi_Q = - \sum_{M < Q, |M:N|=p} \sum_{|Z|=p, Z \not< Z(Q)} (Q/N)/(M/N).$$

Or si $|M : N| = p$, alors

$$|\{Z < Z(Q) \text{ tels que } Z \not\subseteq N, Z \cdot N = M \text{ et } |Z| = p\}| = n_1 - n_2$$
où
\[ n_1 = |\{ Z \prec \mathcal{Z}(Q) \cap M \text{ tels que } |Z| = p\}| \]
\[ n_2 = |\{ Z \prec \mathcal{Z}(Q) \cap N \cap M \text{ tels que } |Z| = p\}| \]
avec \( n_1 \equiv 1(p) \) et \( n_2 \equiv 1(p) \), donc
\[ |\{ Z \prec \mathcal{Z}(Q) \text{ tels que } Z \not\subset N, Z \cdot N = M \text{ et } |Z| = p\}| \equiv 0 \ (p) \]
d'où \( \text{Def}^Q_{Q/N} \xi_Q = 0 \).

- D'après la formule de Mackey on a
\[
\text{Res}^Q_P \xi_Q = \sum_{x \in (P \setminus Q/1)} P/(P \cap x) - \sum_{|Z| = p} \sum_{Z \prec \mathcal{Z}(Q)} P/(P \cap Z)
\]
\[ = |Q : P| \cdot P/1 - \sum_{|Z| = p} |Q : P \cdot Z| \cdot P/(P \cap Z). \]

Pour cela on distinguera deux cas:
1. Si \( |Q : P| \geq p^2 \), alors
\[ \forall Z \prec \mathcal{Z}(Q), |Z| \leq p, \text{ on a } |Q : P \cdot Z| \geq p. \]
Donc \( \text{Res}^Q_P \xi_Q = 0 \).
2. Si \( |Q : P| = p \), alors
\[ \text{Res}^Q_P \xi_Q = - \sum_{|Z| = p} P/(P \cap Z)
\]
\[ = -|\{ Z \prec \mathcal{Z}(Q) \text{ tels que } Z \not\subset P \text{ et } |Z| = p\}| \cdot P/1. \]
Comme \( |Q : P| = p \) et \( |Q| \geq p^2 \), alors le sous-groupe \( P \) est non-trivial. Comme de plus \( P \) est normal dans \( Q \), alors \( P \cap \mathcal{Z}(Q) \neq 1 \), et par suite
\[ |\{ Z \prec \mathcal{Z}(Q) \text{ tels que } Z \not\subset P \text{ et } |Z| = p\}| \equiv 0 \ (p). \]
D'où \( \text{Res}^Q_P \xi_Q = 0 \).

\[ \square \]

**Lemme 3.** Dans \( k \otimes_{\mathbb{Z}} B(Q) \), on a \( \text{End}(Q) \times_Q \xi_Q \) est isomorphe à \( k \). C'est un \( k \text{Out} (Q) \)-module trivial.

**Preuve.** Pour tout sous-groupe \( L \) de \( Q \times Q \), on a:
\[
(Q \times Q)/L = \text{Ind}_{p_1(L)}^{Q} \text{Ind}_{p_1(L)/k_1(L)}^{p_1(L)} \text{Ind}_{p_2(L)/k_2(L)}^{p_2(L)} \text{Def}_{p_2(L)/k_2(L)}^{p_2(L)} \text{Res}_{p_2(L)}^{Q}
\]
et
\[ L = \{(g, h) \in Q \times Q| s(g) = t(h)\} \]
où \( s \) est une surjection d’un sous-groupe \( P_1 \) de \( Q \) dans un groupe \( Q' \), et \( t \) une surjection d’un sous-groupe \( P_2 \) de \( Q \) dans \( Q' \).

D’après le Lemme 2, on a

\[
(Q \times Q/L) \times Q \xi_Q \neq 0 \Rightarrow p_2(L) = Q \text{ et } k_2(L) = 1.
\]

Il en résulte que \( Q' \cong Q \).

Or \( Q' \cong p_1(L)/k_1(L) \cong p_2(L)/k_2(L) \), donc \( p_1(L) = Q \) et \( k_1(L) = 1 \), par suite \( s \) et \( t \) sont des isomorphismes.

Ainsi \( L \) devient

\[
L = \Delta_{s^{-1} t}(Q) = \Delta_{\varphi}(Q)
\]

où

\[
\varphi = s^{-1} \circ t \in \text{Aut}(Q).
\]

Dans ce cas

\[
(Q \times Q/\Delta_{\varphi}(Q)) \times Q \xi_Q = Q/\varphi(1) - \sum_{|Z| = p} Q/\varphi(Z).
\]

Donc

\[
(Q \times Q/\Delta_{\varphi}(Q)) \times Q \xi_Q = Q/1 - \sum_{|Z| = p} Q/Z
\]

\[
= \xi_Q
\]

et par suite

\[
\text{End}(Q) \times Q \xi_Q = k\xi_Q.
\]

D’autre part, on a

\[
\text{End}(Q) = k\text{Out}(Q) \oplus I_Q \quad \text{(cf. [1], Corollaire, p. 677)}
\]

où \( I_Q \) est l’idéal de \( \text{End}(Q) \) engendré par des bi-ensembles factorisant par des groupes \( K \) d’ordre strictement inférieur à celui de \( Q \).

On a \( I_Q \times Q \xi_Q = 0 \), car les générateurs de \( I_Q \) sont de la forme

\[
(Q \times K)/L \times_K (K \times Q)/M.
\]

Puisque \(|K| < |Q|\), alors par le Lemme 2 on a \([(K \times Q)/M] \times Q \xi_Q = 0:\)

En effet

\[
|p_2(M)|/|k_2(M)| = |p_1(M)|/|k_1(M)| \leq |K| < |Q|
\]

donc

\[
\text{Def}_{p_2(M)/k_2(M)}^{p_2(M)} \text{Res}_{p_2(M)}^{Q} \xi_Q = 0
\]

il s’ensuit que

\[
[(K \times Q)/M] \times Q \xi_Q = 0
\]

et par conséquent

\[
[(Q \times K)/L \times_K (K \times Q)/M] \times Q \xi_Q = 0.
\]
Ainsi $I_Q \times_Q \xi_Q = 0$ et $\text{End}(Q) \times_Q \xi_Q$ est un $k\text{Out}(Q)$-module trivial. □

**Proposition 1.** Soit $Q$ un $p$-groupe d’ordre au moins $p^2$. Considérons le sous-foncteur $F_Q$ de $k \otimes B$ défini, pour un $p$-groupe fini $P$, par:

$$F_Q(P) = \text{Hom}_{C_k}(Q, P) \times_Q \xi_Q$$

et le sous-foncteur $J_Q$ de $F_Q$ défini, pour un $p$-groupe fini $P$, par:

$$J_Q(P) = \{ u \in F_Q(P) | \forall \varphi \in \text{Hom}_{C_k}(P, Q), \varphi \times_P u = 0 \}.$$

Alors $J_Q$ est l’unique sous-foncteur maximal de $F_Q$, et on a $F_Q/J_Q \simeq S_{Q,k}$.

Puis considérons le sous-foncteur $J_{1,k}$ de $k \otimes B$ défini, pour un $p$-groupe fini $P$, par:

$$J_{1,k}(P) = \{ u \in k \otimes B(P) | \forall \varphi \in \text{Hom}_{C_k}(P, 1), \varphi \times_P u = 0 \}$$

i.e.,

$$J_{1,k}(P) = \{ X \in k \otimes B(P) | \forall U \text{ sous-groupe de } P, |U \setminus X| = 0 \}.$$

Alors $J_{1,k}$ est l’unique sous-foncteur maximal de $k \otimes B$, et on a $(k \otimes B)/J_{1,k} \simeq S_{1,k}$.

**Preuve.** L’existence et l’unicité du foncteur $J_Q$ dans $F_Q$ résultent du Lemme 1 et du Lemme 3. On sait aussi que $F_Q/J_Q$ est simple, il reste à vérifier que $F_Q/J_Q \simeq S_{Q,k}$:

On a $F_Q(Q)/J_Q(Q) \simeq \text{End}(Q) \times_Q \xi_Q$ qui est un $k\text{Out}(Q)$-module trivial [cf. Lemme 3]. De plus

$$\forall K p \text{-groupe tel que } |K| < |Q|, F_Q(K)/J_Q(K) = 0.$$

En effet, pour tout sous-groupe $L$ de $K \times Q$, on a $(K \times Q)/L \times_Q \xi_Q = 0$.

Donc

$$F_Q(K) = 0, \text{ si } |K| < |Q|.$$ 

D’autre part, on a

$$k \otimes B(P) = \text{Hom}_{C_k}(1, P) \times_1 \text{End}(1).$$

Sachant que $\text{End}(1)$ est réduit à $k$, alors par le Lemme 1 $k \otimes B$ a un unique sous-foncteur maximal $J_{1,k}$ défini, pour un $p$-groupe fini $P$, par:

$$J_{1,k}(P) = \{ u \in k \otimes B(P) | \forall \varphi \in \text{Hom}_{C_k}(P, 1), \varphi \times_P u = 0 \}.$$

De plus $(k \otimes B)(1)/J_{1,k}(1)$ est le $k\text{Out}(Q)$-module trivial $k$. D’où $(k \otimes B)/J_{1,k} \simeq S_{1,k}$.

Finalement, pour tout sous-groupe $U$ de $P$, si $X$ est un élément de $k \otimes B(P)$, alors

$$((1 \times P)/(1 \times U)) \times_P X = |U \setminus X|.$$ 

Par suite

$$J_{1,k}(P) = \{ X \in k \otimes B(P) | \forall U \text{ sous-groupe de } P, |U \setminus X| = 0 \}.$$

D’où l’équivalence entre les deux formules de $J_{1,k}(P)$.
2.1. Le cas des foncteurs simples $S_{Q,k}$, avec $|Q| \geq p^2$. Soit $K$ un sous-groupe d’un groupe fini $P$, et $N$ un sous-groupe normal de $K$, on dit que $(K,N)$ est une section de $P$. Soit $H$ un autre groupe, on note $\Sigma_H(P)$ l’ensemble des sections $(K,N)$ de $P$ telles que $K/N \simeq H$.

Rappelons la définition de deux sections liées (cf. [1], p. 685):

Soient $K \supset N$ et $L \supset M$ des sous-groupes de $P$, les sections $(K,N)$ et $(L,M)$ sont liées, si

$$(K \cap L) \cdot N = K, \quad (K \cap L) \cdot M = L, \quad K \cap M = L \cap N.$$ 

On note $B_{Q,k}(P)$ l’espace vectoriel sur $k$ ayant pour base les classes de conjugaison par $P$ de sections $(K,N)$, éléments de $\Sigma_Q(P)$. D’après ([1], p. 717), $\dim_k S_{Q,k}(P)$ est égale au rang de la forme bilinéaire sur $B_{Q,k}(P)$ à valeurs dans $k$ définie par:

$$\langle (K,N), (L,M) \rangle_P = |\{x \in (K \backslash P/L)|(K,N) \text{ et } (xL,xM) \text{ sont liées}\}|.$$

**Proposition 2.** Soient $Q$ et $P$ deux $p$-groupes finis tels que l’ordre de $Q$ est au moins $p^2$, alors

$$F_Q(P) = \left< \frac{P}{R} - \sum_{|Z/R|=p, Z/R \prec Z(S/R)} \frac{P}{Z}/(S,R) \in \Sigma_Q(P) \right>$$ 

où $F_Q$ est le sous-foncteur défini dans Proposition 1.

**Preuve.** Soit $(S,R)$ un élément de $\Sigma_Q(P)$, alors

$$\left( \frac{P}{R} - \sum_{|Z/R|=p, Z/R \prec Z(S/R)} \frac{P}{Z} \right) = (\text{Ind}_{S \leftarrow R}^{P} \text{Inf}_{S/R}^{S} \xi_{S/R}) \in F_Q(P)$$

donc

$$\left< \frac{P}{R} - \sum_{|Z/R|=p, Z/R \prec Z(S/R)} \frac{P}{Z}/(S,R) \in \Sigma_Q(P) \right> \subseteq F_Q(P).$$

Inversement, un générateur de $F_Q(P)$ est de la forme $(P \times Q)/L \times Q \xi_Q$, où $L$ est un sous-groupe du produit $P \times Q$, alors en utilisant la décomposition de $(P \times Q)/L$ puis en appliquant le Lemme 2 à $(P \times Q)/L \times Q \xi_Q$, on obtient

$$(P \times Q)/L \times Q \xi_Q = \text{Ind}_{S \leftarrow R}^{P} \text{Inf}_{S/R}^{S} \text{Iso}_{S/R}^{Q} \xi_Q = \frac{P}{R} - \sum_{|Z/R|=p, Z/R \prec Z(S/R)} \frac{P}{Z}$$

pour une section $(S,R)$ convenable, d’où la proposition. \qed
Lemme 4. Soit $P$ un $p$-groupe d’ordre au moins $p^2$, et $M_0$ un sous-groupe maximal de $P$, alors

$$F_{M_0}(P) = J_{M_0}(P) \oplus \left( \frac{P}{Z} - \sum_{E/Z \simeq (P/Z) \in \Sigma_{M_0}(P)} \frac{P/E}{(P/Z) \in \Sigma_{M_0}(P)} \right)$$

et

$$\dim_k S_{M_0,k}(P) = |\{ Z < Z(P) | (P/Z) \cong M_0 \}|.$$

Preuve. Calculons $\dim_k S_{M_0,k}(P)$:

On a $\dim_k S_{M_0,k}(P)$ est égal au rang de la matrice suivante

$$I = \left( ((K, N), (L, M)) \right)_{(K, N), (L, M) \in B_{M_0}(P)}$$

ou encore

$$I = \left( I_r \quad A \quad O \right).$$

On désigne par $^t A$ la matrice transposée de $A$, qui est obtenue par la restriction de la forme bilinéaire de $B_{M_0}(P)$ aux sections $[(P, U), (M, 1)]$, alors que la matrice $I_r$ est obtenue par la restriction aux sections $[(P, U), (P, V)]$. Il en résulte que $I_r$ est la matrice unité de rang $r$, où $r = |\{ Z < Z(P) | (P/Z) \cong M_0 \}|$. La matrice $O$ est quant à elle égale à la matrice nulle de rang le nombre de sous-groupes maximaux de $P$ isomorphes à $M_0$, car

$$O = \left( ((M, 1), (M', 1)) \right)_{(M, 1), (M', 1) \in B_{M_0}(P)}.$$

On remarque que les coefficients de la matrice $^t A \cdot A$ sont de la forme

$$\lambda_{M, M'} = \sum_{(P, Z) \in \Sigma_{M_0}(P)} \frac{1}{Z \not\subset M \not\subset Z' M'},$$

où la section $(M, 1) \in \Sigma_{M_0}(P)$. Donc

$$\lambda_{M, M'} = \sum_{Z \in \Omega_1(Z(P)) \not\subset M \not\subset Z' M'} \frac{1}{Z \not\subset M \not\subset Z' M'}.$$

Or

$$\sum_{Z \in \Omega_1(Z(P)) \not\subset M \not\subset Z' M'} 1 = \sum_{Z \in \Omega_1(Z(P)) \not\subset M \not\subset Z' M'} 1 - \sum_{Z \in \Omega_1(Z(P)) \not\subset M \not\subset Z' M'} 1 = 0 \quad (p)$$
alors certainement
\[
\sum_{Z \prec \Omega_1(Z(P))} 1 = \sum_{Z \prec \Omega_1(Z(P))} 1 - \sum_{Z \prec \Omega_1(Z(P))} 1 = - \sum_{Z \prec \Omega_1(Z(P))} 1 = 0 (p).
\]
Ainsi \(\lambda_{M,M'} \equiv 0 (p)\), et on trouve \(^TA \cdot A = 0\).

Montrons ensuite que
\[
\text{rg } I = r + \text{rg } (\! ^TA \cdot A\!).
\]

Associons respectivement aux matrices \(I\) et \(^TA \cdot A\), les applications \(k\)-linéaires \(f\) et \(g\). Considérons le vecteur \(T\)
\[
T = \begin{pmatrix} X \\ Y \end{pmatrix} \in \text{Ker}(f).
\]
Donc
\[
I \cdot T = I \begin{pmatrix} X \\ Y \end{pmatrix} = 0.
\]
On obtient le système suivant
\[
\begin{cases}
X + AY = 0 \\
(\! ^TA \cdot A\!)Y = 0
\end{cases}
\]
qui est équivalent au système
\[
\begin{cases}
X = -AY \\
(\! ^TA \cdot A\!)Y = 0
\end{cases}
\]
Ainsi
\[
\text{dim}_k\text{Ker}(f) = \text{dim}_k\text{Ker}(g)
\]
par suite
\[
\text{rg } I = r + \text{rg } (\! ^TA \cdot A\!) = r.
\]
D’où
\[
\text{dim}_kS_{M_0,k}(P) = |\{Z < Z(P)|(P/Z) \cong M_0\}|.
\]
D’autre part, on a:
\[
F_{M_0}(P) = J_{M_0}(P) \oplus \left( P/Z - \sum_{E/Z < Z(P/Z)} P/E |(P,Z) \in \Sigma_{M_0}(P) \right).
\]
En effet: soit
\[ x = \sum_{(P,Z) \in \Sigma_{M_0}(P)} \lambda_Z \left( \frac{P}{Z} - \sum_{\substack{|E/Z| = p \\ E/Z < Z(P/Z)}} \frac{P}{E} \right) \in J_{M_0}(P). \]

Fixons une section \((P, Z_0) \in \Sigma_{M_0}(P)\) et appliquons Def\(_{P/Z_0}\) à \(x\). On a donc
\[ \text{Def}_{P/Z_0}(x) = 0, \] dans \(k \otimes_Z B(P/Z_0)\)
i.e.,
\[ 0 = \lambda_{Z_0} \left[ \left( \frac{P}{Z_0} \right)/(Z_0/Z_0) - \sum_{\substack{|E/Z_0| = p \\ E/Z_0 < Z(P/Z_0)}} \left( \frac{P}{Z_0} \right)/(E \cdot Z_0/Z_0) \right] \]
\[ + \sum_{(P,Z') \in \Sigma_{M_0}(P)} \lambda_{Z'} \left[ \left( \frac{P}{Z_0} \right)/(Z' \cdot Z_0/Z_0) \right] \]
\[ - \sum_{\substack{|E'/Z'| = p \\ E'/Z' < Z(P/Z')}} \left( \frac{P}{Z_0} \right)/(E' \cdot Z_0/Z_0) \].

Dans cette égalité le terme \(\left( \frac{P}{Z_0} \right)/(Z_0/Z_0)\) est unique, donc \(\lambda_{Z_0} = 0\).
Comme la section \((P, Z_0)\) élément de \(\Sigma_{M_0}(P)\), est arbitraire, on conclut que \(x = 0\). Par suite, dans \(F_{M_0}(P)\) on a
\[ J_{M_0}(P) \cap \left\{ \frac{P}{Z} - \sum_{\substack{|E/Z| = p \\ E/Z < Z(P/Z)}} \frac{P}{E} \big| (P, Z) \in \Sigma_{M_0}(P) \right\} = \{0\}. \]
Or
\[ \dim_k S_{M_0,k}(P) = \dim_k (F_{M_0}(P)/J_{M_0}(P)) \]
\[ = |\{ Z \prec Z(P) \big| (P/Z) \cong M_0 \}| \]
ce qui prouve que
\[ F_{M_0}(P) = J_{M_0}(P) \oplus \left\{ \frac{P}{Z} - \sum_{\substack{|E/Z| = p \\ E/Z < Z(P/Z)}} \frac{P}{E} \big| (P, Z) \in \Sigma_{M_0}(P) \right\}. \]

□
Proposition 3. Soient $P$ et $Q$ deux $p$-groupes finis, tel que l’ordre de $Q$ est au moins $p^2$, alors

$$F_Q(P) = J_Q(P) + \left\langle P/R - \sum_{|Z/R|=p, Z/R < Z(Q)} P/Z | (N_P(R)/R) \cong Q \right\rangle$$

$$\supseteq J_Q(P) \oplus \left\langle P/R - \sum_{|Z/R|=p, Z/R < Z(Q)} P/Z | (P, R) \in \Sigma_Q(P) \right\rangle$$

et on a

$$|\{\text{sous-groupes normaux } R \text{ de } P | (P/R) \cong Q\}|$$

$$\leq \dim_k S_{Q,k}(P)$$

$$\leq |\{\text{classes de conjugaison de sous-groupes } R \text{ de } P | (N_P(R)/R) \cong Q\}|.$$

Preuve. Soit $(S_0, R_0)$ un élément de $\Sigma_Q(P)$, montrons par récurrence sur l’indice $[P : S_0]$, que le couple $(S_0, R_0)$ vérifie la propriété $\mathcal{P}$ suivante:

$$\left\langle P/R_0 - \sum_{|Z/R_0|=p, Z/R_0 < Z(S_0/R_0)} P/Z \right\rangle \subseteq J_Q(P) + \left\langle P/R - \sum_{|Z/R|=p, Z/R < Z(Q)} P/Z | (N_P(R)/R) \cong Q \right\rangle.$$

Si $[P/S_0] = 1$, alors $R_0$ est normal dans $P$, par suite la propriété $\mathcal{P}$ est vérifiée.

Soit $(S_0, R_0)$ un élément de $\Sigma_Q(P)$ tel que $[P/S_0] = p^{r+1}$. Supposons que la propriété $\mathcal{P}$ est vérifiée pour les sections $(S, R) \in \Sigma_Q(P)$ telles que $[P/S] = p^s$, et montrons que la section $(S_0, R_0)$ vérifie aussi la propriété $\mathcal{P}$.

Si $S_0 = N_P(R_0)$ alors la propriété $\mathcal{P}$ est vérifiée. Si $S_0$ est un sous-groupe propre de $N_P(R_0)$, il existe un sous-groupe $T$ de $P$ tel que $T$ contient $S_0$, le sous-groupe $R_0$ est normal dans $T$ et $[T/S_0] = p$, alors par la Proposition 2 on a

$$y = (T/R_0)/(R_0/R_0) - \sum_{|Z/R_0|=p, Z/R_0 < Z(S_0/R_0)} (T/R_0)/(Z/R_0) \in F_Q(T/R_0).$$
Et par le Lemme 4, on a
\[ F_{Q}(T/R_{0}) = J_{Q}(T/R_{0}) \oplus \left( (T/R_{0})/(R_{T}/R_{0}) - \sum_{|Z/R_{T}|=p}^{\sum_{Z/R_{T} <Z(T/R_{T})}} (T/R_{0})/(Z/R_{0})|T, R_{T} \in \Sigma_{Q}(T) \right) \]
donc
\[ \text{Ind}_{T}^{P} \text{Inf}_{T/R_{0}}^{T} y = \left( P/R_{0} - \sum_{|Z/R_{0}|=p}^{\sum_{Z/R_{0} < Z(S_{0}/R_{0})}} P/Z \right) \in J_{Q}(P) + \left( P/R_{T} - \sum_{|Z/R_{T}|=p}^{\sum_{Z/R_{T} < Z(S_{0}/R_{0})}} P/Z, \text{ avec } |R_{T}/R_{0}| = p \text{ et } (T, R_{T}) \in \Sigma_{Q}(T) \right) \].

On remarque qu’on est passé d’une section \((S_{0}, R_{0})\) de \(P\) telle que \((S_{0}/R_{0}) \cong Q\) à une autre section \((T, R_{T})\) de \(P\) telle que \((T/R_{T}) \cong Q\), et \(|R_{T}/R_{0}| = p\). Comme \(|P/T| = p^{n}\), alors par hypothèse de récurrence on a
\[ \left( P/R_{T} - \sum_{|Z/R_{T}|=p}^{\sum_{Z/R_{T} < Z(T/R_{T})}} P/Z \right) \in J_{Q}(P) + \left( P/R - \sum_{|Z/R| = p}^{\sum_{Z/R < Z(N_{P}(R)/R)}}, \text{ si } |N_{P}(R)/R| \cong Q \right) \].

Ainsi la section \((S_{0}, R_{0})\) vérifie la propriété \(\mathcal{P}\), et on a
\[ F_{Q}(P) = J_{Q}(P) + \left( P/R - \sum_{|Z/R| = p}^{\sum_{Z/R < Z(N_{P}(R)/R)}}, P/Z |(N_{P}(R)/R) \cong Q \right) . \]

On en déduit alors que
\[ \dim_{k} S_{Q,k}(P) \]
\[ \leq |\{ \text{classes de conjugaison de sous-groupes } R \text{ de } P | (N_{P}(R)/R) \cong Q \}| . \]

Finalement montrons que dans \(F_{Q}(P)\):
\[ J_{Q}(P) \cap \left( P/R - \sum_{|Z/R| = p}^{\sum_{Z/R < Z(P/R)}}, P/Z |(P, R) \in \Sigma_{Q}(P) \right) = \{0\} . \]
En effet: soit
\[x = \sum_{(P,R) \in \Sigma_Q(P)} \lambda_R \left( \frac{P}{R} - \sum_{\frac{|Z/R|}{p} \in \mathbb{Z} \wedge \mathbb{Z}(P/R)} \frac{P}{Z} \right) \in J_Q(P)\]

vérifions que
\[\lambda_R = 0, \ \forall (P,R) \in \Sigma_Q(P).\]

Fixons une section \((P,R) \in \Sigma_Q(P)\), et appliquons le foncteur \(\text{Def}_{P/R}\) à \(x\).
On a donc
\[\text{Def}_{P/R}(x) = 0, \ \text{dans} \ k \otimes_{\mathbb{Z}} B(P/R)\]
i.e.,
\[0 = \lambda_R \left[ \frac{(P/R)}{(R/R)} - \sum_{\frac{|Z/R|}{p} \in \mathbb{Z} \wedge \mathbb{Z}(P/R)} \frac{(P/R)}{(Z \cdot R/R)} \right] + \sum_{(P,R') \in \Sigma_Q(P)} \lambda_{R'} \left[ \frac{(P/R)}{(R' \cdot R/R)} - \sum_{\frac{|Z'/R'|}{p} \in \mathbb{Z} \wedge \mathbb{Z}(P/R)} \frac{(P/R)}{(Z' \cdot R/R)} \right].\]
Dans cette égalité le terme \((P/R)/(R/R)\) est unique, donc \(\lambda_R = 0\). Comme la section \((P,R)\) est arbitraire dans \(\Sigma_Q(P)\), on obtient \(x = 0\). Par suite
\[J_Q(P) \cap \left\{ \frac{P}{R} - \sum_{\frac{|Z/R|}{p} \in \mathbb{Z} \wedge \mathbb{Z}(P/R)} \frac{P}{Z} | (P,R) \in \Sigma_Q(P) \right\} = \{0\}.\]
Ainsi
\[\dim_k S_{Q,k}(P) \geq \left| \{\text{sous-groupes } R \text{ de } P | (P/R) \cong Q\} \right|.\]

\[\square\]

**Corollaire 1.** Soient \(Q\) et \(P\) deux \(p\)-groupes finis, tel que \(Q\) est d’ordre au moins \(p^2\). Si tous les sous-groupes de \(P\) sont normaux (par exemple si \(P\) est abélien), alors \(\dim_k S_{Q,k}(P)\) est égale au nombre de sous-groupes \(R\) de \(P\) tels que \(P/R \cong Q\).

**Preuve.** Si tous les sous-groupes de \(P\) sont normaux, alors par la Proposition 3 on a
\[F_Q(P) = J_Q(P) \oplus \left\{ \frac{P}{R} - \sum_{\frac{|Z/R|}{p} \in \mathbb{Z} \wedge \mathbb{Z}(P/R)} \frac{P}{Z} | (P,R) \in \Sigma_Q(P) \right\}\]
et on conclut
\[ \dim_k S_{Q,k}(P) = |\{\text{sous-groupes } R \text{ de } P | (P/R) \cong Q\}|. \]
\qed

2.2. Le cas des foncteurs simples \( S_{1,k} \). D’après ([1], 7.2.1), \( \dim_k S_{1,k}(P) \) est égale au rang de la forme bilinéaire sur \( k \otimes \mathbb{Z} B(P) \) à valeurs dans \( k \) qui au couple \( (P/U, P/V) \) associe
\[ \langle P/U, P/V \rangle_p = |U \setminus P/V|. \]

\textbf{Lemme 5.} Soit \( P \) un \( p \)-groupe fini d’ordre \( p^2 \), alors
\[ k \otimes \mathbb{Z} B(P) = J_{1,k}(P) + \langle P/P, P/R \text{ avec } |P/R| = p \rangle. \]

\textit{Preuve.} On a
\[ k \otimes \mathbb{Z} B(P) = \{P/P, P/R \text{ avec } |P/R| = p, P/1\}. \]

Soit
\[ x = P/1 - \sum_{R < P \atop |P/R| = p} P/R. \]

Vérifions que pour tout sous-groupe \( K \) de \( P \) on a \( \langle P/K, x \rangle = 0 \), i.e.,
\[ |K \setminus P/1| - \sum_{R < P \atop |P/R| = p} |K \setminus P/R| = 0. \]

Il y a trois cas à envisager:

Cas 1. Si \( K = 1 \), alors
\[ |1 \setminus P/1| - \sum_{R < P \atop |P/R| = p} |1 \setminus P/R| = |P| - \sum_{R < P \atop |P/R| = p} |P/R| \]
\[ = 0 \text{ dans } k. \]

Cas 2. Si \( K = P \), alors
\[ |P \setminus P/1| - \sum_{R < P \atop |P/R| = p} |P \setminus P/R| = 1 - \left| \{R < P \text{ avec } |P/R| = p\} \right| \]
\[ = 0 \text{ dans } k. \]
Cas 3. Si $K = R_0$ avec $|P/R_0| = p$, alors on a

\[
|R_0/P| - \sum_{R < P \atop |P|R| = p} |R_0/P/R| = |P/R_0| - \sum_{R < P \atop |P|R| = p} |P/(R_0 \cdot R)|
\]

\[
= \sum_{R < P \atop |P|R| = p, R \neq R_0} |P/P|
\]

\[
= 0 \text{ dans } k.
\]

Donc $x \in J_{1,k}(P)$, par suite

\[
P/1 \in \left( J_{1,k}(P) + \langle P/P, P/R \text{ avec } |P/R| = p \rangle \right)
\]

d'où le lemme. □

**Proposition 4.** Soit $P$ un $p$-groupe fini, alors

\[
k \otimes \Z B(P) = J_{1,k}(P) + \langle P/P, P/R \text{ avec } |N_P(R)/R| = p \rangle
\]

\[
\supseteq J_{1,k}(P) + \langle P/P, P/M \text{ avec } |P/M| = p \rangle
\]

et on a

\[
|\text{sous-groupes } R \text{ de } P \text{ tels que } |P/R| \leq p|\ 
\leq \dim_k S_{1,k}(P)
\leq |\text{classes de conjugaison de sous-groupes } R \text{ de } P
\text{ tels que } |N_P(R)/R| = p|.
\]

**Preuve.**

- On sait que

\[
k \otimes \Z B(P) = J_{1,k}(P) + \langle P/P, P/R \text{ avec } |N_P(R)/R| = p \rangle.
\]

En effet, soit $P/R_0$ un élément de $k \otimes \Z B(P)$, montrons par récurrence sur l’indice $|P : R_0|$, que la section $(P, R_0)$ vérifie la propriété $\mathcal{P}$ suivante:

\[
P/R_0 \in \left( J_{1,k}(P) + \langle P/P, P/R \text{ avec } |N_P(R)/R| = p \rangle \right).
\]

Si $|P/R_0| = 1$, i.e., $R_0 = P$, alors la propriété $\mathcal{P}$ est vérifiée.

Soit $P/R_0$ un élément de $k \otimes \Z B(P)$ tel que $|P/R_0| = p^{n+1}$. Supposons que la propriété $\mathcal{P}$ est vérifiée pour les sections $(P, R)$ telles que l’indice $|P/R|$ est inférieur à $p^n$, et montrons que la section $(P, R_0)$ vérifie aussi la propriété $\mathcal{P}$:

Si $|N_P(R_0)/R_0| \leq p$, alors

\[
P/R_0 \in \left( J_{1,k}(P) + \langle P/P, P/R \text{ avec } |N_P(R)/R| = p \rangle \right).
\]
Et si $|N_P(R_0)/R| \geq p^2$, alors il existe un sous-groupe $T$ de $P$ tel que $R_0$ est normal dans $T$ et $|T/R_0| = p^2$. D’après le Lemme 5 on a

$$x = (T/R_0)/(R_0/R_0) \in \left(J_{1,k}(T/R_0)ight.$$ 

$$+ \langle(T/R_0)/(T/R_0), (T/R_0)/(M/R_0) \text{ avec } |T/M| = p \rangle \left.$$ 

donc

$$\text{Ind}_T^P \text{Inf}_{T/R_0}^T x = P/R_0 \in \left(J_{1,k}(P) + \langle P/T, P/M \text{ avec } |T/M| = p \rangle \right).$$

On remarque qu’on est passé d’une section $(T, R_0)$ de $P$ telle que $|T/R_0| = p^2$ à deux autres sections de $P$: $(P, M)$ et $(P, T)$ telles que $|M/R_0| = p$ et $|T/M| = p$. Comme $|P/M| = p^n$ et $|P/T| = p^{n-1}$, alors par hypothèse de récurrence on a

$$P/M \in \left(J_{1,k}(P) + \langle P/P, P/R \text{ avec } |N_P(R)/R| = p \rangle \right)$$

et

$$P/T \in \left(J_{1,k}(P) + \langle P/P, P/R \text{ avec } |N_P(R)/R| = p \rangle \right).$$

Ainsi la section $(P, R_0)$ vérifie la propriété $\mathcal{P}$, et on a

$$k \otimes \mathbb{Z} B(P) = J_{1,k}(P) + \langle P/P, P/R \text{ avec } |N_P(R)/R| = p \rangle.$$

Comme $S_{1,k}(P) = k \otimes \mathbb{Z} B(P)/J_{1,k}(P)$, il en résulte que

$$\dim_k S_{1,k}(P) \leq |\{\text{classes de conjugaison de sous-groupes } R \text{ de } P \} \text{ tels que } |N_P(R)/R| = p|.$$

• Dans $k \otimes \mathbb{Z} B(P)$, on a

$$J_{1,k}(P) \cap \langle P/P, P/M \text{ avec } |P/M| = p \rangle = \{0\}.$$

Soit

$$x = \lambda_P P/P + \sum_{\substack{M \prec P \quad |P/M| = p}} \lambda_M P/M \in J_{1,k}(P).$$

Alors, pour tout sous-groupe $K$ de $P$ on a $\langle P/K, x \rangle = 0$, i.e.,

$$\lambda_P |K \setminus P/P| + \sum_{\substack{M \prec P \quad |P/M| = p}} \lambda_M |K \setminus P/M| = 0.$$

En particulier pour $K = 1$, l’égalité (♣) devient

$$\lambda_P |1 \setminus P/P| = 0.$$
Donc $\lambda_P = 0$, et par suite

$$x = \sum_{M < P \atop |P/M|=p} \lambda_M \cdot P/M \in J_{1,k}(P).$$

Appliquons l’égalité $(♣)$ pour le sous-groupe $K = P$, d’où

$$(♠_1) \quad \sum_{M < P \atop |P/M|=p} \lambda_M = 0.$$ Fixons un sous-groupe $M_0$ de $P$ tel que $|P/M_0| = p$, alors en appliquant l’égalité $(♣)$ pour le sous-groupe $K = M_0$, on obtient

$$(♠_2) \quad \sum_{M < P \atop M \neq M_0 \atop |P/M|=p} \lambda_M = 0.$$ En faisant la différence $(♠_1 - ♠_2)$, on tire: $\lambda_{M_0} = 0$. Puisque $M_0$ est arbitraire, alors $x = 0$, et par suite dans $k \otimes_{\mathbb{Z}} B(P)$ on a

$$J_{1,k}(P) \cap \langle P/P, P/M \text{ avec } |P/M| = p \rangle = \{0\}.$$ Il vient donc

$$\dim_k S_{1,k}(P) \geq |\{\text{sous-groupes } R \text{ de } P \text{ tels que } |P/R| \leq p\}|.$$ Ce qui achève la démonstration de la proposition.

Corollaire 2. Soit $P$ un $p$-groupe fini. Si tous les sous-groupes de $P$ sont normaux (par exemple si $P$ est abélien), alors $\dim_k S_{1,k}(P)$ est égale au nombre de sous-groupes $R$ de $P$ tels que $|P/R| \leq p$.

Preuve. Dans le cas où tous les sous-groupes de $P$ sont normaux, alors par la Proposition 4 on a

$$k \otimes_{\mathbb{Z}} B(P) = J_{1,k}(P) \oplus \langle P/P, P/M \text{ avec } |P/M| = p \rangle$$ ce qui donne

$$\dim_k S_{1,k}(P) = |\{\text{sous-groupes } R \text{ de } P \text{ tels que } |P/R| \leq p\}|.$$ Acknowledgements. I would like to thank Serge Bouc for perseverance in answering my queries as well as for useful comments.
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UFR de Mathématiques
Université Paris 7, Case 7012
2, place Jussieu, 75251 Paris Cedex 05
France
E-mail address: bourizk@ccr.jussieu.fr
EVERY TIGHT IMMERSION IN THREE-SPACE OF THE PROJECTIVE PLANE WITH ONE HANDLE IS ASYMMETRIC

DAVIDE P. CERVONE

Kuiper’s original analysis of tight surfaces showed that every surface has a tight immersion in three-space except for the Klein bottle and the projective plane, which have none, and the projective plane with one handle, for which he was unable to determine whether a tight immersion was possible. The latter obtained a unique position among surfaces when it was shown that no smooth tight immersion of it can be formed, while a polyhedral one does exist. Continuing in its role as an unusual example, this surface has another unexpected property, demonstrated here: Any tight immersion is necessarily asymmetric, while every other surface can be immersed tightly and symmetrically in space.

1. Introduction

The real projective plane with one handle has proven to be an unusual surface in more than one way. Initially, when Kuiper determined which surfaces admit tight immersions in space ([13], [14] and [15]), it was the sole surface for which the answer was unknown. This situation persisted for thirty years until two important results appeared in rapid succession. First, in 1992, Haab showed [10] that no smooth tight immersion of the real projective plane with one handle is possible. The methods he used rely heavily on the smoothness of the immersion, and so the question remained open for polyhedral surfaces. Two years later, in an unexpected result [8], the author produced a polyhedral example of a tight immersion of this surface, representing one of only a handful of low-dimensional examples where the polyhedral and smooth theories differ in a significant way.

The reason why a polyhedral immersion exists while no smooth one does is not well understood. In [7] the author shows that the obstruction to smoothing the polyhedral example is not a local one, but further study is still warranted. In an attempt to understand this surface better, the author began a search for a more symmetric tight immersion, but was unable to produce one. As it turns out, there is no symmetric tight immersion, which is the main result of this paper.
Theorem 1.1. Any tight immersion of the real projective plane with one handle in three-space is necessarily asymmetric.

The proof begins in Section 4, and is broken into several parts. The reflections and rotation-reflections are ruled out easily, leaving only rotations as possible symmetries for this surface. The cases of an \( n \)-fold symmetry with \( n > 2 \) and \( n = 2 \) are analyzed separately. Section 4.1 shows that, in a tight immersion with \( n \)-fold symmetry where \( n \geq 3 \), the central core of the surface can be used to make an immersion of the projective plane having exactly one maximum, one minimum and one saddle; but in Section 4.2, a close study of the requirements for the central critical level show that no such symmetric projective plane exists. Finally, in Section 4.4, we prove that no closed surface of odd Euler characteristic can have a polyhedral immersion (tight or otherwise) with 2-fold rotational symmetry. This exhausts all possible symmetries, and thus completes the proof of the main theorem.

The fact that there are no symmetric tight immersions for the projective plane with one handle again makes this surface unique, in light of the following theorem proven in Section 3:

Theorem 1.2. Every orientable surface, and every non-orientable surface with Euler characteristic strictly less than \(-1\), has a symmetric tight immersion in three-space.

The three surfaces that are excluded are the Klein bottle and the real projective plane, for which no tight immersion is possible, and the real projective plane with one handle, which is excluded by the previous theorem. All other surfaces have symmetric tight immersions, and frequently several different ones. Thus the projective plane with one handle continues to play a role as an important example of a tight surface.

2. Definitions and basic results

Given an abstract polyhedral surface \( M \), a polyhedral map of \( M \) is a function \( f: M \to \mathbb{R}^3 \) that maps the faces and edges of \( M \) into linear subspaces of \( \mathbb{R}^3 \) (i.e., as planar triangles with straight edges). We assume that \( f \) is nondegenerate, meaning that it does not reduce the dimension of any face or edge of \( M \). The star of a vertex, \( v \), is the union of the faces and edges that contain \( v \).

A polyhedral mapping \( f: M \to \mathbb{R}^3 \) is an embedding if it is a one-to-one map. It is an immersion if it is locally one-to-one; that is, for every point \( p \) of \( M \), there is a neighborhood \( U_p \) of \( p \) where the restriction of \( f \) to \( U_p \) is one-to-one. (For smooth surfaces, there are additional requirements that guarantee the existence of a tangent plane at every point, but these are not necessary in the polyhedral case.) The interiors of faces are always immersed, and the interiors of edges are immersed provided the adjacent faces don’t overlap,
so \( f \) is an immersion provided the vertices are immersed. In a polyhedral map, a small neighborhood of a vertex is effectively the same as the star of the vertex, so we have the following lemma:

**Lemma 2.1.** A polyhedral map \( f : M \to \mathbb{R}^3 \) is an immersion if, and only if, the star of every vertex of \( M \) is embedded by \( f \).

A mapping \( f : M \to \mathbb{R}^3 \) is said to be *tight* provided that the preimage of every half-space of \( \mathbb{R}^3 \) is connected in \( M \); that is, every plane cuts the image of \( M \) into at most two pieces. This is also called the *two-piece property*. Several other interpretations of tightness can be found in the literature, e.g., [3]. Tightness is a property of the mapping \( f \), not the surface itself, but it is common to speak of \( M \) in place of \( f(M) \) and let the mapping be implied. In practice, this ambiguity is resolved naturally by the context.

For polyhedral surfaces, tightness can be characterized as follows: A vertex \( v \) of \( M \) is a *local extreme vertex* if \( f(v) \) is a vertex of the convex hull of the image of the star of \( v \) (i.e., it is an isolated local maximum for the height function on \( f(M) \) in some direction). A vertex is a *global extreme vertex* if its image is a vertex of the convex hull of \( f(M) \). Note that \( v \) will not be an extreme vertex (local or global) if it lies in the interior of the convex hull of some subset of its adjacent vertices; for example, if \( v \) lies on the line segment between two of its neighbors, then \( v \) can not be locally or globally extreme. With these definitions, we can state:

**Lemma 2.2.** A polyhedral map \( f : M \to \mathbb{R}^3 \) of a closed, compact, connected surface \( M \) is tight if, and only if:

i) Every local extreme vertex is a global extreme vertex,

ii) every edge of the convex hull of \( f(M) \) is contained in \( f(M) \), and

iii) every vertex of the convex hull of \( f(M) \) is the image of a single vertex of \( M \).

This lemma can be found in the literature ([3] or [11], for example) as a result for embedded surfaces, without the third condition. See [6] for an example of why this condition is needed for immersions.

The intersection of a face of the convex hull of \( f(M) \) with \( f(M) \) is called a *top set* of \( M \). (This is sufficient for our needs, but top sets are more general than this; see [3].) The lemma tells us that in a tight immersion, every edge of the convex hull is in the image of \( M \), so the inverse image of the boundary of a face of the convex hull is a cycle in \( M \). If this cycle does not separate \( M \), then the cycle (or its image) is called a *top cycle* of \( M \) (or \( f(M) \)).

The number of top cycles in a tight immersion of a surface is related to its Euler characteristic, \( \chi(M) \). For a sphere, there are none; for other surfaces suppose \( \alpha(M) \) is the number of top cycles in \( M \). Then \( 2 \leq \alpha(M) \leq 2 - \chi(M) \) when \( \chi(M) \) is even, and \( 2 \leq \alpha(M) \leq 1 - \chi(M) \) when \( \chi(M) \) is odd. See [4] for further details.
Note that if the boundary, \( B \), of a face of the convex hull of \( f(M) \) is not a top cycle, then \( B \) separates \( M \). In this case, one of the two regions in \( M \) maps onto the complete face of the convex hull; if it didn’t, then the height function in the direction perpendicular to the face (and toward the rest of the surface) would contain two maxima (one for each of the two regions of \( M \)). So moving the plane slightly in this direction would cut the surface into three pieces (one for each maximum, and a third containing \( B \)), contradicting the fact that the immersion is tight. So any face whose boundary is not a top cycle is entirely contained in \( f(M) \). From a topological standpoint, this means that the intersection of \( f(M) \) with its convex envelope is a sphere minus \( \alpha(M) \) planar convex disks.

By a symmetry of \( M \), we mean a rigid motion of \( \mathbb{R}^3 \) that carries \( f(M) \) onto itself. For compact surfaces, such a motion will have to fix at least one point of \( \mathbb{R}^3 \), which we can take to be the origin. Any symmetry will be induced by one of three kinds of motions in \( \mathbb{R}^3 \): A reflection across a plane, a rotation about a line, or a rotation-reflection (a rotation about a line followed by a reflection across a plane perpendicular to that line). These are distinguished by the fact that they fix a plane, a line, and a point, respectively.

If \( M \) is a symmetric polyhedral surface, non-triangular faces can be subdivided while maintaining the symmetry, provided faces that are symmetric to each other are subdivided in the same way, and self-symmetric faces are divided so that the pieces preserve the underlying symmetry. Thus in the sections below, we can assume, without loss of generality, that \( M \) is a triangulated surface.

### 3. Symmetric tight immersions

Closed surfaces can be organized into three families of related surfaces: Spheres with some number of handles (possibly zero), Klein bottles with some number of handles, and real projective planes with some number of handles. Kuiper’s original proof ([13] and [14]) that all but three surfaces have tight immersions involved showing that the sphere, the Klein bottle with a handle, and the real projective plane with two handles all have tight immersions, and that a handle can be added to a tight surface while maintaining tightness. We use a similar approach below to show that all these surfaces can be tightly immersed with symmetry.

For the orientable surfaces, we start with a sphere, or rather its polyhedral counterpart, a rectangular box. Drilling a tube through the box will add a handle to it, so drilling \( n \) holes adds \( n \) handles. If we place them equally spaced down the middle of the box, as in Figure 1, then the resulting surface will have three planes of reflective symmetry (parallel to the sides of the box) and three axes of 2-fold rotational symmetry (through the centers of
the sides of the box). Combining a rotation with the reflection in the plane perpendicular to the axis of rotation produces a rotation-reflection that is also a symmetry of the surface. Thus the sphere with \( n \) handles can be made with many types of symmetries.

![Figure 1](image)

**Figure 1.** A tight torus with two handles can be made so that it has three planes of reflective symmetry and three axes of rotational symmetry. Any number of handles can be added in a similar way.

The tight Klein bottle with one handle originally described by Kuiper starts with a tight torus and adds a tube that connects the inside of the surface to the outside, as shown in Figure 2 (left). We can then add more handles as we did above. For an even number of additional handles, half can be placed on each side of the non-orientable handle to produce a tight surface with two planes of reflective symmetry and one axis of rotational symmetry. To make an odd number of handles, one more can be added “inside” the initial non-orientable handle (starting at the face through which the original handle passes and ending at the bottom face of the box). This retains the reflective and rotational symmetries.

Note, however, that this immersion does not have a reflection-rotation as a symmetry. We can obtain versions of the Klein bottle with two or more handles that do have such symmetries in the following way: Make the two attachment squares for the initial handle be of different sizes. Then for the next handle, make it in a symmetric position as shown in Figure 2 (right). This has the same three reflective symmetries and 2-fold rotational symmetries as the tori above, so it also has reflection-rotation symmetries. An even number of handles can be added by putting half on each side of the central handles, as before. An odd number can be added by putting the final one horizontally between the two side walls of the original box. In this way, a Klein bottle with at least two handles can be made with reflective, rotational and reflective-rotational symmetries.

Finally, consider the real projective plane. In [12], Kühnel and Pinkall provide a tight polyhedral immersion of the real projective plane with two
Figure 2. A tight Klein bottle with one handle (left) having one axis of rotational symmetry and two planes of reflective symmetry. After widening one end of the handle, a second handle can be added (right) that passes through the first, increasing the symmetries to include three axes of rotational and three planes of reflective symmetry. Self-intersection is indicated where two faces meet without a heavy black line.

handles that has 3-fold rotational symmetry, and they show how any number of handles can be added to this surface while maintaining the symmetry. Rotational and reflective-rotational symmetries are not possible for these surfaces, for the reasons discussed in the next section, so rotational symmetries are the only possibilities.

All the polyhedral surfaces provided in this section have smooth counterparts. A smoothing algorithm is given in [12] that can be applied to all these surfaces while preserving the symmetries, so Theorem 1.2 is true for both smooth and polyhedral surfaces.

4. Symmetry and the projective plane with one handle

Unlike the surfaces described above, which have both smooth and polyhedral tight immersions, the real projective plane has no smooth tight immersion [10] but does have polyhedral ones [8], so we need only consider polyhedral representations of this surface when proving Theorem 1.1.

Suppose that $M$ is a symmetric tight immersion of the projective plane with one handle (necessarily a polyhedral one). Then the symmetry is induced by a symmetry of $\mathbb{R}^3$, and must be either a rotation-reflection, a rotation, or a reflection, depending on whether it fixes a point, a line, or a plane in $\mathbb{R}^3$. Pinkall [16] classified all possible immersions of surfaces, up to image homotopy, and showed that for the projective plane, there are two distinct classes: Right- and left-handed versions of Boy’s surface, which are mirror images of each other. Similarly, the projective plane with one handle has distinct left- and right-handed versions. Thus the mirror reflection of an immersion of the projective plane with one handle is in a different image.
homotopy class, and in particular, it is not identical to the original. This means no reflective symmetry is possible for that surface, and also rules out rotation-reflections, since these reverse the handedness as well. Thus the only possible symmetries for $M$ are rotations. We will eliminate these in the following sections. In 4.1, we show that no symmetries of order $n \geq 3$ are possible, and in 4.4, we show there are no tight immersions having 2-fold rotational symmetry.

4.1. Rotations of order 3 or more. Suppose $M$ has an $n$-fold rotational symmetry with $n \geq 3$. We begin by looking at the top cycles for the immersion. Cecil and Ryan [5] showed that the number of top cycles in a tight immersion is bounded by a formula dependent on the Euler characteristic of the surface. In particular, for a surface of odd Euler characteristic, if the number of top cycles is $\alpha$, then $2 \leq \alpha \leq 1 - \chi(M)$. For the projective plane with one handle, this becomes $2 \leq \alpha \leq 1 - (-1) = 2$, so $\alpha = 2$. Thus there are exactly two top cycles for $M$. Any symmetry must take a top cycle either to itself or to another top cycle. Since $n \geq 3$ in our case, each top cycle must map to itself, as there are only two of them. Thus each top cycle must itself be $n$-fold symmetric and must be perpendicular to the axis of rotational symmetry, which means that the two planes containing the top cycles are parallel. We can assume the axis of symmetry is the $z$ axis and the planes for the top cycles are at $z = -1$ and $z = 1$.

The intersection of $M$ with its convex envelope is all of the envelope minus the interiors of the two faces bounded by the top cycles. Topologically, this is a cylinder. If we remove this cylinder, we are left with the interior portion of $M$. The Euler characteristic of the remainder is $-1$ since the Euler characteristic of the removed cylinder is 0, and the sum of the values must be $\chi(M) = -1$. To close the surface again, we can add a topological disk to the two top cycles that form the boundary of the interior core. These can be formed by taking the cone over these curves to two new vertices, one above the upper top cycle at $(0,0,2)$ and one below the lower one at $(0,0,-2)$. Call this surface $\overline{M}$. Each disk has Euler characteristic 1, so $\chi(\overline{M}) = -1 + 1 + 1 = 1$. As $\overline{M}$ is now a closed surface, this means it is a real projective plane. Since $M$ was contained between heights $-1$ and 1, and since the top cycles are convex polygons, $\overline{M}$ is an immersion.

Consider the height function on $\overline{M}$ in the direction of the positive $z$ axis. Since the original surface is tight, the interior core has no local extreme vertices, hence the height function has exactly one maximum and one minimum (the two cone points). Since $\chi(\overline{M}) = -1$, and the Euler characteristic equals the sum of the maxima and minima minus the saddles, this means there is one additional critical level between the maximum and minimum.

At this point, we would like to make the following argument: The critical level must contain a single saddle point, in order for the sum of the critical points to add up to the Euler characteristic. The critical level is a slice
through the surface perpendicular to the axis of symmetry, so the level set at this height must also exhibit the \( n \)-fold rotational symmetry. Under this symmetry, a saddle point will have to map to another saddle, and since we have only one, it must map to itself. Thus the saddle point is on the axis of symmetry, and so the saddle itself must be \( n \)-fold symmetric. A simple saddle can exhibit at most a 2-fold symmetry, however, since a small neighborhood of the point has two regions where the surface lies above the critical level, and a nontrivial symmetry would have to map these to each other. Our \( n \) is at least 3, so this is a contradiction, and we should conclude that no \( n \)-fold symmetric tight immersion of \( M \) exists for \( n \geq 3 \).

There is a problem with this argument, however: The claim that there is a critical “point” assumes a level of genericity that we are not guaranteed. It is possible for a saddle to occur along an edge or face contained in the critical level, for example (see Figure 3). Indeed, the critical levels of a polyhedral surface may be quite complex, and it may not be easy to localize the critical behavior to a single area of the level set. Furthermore, we can not simply say “put the vertices in general position” as this may not be possible while still maintaining the symmetry.

![Figure 3](image-url)

**Figure 3.** A saddle “point” can occur along an edge (left) or even at a face (right). If these are symmetric about the axis of 2-fold rotation, we can not force the saddle to a vertex without destroying the symmetry.

A more sophisticated argument is needed in order to overcome these problems. The next section investigates the critical level more carefully, and shows that a level set with the desired properties cannot be obtained.

### 4.2. The critical level in detail

Since there is only one minimum for the height function along the axis of symmetry, this means that the region below this critical level is a topological disk, and similarly for the region above. These disks each have Euler characteristic 1, so the Euler characteristic of the critical level must be \(-1\) in order to have \( \chi(M) = 1 \).

Let \( C \) be the intersection of \( \overline{M} \) with the plane perpendicular to the axis of symmetry at the critical height; i.e., \( C \) is the critical level. By refining the triangulation of \( \overline{M} \), if necessary, we can guarantee that the level set is made
up of vertices, edges and faces of $M$ (taking care to subdivide symmetric simplices in the same way). Since the level set is perpendicular to the axis of symmetry, $C$ must also show $n$-fold rotational symmetry, with the center of rotation at the point $P$ where the axis of rotation intersects the plane. If $P$ lies in the interior of any face or edge of $C$, we can subdivide it by placing a new vertex at $P$ and making edges to the vertices of the original simplex. This divides the simplex symmetrically.

In this way, we can assume that simplices of $C$ meet the axis of rotation only at vertices. In particular, this means no face or edge of $C$ is mapped to itself under the rotational symmetry, and also that each simplex (other than the vertices at $P$) is repeated $n$ times by the rotational symmetry about $P$. One consequence of this is, since the Euler characteristic is the number of vertices minus the number of edges plus the number of faces, the Euler characteristic of $C$ minus the number of vertices at $P$ is a multiple of $n$.

We now modify $C$ so as to remove any faces that lie in the level set, leaving only edges and vertices remaining in $C$. We will do this in such a way that symmetry is maintained, as well as the topology of $C$. That is, we will show:

**Lemma 4.1.** Any polyhedral immersion of the projective plane having $n$-fold rotational symmetry for $n \geq 3$ that is formed by two disks attached along a planar set $C$ of Euler characteristic $-1$ can be modified to produce a new immersion, again formed by two disks meeting at a planar set $\bar{C}$ having the same symmetry, but where $\bar{C}$ contains no faces.

The modification proceeds as follows: Suppose that $C$ contains a face. Then $C$ must contain at least one triangle with an edge that is not attached to another triangle in $C$ (otherwise every edge of every face in $C$ would be matched by another face in $C$, and so the triangles would form a closed surface completely contained in $C$; thus all of the projective plane would be in $C$, contradicting that it is formed by two disks meeting at $C$, one below and one above $C$). Let $T$ be such a triangle in $C$. Then at most two edges of $T$ are shared with other triangles of $C$, and at least one edge is part of the boundary of $C$. There are seven possible configurations for $T$, depending on how many edges meet other triangles of $C$, and on whether the remaining vertices of $T$ meet other edges or faces of $C$, as shown in Figure 4.

Each of these can be modified so as to remove the triangle from $C$. For (a), (b), (c), and (e), there is a vertex $v$ of $T$ that is not connected to other simplices of $C$, thus the rest of the star of $v$ must lie entirely on one side of the plane containing $C$. We can move $v$ slightly to this side of the plane, removing $T$ from $C$ but leaving the opposite edge of $T$ in $C$. For (d), (f) and (g), triangle $T$ has at least one edge that is on the boundary of $C$, so the other face containing this edge is either below or above the plane of $C$; assume that is it below $C$. We can subdivide the edge (together with $T$ and
its neighbor) and move the new vertex slightly downward. This removes $T$ from $C$, leaving the remaining two edges in $C$.

Note that the topology of the surface is not changed by these modifications, and so it is still formed by two disks meeting along (the modified) $C$. In each case the Euler characteristic is unchanged, so $\chi(C)$ continues to be $-1$. Finally, provided that we modify every face that is symmetric to $T$ in the same fashion as we did $T$, the symmetry will be maintained. Note that since we changed $T$ only along edges where it did not meet $C$, and since no face is symmetric to itself by construction, the modifications for $T$ and its symmetric copies do not interfere with each other, so all can be modified simultaneously without trouble.

Performing this process on a triangle and its symmetric ones removes $n$ faces from $C$. If there are no more triangles in $C$, we are done; otherwise we go back and do it all again. Eventually, we will remove all the triangles from $C$, and this proves the lemma.
In a similar fashion, if $C$ contains any edges with a vertex that is not incident to another edge, that vertex can be pulled to one side of the plane of $C$, removing that vertex and edge from $C$. This does not change the Euler characteristic, and if all the symmetric copies of the edge are modified in the same way simultaneously, the symmetry will be maintained. Again, this is possible to do since the symmetric copies will not interfere with each other. Repeatedly performing this operation yields a $\bar{C}$ such that every vertex meets either zero or at least two edges. But since $\bar{C}$ is attached to a disk (both below and above), and the boundary of a disk is a circle, and hence connected, $\bar{C}$ must also be connected, so an isolated vertex is not allowed. Thus every vertex of $\bar{C}$ meets at least two edges of $\bar{C}$.

We can assume, then, that for our surface $\bar{M}$, the critical level $C$ has the properties listed above; namely that it contains no faces, every vertex has at least two edges, it has $n$-fold rotational symmetry, and it is attached to a single topological disk above and another below. Note that our modifications to $\bar{M}$ may have introduced other critical points for the height function; but this is immaterial, as tightness is no longer an issue. The presence of such a set $C$ is all that we need.

We are now in a position to analyze the possible structures for $C$. It turns out that there is only one set $C$ with the proper symmetries and Euler characteristic, but that it can not be attached to two disks. Showing this will prove that the required immersion $\bar{M}$ of the projective plane does not exist. Hence our original tight immersion of the projective plane with one handle having $n$-fold rotational symmetry can not exist. This will complete the proof that there is no tight immersion of the projective plane with one handle having rotational symmetry of order $n \geq 3$.

4.3. The possible critical levels. Suppose we have decomposed $\bar{M}$ into two disks meeting at a critical level $C$, where $C$ has the properties described in the previous section: It contains no faces, every vertex meets at least two edges, it has $n$-fold rotational symmetry for $n \geq 3$ about a center of rotation, and $\chi(C) = -1$. If $C$ has no vertices on the center of rotation, then each edge and vertex of $C$ is repeated $n$ times by the symmetry, so $\chi(C) = kn$ for some $k \in \mathbb{Z}$. But since $n \geq 3$, $kn \neq -1$, a contradiction. Thus some vertices must lie on the center of rotation.

Figure 5. A configuration having 6-fold rotational symmetry that consists of two distinct vertices on the axis of rotation, each with three edges.

Every such vertex must be mapped by the rotation either to itself or some other vertex on the center of rotation. For example, if $n = 6$, Figure 5 gives
one possible arrangement. The orbit of a vertex must be some \( v \) vertices where \( v \mid n \). Each of these \( v \) vertices is attached to the rest of \( C \) by some number \( e \) of edges, where \( e \geq 2 \) (since every vertex meets at least two edges), and so the total number of edges involved is \( ve \). By symmetry, this must be divisible by \( n \), i.e., \( ve = kn \) for some \( k \in \mathbb{Z}, k \geq 1 \). In this way, the vertices on the axis can be broken down into a collection of \( m \) distinct orbits, with \( v_i \) vertices in the \( i \)-th orbit, each having \( e_i \) edges such that \( v_i e_i = k_i n \) for \( i = 1, \ldots, m \).

Suppose \( p \) is a vertex at the center of rotation, and \( pp_1 \) is an edge of \( C \). Then since every vertex meets at least two edges, there is at least one more edge containing \( p_1 \). If there is exactly one, then let \( p_1p_2 \) be this edge. We can continue to move along edges in this way until we come to an edge \( p_{i-1}p_i \) where one of two things happen: Either \( p_i \) has more than one other edge (so there is no obvious choice for continuing our path), or \( p_i \) is on the center of rotation (so we have returned to the center, but not necessarily to the same vertex where we started). Call the first type of path a terminal path and the second a returning path. Each edge attached to a vertex on the center of rotation is part of either a terminal or a returning path.

Let \( C \) be the portion of \( C \) consisting of the vertices on the center of rotation together with the terminal and returning paths (not including the final vertex of the terminal paths); see Figure 6. Since \( C \) has \( n \)-fold rotational symmetry, so does \( \overline{C} \).

![Figure 6](image.png)

**Figure 6.** An example critical level (left) having 3-fold rotational symmetry. Here, there are two vertices at the center, so \( m = 2, v_1 = 1, e_1 = 6, k_1 = 2, v_2 = 1, e_2 = 3, k_2 = 1 \) and \( n = 3 \). The portion that becomes \( \overline{C} \) is shown at the right. There are three terminal paths and three returning paths.

Note that since the terminal paths end at vertices having at least two other edges, each vertex of \( C - \overline{C} \) still has at least two edges, and it still is connected. So if \( V \) is the number of vertices of \( C - \overline{C} \) and \( E \) the number of edges, this means \( E \geq V \), so \( \chi(C - \overline{C}) = V - E \leq 0 \). Since \( C \) and \( \overline{C} \) have \( n \)-fold rotational symmetry, so does \( C - \overline{C} \). Since \( C - \overline{C} \) has no vertices on
the center of rotation by construction, every edge and vertex is repeated \( n \) times, so \( \chi(C - C) = qn \) for some \( q \in \mathbb{Z} \) with \( q \leq 0 \).

Let \( v = \sum v_i \) be the total number of vertices at the center of rotation, and let \( e = \sum v_i e_i = \sum k_i n = n \sum k_i \) be the total number of edges meeting the vertices at the center. Let \( k = \sum k_i \), so \( e = nk \). If we ignore the vertices on the center, each path (either returning or terminal) contains one more edge than vertex, so contributes \(-1\) to \( \chi(C) \). How many such paths are there? Let \( p \) be the number of returning paths; then since each returning path accounts for two edges at the axis, the total number of paths is \( e - p \).

Thus \( \chi(C) = v - (e - p) = v - e + p \).

Note that since each \( e_i \geq 2 \) and \( v_i e_i = nk_i \), this means \( v_i \leq nk_i/2 \), so \( v = \sum v_i \leq (n/2) \sum k_i = nk/2 = e/2 \). Thus \( v \leq e/2 \). Also note that since each returning path uses two edges at the center, \( p \leq e/2 \). So \( v - e + p \leq e/2 - e + e/2 = 0 \), so \( \chi(C) \leq 0 \).

We already know \( \chi(C - C) \leq 0 \) and in fact \( \chi(C - C) = qn \) for some integer \( q \leq 0 \). We also know \( \chi(C) - \chi(C) = \chi(C - C) \) by construction. If \( q \neq 0 \), then \( q \leq -1 \) so \( \chi(C - C) \leq -n \), so \( -1 - \chi(C) \leq -n \), or \( \chi(C) \geq n - 1 \geq 2 \) since \( n \geq 3 \). This is a contradiction since \( \chi(C) \leq 0 \) from above. Thus it must be that \( q = 0 \), so \( \chi(C - C) = 0 \). This means \( \chi(C) = \chi(C) \), so \( \chi(C) = -1 \). That is, \( v - e + p = -1 \). Now we saw above that \( v \leq e/2 \), so \( e/2 - e + p \geq -1 \), or \( p \geq e/2 - 1 \). On the other hand, \( p \leq e/2 \) so \( e/2 - 1 \leq p \leq e/2 \).

If \( e \) is odd, then this means \( p = (e - 1)/2 \) as the only possibility. Now the number of edges that lead to terminal paths is \( e - 2p \), so all but one edge is used in returning paths; but this one edge must be matched by symmetry with \( n - 1 \geq 2 \) other such edges, a contradiction.

So \( e \) must be even, and either \( p = e/2 \) or \( p = e/2 - 1 \). It can’t be the latter, as this would mean that all the edges but two are used in returning paths. Again, by the \( n \)-fold symmetry, if there is one, there must be at least \( n \geq 3 \) of these, a contradiction. So \( p = e/2 \), and every edge is used in a returning path. Since the returning paths do not connect to any edges not in \( C \), and since \( C \) itself is connected, this means \( C = C \); i.e., \( C \) is the entire level set.

Suppose \( e_i = 2 \) for some \( i \). A returning path starting at one of these can’t come back to such a vertex, as this would form a closed loop, which would disconnect \( C \), a contradiction. So we can think of the two returning paths through this point as one long returning path. Removing such a vertex from \( v \) also removes two edges from \( e \) while reducing \( p \) by 1. Thus \( v - e + p \) will remain unchanged if we make this adjustment.

We can assume, without loss of generality, that the \( e_i \) equal to 2 are all listed last; i.e., for some \( \bar{m} \leq m \), \( e_i \geq 3 \) for \( i \leq \bar{m} \), and \( e_i = 2 \) for \( i > \bar{m} \). Let \( \bar{v} = \sum_{i=1}^{\bar{m}} v_i \), \( \bar{k} = \sum_{i=1}^{\bar{m}} k_i \), \( \bar{e} = \sum_{i=1}^{\bar{m}} v_i e_i = \sum_{i=1}^{\bar{m}} nk_i = n \bar{k} \), and let \( \bar{p} = \bar{e}/2 \). Then the observations above indicate that \( \bar{v} - \bar{e} + \bar{p} = v - e + p = -1 \). But since \( e_i \geq 3 \) for \( i \leq \bar{m} \), \( v_i e_i = nk_i \) means \( v_i \leq nk_i/3 \), so \( \bar{v} \leq n \bar{k}/3 = \bar{e}/3 \). So
\[-1 = \sigma - \varepsilon + p < \varepsilon/3 - \varepsilon + \varepsilon/2 = -\varepsilon/6; \text{ i.e., } \varepsilon \leq 6, \text{ and so } n\bar{k} \leq 6. \text{ Since } n \geq 3, \text{ this means } \bar{k} \leq 2. \text{ If } \bar{k} = 1, \text{ then } \bar{\varepsilon} = n\bar{k} = n \text{ means } \bar{p} = \bar{\varepsilon}/2 = n/2; \text{ but by symmetry, each path maps to another path, hence } p \geq n, \text{ a contradiction. } \text{ So } \bar{k} = 2. \text{ This means } 6 \geq n\bar{k} = 2n \geq 2 \cdot 3 = 6, \text{ so } 2n = 6, \text{ and hence } n = 3.

So we know \( n = 3 \), \( \bar{k} = 2 \), \( \bar{\varepsilon} = 6 \) and \( \bar{p} = 3 \). \text{ Then } \bar{\sigma} - \bar{\varepsilon} + \bar{p} = -1 \text{ implies } \bar{\sigma} = 2. \text{ Since } n = 3 \text{ and there are only two vertices at the center of rotation, each vertex must map to itself (as there can’t be an orbit of length three). So } \bar{m} = 2, \text{ and } v_1 = 1, v_2 = 1. \text{ Since } 2 = \bar{k} = k_1 + k_2, \text{ this means } k_i = 1, \text{ so from } v_i e_i = nk_i, \text{ we conclude } e_i = 3. \text{ Thus there are two vertices each with three edges (as in Figure 5) connected by three returning paths. A returning path can not return to the same vertex, since the vertex must exhibit 3-fold symmetry, so the returning paths at each edge of one vertex must lead to the other vertex. Thus there is only one possible configuration for } \bar{C} \text{ (up to topological type), as seen in Figure 7 (left).}

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure7}
\caption{The only possible critical set has 3-fold symmetry, two vertices on the axis of rotation, each with three edges, and three returning paths (left). By “fattening” the edges, we see that this arrangement has only one boundary curve, so it can’t be connected to two disks to form a projective plane; in fact, it is a torus minus a disk.}
\end{figure}

Recall that \( C = \bar{C} \), and \( C \) is a level set that is attached to a disk above \( C \) and another disk below \( C \). \text{ Note, however, that this configuration can not be attached to two disks. One way to see this is to widen the paths slightly and note that the result has only one boundary curve (see Figure 7, right), so can be attached to only one disk.}

Since this was the only possible symmetric level set with Euler characteristic \(-1\), and it can’t attach to the disks above and below, this proves that there is no tight immersion of the projective plane with one handle having \( n \)-fold rotational symmetry for \( n \geq 3 \). \text{ Thus we have eliminated all possibilities except 2-fold rotations.}

4.4. Rotations of order 2. \text{ The previous sections considered the case of an } n \text{-fold rotational symmetry for } n \geq 3 \text{ and showed that no tight immersion}
of the projective plane with one handle can be formed with such a symmetry. This leaves the case for \( n = 2 \) to be considered. We will address this last situation by proving the following, more general, result:

**Lemma 4.2.** There is no immersion (tight or otherwise) of a closed, compact polyhedral surface with odd Euler characteristic that has a rotational symmetry of order 2.

While our proof is for polyhedral surfaces, the technique should carry over to the smooth case as well. Once we have proved this lemma, we will have completed our proof of Theorem 1.1, since the real projective plane with one handle has odd Euler characteristic and we have already eliminated all other possible symmetries.

The proof of this lemma rests on two facts. The first is a result of Banchoff [2] that the number of triple points in an immersion of a surface in space is equivalent, modulo 2, to the Euler characteristic of the surface. This means that a surface of odd Euler characteristic has an odd number of triple points (and in particular, has at least one). The second is the following: In an immersion of a triangulated surface with 2-fold rotational symmetry, no triangle is self-symmetric; that is, the symmetry takes each triangle to a different one.

To see this, suppose some triangle \( abc \) is mapped to itself by the symmetry. Then the rotation must interchange two vertices, say \( a \) and \( b \), and leave the other one, \( c \), fixed. Then \( c \) lies on the axis of symmetry, and so does the midpoint of \( a \) and \( b \), which means that the axis of symmetry passes through the plane of \( abc \). Consider the slices of the surface by planes perpendicular to the axis of symmetry. Note that each slice inherits the 2-fold rotational symmetry, and that the slices that pass through \( abc \) contain self-symmetric edges that intersect the center of rotation for the slice.

As we pass through the various slices, the level curves change continuously except at critical levels. Since these slices must also maintain the 2-fold symmetry, a portion of a level curve that is a self-symmetric section must remain self-symmetric in nearby levels, and hence must continue to pass through the axis of rotation. This can change at a critical level, but only if another portion of the level set comes in contact with the self-symmetric curve. By the 2-fold symmetry, however, any contacts must come in pairs, and so an odd number of level curves must come together at such a point. After passing the critical level, there will still be an odd number of level curves, so one of these still must be self-symmetric, hence it must still pass through the axis of symmetry.

Thus there is no way to eliminate a segment that passes through the axis. Since the surface is compact, high enough level sets must be empty (every level curve will shrink to a point and disappear at a maximum), but since a segment of a level curve passing through the axis can’t be removed, no
level set can contain such a segment. This contradicts the fact that we must have such a segment when there is a self-symmetric triangle, hence there is no such triangle.

One last subtlety must be considered, however: Suppose there were several self-symmetric segments passing through the axis. Could these be used to eliminate each other? For example, could two such segments combine at a critical point to form two curves that do not pass through the axis? The answer is “yes, but not in an immersion”. Such a configuration is shown in Figure 8, but any such arrangement would require that the self-symmetric segments would meet at a vertex on the axis, and the star of this vertex would contain self-intersection, and so this vertex star would not be embedded, hence the surface would not be immersed.

![Figure 8. Two self-symmetric level curves (the two lines crossing at the bottom) combine to form a symmetric pair (the two separated lines at the top); but in doing so, they create a Whitney umbrella, where the surface is not immersed.](image)

A similar argument shows that no self-symmetric edge can lie entirely on the axis of rotation, since then the two faces meeting at that edge would be paired by the symmetry; slices perpendicular to the axis of rotation through this pair would form self-symmetric level curves like the ones in the previous proof, but we have just seen that such curves are not allowed.

With these results in hand, we can now prove Lemma 4.2. Suppose $M$ is an immersion of a polyhedral surface of odd Euler characteristic, and suppose that $M$ has 2-fold rotational symmetry. Note that Banchoff’s result tells us that there are an odd number of triple points. The symmetry must map each triple point to another triple point (either itself or a different one), so any triple point not mapped to itself is paired with a second one by the symmetry. This accounts for an even number of triple points, hence there must be at least one that is self-symmetric. This means it lies on the axis of rotation, and so a neighborhood of this triple point must show 2-fold rotational symmetry.

A triple point is formed by three faces intersecting, and so, to have 2-fold symmetry, the rotation must interchange two of these faces and map the third to itself. We have already shown above that no triangle can be self-symmetric in an immersion, so this is a contradiction. Thus $M$ can not have 2-fold rotational symmetry after all.

At first glance, this argument seems reasonable, but it has a problem as it currently stands. It assumes that the triple points are generic, that is,
that they fall in the interior of faces, and occur where exactly three faces meet. This need not be the case, however; for example, one could have four faces meeting at a common point arranged so as to have 2-fold rotational symmetry (see Figure 9). Such a “quadruple point” actually represents several triple points combined at one location. In order for Banchoff’s theorem to count them properly, this degenerate configuration must be perturbed to break up the self-intersection into generic triple points.

![Figure 9](image)

**Figure 9.** Four faces can meet at a single point (left) with 2-fold (actually 4-fold) rotational symmetry about an axis vertical to the page. Moving the top and bottom vertices toward the viewer and the left and right ones away (right), we maintain the 2-fold symmetry and break up the quadruple point into four triple points; two are visible here.

The original point has six lines of self-intersection meeting at the quadruple point (one for each pair of the four faces). As we adjust two of the faces slightly (and symmetrically), these lines move apart, forming the edges of a tetrahedron whose vertices are the triple points where groups of three faces meet (there are four such groups of three since there are four faces in total). Thus this configuration actually represents *four* triple points, and there still must be a self-symmetric triple point elsewhere in the surface.

It would be tempting to say “put the surface into general position”, but we want our modified surface to retain the original 2-fold symmetry, so this must be handled with some care. In the next section, we show how this can be done, and thus complete the proof.

**4.5. Making the immersion generic.** Our argument in the previous section relies on counting the number of triple points in the immersion, but there are a number of non-generic behaviors that can complicate this process. For example, if three faces meet along a common line (rather than at a point), there are an infinite number of triple points. As we have already seen (Figure 9), several triple points can be combined to form crossings of four or more faces. We need our triple points to occur in the interiors of faces where exactly three faces meet.
In order to overcome these problems, we would like to move the surface into general position, but doing so might disrupt the symmetry. Instead, we need to look at the possible problem configurations and show that they can be removed while maintaining the 2-fold rotational symmetry.

We have already noted that every face is paired by the symmetry with another (different) face of the triangulation. As we adjust one triangle, we need to make corresponding changes in the other in order to preserve the symmetry. A triangle and its pair intersect each other if, and only if, they meet the axis of symmetry. (If a triangle intersects the axis of symmetry, then that point is mapped to itself by the rotation so it also must be on the other triangle. If a triangle and its pair intersect at a point not on the axis, they also intersect at the point symmetric to this, and since the triangles are convex, they intersect all along the line segment between these two points. This line passes through the axis of symmetry, since it is a 2-fold rotation).

If a triangle does not meet the axis of rotation, then it and its pair can be moved slightly (in symmetric ways) to remove any non-generic behavior from both triangles. Since the paired triangles don’t intersect, these changes will not interfere with each other. Thus any unwanted configurations that are away from the axis can be removed while keeping the symmetry intact. So the only troublesome non-generic behavior occurs at the axis of symmetry. Since triangles that intersect the axis also intersect their pairs, these configurations always involve sets of symmetric pairs of faces.

There are two distinct kinds of pairs of faces: Those that share vertices and those that don’t. If a face and its pair have a vertex or edge in common, then they don’t intersect elsewhere, since the vertex stars are embedded in an immersion (Lemma 2.1). If they share a single vertex, that vertex must be on the axis of rotation, and it can be moved along the axis while maintaining the symmetry, thus removing it from any non-generic configuration.

If the faces share an edge, then the vertices of this edge map either to themselves or to each other. The first case is not possible, however, since if they map to themselves, then the edge lies along the axis of symmetry; but we saw in Section 4.4 that no self-symmetric edge lies on the axis. So if the faces share an edge, the vertices of the edge map to each other, and hence that edge must be perpendicular to the axis of rotation. We can move that edge parallel to the axis without disrupting the symmetry, thus removing the faces from the non-generic configuration.

In this way, we can guarantee that any problem area involves only symmetric pairs of intersecting faces that have no vertices in common. Such a pair can be made to intersect either along a segment in their interiors or not at all, as follows: If one of the vertices of the face lies on the axis, so does its counterpart in the symmetric triangle; both can be moved away from the axis while maintaining symmetry. If the faces no longer intersect, we are done. If an edge of one of the triangles intersects the axis, then so does
it symmetric counterpart, and at the same point on the axis. In this case, symmetric vertices of each edge can be moved slightly (and symmetrically) so that the edges no longer intersect each other or the axis. Again, if the faces of the symmetric pair no longer meet, then we are done, otherwise they meet at a point on the axis that is in their interiors.

At this point, if the faces meet in their interiors, they do so either along a line segment in their interiors, or (if they lie in a common plane) in a 2-dimensional region. In the latter case, any vertex and its pair can be moved so as to change the plane of the triangles slightly, causing the planes to intersect in a line segment or not at all. The planes of the paired triangles will change in opposite ways due to the symmetry, so they will not remain parallel after the adjustment, and so can’t intersect in a 2-dimensional region.

Note that the line of intersection of two such faces is perpendicular to the axis of symmetry. To see this, we note that if the line of intersection were contained in the axis, then the two symmetric faces would have to lie in a common plane, and so would have a 2-dimensional overlap that would have been removed above. So some point on the line of intersection is not on the axis; but any point of intersection must have its corresponding point on the paired triangle, and since the symmetry is 2-fold, this point lies directly opposite the original. Thus the line of intersection is the line between these two points, and so is perpendicular to the axis of symmetry.

Finally, note that moving a vertex of one of the faces together with its symmetric pair on the other will cause the line of intersection either to move parallel to that axis or to rotate around it (or both). The movements that cause the intersection to rotate in place are a set of measure zero, so the line of intersection always can be translated along the axis by a slight movement of vertices of the paired faces.

In this way, any non-generic behavior at the axis of symmetry can be broken up while still maintaining the symmetry. Thus the intersections within the surface can be made generic while still keeping the immersion a symmetric one, so we can guarantee that the triple points all occur in the interiors of faces where only three triangles meet. These are the conditions required by Banchoff’s theorem, and so our arguments in Section 4.4 now can be made without difficulty, and we have proven Lemma 4.2.

This completes the proof of the main result, since the lemma shows that there is no tight immersion of the real projective plane with one handle having 2-fold rotation, and Section 4.1 ruled out rotations of higher order.

5. Conclusion

In Section 3 we showed that the surfaces of even Euler characteristic all can be tightly immersed with multiple symmetries. We produced only rotations and rotation-reflections of order 2, however. What higher order rotations
and rotation-reflections are possible? Since top-cycles must be mapped to top-cycles, and these are bounded by the Euler characteristic, the number of handles certainly will play a role in the answer. This is addressed in [9].

For surfaces of odd Euler characteristic, Pinkall's classification of surfaces under image homotopy [16] can be used as in Section 4 to show that only rotational symmetries are possible. Arguments similar to those used in Lemma 4.2 should show that \( n \)-fold rotations are possible only for odd \( n \). Again, however, the number of handles will control which odd rotations are possible. This issue also is considered in [9].

The real projective plane with one handle continues to be an unusual example among tight surfaces. It is the only one that has a polyhedral tight immersion but no smooth one, and now we see that it is the only surface that can be made tight but not symmetrically. Further study of this peculiar object may yield additional features that distinguish it from other surfaces.

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Received July 2, 2001 and revised April 13, 2002.

Department of Mathematics
Union College
Schenectady, NY 12308

E-mail address: dpvc@union.edu
**Z₃ SYMMETRY AND W₃ ALGEBRA IN LATTICE VERTEX OPERATOR ALGEBRAS**

Chongying Dong, Ching Hung Lam, Kenichiro Tanabe, Hiromichi Yamada, and Kazuhiro Yokoyama

The $W_3$ algebra of central charge $6/5$ is realized as a subalgebra of the vertex operator algebra $V_{\sqrt{2}A_2}$ associated with a lattice of type $\sqrt{2}A_2$ by using both coset construction and orbifold theory. It is proved that $W_3$ is rational. Its irreducible modules are classified and constructed explicitly. The characters of those irreducible modules are also computed.

1. Introduction

The vertex operator algebras associated with positive definite even lattices afford a large family of known examples of vertex operator algebras. An isometry of the lattice induces an automorphism of the lattice vertex operator algebra. The subalgebra of fixed points of an automorphism is the so-called orbifold vertex operator algebra. In this paper we deal with the case where the lattice $L = \sqrt{2}A_2$ is $\sqrt{2}$ times an ordinary root lattice of type $A_2$ and the isometry $\tau$ is an element of the Weyl group of order 3. We use this algebra to study the $W_3$ algebra of central charge 6/5. In fact, by using both coset construction and orbifold theory we construct the $W_3$ algebra of central charge 6/5 inside $V_L$ and classify its irreducible modules. We also prove that the $W_3$ algebra is rational and compute the characters of the irreducible modules.

The vertex operator algebra $V_L$ associated with $L = \sqrt{2}A_2$ contains three mutually orthogonal conformal vectors $\omega^1$, $\omega^2$, $\omega^3$ with central charge $c = 1/2$, 7/10, or 4/5 respectively [10]. The subalgebra $\text{Vir}(\omega^i)$ generated by $\omega^i$ is the Virasoro vertex operator algebra $L(c, 0)$, which is the irreducible unitary highest weight module for the Virasoro algebra with central charge $c$ and highest weight 0. The structure of $V_L$ as a module for $\text{Vir}(\omega^1) \otimes \text{Vir}(\omega^2) \otimes \text{Vir}(\omega^3)$ was discussed in [23]. Among other things it was shown that $V_L$ contains a subalgebra of the form $L(4/5, 0) \oplus L(4/5, 3)$. Such a vertex operator algebra is called a 3-state Potts model. This subalgebra is contained in the subalgebra $(V_L)^\tau$ of fixed points of $\tau$. There is another
subalgebra $M$ in $V_L$, which is of the form

$$L\left(\frac{1}{2},0\right) \otimes L\left(\frac{7}{10},0\right) \oplus L\left(\frac{1}{2},\frac{1}{2}\right) \otimes L\left(\frac{7}{10},\frac{3}{2}\right)$$

and is invariant under $\tau$. The representation theory of $M$ was studied in [21] and [24].

We are interested in the subalgebra $M^\tau$ of fixed points of $\tau$ in $M$. Its Virasoro element is $\omega = \omega^1 + \omega^2$. The central charge of $\omega$ is $1/2 + 7/10 = 6/5$. We find an element $J$ of weight 3 in $M^\tau$ such that the component operators $L(n) = \omega_{n+1}$ and $J(n) = J_{n+2}$ satisfy the same commutation relations as in $[3, (2.1), (2.2)]$ for $W_3$. Thus the vertex operator subalgebra $\mathcal{W}$ generated by $\omega$ and $J$ is a $W_3$ algebra with central charge $6/5$.

We construct 20 irreducible $M^\tau$-modules. 8 of them are inside irreducible untwisted $M$-modules, while 6 of them are inside irreducible $\tau$-twisted $M$-modules and the remaining 6 are inside irreducible $\tau^2$-twisted $M$-modules.

There are exactly two inequivalent irreducible $\tau^i$-twisted $M$-modules $MT(\tau^i)$ and $WT(\tau^i)$, $i = 1, 2$. We investigate the irreducible $\tau^i$-twisted $V_L$-modules constructed in [7] and obtain $MT(\tau^i)$ and $WT(\tau^i)$ inside them.

We classify the irreducible modules for $\mathcal{W}$ by determining the Zhu algebra $A(\mathcal{W})$ (cf. [36]). The method used here is similar to that in [35], where the Zhu algebra of a $W_3$ algebra with central charge $-2$ is studied. We can define a map of the polynomial algebra $\mathbb{C}[x, y]$ with two variables $x, y$ to $A(\mathcal{W})$ by $x \mapsto [\omega]$ and $y \mapsto [J]$, which is a surjective algebra homomorphism. Thus it is sufficient to determine its kernel $\mathcal{I}$. The key point is the existence of a singular vector $v$ for the $W_3$ algebra $\mathcal{W}$ of weight 12. A positive definite invariant Hermitian form on $V_L$ implies that $v$ is in fact 0. Thus $[v] = 0$. Moreover, $[J(-1)v] = [J(-2)v] = [J(-1)^2v] = 0$. Hence the corresponding polynomials in $\mathbb{C}[x, y]$ must be contained in the ideal $\mathcal{I}$. It turns out that $\mathcal{I}$ is generated by those four polynomials and the classification of irreducible $\mathcal{W}$-modules is established by Zhu’s theory ([36]). That is, there are exactly 20 inequivalent irreducible $\mathcal{W}$-modules. The calculation of explicit form of the singular vector $v$ and the calculation of the ideal $\mathcal{I}$ were done by a computer algebra system Risa/Asir.

By the classification of irreducible $\mathcal{W}$-modules and a positive definite invariant Hermitian form, we can show that $M^\tau = \mathcal{W}$. The eigenvalues of the action of weight preserving operators $L(0) = \omega_1$ and $J(0) = J_2$ on the top levels of those 20 irreducible $M^\tau$-modules coincide with the values $\Delta\left(\frac{m}{n}, \frac{m}{n'}\right)$ and $w\left(\frac{m}{n}, \frac{m}{n'}\right)$ of [14, (1.2), (5.6)] with $p = 5$. Hence our $M^\tau$ is an algebra denoted by $[Z_3^{(5)}]$ in [14].

We prove that $\mathcal{W}$ is $C_2$-cofinite and rational by using the singular vector $v$ of weight 12 and the irreducible modules for $\mathcal{W}$. In the course of the proof we use a result about a general vertex operator algebra $V$. It says that if $V$
is $C_2$-cofinite, then $V$ is rational if and only if $A(V)$ is semisimple and any simple $A(V)$-module generates an irreducible $V$-module. This result will certainly be useful in the future study of relationship between rationality and $C_2$-cofiniteness.

We also study the characters of those irreducible $M^\tau$-modules. Using the modular invariance of trace functions in orbifold theory (cf. [9]), we describe the characters of the 20 irreducible $M^\tau$-modules in terms of the characters of irreducible unitary highest weight modules for the Virasoro algebras.

The results in this paper have applications to the Monster simple group. Recently, it was shown in [22] that the $Z_3$ symmetry of a 3-state Potts model in $(V_L)^\tau$ affords $3A$ elements of the Monster simple group. Such a result has been suggested by [28]. It is expected that the $Z_3$ symmetry of $M^\tau$ affords $3B$ elements.

The organization of the paper is as follows: In Section 2 we review some properties of $M$ for later use. In Section 3 we define the vector $J$ and compute the commutation relations among the component operators $L(n) = \omega_{n+1}$ and $J(n) = J_{n+2}$. In Section 4 we construct 20 irreducible $M^\tau$-modules and discuss their properties. In Section 5 we determine the Zhu algebra of the vertex operator subalgebra $W$ generated by $\omega$ and $J$ and show that $M^\tau = W$. Thus we conclude that $M^\tau$ has exactly 20 inequivalent irreducible $M^\tau$-modules. Finally, in Section 6 we study the characters of those irreducible $M^\tau$-modules.

2. Subalgebra $M$ of $V_{\sqrt{2}A_2}$

In this section we fix notation. For basic definitions concerning lattice vertex operator algebras we refer to [7] and [17]. We also recall certain properties of the vertex operator algebra $V_{\sqrt{2}A_2}$ (cf. [23]).

Let $\alpha_1, \alpha_2$ be the simple roots of type $A_2$ and set $\alpha_0 = -(\alpha_1 + \alpha_2)$. Then $\langle \alpha_i, \alpha_i \rangle = 2$ and $\langle \alpha_i, \alpha_j \rangle = -1$ if $i \neq j$. Set $\beta_i = \sqrt{2}\alpha_i$ and let $L = \mathbb{Z}\beta_1 + \mathbb{Z}\beta_2$ be the lattice spanned by $\beta_1$ and $\beta_2$. We usually denote $L$ by $\sqrt{2}A_2$.

We follow Sections 2 and 3 of [7] with $L = \sqrt{2}A_2$, $p = 3$, and $q = 6$. In our case $\langle \alpha, \beta \rangle \in 2\mathbb{Z}$ for all $\alpha, \beta \in L$, so that the alternating $\mathbb{Z}$-bilinear map $c_0 : L \times L \to \mathbb{Z}/6\mathbb{Z}$ defined by [7, (2.9)] is trivial. Thus the central extension

\begin{equation}
1 \longrightarrow \langle \kappa_6 \rangle \longrightarrow \hat{L} \longrightarrow L \longrightarrow 1
\end{equation}

determined by the commutator condition $aba^{-1}b^{-1} = \kappa_6^c(\bar{a}, \bar{b})$ splits. Then for each $\alpha \in L$, we can choose an element $e^\alpha$ of $\hat{L}$ so that $e^\alpha e^\beta = e^{\alpha + \beta}$. The twisted group algebra $\mathbb{C}\{L\}$ is isomorphic to the ordinary group algebra $\mathbb{C}[L]$.
We adopt the same notation as in [21] to denote cosets of \( L \) in the dual lattice \( L^\perp = \{ \alpha \in \mathbb{Q} \otimes \mathbb{Z} \mid \langle \alpha, L \rangle \subset \mathbb{Z} \} \), namely,

\[
\begin{align*}
L^0 &= L, \\
L^1 &= -\frac{\beta_1 + \beta_2}{3} + L, \\
L^2 &= \frac{\beta_1 - \beta_2}{3} + L,
\end{align*}
\]

\[
L_0 = L, \quad L_a = \frac{\beta_2}{2} + L, \quad L_b = \frac{-\beta_0}{2} + L, \quad L_c = \frac{\beta_1}{2} + L,
\]

and

\[L^{(i,j)} = L_i + L^j\]

for \( i = 0, a, b, c \) and \( j = 0, 1, 2 \), where \( \{0, a, b, c\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \). Then, \( L^{(i,j)}, i \in \{0, a, b, c\}, j \in \{0, 1, 2\} \) are all the cosets of \( L \) in \( L^\perp \) and \( L^\perp / L \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \).

Our notation for the vertex operator algebra \((V_L, Y(\cdot, z))\) associated with \( L \) is standard [17]. In particular, \( \mathfrak{h} = \mathbb{C} \otimes \mathbb{Z} L \) is an abelian Lie algebra, \( \mathfrak{h} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \) is the corresponding affine Lie algebra, \( M(1) = \mathbb{C}[\alpha(n); \alpha \in \mathfrak{h}, n < 0] \), where \( \alpha(n) = \alpha \otimes t^n \), is the unique irreducible \( \mathfrak{h} \)-module such that \( \alpha(n)1 = 0 \) for all \( \alpha \in \mathfrak{h} \) and \( n > 0 \), and \( c = 1 \). As a vector space \( V_L = M(1) \otimes \mathbb{C}[L] \) and for each \( v \in V_L \), a vertex operator \( Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1} \in \text{End}(V_L)([z, z^{-1}]) \) is defined. The coefficient \( v_n \) of \( z^{-n-1} \) is called a component operator. The vector \( 1 = 1 \otimes 1 \) is called the vacuum vector.

By Dong [5], there are exactly 12 isomorphism classes of irreducible \( V_L \)-modules, which are represented by \( V_{L(\iota,j)} \), \( i = 0, a, b, c \) and \( j = 0, 1, 2 \). We use the symbol \( e^\alpha, \alpha \in L^\perp \) to denote a basis of \( \mathbb{C}\{L^\perp\} \).

To describe certain weight 2 elements in \( V_L \), we introduce the following notation:

\[
\begin{align*}
x(\alpha) &= e^{\sqrt{2}\alpha} + e^{-\sqrt{2}\alpha}, \\
y(\alpha) &= e^{\sqrt{2}\alpha} - e^{-\sqrt{2}\alpha}, \\
w(\alpha) &= \frac{1}{2}\alpha(-1)^2 - x(\alpha)
\end{align*}
\]

for \( \alpha \in \{\pm \alpha_0, \pm \alpha_1, \pm \alpha_2\} \). We have

\[
w(\alpha_i)w(\alpha_j) = \begin{cases} 
8w(\alpha_i) & \text{if } i = j \\
w(\alpha_i) + w(\alpha_j) - w(\alpha_k) & \text{if } i \neq j,
\end{cases}
\]

where \( k \) is such that \( \{i, j, k\} = \{0, 1, 2\} \). Moreover, \( w(\alpha_i)w(\alpha_j) = 0 \) and

\[
w(\alpha_i)w(\alpha_j) = \begin{cases} 
41 & \text{if } i = j \\
\frac{1}{2}1 & \text{if } i \neq j.
\end{cases}
\]
Let

\[
\omega = \frac{1}{5} (w(\alpha_1) + w(\alpha_2) + w(\alpha_0)),
\]

\[
\tilde{\omega} = \frac{1}{6} (\alpha_1(-1)^2 + \alpha_2(-1)^2 + \alpha_0(-1)^2),
\]

\[
\omega^1 = \frac{1}{4} w(\alpha_1), \quad \omega^2 = \omega - \omega^1, \quad \omega^3 = \tilde{\omega} - \omega.
\]

Then \(\tilde{\omega}\) is the Virasoro element of \(V_L\) and \(\omega^1, \omega^2, \omega^3\) are mutually orthogonal conformal vectors of central charge \(1/2, 7/10, 4/5\) respectively (cf. [10]). The subalgebra \(\text{Vir}(\tilde{\omega})\) generated by \(\tilde{\omega}\) is isomorphic to the Virasoro vertex operator algebra of given central charge, and \(\omega^1, \omega^2, \omega^3\) generate

\[
\text{Vir}(\omega^1) \otimes \text{Vir}(\omega^2) \otimes \text{Vir}(\omega^3) \cong L \left( \frac{1}{2}, 0 \right) \otimes L \left( \frac{7}{10}, 0 \right) \otimes L \left( \frac{4}{5}, 0 \right).
\]

We study certain subalgebras, and also submodules for them in \(V_{L_i}, i = 0, a, b, c\) and \(V_{L_j}, j = 0, 1, 2\). Set

\[
M^i_k = \{ v \in V_{L_i} | (\omega^3)_1 v = 0 \},
\]

\[
W^i_k = \left\{ v \in V_{L_i} | (\omega^3)_1 v = \frac{2}{5} v \right\}, \quad \text{for } i = 0, a, b, c,
\]

and

\[
M^j_l = \{ v \in V_{L_j} | (\omega^1)_1 v = (\omega^2)_1 v = 0 \},
\]

\[
W^j_l = \left\{ v \in V_{L_j} | (\omega^1)_1 v = 0, \quad (\omega^2)_1 v = \frac{3}{5} v \right\}, \quad \text{for } j = 0, 1, 2.
\]

Then \(M^0_k\) and \(M^0_i\) are simple vertex operator algebras. Furthermore, \(\{M^i_k, W^i_k, i = 0, a, b, c\}\) and \(\{M^j_l, W^j_l, j = 0, 1, 2\}\) are the sets of all inequivalent irreducible modules for \(M^0_k\) and \(M^0_i\), respectively ([21], [23] and [24]). We also have

\[
M^0_k \cong L \left( \frac{1}{2}, 0 \right) \otimes L \left( \frac{7}{10}, 0 \right) \oplus L \left( \frac{1}{2}, \frac{1}{2} \right) \otimes L \left( \frac{7}{10}, \frac{3}{2} \right),
\]

\[
W^0_k \cong L \left( \frac{1}{2}, 0 \right) \otimes L \left( \frac{7}{10}, \frac{3}{5} \right) \oplus L \left( \frac{1}{2}, \frac{1}{2} \right) \otimes L \left( \frac{7}{10}, \frac{1}{10} \right),
\]

\[
M^a_k \cong M^b_k \cong L \left( \frac{1}{2}, \frac{1}{16} \right) \otimes L \left( \frac{7}{10}, \frac{7}{16} \right),
\]

\[
W^a_k \cong W^b_k \cong L \left( \frac{1}{2}, \frac{1}{16} \right) \otimes L \left( \frac{7}{10}, \frac{3}{80} \right),
\]

\[
M^c_k \cong L \left( \frac{1}{2}, \frac{1}{2} \right) \otimes L \left( \frac{7}{10}, 0 \right) \oplus L \left( \frac{1}{2}, 0 \right) \otimes L \left( \frac{7}{10}, \frac{3}{2} \right).
\]
\[ W_k^c \cong L \left( \frac{1}{2}, \frac{1}{2} \right) \otimes L \left( \frac{7}{10}, \frac{3}{5} \right) \oplus L \left( \frac{1}{2}, 0 \right) \otimes L \left( \frac{7}{10}, \frac{1}{10} \right) \]

as \( L(1/2, 0) \otimes L(7/10, 0) \)-modules and

\[ M^0_t \cong L \left( \frac{4}{5}, 0 \right) \oplus L \left( \frac{4}{5}, 3 \right), \quad W^0_t \cong L \left( \frac{4}{5}, \frac{2}{5} \right) \oplus L \left( \frac{4}{5}, \frac{7}{5} \right), \]

\[ M^1_t \cong M^2_t \cong L \left( \frac{4}{5}, \frac{2}{3} \right), \quad W^1_t \cong W^2_t \cong L \left( \frac{4}{5}, \frac{1}{15} \right) \]

as \( L(4/5, 0) \)-modules.

Note also that

\[ V \cong \frac{M^0_t}{M^0_k} \cong \frac{M^1_t}{M^0_k} \cong \frac{M^2_t}{M^0_k} \]

as an \( M^0_k \otimes M^0_k \)-module. We consider the following three isometries of \((L, \langle \cdot, \cdot \rangle)\):

\begin{align*}
\tau & : \beta_1 \to \beta_2 \to \beta_0 \to \beta_1, \\
\sigma & : \beta_1 \to \beta_2, \quad \beta_2 \to \beta_1, \\
\theta & : \beta_i \to -\beta_i, \quad i = 1, 2.
\end{align*}

Note that \( \tau \) is fixed-point-free and of order 3. Note also that \( \sigma \tau \sigma = \tau^{-1}. \) The isometries \( \tau, \sigma, \) and \( \theta \) of \( L \) can be extended to isometries of \( L^\perp. \) Then they induce permutations on \( L^\perp/L. \) Since \( \hat{L} \) is a split extension, the isometry \( \tau \) of \( L \) lifts naturally to an automorphism of \( \hat{L}. \) Then it induces an automorphism of \( V_L: \)

\[ \alpha^1(-n_1) \cdots \alpha^k(-n_k)e^\beta \mapsto (\tau \alpha^1)(-n_1) \cdots (\tau \alpha^k)(-n_k)e^\tau \beta. \]

By abuse of notation, we denote it by \( \tau \) also. Moreover, we can consider the action of \( \tau \) on \( V_{L^\perp/L} \) in a similar way. We apply the same argument to \( \sigma \) and \( \theta. \)

Set \( M = M^0_k. \) The vertex operator algebra \( M \) plays an important role in this paper. Recall that

\[ M \cong L \left( \frac{1}{2}, 0 \right) \otimes L \left( \frac{7}{10}, 0 \right) \oplus L \left( \frac{1}{2}, \frac{1}{2} \right) \otimes L \left( \frac{7}{10}, \frac{3}{2} \right) \]

as \( \text{Vir}(\omega^1) \otimes \text{Vir}(\omega^2) \)-modules. Note that \( \omega \) is the Virasoro element of \( M \) whose central charge is 6/5. For \( u \in M, \) we have \( \omega_1 u = hu \) for some \( h \in \mathbb{Z} \) if and only if \( \bar{\omega}_1 u = hu. \) In such a case \( h \) is called the weight of \( u. \) Note also that \( M \) is generated by \( w(\alpha_1), w(\alpha_2), \) and \( w(\alpha_0). \) In particular, \( M \) is invariant under \( \tau, \sigma, \) and \( \theta. \) In fact, \( \theta \) acts on \( M \) as the identity.

We next show that the automorphism group \( \text{Aut}(M) \) of \( M \) is generated by \( \sigma \) and \( \tau. \)
Theorem 2.1.

(1) There are exactly three conformal vectors of central charge 1/2 in \( M \), which are \( \frac{1}{4} w(\alpha_i) \), \( i = 0, 1, 2 \).

(2) \( \text{Aut}(M) = \langle \sigma, \tau \rangle \) is isomorphic to a symmetric group of degree 3.

Proof. We first consider conformal vectors in \( M \). By [27, Lemma 5.1], a weight 2 vector \( v \) is a conformal vector of central charge 1/2 if and only if \( v_1 v = 2v \) and \( v_3 v = \frac{1}{4} \mathbf{1} \). Since \( \{w(0), w(\alpha_1), w(\alpha_2)\} \) is a basis of the weight 2 subspace of \( M \), we may write \( v = \sum_{i=0}^2 a_i w(\alpha_i) \) for some \( a_i \in \mathbb{C} \).

From (2.2) and (2.3) we see that \( v_1 v = 2v \) and \( v_3 v = \frac{1}{4} \mathbf{1} \) hold only if \( (a_0, a_1, a_2) = (1/4, 0, 0), (0, 1/4, 0), \) or \( (0, 0, 1/4) \). This proves (1). Then any automorphism of \( M \) induces a permutation on \( \{w(0), w(\alpha_1), w(\alpha_2)\} \). If an automorphism induces the identity permutation on the set, it must be the identity since \( M \) is generated by \( w(\alpha_1), w(\alpha_2) \), and \( w(0) \). Now

\[
\tau : w(\alpha_1) \rightarrow w(\alpha_2) \rightarrow w(\alpha_0) \rightarrow w(\alpha_1),
\]

and

\[
\sigma : w(\alpha_1) \rightarrow w(\alpha_2), \quad w(\alpha_2) \rightarrow w(\alpha_1), \quad w(\alpha_0) \rightarrow w(\alpha_0).
\]

Hence (2) holds. \( \square \)

Let \( v_h = w(\alpha_2) - w(\alpha_0) \). This vector is a highest weight vector of highest weight \((1/2, 3/2)\) for \( \text{Vir}(\omega^1) \otimes \text{Vir}(\omega^2) \), that is, \((\omega^1)_1 v_h = (1/2) v_h, (\omega^2)_1 v_h = (3/2) v_h, \) and \((\omega^1)_n v_h = (\omega^2)_n v_h = 0 \) for \( n \geq 2 \). Thus the \( \text{Vir}(\omega^1) \otimes \text{Vir}(\omega^2) \)-submodule in \( M \) generated by \( v_h \) is isomorphic to \( L(1/2, 1/2) \otimes L(7/10, 3/2) \). In particular, \( M \) is generated by \( \omega^1, \omega^2 \), and \( v_h \).

We can choose another generator of \( M \). Let

\[
(2.4) \quad u^1 = w(\alpha_1) + \xi^2 w(\alpha_2) + \xi w(\alpha_0),
\]

\[
 u^2 = w(\alpha_1) + \xi w(\alpha_2) + \xi^2 w(\alpha_0),
\]

where \( \xi = \exp(2\pi \sqrt{-1}/3) \) is a primitive cubic root of unity. Then \( \tau u^1 = \xi u^1, \tau u^2 = \xi^2 u^2, \) and \( \sigma u^1 = \xi^2 u^2 \). We also have \((u^1)_1 u^1 = 4u^2 \) and \((u^1)_1 u^1)_1 u^1 = 140 \omega \). Thus \( u^1, (u^1)_1 u^1, \) and \((u^1)_1 u^1)_1 u^1 \) span the weight 2 subspace of \( M \). This implies that \( M \) is generated by a single vector \( u^1 \). A similar assertion holds for \( u^2 \).

The subalgebra \( M^0_3 \cong L(4/5, 0) \oplus L(4/5, 3) \) is called a 3-state Potts model. It plays an important role in Subsection 4.2. The irreducible \( M^0_3 \)-modules and their fusion rules are determined in [23] and [28]. The Virasoro element
of $M^0_\ell$ is $\omega^3$. Let
\[
(2.5) \quad v_t = \frac{1}{9}(\alpha_1 - \alpha_2)(-1)(\alpha_2 - \alpha_0)(-1)(\alpha_0 - \alpha_1)(-1) \\
- \frac{1}{2}(\alpha_1 - \alpha_2)(-1)x(\alpha_0) - \frac{1}{2}(\alpha_2 - \alpha_0)(-1)x(\alpha_1) \\
- \frac{1}{2}(\alpha_0 - \alpha_1)(-1)x(\alpha_2),
\]
which is denoted by $q$ in [23]. The vector $v_t$ is a highest weight vector in $M^0_\ell$ of highest weight 3 for $\text{Vir}(\omega^3)$. Clearly, $\tau v_t = v_t$ and thus $\tau$ fixes every element in $M^0_\ell$. Moreover, $\sigma v_t = -v_t$ and $\theta v_t = -v_t$. Hence $\sigma$ and $\theta$ induce the same automorphism of $M^0_\ell$, namely, 1 on $\text{Vir}(\omega^3) \cong L(4/5,0)$ and $-1$ on the $\text{Vir}(\omega^3)$-submodule generated by $v_t$, which is isomorphic to $L(4/5,3)$. The automorphism group $\text{Aut}(M^0_\ell)$ is of order 2 generated by $\theta$.

3. Subalgebra $\mathcal{W}$ generated by $\omega$ and $J$ in $M^\tau$

For any $\tau$-invariant space $U$, set $U(\epsilon) = \{ u \in U \mid \tau u = \xi^\epsilon u \}$, $\epsilon = 0, 1, 2$, where $\xi = \exp(2\pi \sqrt{-1}/3)$. We usually denote the subspace $U(0)$ of fixed points by $U^0$ also.

We are interested in the subalgebra $M^\tau$. The weight 2 subspace of $M^\tau$ is spanned by $\omega$. In fact, $\omega$ is the Virasoro element of $M$ with central charge $6/5$. This means that the subalgebra $\text{Vir}(\omega)$ generated by $\omega$ is isomorphic to $L(6/5,0)$. Note that $M$ and $M^\tau$ are completely reducible as modules for $\text{Vir}(\omega)$, since $V_L$ possesses a positive definite invariant Hermitian form (see Subsection 5.3). Every irreducible direct summand in $M$ or $M^\tau$ is isomorphic to $L(6/5,h)$ for some nonnegative integer $h$. Note also that $\sigma$ leaves $M^\tau = M(0)$ invariant and interchanges $M(1)$ and $M(2)$. Since $\sigma$ fixes $\omega$, $\sigma$ acts on $\text{Vir}(\omega)$ as the identity. Thus $M(1)$ and $M(2)$ are equivalent $\text{Vir}(\omega)$-modules.

We now count dimensions of homogeneous subspaces of $M$ of small weights. The characters of $L(1/2,h)$, $L(7/10,h)$, and $L(6/5,h)$ are well-known (cf. [19] and [32]). Using them, we have the first several terms of the character of $M$:
\[
\text{ch } M = \text{ch } L\left(\frac{1}{2},0\right) \text{ch } L\left(\frac{7}{10},0\right) + \text{ch } L\left(\frac{1}{2},\frac{1}{2}\right) \text{ch } L\left(\frac{7}{10},\frac{3}{2}\right) \\
= 1 + 3q^2 + 4q^3 + 9q^4 + 12q^5 + 22q^6 + \cdots.
\]
Comparing $\text{ch } M$ with the character of $L(6/5,h)$, we see that
\[
M \cong L\left(\frac{6}{5},0\right) + 2L\left(\frac{6}{5},2\right) + L\left(\frac{6}{5},3\right) + 2L\left(\frac{6}{5},4\right) + L\left(\frac{6}{5},6\right) + \cdots
\]
as $\text{Vir}(\omega)$-modules.
The vectors $u^1$ and $u^2$ of (2.4) are highest weight vectors for $\text{Vir}(\omega)$ of weight 2. Hence the $\text{Vir}(\omega)$-submodule generated by $u^\epsilon$ in $M(\epsilon)$ is isomorphic to $L(6/5, 2)$, $\epsilon = 1, 2$.

Next, we study the weight 3 subspace. The weight 3 subspace of $M$ is of dimension 4 and so there are nontrivial relations among $w(\alpha_i)w(\alpha_j)$, $i, j \in \{0, 1, 2\}$. For example,

$$w(\alpha_1)w(\alpha_2) - w(\alpha_2)w(\alpha_1) = w(\alpha_0)w(\alpha_1) - w(\alpha_1)w(\alpha_0).$$

Set $J = w(\alpha_1)w(\alpha_2) - w(\alpha_2)w(\alpha_1)$. In terms of the lattice vertex operator algebra $V_L$, $J$ can be written as

$$J = \frac{1}{3} \left( \alpha_1(-2)(\alpha_0(-1) - \alpha_2(-1)) + \alpha_2(-2)(\alpha_1(-1) - \alpha_0(-1)) + \alpha_0(-2)(\alpha_2(-1) - \alpha_1(-1)) \right) + \sqrt{2} \left( (\alpha_0(-1) - \alpha_2(-1))y(\alpha_1) + (\alpha_1(-1) - \alpha_0(-1))y(\alpha_2) + (\alpha_2(-1) - \alpha_1(-1))y(\alpha_0) \right).$$

Note that $(u^1)^2 - (u^2)^2 = 3 \sqrt{-3} J$. Note also that $\tau J = J$, $\sigma J = -J$ and $\theta J = J$. The weight 3 subspace of $M^\tau$ is of dimension 2 and it is spanned by $\omega_0 \omega$ and $J$. Furthermore, we have $\omega_1 J = 3J$ and $\omega_n J = 0$ for $n \geq 2$. Hence:

**Lemma 3.1.** $J$ is a highest weight vector for $\text{Vir}(\omega)$ of highest weight 3 in $M^\tau$.

The weight 4 subspace of $M$ is of dimension 9. By a direct calculation, we can verify that $w(\alpha_i)w(\alpha_j)$, $0 \leq i, j \leq 2$ are linearly independent. Hence $w(\alpha_i)w(\alpha_j)$’s form a basis of the weight 4 subspace of $M$. From this it follows that the weight 4 subspace of $M^\tau$ is of dimension 3. Since the weight 4 subspace of $\text{Vir}(\omega) \cong L(6/5, 0)$ is of dimension 2 and since the weight 4 subspace of the $\text{Vir}(\omega)$-submodule generated by $J$, which is isomorphic to $L(6/5, 3)$, is of dimension 1, we conclude that there is no highest weight vector for $\text{Vir}(\omega)$ in the weight 4 subspace of $M^\tau$. We have shown that:

**Lemma 3.2.**

1. $\{w(\alpha_i)w(\alpha_j) | 0 \leq i, j \leq 2\}$ is a basis of the weight 4 subspace of $M$.
2. There is no highest weight vector for $\text{Vir}(\omega)$ of weight 4 in $M^\tau$.

By the above argument, we know all the irreducible direct summands $L(6/5, h)$ with $h \leq 6$ in the decomposition of $M(\epsilon)$ into a direct sum of...
irreducible Vir(\(\omega\))-modules. Namely,

\[
M^\tau = M(0) \cong L\left(\frac{6}{5},0\right) + L\left(\frac{6}{5},3\right) + L\left(\frac{6}{5},6\right) + \cdots,
\]

\[
M(1) \cong M(2) \cong L\left(\frac{6}{5},2\right) + L\left(\frac{6}{5},4\right) + \cdots.
\]

We now consider the vertex operator algebra \(W\) generated by \(\omega\) and \(J\) in \(M^\tau\). Of course \(W\) is a subalgebra of \(M^\tau\). We shall show that \(W\) is, in fact, equal to \(M^\tau\). The basic data are the commutation relations of the component operators \(\omega_m\) and \(J_n\). For the determination of the commutation relation \([J_m, J_n]\), it is sufficient to express \(J_n J_m\), \(0 \leq n \leq 5\), by using \(\omega\). First of all we note that the weight \(\text{wt} J_n J_m = 5 - n\) is at most 5 for \(0 \leq n \leq 5\). Thus \(J_n J_m\) is contained in \(L(6/5, 0) + L(6/5, 3)\), where \(L(6/5, 0)\) and \(L(6/5, 3)\) stand for Vir(\(\omega\)) and the Vir(\(\omega\))-submodule generated by \(J\) respectively. Since \(\sigma\) fixes every element in Vir(\(\omega\)) and \(\sigma J = -J\), \(\sigma\) acts as \(-1\) on the Vir(\(\omega\))-submodule generated by \(J\). Hence \(J_n J_m\) is in fact contained in Vir(\(\omega\)).

By a direct calculation, we have

\[
J_5 J = -841, \\
J_4 J = 0, \\
J_3 J = -420 \omega, \\
J_2 J = -210 \omega \omega, \\
J_1 J = 9 \omega \omega \omega - 240 \omega \omega_1, \\
J_0 J = 22 \omega \omega \omega \omega - 120 \omega \omega_1 \omega_1.
\]

(3.1)

Note that \(\{\omega \omega, J\}, \{\omega \omega \omega, \omega_1, \omega \omega J\}, \) and \(\{\omega \omega \omega \omega, \omega \omega \omega \omega, \omega \omega \omega J, \omega_1 J\}\) are bases of weight 3, 4, and 5 subspaces of \(M^\tau\) respectively.

In terms of the lattice vertex operator algebra \(V_L\), the vectors \(J_2 J, J_1 J,\) and \(J_0 J\) can be written as follows:

\[
J_2 J = -42 \sum_{i=0}^{2} \alpha_i(-2)\alpha_i(-1) + 42 \sqrt{2} \sum_{i=0}^{2} \alpha_i(-1)y(\alpha_i),
\]

\[
J_1 J = -38 \sum_{i=0}^{2} \alpha_i(-3)\alpha_i(-1) - 3 \sum_{i=0}^{2} \alpha_i(-2)^2 - 8 \sum_{i=0}^{2} \alpha_i(-1)^4
\]

\[
+ 6 \sum_{i=0}^{2} \alpha_i(-1)^2 x(\alpha_i) + 51 \sqrt{2} \sum_{i=0}^{2} \alpha_i(-2)y(\alpha_i),
\]
\[ J_0J = -36 \sum_{i=0}^{2} \alpha_i(-4)\alpha_i(-1) - 4 \sum_{i=0}^{2} \alpha_i(-3)\alpha_i(-2) - 16 \sum_{i=0}^{2} \alpha_i(-2)\alpha_i(-1)^3 \\
+ 36 \sum_{i=0}^{2} \alpha_i(-2)\alpha_i(-1)x(\alpha_i) + \sum_{i=0}^{2} \left( 44\sqrt{2}\alpha_i(-3) - 4\sqrt{2}\alpha_i(-1)^3 \right) y(\alpha_i). \]

We need some formulas for vertex operator algebras (cf. [17]), namely,

\[ [u_m, v_n] = \sum_{k=0}^{\infty} \binom{m}{k} (u_k v)_{m-n-k}, \]

\[ (u_m v)_n = \sum_{k=0}^{\infty} (-1)^k \binom{m}{k} (u_{m-k} v_{n+k} - (-1)^m v_{m+n-k} u_k), \]

\[ (\omega_0 v)_n = -nv_{n-1}. \]

Using them we can obtain the commutation relations of the component operators \( \omega_m \) and \( J_n \).

**Theorem 3.3.** Let \( L(n) = \omega_{n+1} \) and \( J(n) = J_{n+2} \) for \( n \in \mathbb{Z} \), so that the weights of these operators are \( \text{wt} L(n) = \text{wt} J(n) = -n \). Then

\[ [L(m), L(n)] = (m - n)L(m + n) + \frac{m^3 - m}{12} \cdot \frac{6}{5} \cdot \delta_{m+n,0}, \]

\[ [L(m), J(n)] = (2m - n)J(m + n), \]

\[ [J(m), J(n)] = (m - n) \left( 22(m + n + 2)(m + n + 3) \right) L(m + n) \\
+ 35(m + 2)(n + 2) L(m + n) \\
- 120(m - n) \left( \sum_{k \leq -2} L(k) L(m + n - k) \right) \\
+ \sum_{k \geq -1} L(m + n - k) L(k) \\
- \frac{7}{10} m(m^2 - 1)(m^2 - 4)\delta_{m+n,0}. \]

**Proof.** The first equation holds since \( \omega \) is the Virasoro element of central charge 6/5. We know that \( \omega_1 J = 3J \) and \( \omega_n J = 0 \) for \( n \geq 2 \). Hence the
second equation holds. Now
\[
(\omega_{-1} \omega)_{n+3} = \sum_{k=0}^{\infty} (-1)^k \binom{-1}{k} \left( \omega_{-1-k} \omega_{n+3+k} - (-1)^{-1} \omega_{n+2-k} \omega_k \right)
\]
\[
= \sum_{k=0}^{\infty} \left( L(-k - 2)L(n + k + 2) + L(n - k + 1)L(k - 1) \right)
\]
\[
= \sum_{k \leq -2} L(k)L(n - k) + \sum_{k \geq -1} L(n - k)L(k).
\]
Thus the last equation follows from (3.1).

\[\square\]

**Remark 3.4.** Let \( L_n = L(n) \) and \( W_n = \sqrt{-1/210} J(n) \). Then the commutation relations in the above theorem coincide with the commutation relations (2.1) and (2.2) of [3]. Thus \( \mathcal{W} \) is a \( W_3 \) algebra of central charge 6/5.

Let \( \lambda(m) = i(i + 1) \) if \( m = 2i + 1 \) is odd and \( \lambda(m) = i^2 \) if \( m = 2i \) is even. Let \( : L(n_1) L(n_2) : \) be the normal ordered product, so that it is equal to \( L(n_1) L(n_2) \) if \( n_1 \leq n_2 \) and \( L(n_2) L(n_1) \) if \( n_1 \geq n_2 \). Then we have another expression of \( (\omega_{-1} \omega)_{n+3} \). That is (cf. [14]),
\[
(\omega_{-1} \omega)_{n+3} = \lambda(n + 3) L(n) + \sum_{k \in \mathbb{Z}} : L(k) L(n - k) :.
\]

**4. 20 irreducible modules for \( M^\tau \)**

In this section we construct 20 irreducible modules for \( M^\tau \). Furthermore, we calculate the action of the weight preserving component operators \( L(0) = \omega_1 \) and \( J(0) = J_2 \) on the top levels of those irreducible modules for \( M^\tau \).

Recall that \( M \) has exactly 8 inequivalent irreducible modules \( M_i^0, W_i^0, i = 0, a, b, c \). Let \( (U, Y_U) \) be one of those irreducible \( M \)-modules. Following [9], we consider a new \( M \)-module \( (U \circ \tau, Y_{U\circ\tau}) \) such that \( U \circ \tau = U \) as vector spaces and
\[
Y_{U\circ\tau}(v, z) = Y_U(\tau v, z) \quad \text{for} \ v \in M.
\]
Then \( U \mapsto U \circ \tau \) induces a permutation on the set of irreducible \( M \)-modules. If \( U \) and \( U \circ \tau \) are equivalent \( M \)-modules, \( U \) is said to be \( \tau \)-stable. By the definition, we have \( U \circ \tau^2 = (U \circ \tau) \circ \tau \). The following lemma is a straightforward consequence of the definition of \( M_i^0 \) and \( W_i^0 \):

**Lemma 4.1.**

1. \( M_k^0 \circ \tau = M_k^0 \) and \( W_k^0 \circ \tau = W_k^0 \).
2. \( M_k^a \circ \tau = M_k^c \), \( M_k^c \circ \tau = M_k^b \), and \( M_k^b \circ \tau = M_k^a \).
3. \( W_k^a \circ \tau = W_k^c \), \( W_k^c \circ \tau = W_k^b \), and \( W_k^b \circ \tau = W_k^a \).
Here $W^0_k \circ \tau = W^0_k$ means that there exists a linear isomorphism $\phi(\tau) : W^0_k \rightarrow W^0_k$ such that $\phi(\tau) Y_{W^0_k}(v, z) \phi(\tau)^{-1} = Y_{W^0_k}(\tau v, z)$ for all $v \in M$. The automorphism $\tau$ of $V_L$ fixes $\omega^3$ and so $W^0_k$ is invariant under $\tau$. Hence we can take $\tau$ as $\phi(\tau)$. Note also that $\tau Y(v, z) \tau^{-1} = Y(\tau v, z)$ for all $v \in M = M^0_k$ since $\tau \in \text{Aut}(M)$.

4.1. Irreducible $M^\tau$-modules in untwisted $M$-modules. We first find 8 irreducible $M^\tau$-modules inside the 8 irreducible modules for $M$. Recall that $M(\epsilon) = \{ v \in M^0_k \mid \tau v = \xi^\epsilon v \}$. Likewise, set $W(\epsilon) = \{ v \in W^0_k \mid \tau v = \xi^\epsilon v \}$. From Lemma 4.1, [11, Theorem 4.4] and [13, Theorem 6.14], we see that $M(\epsilon)$ and $W(\epsilon)$ are inequivalent irreducible $M^\tau$-modules for $\epsilon = 0, 1, 2$. Note that $M_k^\epsilon$, $i = a, b, c$ are equivalent irreducible $M^\tau$-modules and that $W_k^\epsilon$, $i = a, b, c$ are also equivalent irreducible $M^\tau$-modules by [13, Theorem 6.14]. Hence we obtain 8 inequivalent irreducible $M^\tau$-modules.

The top levels, that is, the weight subspaces of the smallest weights of $M(0)$, $M(1)$, and $M(2)$ are $C_1$, $C_u^1$, and $C_u^2$ respectively. The top levels of $W(0)$, $W(1)$, and $W(2)$ are

$$\mathbb{C}(y(\alpha_1) + y(\alpha_2) + y(\alpha_0)), \quad \mathbb{C}(\alpha_1(-1) - \xi \alpha_2(-1)), \quad \text{and} \quad \mathbb{C}(\alpha_1(-1) - \xi^2 \alpha_2(-1))$$

respectively. Moreover, the top levels of $M_k^\epsilon$ and $W_k^\epsilon$ are

$$\mathbb{C}(e^{\beta_1/2} - e^{\beta_1/2}) \quad \text{and} \quad \mathbb{C}(e^{\beta_1/2} + e^{\beta_1/2})$$

respectively. All of those top levels are of dimension one.

Next, we deal with the action of $L(0)$ and $J(0)$ on those top levels. The operator $L(0)$ acts as multiplication by the weight of each top level. For the calculation of the action of $J(0)$, we first notice that

$$[w(\alpha_i), w(\alpha_j)] = (w(\alpha_i) w(\alpha_j))_2 + (w(\alpha_j) w(\alpha_i))_1$$

by (3.2). Since $w(\alpha_i) w(\alpha_j) = w(\alpha_j) w(\alpha_i)$, it follows that

$$J(0) = (w(\alpha_1) w(\alpha_2))_2 - (w(\alpha_2) w(\alpha_1))_2$$

$$= [w(\alpha_1), w(\alpha_2)] - [w(\alpha_2), w(\alpha_1)].$$

Using this formula it is relatively easy to calculate the eigenvalue for the action of $J(0)$ on each of the 8 top levels. The results are collected in Table 1.

4.2. Irreducible $M^\tau$-modules in $\tau$-twisted $M$-modules. Using [9], we show that there are exactly two inequivalent irreducible $\tau$-twisted (resp. $\tau^2$-twisted) $M$-modules. Moreover, we find 3 inequivalent irreducible $M^\tau$-modules in each of the irreducible $\tau$-twisted (resp. $\tau^2$-twisted) $M$-modules. Those irreducible $\tau$-twisted (resp. $\tau^2$-twisted) $M$-modules will in turn be constructed inside irreducible $\tau$-twisted (resp. $\tau^2$-twisted) $V_L$-modules. Basic references to twisted modules for lattice vertex operator algebras are [6], [7] and [25]. The argument here is similar to that in [22, Section 6].
Table 1. Irreducible $M^\tau$-modules in $M^\tau_k$ and $W^\tau_k$.

<table>
<thead>
<tr>
<th>irred. module</th>
<th>top level</th>
<th>$L(0)$</th>
<th>$J(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M(0)$</td>
<td>$\mathbb{C}1$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$M(1)$</td>
<td>$\mathbb{C}u^1$</td>
<td>2</td>
<td>$-12\sqrt{-3}$</td>
</tr>
<tr>
<td>$M(2)$</td>
<td>$\mathbb{C}u^2$</td>
<td>2</td>
<td>$12\sqrt{-3}$</td>
</tr>
<tr>
<td>$W(0)$</td>
<td>$\mathbb{C}(y(\alpha_1) + y(\alpha_2) + y(\alpha_0))$</td>
<td>$\frac{2}{3}$</td>
<td>0</td>
</tr>
<tr>
<td>$W(1)$</td>
<td>$\mathbb{C}(\alpha_1(-1) - \xi\alpha_2(-1))$</td>
<td>$\frac{2}{3}$</td>
<td>$2\sqrt{-3}$</td>
</tr>
<tr>
<td>$W(2)$</td>
<td>$\mathbb{C}(\alpha_1(-1) - \xi^2\alpha_2(-1))$</td>
<td>$\frac{3}{5}$</td>
<td>$-2\sqrt{-3}$</td>
</tr>
<tr>
<td>$M^\tau_k$</td>
<td>$\mathbb{C}(e^{\beta_1/2} - e^{-\beta_1/2})$</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
</tr>
<tr>
<td>$W^\tau_k$</td>
<td>$\mathbb{C}(e^{\beta_1/2} + e^{-\beta_1/2})$</td>
<td>$\frac{1}{10}$</td>
<td>0</td>
</tr>
</tbody>
</table>

We follow $[7]$ with $L = \sqrt{2}\mathbb{A}_2$, $p = 3$, $q = 6$, and $\nu = \tau$. Let $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$ and extend the $\mathbb{Z}$-bilinear form $\langle \cdot, \cdot \rangle$ on $L$ to $\mathfrak{h}$ linearly. Set

$$ h_1 = \frac{1}{3}(\beta_1 + \xi^2\beta_2 + \xi\beta_0), \quad h_2 = \frac{1}{3}(\beta_1 + \xi\beta_2 + \xi^2\beta_0). $$

Then $\tau h_j = \xi^j h_j$, $\langle h_1, h_1 \rangle = \langle h_2, h_2 \rangle = 0$, and $\langle h_1, h_2 \rangle = 2$. Moreover, $\beta_i = \xi^{i-1} h_1 + \xi^{2(i-1)} h_2$, $i = 0, 1, 2$. For $n \in \mathbb{Z}$, set

$$ \mathfrak{h}_{(n)} = \{ \alpha \in \mathfrak{h} \mid \tau \alpha = \xi^n \alpha \}. $$

Since $\tau$ is fixed-point-free on $L$, it follows that $\mathfrak{h}_{(0)} = 0$. Furthermore, $\mathfrak{h}_{(1)} = \mathbb{C} h_1$ and $\mathfrak{h}_{(2)} = \mathbb{C} h_2$. For $\alpha \in \mathfrak{h}$, we denote by $\alpha_{(n)}$ the component of $\alpha$ in $\mathfrak{h}_{(n)}$. Thus $(\beta_{(1)})_{(1)} = \xi^{i-1} h_1$ and $(\beta_{(1)})_{(2)} = \xi^{2(i-1)} h_2$ for $i = 0, 1, 2$.

Define the $\tau$-twisted affine Lie algebra to be

$$ \hat{\mathfrak{h}}[\tau] = \left( \bigoplus_{n \in \mathbb{Z}} \mathfrak{h}_{(n)} \otimes t^{n/3} \right) \oplus \mathbb{C} c $$

with the bracket

$$ [x \otimes t^m, y \otimes t^n] = m \langle x, y \rangle \delta_{m+n,0} c $$

for $x \in \mathfrak{h}_{(3m)}$, $y \in \mathfrak{h}_{(3n)}$, $m, n \in (1/3)\mathbb{Z}$, and $[c, \mathfrak{h}[g]] = 0$. The isometry $\tau$ acts on $\hat{\mathfrak{h}}[\tau]$ by $\tau(x \otimes t^{m/3}) = \xi^m x \otimes t^{m/3}$ and $\tau(c) = c$. Set

$$ \hat{\mathfrak{h}}[\tau]^+ = \bigoplus_{n>0} \mathfrak{h}_{(n)} \otimes t^{n/3}, \quad \hat{\mathfrak{h}}[\tau]^− = \bigoplus_{n<0} \mathfrak{h}_{(n)} \otimes t^{n/3}, $$

and consider the $\hat{\mathfrak{h}}[\tau]$-module

$$ S[\tau] = U(\hat{\mathfrak{h}}[\tau]) \otimes_{U(\hat{\mathfrak{h}}[\tau]^+ \oplus \hat{\mathfrak{h}}[\tau]^0)} \mathbb{C} $$
induced from the $\hat{h}[\tau]^{+} \oplus \hat{h}[\tau]^{0}$-module $\mathbb{C}$, where $\hat{h}[\tau]^{+}$ acts trivially on $\mathbb{C}$ and $c$ acts as 1 on $\mathbb{C}$.

We define the weight in $S[\tau]$ by

$$
wt(x \otimes t^{n}) = -n \quad \text{and} \quad wt 1 = \frac{1}{9},
$$

where $n \in (1/3)\mathbb{Z}$ and $x \in h((3n))$ (cf. [7, (4.6), (4.10)]). By the weight gradation $S[\tau]$ becomes a $(1/3)\mathbb{Z}$-graded space. Its character is

$$
ch S[\tau] = q^{1/9} \prod_{n=1}^{\infty} (1 - q^{n}) \prod_{n=1}^{\infty} (1 - q^{n/3}).
$$

(4.1)

For $\alpha \in h$ and $n \in (1/3)\mathbb{Z}$, denote by $\alpha(n)$ the operator on $S[\tau]$ induced by $\alpha((3n)) \otimes t^{n}$. Then, as a vector space $S[\tau]$ can be identified with a polynomial algebra with variables $h_{1}(1/3 + n)$ and $h_{2}(2/3 + n)$, $n \in \mathbb{Z}$. The weight of the operator $h_{j}(j/3 + n)$ is $-j/3 - n$.

The alternating $\mathbb{Z}$-bilinear map $c_{0}^{\tau} : L \times L \rightarrow \mathbb{Z}/6\mathbb{Z}$ defined by [7, (2.10)] is such that

$$
c_{0}^{\tau}(\alpha, \beta) = \sum_{r=0}^{2} (3 + 2r) \langle \tau^{r} \alpha, \beta \rangle + 6\mathbb{Z}.
$$

In our case $\sum_{r=0}^{2} \tau^{r} \alpha = 0$, since $\tau$ is fixed-point-free on $L$. Moreover, we can verify that

$$
\sum_{r=0}^{2} r \langle \tau^{r} \beta_{i}, \beta_{j} \rangle = \begin{cases} 
\pm 6 & \text{if } \tau \beta_{i} \neq \beta_{j} \\
0 & \text{if } \tau \beta_{i} = \beta_{j}.
\end{cases}
$$

Hence $c_{0}^{\tau}(\alpha, \beta) = 0$ for all $\alpha, \beta \in L$. This means that the central extension

$$
1 \rightarrow \langle \kappa_{6} \rangle \rightarrow \hat{L}_{\tau} \rightarrow L \rightarrow 1
$$

(4.2)

determined by the commutator condition $aba^{-1}b^{-1} = \kappa_{6}^{c_{0}^{\tau}(\alpha, \beta)}$ splits.

We consider the relation between two central extensions $\hat{L}$ of (2.1) and $\hat{L}_{\tau}$ of (4.2). Since both of $\hat{L}$ and $\hat{L}_{\tau}$ are split extensions, we use the same symbol $e^{\alpha}$ to denote both of an element in $\hat{L}$ and an element in $\hat{L}_{\tau}$ which correspond naturally to $\alpha \in L$. Actually, in Section 2 we choose $e^{\alpha} \in \hat{L}$ so that the multiplication in $\hat{L}$ is $e^{\alpha} \times e^{\beta} = e^{\alpha + \beta}$. Also we can choose $e^{\alpha} \in \hat{L}_{\tau}$ such that the multiplication $e^{\alpha} \times_{\tau} e^{\beta}$ in $\hat{L}_{\tau}$ is related to the multiplication in $\hat{L}$ by (cf. [7, (2.4)])

$$
e^{\alpha} \times e^{\beta} = \kappa_{6}^{\varepsilon_{0}(\alpha, \beta)} e^{\alpha} \times_{\tau} e^{\beta},
$$

(4.3)

where the $\mathbb{Z}$-linear map $\varepsilon_{0} : L \times L \rightarrow \mathbb{Z}/6\mathbb{Z}$ is defined by [7, (2.13)]. In our case

$$
\varepsilon_{0}(\alpha, \beta) = -\langle \tau^{-1} \alpha, \beta \rangle + 6\mathbb{Z}.
$$

(4.4)
As in Section 2, we usually write $e^\alpha e^\beta = e^{\alpha+\beta}$ to denote the product of $e^\alpha$ and $e^\beta$ in $\hat{L}$. Note, for example, that the inverse of $e^{\beta_1}$ in $\hat{L}$ is $e^{-\beta_1}$, while the inverse of $e^{\beta_1}$ in $\hat{L}_\tau$ is $\kappa_3^2 e^{-\beta_1}$.

The automorphism $\tau$ of $L$ lifts to an automorphism $\hat{\tau}$ of $\hat{L}$ such that $\hat{\tau}(e^\alpha) = e^{\tau \alpha}$ and $\hat{\tau}(\kappa_6) = \kappa_6$. Since $\varepsilon_0$ is $\tau$-invariant, we can also think $\hat{\tau}$ to be an automorphism of $\hat{L}_\tau$ in a similar way. By abuse of notation we shall denote $\hat{\tau}$ by simply $\tau$ also.

We have $(1-\tau)L = \text{span}_Z \{\beta_1 - \beta_2, \beta_1 + 2\beta_2\}$. The quotient group $L/(1-\tau)L$ is of order 3 and generated by $\beta_1 + (1-\tau)L$. Now $K = \{a^{-1} \tau(a) \mid a \in \hat{L}_\tau\}$ is a central subgroup of $\hat{L}$ with $K = (1-\tau)L$ and $K \cap \langle \kappa_6 \rangle = 1$. Here note that $a^{-1}$ is the inverse of $a$ in $\hat{L}_\tau$ and $a^{-1} \tau(a)$ is the product $a^{-1} \times_\tau \tau(a)$ in $\hat{L}_\tau$. In $\hat{L}_\tau$ we can verify that

$$e^{3\beta_1} = (e^{\beta_0 - \beta_1})^{-1} \times_\tau \tau(e^{\beta_0 - \beta_1}) \in K.$$ 

Since

$$\kappa_3 e^{\beta_1} \times_\tau \kappa_3 e^{\beta_1} \times_\tau \kappa_3 e^{-\beta_1} = e^{3\beta_1} \quad \text{and} \quad \kappa_3 e^{\beta_1} \times_\tau \kappa_3 e^{-\beta_1} = 1,$$

it follows that

$$\hat{L}_\tau/K = \{K, \kappa_3 e^{\beta_1} K, \kappa_3 e^{-\beta_1} K\} \times \langle \kappa_6 \rangle K/K \cong \mathbb{Z}_3 \times \mathbb{Z}_6.$$ 

For $j = 0, 1, 2$, define a linear character $\chi_j : \hat{L}_\tau/K \to \mathbb{C}^\times$ by

$$\chi_j(\kappa_6) = \xi_6, \quad \chi_j(\kappa_3 e^{\beta_1} K) = \xi^j, \quad \text{and} \quad \chi_j(\kappa_3 e^{-\beta_1} K) = \xi^{-j},$$

where $\xi_6 = \exp(2\pi\sqrt{-1}/6)$. Let $T_{\chi_j}$ be the one-dimensional $\hat{L}_\tau/K$-module affording the character $\chi_j$. As an $\hat{L}_\tau$-module, $K$ acts trivially on $T_{\chi_j}$. Since $\sum_{\alpha=0}^{2} \tau^\alpha = 0$ for $\alpha \in L$, those $T_{\chi_j}$, $j = 0, 1, 2$, are the irreducible $\hat{L}_\tau$-modules constructed in [25, Section 6].

Let

$$V_L^{T_{\chi_j}} = V_L^{T_{\chi_j}}(\tau) = S[\tau] \otimes T_{\chi_j}$$

and define the $\tau$-twisted vertex operator $Y^{\tau}(\cdot, z) : V_L \to \text{End}(V_L^{T_{\chi_j}}(z))$ as in [7]. For $a \in \hat{L}$, define

$$Y^{\tau}(a, z) = 3^{-\frac{(\langle \alpha, \alpha \rangle - \phi(\tilde{\alpha}))}{2}} \phi(\tilde{\alpha}) E^{-}(\tilde{\alpha}, z) E^{+}(-\tilde{\alpha}, z) a z^{-\frac{(\langle \alpha, \alpha \rangle)}{2}},$$

where

$$E^\pm(\alpha, z) = \exp \left( \sum_{n \in (1/3)\mathbb{Z} \pm} \frac{\alpha(n)}{n} z^{-n} \right),$$

$$\phi(\alpha) = (1 - \xi^2)^{\langle \tau \alpha, \alpha \rangle},$$
and $a \in \hat{L}$ acts on $T_{x_j}$ through the set theoretic identification between $\hat{L}$ and $\hat{L}_\tau$. Here we denote $\sigma(\alpha)$ of $[7, (4.35)]$ by $\phi(\alpha)$. For $v = \alpha^1(-n_1) \cdots \alpha^k(-n_k)$, $\iota(a) \in V_L$ with $\alpha^1, \ldots, \alpha^k \in h$ and $n_1, \ldots, n_k \in \mathbb{Z}_{>0}$, set

$$W(v, z) = \phi \left( \frac{1}{(n_k - 1)!} \left( \frac{d}{dz} \right)^{n_k - 1} \alpha^k(z) \right) \cdots \left( \frac{1}{(n_1 - 1)!} \left( \frac{d}{dz} \right)^{n_1 - 1} \alpha^1(z) \right) Y^\tau(a, z)^0,$$

where $\alpha(z) = \sum_{n \in (1/3)\mathbb{Z}} \alpha(n) z^{-n - 1}$. Define constants $c^i_{mn} \in \mathbb{C}$ for $m, n \geq 0$ and $i = 0, 1, 2$ by

$$\sum_{m, n \geq 0} c^0_{mn} x^m y^n = -\frac{1}{2} \sum_{r=1}^2 \log \left( \frac{(1 + x)^{1/3} - \xi^{-r}(1 + y)^{1/3}}{1 - \xi^{-r}} \right),$$

$$\sum_{m, n \geq 0} c^i_{mn} x^m y^n = \frac{1}{2} \log \left( \frac{(1 + x)^{1/3} - \xi^{-i}(1 + y)^{1/3}}{1 - \xi^{-i}} \right) \text{ for } i \neq 0.$$

Let $\{\gamma_1, \gamma_2\}$ be an orthonormal basis of $h$ and set

$$\Delta_z = \sum_{m, n \geq 0} \sum_{i=0}^2 \sum_{j=1}^2 c^i_{mn} (\tau^{-i} \gamma_j)(m) \gamma_j(n) z^{-m - n}.$$

Then for $v \in V_L$, $Y^\tau(v, z)$ is defined by

$$Y^\tau(v, z) = W(e^{\Delta_z} v, z).$$

We extend the action of $\tau$ to $V_L^{T_{x_j}}$ so that $\tau$ is the identity on $T_{x_j}$. The weight of every element in $T_{x_j}$ is defined to be 0. Then the character of $V_L^{T_{x_j}}$ is identical with that of $S[\tau]$.

By $[7, \text{Theorem 7.1}]$, $(V_L^{T_{x_j}}(\tau), Y^\tau(\cdot, z))$, $j = 0, 1, 2$ are inequivalent irreducible $\tau$-twisted $V_L$-modules. Now among the 12 irreducible $V_L$-modules $V_{L(i, j)}$, $i \in \{0, a, b, c\}$ and $j \in \{0, 1, 2\}$, the $\tau$-stable irreducible modules are $V_{L(0, j)}$, $j \in \{0, 1, 2\}$. Hence by $[9, \text{Theorem 10.2}]$, we conclude that $(V_L^{T_{x_j}}(\tau), Y^\tau(\cdot, z))$, $j = 0, 1, 2$, are all the inequivalent irreducible $\tau$-twisted $V_L$-modules. The isometry $\theta$ of $(L, \langle \cdot, \cdot \rangle)$ induces a permutation on $V_L^{T_{x_j}}(\tau)$, $j = 0, 1, 2$. In fact, the permutation leaves $V_L^{T_{x_0}}(\tau)$ invariant and interchanges $V_L^{T_{x_1}}(\tau)$ and $V_L^{T_{x_2}}(\tau)$.

Since $M^\tau \otimes M^0_\omega$ is contained in the subalgebra $(V_L)^\tau$ of fixed points of $\tau$ in $V_L$, we can deal with $(V_L^{T_{x_j}}(\tau), Y^\tau(\cdot, z))$ as an $M^\tau \otimes M^0_\omega$-module. We shall find 6 irreducible $M^\tau$-modules inside $V_L^{T_{x_j}}(\tau)$. Recall that $\omega$, $\omega^3$, and
$\tilde{\omega} = \omega + \omega^3$ are the Virasoro element of $M^\tau$, $M^0_\tau$, and $V_L$ respectively. Our main tool is a careful study of the action of $\omega_1$ on homogeneous subspaces of $V_L^{T_{\chi_j}}(\tau)$ of small weights. Here we denote by $u_n$ the coefficient of $z^{-n-1}$ in the twisted vertex operator $Y^\tau(u, z) = \sum u_n z^{-n-1}$ associated with a vector $u$ in $V_L$. The weight in $V_L^{T_{\chi_j}}(\tau)$ defined above is exactly the eigenvalue for $\tilde{\omega}_1$ (cf. [7, (6.10), (6.28)]).

The character of $V_L^{T_{\chi_j}}(\tau)$ is equal to the character of $S[\tau]$ (cf. (4.1)). Its first several terms are

$$\text{ch} V_L^{T_{\chi_j}}(\tau) = \text{ch} S[\tau] = q^{1/9} + q^{1/9+1/3} + 2q^{1/9+2/3} + 2q^{1/9+1} + 4q^{1/9+4/3} + \cdots.$$ 

Fix a nonzero vector $v \in T_{\chi_j}$. We can choose a basis of each homogeneous subspace of $V_L^{T_{\chi_j}}(\tau)$ of weight at most $1/9 + 4/3$ as in Table 2.

<table>
<thead>
<tr>
<th>weight</th>
<th>basis</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{9}$</td>
<td>$1 \otimes v$</td>
</tr>
<tr>
<td>$\frac{1}{9} + \frac{1}{3}$</td>
<td>$h_2(-\frac{1}{3}) \otimes v$</td>
</tr>
<tr>
<td>$\frac{1}{9} + \frac{2}{3}$</td>
<td>$h_1(-\frac{2}{3}) \otimes v, \quad h_2(-\frac{1}{3})^2 \otimes v$</td>
</tr>
<tr>
<td>$\frac{1}{9} + 1$</td>
<td>$h_1(-\frac{2}{3})h_2(-\frac{1}{3}) \otimes v, \quad h_2(-\frac{1}{3})^3 \otimes v$</td>
</tr>
<tr>
<td>$\frac{1}{9} + \frac{4}{3}$</td>
<td>$h_2(-\frac{4}{3}) \otimes v, \quad h_1(-\frac{2}{3})^2 \otimes v, \quad h_1(-\frac{2}{3})h_2(-\frac{1}{3})^2 \otimes v, \quad h_2(-\frac{1}{3})^4 \otimes v$</td>
</tr>
</tbody>
</table>

We need to know the action of $\omega_1$ on those bases. For this purpose, notice that

$$Y^\tau(e^{\pm \beta_i}, z) = -\frac{1}{27} E^-(\mp \beta_i, z) E^+(\mp \beta_i, z) \xi^\pm j z^{-2}, \quad i = 0, 1, 2,$$

since $\phi(\pm \beta_i) = -\xi/3$ and since $e^{\pm \beta_i}$ acts on $T_{\chi_j}$ as a multiplication by $\chi_j(e^{\pm \beta_i}) = \xi^\pm j$ for $i, j = 0, 1, 2$. The image of the vectors in Table 2 under the operator $\omega_1$ are calculated as follows:
\[
\omega_1(1 \otimes v) = \left( \frac{1}{15} + \frac{1}{45}(\xi^j + \xi^{-j}) \right) 1 \otimes v,
\]
\[
\omega_1\left( h_2 \left( -\frac{1}{3} \right) \otimes v \right) = \left( \frac{4}{15} - \frac{1}{9}(\xi^j + \xi^{-j}) \right) h_2 \left( -\frac{1}{3} \right) \otimes v,
\]
\[
\omega_1\left( h_1 \left( -\frac{2}{3} \right) \otimes v \right) = \left( \frac{7}{15} - \frac{2}{45}(\xi^j + \xi^{-j}) \right) h_1 \left( -\frac{2}{3} \right) \otimes v
\]
\[
- \frac{1}{5}(\xi^j - \xi^{-j})h_2 \left( -\frac{1}{3} \right)^2 \otimes v,
\]
\[
\omega_1\left( h_2 \left( -\frac{1}{3} \right)^2 \otimes v \right) = \left( \frac{7}{15} + \frac{7}{45}(\xi^j + \xi^{-j}) \right) h_2 \left( -\frac{1}{3} \right)^2 \otimes v
\]
\[
+ \frac{2}{15}(\xi^j - \xi^{-j})h_1 \left( -\frac{2}{3} \right)^2 \otimes v,
\]
\[
\omega_1\left( h_1 \left( -\frac{2}{3} \right) h_2 \left( -\frac{1}{3} \right) \otimes v \right) = \left( \frac{2}{3} + \frac{2}{9}(\xi^j + \xi^{-j}) \right) h_1 \left( -\frac{2}{3} \right) h_2 \left( -\frac{1}{3} \right) \otimes v
\]
\[
+ \frac{1}{5}(\xi^j - \xi^{-j})h_2 \left( -\frac{1}{3} \right)^3 \otimes v,
\]
\[
\omega_1\left( h_2 \left( -\frac{1}{3} \right)^3 \otimes v \right) = \left( \frac{2}{3} + \frac{1}{45}(\xi^j + \xi^{-j}) \right) h_2 \left( -\frac{1}{3} \right)^3 \otimes v
\]
\[
- \frac{2}{5}(\xi^j - \xi^{-j})h_1 \left( -\frac{2}{3} \right) h_2 \left( -\frac{1}{3} \right)^2 \otimes v,
\]
\[
\omega_1\left( h_2 \left( -\frac{4}{3} \right) \otimes v \right)
\]
\[
= \frac{13}{15} h_2 \left( -\frac{4}{3} \right) \otimes v
\]
\[
+ (\xi^j + \xi^{-j}) \left( -\frac{1}{90} h_2 \left( -\frac{4}{3} \right) - \frac{3}{10} h_1 \left( -\frac{2}{3} \right) h_2 \left( -\frac{1}{3} \right)^2 \right) \otimes v
\]
\[
+ (\xi^j - \xi^{-j}) \left( -\frac{1}{20} h_1 \left( -\frac{2}{3} \right)^2 - \frac{3}{20} h_2 \left( -\frac{1}{3} \right)^4 \right) \otimes v,
\]
\[ \omega_1 \left( h_1 \left( -\frac{2}{3} \right)^2 \otimes v \right) \]
\[ = \frac{13}{15} h_1 \left( -\frac{2}{3} \right)^2 \otimes v \]
\[ + (\xi^j + \xi^{-j}) \left( -\frac{1}{90} h_1 \left( -\frac{2}{3} \right)^2 + \frac{3}{10} h_2 \left( -\frac{1}{3} \right)^4 \right) \otimes v \]
\[ + (\xi^j - \xi^{-j}) \left( \frac{1}{15} h_2 \left( -\frac{4}{3} \right) + \frac{1}{5} h_1 \left( -\frac{2}{3} \right) h_2 \left( -\frac{1}{3} \right)^2 \right) \otimes v, \]
\[ \omega_1 \left( h_1 \left( -\frac{2}{3} \right) h_2 \left( -\frac{1}{3} \right)^2 \otimes v \right) \]
\[ = \frac{13}{15} h_1 \left( -\frac{2}{3} \right) h_2 \left( -\frac{1}{3} \right)^2 \otimes v \]
\[ + (\xi^j + \xi^{-j}) \left( \frac{2}{15} h_2 \left( -\frac{4}{3} \right) - \frac{14}{45} h_1 \left( -\frac{2}{3} \right) h_2 \left( -\frac{1}{3} \right)^2 \right) \otimes v \]
\[ - \frac{1}{15} (\xi^j - \xi^{-j}) h_1 \left( -\frac{2}{3} \right)^2 \otimes v, \]
\[ \omega_1 \left( h_2 \left( -\frac{1}{3} \right)^4 \otimes v \right) = \frac{13}{15} h_2 \left( -\frac{1}{3} \right)^4 \otimes v \]
\[ + (\xi^j + \xi^{-j}) \left( \frac{2}{5} h_1 \left( -\frac{2}{3} \right) - \frac{1}{9} h_2 \left( -\frac{1}{3} \right)^4 \right) \otimes v \]
\[ + \frac{4}{15} (\xi^j - \xi^{-j}) h_2 \left( -\frac{4}{3} \right) \otimes v. \]

The decomposition of \( V_{L}^{T_{x_j}}(\tau) \) as a \( \tau \)-twisted \( M \otimes M^{0}_{0} \)-module was studied in [22]. The outline of the argument is as follows: For \( j = 0, 1, 2 \), the vectors
\[ 1 \otimes v, \ h_1 \left( -\frac{2}{3} \right) \otimes v + (\xi^j - \xi^{-j}) h_2 \left( -\frac{1}{3} \right)^2 \otimes v, \]
\[ h_2 \left( -\frac{1}{3} \right)^2 \otimes v + \frac{2}{3} (\xi^j - \xi^{-j}) h_1 \left( -\frac{2}{3} \right) \otimes v \]
are simultaneous eigenvectors for \( \omega_1 \) and \( (\omega^3)_1 \). Denote by \( k_1 \) and \( k_2 \) the eigenvalues for \( \omega_1 \) and \( (\omega^3)_1 \) respectively. Then the pairs \( (k_1, k_2) \) are
We first discuss the decomposition of $V_{L,\tau}^{X_y}(\tau)$ into a direct sum of irreducible $M_0^\omega$-modules. We use the classification of irreducible $M_0^\omega$-modules [23] and their fusion rules [28]. Note also that the vector $y(\alpha_1) + y(\alpha_2) + y(\alpha_0)$ in $(V_L)^\tau$ is an eigenvector for $\omega_1$ of eigenvalue $8/5$. Hence $(V_L)^\tau$ contains the $\text{Vir}(\omega) \otimes M_0^\omega$-submodule generated by the vector, which is isomorphic to

$$L \left( \frac{6}{5}, \frac{8}{5} \right) \otimes \left( L \left( \frac{4}{5}, \frac{2}{5} \right) + L \left( \frac{4}{5}, \frac{7}{5} \right) \right).$$

Set

$$M_0^\omega_T(\tau) = \left\{ u \in V_{L,\tau}^{T^0}(\tau) \mid (\omega^3)_{1}u = 0 \right\},$$

$$W_0^\omega_T(\tau) = \left\{ u \in V_{L,\tau}^{T^0}(\tau) \mid (\omega^3)_{1}u = \frac{2}{5}u \right\}.$$

Moreover, for $j = 1, 2$ set

$$M_j^\omega_T(\tau) = \left\{ u \in V_{L,\tau}^{T^j}(\tau) \mid (\omega^3)_{1}u = \frac{2}{3}u \right\},$$

$$W_j^\omega_T(\tau) = \left\{ u \in V_{L,\tau}^{T^j}(\tau) \mid (\omega^3)_{1}u = \frac{1}{15}u \right\}.$$

Then, by [22, Proposition 6.8], $M_j^\omega_T(\tau)$ and $W_j^\omega_T(\tau)$, $j = 0, 1, 2$, are irreducible $\tau$-twisted $M$-modules. Furthermore, for $j = 0, 1, 2$,

$$V_{L,\tau}^{T^j}(\tau) \cong M_j^\omega_T(\tau) \otimes M_j^\omega_T(\tau) + W_j^\omega_T(\tau) \otimes W_j^\omega_T(\tau)$$

as $\tau$-twisted $M \otimes M_0^\omega$-modules.

There are at most two inequivalent irreducible $\tau$-twisted $M$-modules by Lemma 4.1 and [9, Theorem 10.2]. Then, looking at the smallest weight of $M_j^\omega_T(\tau)$ and $W_j^\omega_T(\tau)$, we have that $M_0^\omega_T(\tau) \cong M_1^\omega_T(\tau) \cong M_2^\omega_T(\tau)$ and $W_0^\omega_T(\tau) \cong W_1^\omega_T(\tau) \cong W_2^\omega_T(\tau)$ and that $M_0^\omega_T(\tau) \not\cong W_0^\omega_T(\tau)$ as $\tau$-twisted $M$-modules. We denote $M_0^\omega_T(\tau)$ by $M_T(\tau)$ and $W_0^\omega_T(\tau)$ by $W_T(\tau)$. We conclude that there are exactly two inequivalent irreducible $\tau$-twisted $M$-modules, which are represented by $M_T(\tau)$ and $W_T(\tau)$. As $\tau$-twisted $M \otimes \text{Vir}(\omega^3)$-modules, we
have

\begin{equation}
V_{L}^{T_{x_{0}}} (\tau) \cong M_{T} (\tau) \otimes \left( L \left( \frac{4}{5}, 0 \right) + L \left( \frac{4}{5}, 3 \right) \right) \\
\quad \oplus W_{T} (\tau) \otimes \left( L \left( \frac{4}{5}, \frac{2}{3} \right) + L \left( \frac{4}{5}, \frac{7}{5} \right) \right),
\end{equation}

\begin{equation}
V_{L}^{T_{x_{1}}} (\tau) \cong V_{L}^{T_{x_{2}}} (\tau) \cong M_{T} (\tau) \otimes L \left( \frac{4}{5}, \frac{2}{3} \right) \oplus W_{T} (\tau) \otimes L \left( \frac{4}{5}, \frac{1}{15} \right).
\end{equation}

The first several terms of the characters of $M_{T} (\tau)$ and $W_{T} (\tau)$ are

\begin{align*}
\text{ch } M_{T} (\tau) &= q^{\frac{1}{5}} + q^{\frac{1}{5} + \frac{2}{3}} + q^{\frac{1}{5} + 1 + \frac{2}{3}} + q^{\frac{1}{5} + 2 + \frac{2}{3}} + \cdots, \\
\text{ch } W_{T} (\tau) &= q^{\frac{2}{5}} + q^{\frac{2}{5} + \frac{2}{3}} + q^{rac{2}{5} + 1 + \frac{2}{3}} + q^{rac{2}{5} + 2 + \frac{2}{3}} + 2q^{\frac{2}{5} + \frac{4}{3}} + \cdots.
\end{align*}

For $\epsilon = 0, 1, 2$, let

\begin{align*}
M_{T} (\tau) (\epsilon) &= \{ u \in M_{T} (\tau) \mid \tau u = \xi^{\epsilon} u \}, \\
W_{T} (\tau) (\epsilon) &= \{ u \in W_{T} (\tau) \mid \tau u = \xi^{\epsilon} u \}.
\end{align*}

Those 6 modules for $M^{\tau}$ are inequivalent irreducible modules by [30, Theorem 2]. Their top levels are of dimension one. Those top levels and the eigenvalues for the action of $L^{\tau} (0) = \omega_1$ and $J^{\tau} (0) = J_2$ are collected in Table 3.

**Table 3.** Irreducible $M^{\tau}$-modules in $M_{T} (\tau)$ and $W_{T} (\tau)$.

<table>
<thead>
<tr>
<th>irred. module</th>
<th>top level</th>
<th>$L^{\tau} (0)$</th>
<th>$J^{\tau} (0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_{T} (\tau) (0)$</td>
<td>$C_1 \otimes v$</td>
<td>$\frac{1}{5}$</td>
<td>$\frac{1}{81} \sqrt{-3}$</td>
</tr>
<tr>
<td>$M_{T} (\tau) (1)$</td>
<td>$Ch_2 (-\frac{1}{3})^2 \otimes v$</td>
<td>$\frac{1}{5} + \frac{2}{3}$</td>
<td>$-\frac{238}{81} \sqrt{-3}$</td>
</tr>
<tr>
<td>$M_{T} (\tau) (2)$</td>
<td>$C (\frac{4}{3} h_1 (-\frac{2}{3})^2 \otimes v + h_2 (-\frac{1}{3})^4 \otimes v)$</td>
<td>$\frac{1}{5} + \frac{4}{3}$</td>
<td>$\frac{374}{81} \sqrt{-3}$</td>
</tr>
<tr>
<td>$W_{T} (\tau) (0)$</td>
<td>$Ch_2 (-\frac{1}{3})^3 \otimes v$</td>
<td>$\frac{2}{35} + \frac{2}{3}$</td>
<td>$\frac{176}{81} \sqrt{-3}$</td>
</tr>
<tr>
<td>$W_{T} (\tau) (1)$</td>
<td>$Ch_1 (-\frac{2}{3}) \otimes v$</td>
<td>$\frac{2}{35} + \frac{1}{3}$</td>
<td>$-\frac{22}{81} \sqrt{-3}$</td>
</tr>
<tr>
<td>$W_{T} (\tau) (2)$</td>
<td>$Ch_2 (-\frac{1}{3}) \otimes v$</td>
<td>$\frac{2}{35}$</td>
<td>$-\frac{4}{81} \sqrt{-3}$</td>
</tr>
</tbody>
</table>

4.3. Irreducible $M^{\tau}$-modules in $\tau^2$-twisted $M$-modules. Finally, we find 6 irreducible $M^{\tau}$-modules in $\tau^2$-twisted $M$-modules. The argument is parallel to that in Subsection 4.2. Instead of $\tau$, we take $\tau^2$. Thus we follow [7] with $\nu = \tau^2$. Set $h'_1 = h_2$, $h'_2 = h_1$, and

$h'_{\alpha} = \{ \alpha \in \mathfrak{h} \mid \tau^2 \alpha = \xi^n \alpha \}$. 
Then \( h'_0 = 0, h'_{1(1)} = C h'_1, \) and \( h'_{2(2)} = C h'_2. \) Consider a split central extension
\[
1 \longrightarrow \langle \kappa_6 \rangle \longrightarrow \hat{L}_{r^2} \longrightarrow L \longrightarrow 1
\]
and choose linear characters \( \chi'_j : \hat{L}_{r^2} / K \rightarrow \mathbb{C}^*, j = 0, 1, 2, \) such that
\[
\chi'_j(\kappa_6) = \xi_6, \quad \chi'_j(\kappa_3 e^{\beta_1} K) = \xi^j, \quad \text{and} \quad \chi'_j(\kappa_3 e^{-\beta_1} K) = \xi^{-j},
\]
where \( K = \{ a^{-1} r^2(a) \mid a \in \hat{L}_{r^2} \}. \) Let \( T_{\chi'_j} \) be the one-dimensional \( \hat{L}_{r^2} / K \)-module affording the character \( \chi'_j. \) Then the irreducible \( r^2 \)-twisted \( V_L \)-module associated with \( T_{\chi'_j} \) is
\[
V_L^{T_{\chi'_j}}(r^2) = S[r^2] \otimes T_{\chi'_j}.
\]
As a vector space \( S[r^2] \) is isomorphic to a polynomial algebra with variables \( h'_1(1/3 + n) \) and \( h'_2(2/3 + n), n \in \mathbb{Z}. \) The weight on \( S[r^2] \) is given by wt \( 1 = 1/9 \) and wt \( h'_j(j/3 + n) = -j/3 - n. \) Moreover, wt \( v = 0 \) for \( v \in T_{\chi'_j}. \) Set
\[
M_T(r^2) = \left\{ u \in V_L^{T_{\chi'_0}}(r^2) \mid (\omega^3)_1 u = 0 \right\},
\]
\[
W_T(r^2) = \left\{ u \in V_L^{T_{\chi'_0}}(r^2) \mid (\omega^3)_1 u = \frac{2}{5} u \right\}.
\]
Then \( M_T(r^2) \) and \( W_T(r^2) \) are the inequivalent irreducible \( r^2 \)-twisted \( M \)-modules. Furthermore, we have
\[
V_L^{T_{\chi'_0}}(r^2) \cong M_T(r^2) \otimes \left( L \left( \frac{4}{5}, 0 \right) + L \left( \frac{4}{5}, 3 \right) \right)
\]
\[
\oplus W_T(r^2) \otimes \left( L \left( \frac{4}{5}, 2 \right) + L \left( \frac{4}{5}, \frac{7}{5} \right) \right),
\]
\[
V_L^{T_{\chi'_1}}(r^2) \cong V_L^{T_{\chi'_2}}(r^2) \cong M_T(r^2) \otimes L \left( \frac{4}{5}, 2 \right) \otimes W_T(r^2) \otimes L \left( \frac{4}{5}, \frac{1}{15} \right)
\]
as \( r^2 \)-twisted \( M \otimes \text{Vir}(\omega^3) \)-modules. The character of \( M_T(r^2) \) or \( W_T(r^2) \) is equal to that of \( M_T(\tau) \) or \( W_T(\tau) \) respectively. For \( \epsilon = 0, 1, 2, \) let
\[
M_T(r^2)(\epsilon) = \{ u \in M_T(r^2) \mid r^2 u = \xi^\epsilon u \},
\]
\[
W_T(r^2)(\epsilon) = \{ u \in W_T(r^2) \mid r^2 u = \xi^\epsilon u \}.
\]
Those 6 modules for \( M^r \) are inequivalent irreducible modules by [30, Theorem 2]. Their top levels and the eigenvalues for the action of \( L^\tau(0) = \omega_1 \) and \( J^\tau(0) = J_2 \) are collected in Table 4.
Table 4. Irreducible $M^\tau$-modules in $M_T(\tau^2)$ and $W_T(\tau^2)$.

<table>
<thead>
<tr>
<th>irred. module</th>
<th>top level</th>
<th>$L^{\tau^2}(0)$</th>
<th>$J^{\tau^2}(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_T(\tau^2)(0)$</td>
<td>$\mathbb{C}1 \otimes v$</td>
<td>$\frac{1}{5}$</td>
<td>$-\frac{14}{81}\sqrt{-3}$</td>
</tr>
<tr>
<td>$M_T(\tau^2)(1)$</td>
<td>$\mathcal{C}h'_2(-\frac{1}{3})^2 \otimes v$</td>
<td>$\frac{1}{9} + \frac{2}{3}$</td>
<td>$\frac{238}{81}\sqrt{-3}$</td>
</tr>
<tr>
<td>$M_T(\tau^2)(2)$</td>
<td>$\mathcal{C}(\frac{4}{3}h'_1(-\frac{2}{3})^2 \otimes v + h'_2(-\frac{1}{3})^4 \otimes v)$</td>
<td>$\frac{1}{9} + \frac{4}{3}$</td>
<td>$-\frac{374}{81}\sqrt{-3}$</td>
</tr>
<tr>
<td>$W_T(\tau^2)(0)$</td>
<td>$\mathcal{C}h'_2(-\frac{1}{3})^3 \otimes v$</td>
<td>$\frac{2}{35} + \frac{2}{3}$</td>
<td>$-\frac{176}{81}\sqrt{-3}$</td>
</tr>
<tr>
<td>$W_T(\tau^2)(1)$</td>
<td>$\mathcal{C}h'_1(-\frac{2}{3}) \otimes v$</td>
<td>$\frac{2}{35} + \frac{1}{3}$</td>
<td>$\frac{22}{81}\sqrt{-3}$</td>
</tr>
<tr>
<td>$W_T(\tau^2)(2)$</td>
<td>$\mathcal{C}h'_2(-\frac{1}{3}) \otimes v$</td>
<td>$\frac{2}{35}$</td>
<td>$\frac{1}{81}\sqrt{-3}$</td>
</tr>
</tbody>
</table>

4.4. Remarks on 20 irreducible $M^\tau$-modules. We have obtained 20 irreducible $M^\tau$-modules in Subsections 4.1, 4.2, and 4.3. Note that the top levels of them are of dimension one and they can be distinguished by the eigenvalues for $\omega_1$ and $J_2$.

The isometry $\sigma$ of the lattice $(L, (\cdot, \cdot))$ induces a permutation of order 2 on those 20 irreducible $M^\tau$-modules. Clearly, $\sigma$ leaves $M(0)$ and $W(0)$ invariant and transforms $M_k^c$ (resp. $W_k^c$) into an irreducible $M^\tau$-module equivalent to $M_k^c$ (resp. $W_k^c$). Moreover, $\sigma$ interchanges irreducible $M^\tau$-modules as follows:

$$
M(1) \leftrightarrow M(2), \quad M_T(\tau)(\epsilon) \leftrightarrow M_T(\tau^2)(\epsilon), \quad W(1) \leftrightarrow W(2), \quad W_T(\tau)(\epsilon) \leftrightarrow W_T(\tau^2)(\epsilon)
$$

for $\epsilon = 0, 1, 2$. The top level of $M_T(\tau^2)(\epsilon)$ can be obtained by replacing $h_j(j/3 + n)$ with $h'_j(j/3 + n)$ for $j = 1, 2$ in the top level of $M_T(\tau)(\epsilon)$. Similar symmetry holds for $W_T(\tau^2)(\epsilon)$ and $W_T(\tau)(\epsilon)$. The action of $J(0)$ on the top level of $M_T(\tau^2)(\epsilon)$ (resp. $W_T(\tau^2)(\epsilon)$) is negative of the action on the top level of $M_T(\tau)(\epsilon)$ (resp. $W_T(\tau)(\epsilon)$). These symmetries are consequences of the fact that $\sigma T \sigma = \tau^2$ and $\sigma J = -J$.

In [14] an infinite series of $2D$ conformal field theory models with $\mathbb{Z}_3$ symmetry was studied. In the case $p = 5$ of [14], 20 irreducible representations are discussed [14, (5.5)]. If we multiply the values $w \left(\begin{smallmatrix} n & m \\ n' & m' \end{smallmatrix} \right)$ of [14, (5.6)] by $\sqrt{-105/2}$, then the pairs

$$
\Delta \left(\begin{smallmatrix} n & m \\ n' & m' \end{smallmatrix} \right), \sqrt{-105/2} w \left(\begin{smallmatrix} n & m \\ n' & m' \end{smallmatrix} \right)
$$

coincide with the pairs of the eigenvalues for $\omega_1$ and $J_2$ of the top levels of the 20 irreducible $M^\tau$-modules listed in Tables 1, 3 and 4. Here $\Delta \left(\begin{smallmatrix} n & m \\ n' & m' \end{smallmatrix} \right)$ is given by [14, (1.3)].
5. Classification of irreducible modules for $M^\tau$

We show in this section that the 20 irreducible modules discussed in Section 4 are all the inequivalent irreducible modules for $M^\tau$. This is achieved by determining the Zhu algebra $A(W)$ of the vertex operator subalgebra $W$ in $M^\tau$ generated by $\omega$ and $J$. It turns out that $A(W)$ is isomorphic to a quotient algebra of the polynomial algebra $\mathbb{C}[x, y]$ with two variables $x$ and $y$ by a certain ideal $I$ and that $A(W)$ is of dimension 20. We shall also prove that $M^\tau = W$ and $W$ is rational.

As in Theorem 3.3, let $L(n) = \omega_{n+1}$ and $J(n) = J_{n+2}$ for $n \in \mathbb{Z}$. The action of those operators on the vacuum vector $1$ is such that

\begin{align}
L(n)1 = 0 & \quad \text{for } n \geq -1, \\
J(n)1 = 0 & \quad \text{for } n \geq -2, \\
J(-2)1 = \omega, \\
J(-3)1 = J.
\end{align}

5.1. A spanning set for $W$. For a vector expressed in the form $u^1_{n_1} \cdots u^k_{n_k} 1$ with $u^i \in \{\omega, J\}$ and $n_i \in \mathbb{Z}$, we denote by $l_\omega(u^1_{n_1} \cdots u^k_{n_k} 1)$ or $l_J(u^1_{n_1} \cdots u^k_{n_k} 1)$ the number of $i$, $1 \leq i \leq k$ such that $u^i = \omega$ or $u^i = J$ respectively. We shall call these numbers the $\omega$-length or the $J$-length of the expression $u^1_{n_1} \cdots u^k_{n_k} 1$. Since each vector in $W$ is not necessarily expressed uniquely in such a form, the $\omega$-length and the $J$-length are not defined for a vector. They depend on a specific expression in the form $u^1_{n_1} \cdots u^k_{n_k} 1$.

Lemma 5.1. Let the $\omega$-length and the $J$-length of $u^1_{n_1} \cdots u^k_{n_k} 1$ be $s$ and $t$ respectively. Then $u^1_{n_1} \cdots u^k_{n_k} 1$ can be written as a linear combination of vectors of the form

$$L(-m_1) \cdots L(-m_p)J(-n_1) \cdots J(-n_q)1$$

such that:

(1) $m_1 \geq \cdots \geq m_p \geq 2, \quad n_1 \geq \cdots \geq n_q \geq 3$,

(2) $q \leq t$,

(3) $p + q \leq s + t$,

(4) $m_1 + \cdots + m_p + n_1 + \cdots + n_q = \text{wt}(u^1_{n_1} \cdots u^k_{n_k} 1)$.

Proof. We proceed by induction on $t$. If $t = 0$, the assertion follows from the commutation relation (3.5) and the action of $L(n)$ on the vacuum vector (5.1).

Suppose the assertion holds for the case where the $J$-length of $u^1_{n_1} \cdots u^k_{n_k} 1$ is at most $t - 1$ and consider the case where the $J$-length is $t$. By (3.6), we can replace $J(-n)L(-m)$ with $L(-m)J(-n)$ or $J(-m-n)$. Hence we may assume that $u^1_{n_1} \cdots u^k_{n_k} 1$ is of the form

$$L(-m_1) \cdots L(-m_s)J(-n_1) \cdots J(-n_t)1$$

for some $m_i, n_j \in \mathbb{Z}$.
By (5.2), we may assume that \( n_t \geq 3 \). Suppose \( n_i < n_{i+1} \) for some \( i \). Then by the commutation relation (3.7), the vector (5.3) can be written as a linear combination of the vectors which are obtained by replacing \( J(-n_i)J(-n_{i+1}) \) with:

(i) \( J(-n_{i+1})J(-n_i) \),
(ii) \( L(-n_i - n_{i+1}) \),
(iii) \( L(k)L(-n_i - n_{i+1} - k) \) or \( L(-n_i - n_{i+1} - k)L(k) \) for some \( k \in \mathbb{Z} \), or
(iv) a constant.

In Cases (ii), (iii), or (iv), we get an expression whose \( J \)-length is at most \( t - 2 \), and so we can apply the induction hypothesis. Therefore, in (5.3) we may assume that \( n_1 \geq \cdots \geq n_t \geq 3 \).

Now we argue by induction on the \( \omega \)-length \( s \) of the expression (5.3). If \( s = 0 \), the assertion holds. Suppose the assertion holds for the case where the \( \omega \)-length is at most \( s - 1 \). By (3.6), we can replace \( L(-m_s)J(-n_1) \) with:

(i) \( J(-n_1)L(-m_s) \) or
(ii) \( J(-m_s - n_1) \).

In Case (ii), we get an expression of \( \omega \)-length at most \( s - 1 \), so that we can apply the induction hypothesis. Arguing similarly, we can reach

\[
L(-m_1) \cdots L(-m_{s-1})J(-n_1) \cdots J(-n_t)L(-m_s)1.
\]

Hence we may assume that \( m_s \geq 2 \) by (5.1). Suppose \( m_i < m_{i+1} \) for some \( i \). Then by (3.5), the vector (5.3) can be written as a linear combination of the vectors which are obtained by replacing \( L(-m_i)L(-m_{i+1}) \) with:

(i) \( L(-m_{i+1})L(-m_i) \),
(ii) \( L(-m_i - m_{i+1}) \), or
(iii) a constant.

Since Case (ii) or (iii) yields an expression whose \( \omega \)-length is at most \( s - 1 \), we can apply the induction hypothesis. This completes the proof. \( \square \)

A vector of the form

\[
(5.4) \quad L(-m_1) \cdots L(-m_p)J(-n_1) \cdots J(-n_q)1
\]

with \( m_1 \geq \cdots \geq m_p \geq 2 \) and \( n_1 \geq \cdots \geq n_q \geq 3 \) will be called of normal form.

**Corollary 5.2.** \( \mathcal{W} \) is spanned by the vectors of normal form

\[
L(-m_1) \cdots L(-m_p)J(-n_1) \cdots J(-n_q)1
\]

with \( m_1 \geq \cdots \geq m_p \geq 2, \ n_1 \geq \cdots \geq n_q \geq 3, \ p = 0, 1, 2, \ldots, \) and \( q = 0, 1, 2, \ldots \).

**Proof.** As a vector space \( \mathcal{W} \) is spanned by the vectors \( u_{n_1}^{1} \cdots u_{n_k}^{k} 1 \) with \( u^i \in \{ \omega, J \}, \ n_i \in \mathbb{Z}, \) and \( k = 0, 1, 2, \ldots \) Hence the assertion follows from Lemma 5.1. \( \square \)
A similar argument for a spanning set can be found in [12, Section 3]. See also [3, Section 2.2].

**Remark 5.3.** Let $U$ be an admissible $W$-module generated by $u \in U$ such that $L(n)u = J(n)u = 0$ for $n > 0$ and $L(0)u = hu, J(0)u = ku$ for some $h, k \in \mathbb{C}$. It can be proved in a same way that $U$ is spanned by

$$L(-m_1) \cdots L(-m_p)J(-n_1) \cdots J(-n_q)u$$

with $m_1 \geq \cdots \geq m_p \geq 1, n_1 \geq \cdots \geq n_q \geq 1, p = 0, 1, 2, \ldots$, and $q = 0, 1, 2, \ldots$.

**5.2. A singular vector $v^{12}$.** A singular vector $v$ of weight $h$ for $W$ is by definition a vector $v$ which satisfies

1. $L(0)v = hv$,
2. $L(n)v = 0$ and $J(n)v = 0$ for $n \geq 1$.

Note that $v$ is not necessarily an eigenvector for $J(0)$. By commutation relations (3.5) and (3.6), it is easy to show that the condition (2) holds if $v$ satisfies

$$(2') L(1)v = L(2)v = J(1)v = 0.$$ 

We consider $W$ as a space spanned by the vectors of the form (5.4). The weight of such a vector is $m_1 + \cdots + m_p + n_1 + \cdots + n_q$. Let $v$ be a linear combination of the vectors of the form (5.4) of weight $h$. For example, there are 76 vectors of the form (5.4) of weight 12. We use the conditions (5.1) and (5.2) and the commutation relations (3.5), (3.6), and (3.7) to compute $L(1)v, L(2)v, J(1)v$. This computation was done by a computer algebra system Risa/Asir. The result is as follows:

**Lemma 5.4.** Let $v$ be a linear combination of the vectors of the form (5.4) of weight $h$. Under the conditions (5.1) and (5.2) and the commutation relations (3.5), (3.6) and (3.7), we have $L(1)v = L(2)v = J(1)v = 0$ only if $v = 0$ in the case $h \leq 11$. In the case $h = 12$, there exists a unique, up to scalar multiple, linear combination $v^{12}$ which satisfies $L(1)v^{12} = L(2)v^{12} = J(1)v^{12} = 0$. The explicit form of $v^{12}$ is given in Appendix A. We also have $J(0)v^{12} = 0$.

We only use conditions (5.1) and (5.2) and the commutation relations (3.5), (3.6) and (3.7) to obtain $v^{12}$ in the above computation. Since we consider $W$ inside the lattice vertex operator algebra $V_L$, there might exist some nontrivial relations among the vectors of the form (5.4) which are not known so far. This ambiguity will be removed in Subsection 5.3.
5.3. A positive definite invariant Hermitian form on $V_L$. It is well-known that the vertex operator algebra constructed from any positive definite even lattice as in [17] possesses a positive definite Hermitian form which is invariant in a certain sense ([15], [17], [26] and [29]). Following [29, Section 2.5], we review it for our $V_L$.

Set $\tilde{L}(n) = \tilde{\omega}_{n+1}, n \in \mathbb{Z}$, where $\tilde{\omega}$ is the Virasoro element of $V_L$. Then $\tilde{L}(1)(V_L)(1) = 0$ and $(V_L)(0)$ is one dimensional. Thus by [26, Theorem 3.1], there is a unique symmetric invariant bilinear form $(\cdot, \cdot)$ on $V_L$ such that $(1,1) = 1$. That the form is invariant means

$$(5.5) \quad (Y(u,z)v,w) = (v,Y(e^{z\tilde{L}(1)}(-z^{-2}\tilde{L}(0))u,z^{-1})w)$$

for $u,v,w \in V_L$. The value $(u,v)$ is determined by

$$(5.6) \quad (1,1) = 1,$$

$$(5.7) \quad (u,v) = \mbox{Res}_z z^{-1}(1,Y(e^{z\tilde{L}(1)}(-z^{-2}\tilde{L}(0))u,z^{-1})v).$$

From (5.5), we see that $(\tilde{L}(n)u,v) = (u,\tilde{L}(-n)v)$. In case of $n = 0$, this implies $((V_L)(m), (V_L)(n)) = 0$ if $m \neq n$. For $\alpha \in L$ and $u, v \in V_L$,

$$(5.8) \quad (\alpha(n)u,v) = \mbox{Res}_z z^n(Y(\alpha(1),z)u,v) = -(u,\alpha(-n)v).$$

Furthermore, for $\alpha, \beta \in L$ we have

$$(5.9) \quad (e^\alpha, e^\beta) = \delta_{\alpha+\beta,0}.$$

Note that $(-1)^{(\alpha,\alpha)/2} = 1$ since $\alpha \in L$. Consider an $\mathbb{R}$-form $V_{L,\mathbb{R}}$ of $V_L$ as in [17, Section 12.4]. That is, let $M(1)_{\mathbb{R}} = \mathbb{R}[\alpha(n); \alpha \in L, n < 0]$ and $V_{L,\mathbb{R}} = M(1)_{\mathbb{R}} \otimes \mathbb{R}[L]$. Then $\mathbb{C} \otimes_{\mathbb{R}} V_{L,\mathbb{R}} = V_L$. Moreover, $V_{L,\mathbb{R}}$ is invariant under the automorphism $\theta$. Let $V_{L,\mathbb{R}}^\pm = \{v \in V_{L,\mathbb{R}} | \theta v = \pm v\}$. We shall show that the form $(\cdot,\cdot)$ is positive definite on $V_{L,\mathbb{R}}^+$ and negative definite on $V_{L,\mathbb{R}}^-$. Indeed, let $\{\gamma_1, \gamma_2\}$ be an orthonormal basis of $\mathbb{R} \otimes_{\mathbb{Z}} L$. Then using (5.8) and (5.9) we can verify that

$$((\gamma_i(-m_1)\ldots\gamma_i(-m_p)e^\alpha, \gamma_j(-n_1)\ldots\gamma_j(-n_q)e^\beta) \neq 0$$

only if $\gamma_i(-m_1)\ldots\gamma_i(-m_p) = \gamma_j(-n_1)\ldots\gamma_j(-n_q)$ in $M(1)_{\mathbb{R}}$ and $\alpha+\beta = 0$. Furthermore,

$$(5.10) \quad ((\gamma_i(-m_1)\ldots\gamma_i(-m_p)e^\alpha, (-1)^p) = (\alpha \in L, \alpha \in L, \text{ even if } p \text{ even, } \alpha \in L, \alpha \in L, \text{ odd if } p \text{ odd, } \alpha \in L).$$

We can choose a basis of $V_{L,\mathbb{R}}^+$ consisting of vectors of the form

$$\gamma_i(-m_1)\ldots\gamma_i(-m_p)(e^\alpha + e^{-\alpha}), \quad p \text{ even, } \alpha \in L,$$

$$\gamma_i(-m_1)\ldots\gamma_i(-m_p)(e^\alpha - e^{-\alpha}), \quad p \text{ odd, } \alpha \in L.$$
By (5.9) and (5.10), these vectors are mutually orthogonal and the square length of each of them is a positive integer. Hence the form \((\cdot, \cdot)\) is positive definite on \(V_{L,\mathbb{R}}^+\). Likewise, we see that the form \((\cdot, \cdot)\) is negative definite on \(V_{L,\mathbb{R}}^-\).

We also have \((V_{L,\mathbb{R}}^+, V_{L,\mathbb{R}}^-) = 0\). Thus the form \((\cdot, \cdot)\) is positive definite on \(V_{L,\mathbb{R}}^+\) and \(\sqrt{-1}V_{L,\mathbb{R}}^-\). The \(\mathbb{R}\)-vector space \(V_{L,\mathbb{R}}^+ + \sqrt{-1}V_{L,\mathbb{R}}^-\) is an \(\mathbb{R}\)-form of \(V_L\) since \(V_L = \mathbb{C} \otimes \mathbb{R} (V_{L,\mathbb{R}}^+ + \sqrt{-1}V_{L,\mathbb{R}}^-)\). Note that it is invariant under the component operators \(u_n\) of \(Y(u, z)\) for \(u \in V_{L,\mathbb{R}}^+\).

Define a Hermitian form \(((\cdot, \cdot))\) on \(V_L\) by \(((\lambda u, \mu v)) = \lambda \overline{\mu} (u, v)\) for \(\lambda, \mu \in \mathbb{C}\) and \(u, v \in V_{L,\mathbb{R}}^+ + \sqrt{-1}V_{L,\mathbb{R}}^-\). Then the Hermitian form \(((\cdot, \cdot))\) is positive definite on \(V_L\) and invariant under \(V_{L,\mathbb{R}}^+\), that is,

\[
(5.11) \quad ((Y(u, z)v, w)) = \left(\left(v, Y\left(e^{zL(1)}(-z^{-2})\tilde{L}(0)u, z^{-1}\right)\right)w\right)
\]

for \(u \in V_{L,\mathbb{R}}^+\) and \(v, w \in V_L\).

Using the Hermitian form \(((\cdot, \cdot))\), we can show that \(V_L\) is semisimple as a \(W\)-module and that \(W\) is a simple vertex operator algebra. Note that \(\tilde{L}(n)v = L(n)v\) for \(v \in M\). Note also that \(V_{L,\mathbb{R}}^+\) contains \(\omega\) and \(J\). Then by (5.11),

\[
(5.12) \quad ((L(n)u, v)) = ((u, L(-n)v)),
\]

\[
(5.13) \quad ((J(n)u, v)) = -(u, J(-n)v))
\]

for \(n \in \mathbb{Z}\) and \(u, v \in V_L\).

Let \(U\) be a \(W\)-submodule. Denote by \(U^\perp\) the orthogonal complement of \(U\) in \(V_L\) with respect to \(((\cdot, \cdot))\). Then \(V_L = U \oplus U^\perp\) since \(((\cdot, \cdot))\) is positive definite. Moreover, \(U^\perp\) is also a \(W\)-submodule by (5.12) and (5.13). Thus we conclude that:

**Theorem 5.5.** \(V_L\) is semisimple as a \(W\)-module.

Since the weight 0 subspace \(\mathbb{C}1\) of \(W\) is one dimensional and since \(W\) is generated by \(1\) as a \(W\)-module, we have:

**Theorem 5.6.** \(W\) is a simple vertex operator algebra.

Then there is no singular vector in \(W\) of positive weight. Hence:

**Corollary 5.7.** The singular vector \(v^{12} = 0\).

### 5.4. The Zhu algebra \(A(W)\)

Based on the properties of \(W\) we have obtained so far, we shall determine the Zhu algebra \(A(W)\) of \(W\). First we review some notations and formulas for the Zhu algebra \(A(V)\) of an arbitrary vertex operator algebra \((V, Y, 1, \omega)\). The standard reference is [36, Section 2].
For \( u, v \in V \) with \( u \) being homogeneous, define two binary operations

\[
(5.14) \quad u \ast v = \text{Res}_z \left( \frac{(1 + z)^{\text{wt} u}}{z} Y(u, z)v \right) = \sum_{i=0}^{\infty} \binom{\text{wt} u}{i} u_{i-1}v,
\]

\[
(5.15) \quad u \circ v = \text{Res}_z \left( \frac{(1 + z)^{\text{wt} u}}{z^2} Y(u, z)v \right) = \sum_{i=0}^{\infty} \binom{\text{wt} u}{i} u_{i-2}v.
\]

We extend \( \ast \) and \( \circ \) for arbitrary \( u, v \in V \) by linearity. Let \( O(V) \) be the subspace of \( V \) spanned by all \( u \circ v \) for \( u, v \in V \). By a theorem of Zhu [36], \( O(V) \) is a two-sided ideal with respect to the operation \( \ast \). Thus it induces an operation on \( A(V) = V/O(V) \). Denote by \([v]\) the image of \( v \in V \) in \( A(V) \). Then \([u] \ast [v] = [u \ast v]\) and \( A(V)\) is an associative algebra by this operation. Moreover, \([1]\) is the identity and \([\omega]\) is in the center of \( A(V) \). We denote by \([u]^{*p}\) the product of \( p \) copies of \([u]\) in \( A(V) \). For \( u, v \in V \), we write \( u \sim v \) if \([u] = [v]\). For \( f, g \in \text{End} V \), we write \( f \sim g \) if \( fv \sim gv \) for all \( v \in V \). We need some formulas from [36].

\[
(5.16) \quad \text{Res}_z \left( \frac{(1 + z)^{\text{wt}(u)+m}}{z^{2+n}} Y(u, z)v \right) = \sum_{i=0}^{\infty} \binom{\text{wt}(u)+m}{i} u_{i-n-2}v \in O(V)
\]

for \( n \geq m \geq 0 \) and

\[
(5.17) \quad v \ast u \sim \text{Res}_z \left( \frac{(1 + z)^{\text{wt}(u)-1}}{z} Y(u, z)v \right) = \sum_{i=0}^{\infty} \binom{\text{wt}(u)-1}{i} u_{i-1}v.
\]

Moreover (see [34]),

\[
(5.18) \quad L(-n) \sim (-1)^n \{ (n-1)(L(-2) + L(-1)) + L(0) \}
\]

for \( n \geq 1 \) and

\[
(5.19) \quad [\omega] \ast [v] = [(L(-2) + L(-1))v].
\]

It follows from (5.18) and (5.19) that

\[
(5.20) \quad [L(-n)u] = (-1)^n(n-1)[\omega] \ast [u] + (-1)^n[L(0)u]
\]

for \( n \geq 1 \).

For a homogeneous \( u \in V \), set \( o(u) = u_{\text{wt}(u)-1} \), which is the weight zero component operator of \( Y(u, z) \). Extend \( o(u) \) for arbitrary \( u \in V \) by linearity. We call a module in the sense of [36] as an admissible module as in [9]. If \( U = \bigoplus_{n=0}^{\infty} U(n) \) is an admissible \( V \)-module with \( U(0) \neq 0 \), then \( o(u) \) acts on its top level \( U(0) \). Zhu’s theory [36] says:

1. \( o(u) o(v) = o(u \ast v) \) as operators on the top level \( U(0) \) and \( o(u) \) acts as 0 on \( U(0) \) if \( u \in O(V) \). Thus \( U(0) \) is an \( A(V) \)-module, where \([u]\) acts on \( U(0) \) as \( o(u) \).
(2) The map $U \mapsto U(0)$ is a bijection between the set of equivalence classes of irreducible admissible $V$-modules and the set of equivalence classes of irreducible $A(V)$-modules.

We now return to $W$. Since $\text{wt} J = 3$, we have
\begin{equation}
[J(-n - 4)v] = -3[J(-n - 3)v] - 3[J(-n - 2)v] - [J(-n - 1)v]
\end{equation}
for $v \in W$ and $n \geq 0$ by (5.16).

**Lemma 5.8.** The image $[L(-m_1) \cdots L(-m_p)J(-n_1) \cdots J(-n_q)1]$ of the vector (5.4) with $m_1 \geq \cdots \geq m_p \geq 2$ and $n_1 \geq \cdots \geq n_q \geq 3$ in $A(W)$ is contained in
\[
\begin{aligned}
\text{span} \left\{ [\omega]^s * [J]^t \mid 0 \leq s, 0 \leq t \leq q, 2s + 3t \leq m_1 + \cdots + m_p + n_1 + \cdots + n_q \right\}.
\end{aligned}
\]

In particular, $A(W)$ is commutative and every element of $A(W)$ is a polynomial in $[\omega]$ and $[J]$.

**Proof.** We proceed by induction on the $J$-length $q$. By a repeated use of (5.20), we see that $[L(-m_1) \cdots L(-m_p)J(-n_1) \cdots J(-n_q)1]$ is a linear combination of $[\omega]^s * [J(-n_1) \cdots J(-n_q)1]$, $0 \leq s \leq p$. Thus the assertion holds if $q = 0$.

Suppose the assertion holds for vectors of normal form with $J$-length at most $q - 1$ and consider $[J(-n_1) \cdots J(-n_q)1]$. Let $v = J(-n_1) \cdots J(-n_q)1$ and $u = J(-n_2) \cdots J(-n_q)1$, so that $v = J(-n_1)u$. We proceed by induction on the weight. The vector of the smallest weight is the case $n_1 = 3$. In this case $v = J(-3)^q1$ and $u = J(-3)^{q-1}1$. Since $v = J_{-1}u$, it follows from (5.14) that
\[
[v] = [J] * [u] - 3[J(-2)u] - 3[J(-1)u] - [J(0)u].
\]

The weight of $J(-n)u$, $0 \leq n \leq 2$, is less than $\text{wt} v$. By Lemma 5.1, each of these three vectors is a linear combination of vectors of normal form with $J$-length at most $q - 1$. Then we can apply the induction hypothesis on $J$-length and the assertion holds if $n_1 = 3$. Assume that $n_1 \geq 4$. By (5.21), $[v] = [J(-n_1)u]$ is a linear combination of $[J(-n)u]$, $n_1 - 3 \leq n \leq n_1 - 1$. The weight of $J(-n)u$, $n_1 - 3 \leq n \leq n_1 - 1$, is less than $\text{wt} v$. Hence by Lemma 5.1, these three vectors are linear combinations of vectors of normal form with $J$-length at most $q$ and weight less than $\text{wt} v$. The induction is complete. \hfill \Box

The image $[L(-m_1) \cdots L(-m_p)J(-n_1) \cdots J(-n_q)1]$ of the vector of normal form (5.4) with $m_1 \geq \cdots \geq m_p \geq 2$ and $n_1 \geq \cdots \geq n_q \geq 3$ in $A(W)$ can be written explicitly as a polynomial in $[\omega]$ and $[J]$ by the following algorithm:
Since $A(W)$ is commutative, it follows from (5.17) that
\begin{equation}
(5.22) \quad [J(-3)v] = [J] \ast [v] - 2[J(-2)v] - [J(-1)v]
\end{equation}
for $v \in W$. Now we use (5.20), (5.21) and (5.22). Although $J(-n - 4)v$ is of normal form, the vectors $J(-n - 3)v$, $J(-n - 2)v$, and $J(-n - 1)v$ in (5.21) may not be of normal form. However, the weight of any of these three vectors is less than the weight of $J(-n - 4)v$, and so we can apply the argument in the proof of Lemma 5.1. A similar discussion is also needed for the formula (5.22). Thus the algorithm is by induction on the weight. We use formulas (5.20), (5.21), (5.22) and apply Lemma 5.1, that is, use the commutation relations (3.5), (3.6), (3.7) and the conditions (5.1) and (5.2). By induction on the weight and a repeated use of those formulas and conditions, we can write explicitly the image of the vector (5.4) in $A(W)$ as a polynomial in $[\omega]$ and $[J]$. Consider the algebra homomorphism
\[ \mathbb{C}[x, y] \rightarrow A(W); \quad x \mapsto [\omega], \quad y \mapsto [J] \]
of the polynomial algebra $\mathbb{C}[x, y]$ with two variables $x, y$ onto $A(W)$. Denote its kernel by $\mathcal{I}$. Then $\mathbb{C}[x, y]/\mathcal{I} \cong A(W)$. We shall consider $v^{12}$, $J(-1)v^{12}$, $J(-2)v^{12}$, and $J(-1)^2v^{12}$. These vectors are described explicitly as linear combinations of vectors of normal form in Appendix A. Their images $[v^{12}]$, $[J(-1)v^{12}]$, $[J(-2)v^{12}]$, and $[J(-1)^2v^{12}]$ can be written as polynomials in $[\omega]$ and $[J]$ by the above mentioned algorithm. The results are given in Appendix B. Let $F_i(x, y) \in \mathbb{C}[x, y]$, $1 \leq i \leq 4$, be the polynomials which are obtained by replacing $[\omega]$ with $x$ and $[J]$ with $y$ in the polynomials given in Appendix B. Since $v^{12} = 0$ by Corollary 5.7, $F_i(x, y)$'s are contained in $\mathcal{I}$. Let $\mathcal{I}'$ be the ideal in $\mathbb{C}[x, y]$ generated by $F_i(x, y)$, $1 \leq i \leq 4$.

The primary decomposition of $\mathcal{I}'$ is $\mathcal{I}' = \bigcap_{i=1}^{20} \mathcal{P}_i$, where $\mathcal{P}_i$, $1 \leq i \leq 20$ are
\begin{align}
\langle x, y \rangle, & \quad \langle 5x - 8, y \rangle, \\
\langle 2x - 1, y \rangle, & \quad \langle 10x - 1, y \rangle, \\
\langle x - 2, y - 12\sqrt{-3} \rangle, & \quad \langle x - 2, y + 12\sqrt{-3} \rangle, \\
\langle 5x - 3, y - 2\sqrt{-3} \rangle, & \quad \langle 5x - 3, y + 2\sqrt{-3} \rangle, \\
\langle 9x - 1, 81y - 14\sqrt{-3} \rangle, & \quad \langle 9x - 1, 81y + 14\sqrt{-3} \rangle, \\
\langle 9x - 7, 81y - 238\sqrt{-3} \rangle, & \quad \langle 9x - 7, 81y + 238\sqrt{-3} \rangle, \\
\langle 9x - 13, 81y - 374\sqrt{-3} \rangle, & \quad \langle 9x - 13, 81y + 374\sqrt{-3} \rangle, \\
\langle 45x - 2, 81y - 4\sqrt{-3} \rangle, & \quad \langle 45x - 2, 81y + 4\sqrt{-3} \rangle, \\
\langle 45x - 17, 81y - 22\sqrt{-3} \rangle, & \quad \langle 45x - 17, 81y + 22\sqrt{-3} \rangle, \\
\langle 45x - 32, 81y - 176\sqrt{-3} \rangle, & \quad \langle 45x - 32, 81y + 176\sqrt{-3} \rangle.
\end{align}

These primary ideals correspond to the 20 irreducible $M^{\tau}$-modules listed in Tables 1, 3 and 4 in Section 4. The correspondence is given by substituting $x$ and $y$ with the eigenvalues for $L(0)$ and $J(0)$ on the top levels of 20 irreducible modules. The eigenvalues are the zeros of those primary ideals.
Note that the 20 pairs of those eigenvalues for $L(0)$ and $J(0)$ on the top levels are different from each other. Since the top levels of the 20 irreducible $M^\tau$-modules are one dimensional and since $\mathcal{W}$ is contained in $M^\tau$, there are at least 20 inequivalent irreducible $\mathcal{W}$-modules whose top levels are the same as those of irreducible $M^\tau$-modules. Hence by Zhu’s theory [36], we conclude that $I = I'$ and $A(\mathcal{W}) \cong \bigoplus_{i=1}^{20} \mathbb{C}[x,y]/\mathcal{P}_i$. In particular, $\mathcal{W}$ has exactly 20 inequivalent irreducible modules.

If $\mathcal{W} \neq M^\tau$, then we can take an irreducible $\mathcal{W}$-module $U$ in $M^\tau$ such that $\mathcal{W} \cap U = 0$ by Theorem 5.5. From the classification of irreducible $\mathcal{W}$-modules we see that the smallest weight of $U$ is at most 2. But we can verify that the homogeneous subspaces of $\mathcal{W}$ of weight 0, 1, and 2 coincide with those of $M^\tau$. Therefore, $\mathcal{W} = M^\tau$.

We have obtained the following theorem:

**Theorem 5.9.**

1. $M^\tau = \mathcal{W}$.
2. $A(M^\tau) \cong \bigoplus_{i=1}^{20} \mathbb{C}[x,y]/\mathcal{P}_i$ is a 20-dimensional commutative associative algebra.
3. There are exactly 20 inequivalent irreducible $M^\tau$-modules. Their representatives are listed in Tables 1, 3 and 4 in Section 4, namely, $M(\epsilon)$, $W(\epsilon)$, $M^\epsilon_k$, $W^\epsilon_k$, $M^\tau(\tau^i)(\epsilon)$, and $W^\tau(\tau^i)(\epsilon)$ for $\epsilon = 0, 1, 2$ and $i = 1, 2$.

**Remark 5.10.** The explicit description of $v^{12}$, $J(-1)v^{12}$, $J(-2)v^{12}$, and $J(-1)^2v^{12}$ in Appendix A, the images of these four vectors in $A(\mathcal{W})$ in Appendix B, and the primary ideals (5.23) were obtained by a computer algebra system Risa/Asir.

### 5.5. Rationality of $\mathcal{W}$

Recall that a vertex operator algebra $V$ is called $C_2$-cofinite if $V/C_2(V)$ is finite dimensional where $C_2(V)$ is the subspace of $V$ spanned by $u \cdot 2v$ for $u, v \in V$. The following result about a general vertex operator algebra was essentially proved in [31, Theorem 9.0.1]:

**Proposition 5.11.** Let $V = \bigoplus_{n \geq 0} V_n$ be a $C_2$-cofinite vertex operator algebra such that $V_0$ is one-dimensional. Assume that $A(V)$ is semisimple and any $V$-module generated by an irreducible $A(V)$-module is irreducible. Then $V$ is a rational vertex operator algebra.

**Proof.** By the definition of rationality (cf. [8]), we need to prove that any admissible $\mathcal{W}$-module $Z$ is completely reducible. By [1, Lemma 5.5], $Z$ is a direct sum of generalized eigenspaces for $L(0)$. So it is enough to prove that any submodule generated by a generalized eigenvector for $L(0)$ is completely reducible. We can assume that $Z$ is generated by a generalized eigenvector for $L(0)$. Then $Z = \bigoplus_{n \geq 0} Z_{\lambda+n}$ for some $\lambda \in \mathbb{C}$ where $Z_{\lambda+n}$ is the generalized eigenspace for $L(0)$ with eigenvalue $\lambda+n$ and $Z_{\lambda} \neq 0$. We call $\lambda$ the minimal weight of $Z$. By [4, Theorem 1], each $Z_{\lambda+n}$ is finite dimensional.
Let $X$ be the submodule of $Z$ generated by $Z_\lambda$. Then $X$ is completely reducible by the assumption. So we have an exact sequence

$$0 \to X \to Z \to Z/X \to 0$$

of admissible $V$-modules. Let $Z' = \oplus_{n \geq 0} Z^*_{\lambda+n}$ be the graded dual of $Z$. Then $Z'$ is also an admissible $V$-module (see [15]) and we have an exact sequence

$$0 \to (Z/X)' \to Z' \to X' \to 0$$

of admissible $V$-modules. On the other hand, the $V$-submodule of $Z'$ generated by $Z_1'$ is isomorphic to $X'$. As a result we have $Z'$ is isomorphic to $X' \oplus (Z/X)'$. This implies that $Z \cong X \oplus Z/X$. Clearly, the minimal weight of $Z/X$ is greater than the minimal weight of $Z$. Continuing in this way we prove that $Z$ is a direct sum irreducible modules.

Now we turn our attention to $W$.

**Theorem 5.12.** $W$ is $C_2$-cofinite.

**Proof.** Note from Corollary 5.2 that $W$ is spanned by

$$L(-m_1) \cdots L(-m_p)J(-n_1) \cdots J(-n_q)1$$

with $m_1 \geq \cdots \geq m_p \geq 2$, $n_1 \geq \cdots \geq n_q \geq 3$, $p = 0, 1, 2, \ldots$, and $q = 0, 1, 2, \ldots$. Then $W$ is spanned by $L(-2)^pJ(-3)^q1$ modulo $C_2(W)$. It is well-known that $W/C_2(W)$ is a commutative associative algebra under the product $u \cdot v = u_{-1}v$ for $u, v \in W$ (cf. [36]). So $W$ is spanned by $\omega^p \cdot J^q$ modulo $C_2(W)$ for $p, q \geq 0$.

The key idea to prove that $W$ is $C_2$-cofinite is to use the singular vector $\mathbf{v}^{12}$. By the explicit form of $\mathbf{v}^{12}$, $J(-1)\mathbf{v}^{12}$, and $J(-1)^2\mathbf{v}^{12}$ in Appendix A, we have the following relations in $W/C_2(W)$:

$$-(5968000/3501)\omega^6 - (184400/1167)\omega^3 \cdot J^2 + J^4 = 0,$$

$$-926640\omega^2 \cdot J^3 - 9856000\omega^5 \cdot J = 0,$$

$$21565440000\omega^7 - 680659200\omega^4 \cdot J^2 - 5559840\omega \cdot J^4 = 0.$$ 

Multiplying by $\omega^2, J, \omega$ respectively we get

$$-(5968000/3501)\omega^8 - (184400/1167)\omega^5 \cdot J^2 + \omega^2 \cdot J^4 = 0,$$

$$-926640\omega^2 \cdot J^3 - 9856000\omega^5 \cdot J^2 = 0,$$

$$21565440000\omega^8 - 680659200\omega^5 \cdot J^2 - 5559840\omega^2 \cdot J^4 = 0.$$ 

It follows immediately that

$$\omega^8 = \omega^2 \cdot J^4 = \omega^5 \cdot J^2 = 0.$$ 

Thus

$$J^8 = \left((59680000/3501)\omega^6 + (184400/1167)\omega^3 \cdot J^2\right)^2 = 0.$$ 

As a result, $W/C_2(W)$ is spanned by $\omega^p \cdot J^q$ for $0 \leq p, q \leq 7$, as desired. \qed
Lemma 5.13. Let $U$ be an irreducible $A(W)$-module. Then any $W$-module $Z$ generated by $U$ is irreducible.

Proof. By Theorem 5.9, $A(W)$ has exactly 20 irreducible modules and $\omega$ acts on each irreducible module as a constant in the set

$$P = \{0, 2, 8/5, 3/5, 1/2, 1/10, 1/9, 1/9 + 2/3, 1/9 + 4/3, 2/45, 2/45 + 1/3, 2/45 + 2/3\}.$$

Let $\omega$ act on $U$ as $\lambda$. Assume that $\lambda \neq 0, 3/5$. Then $\lambda$ is maximal in the set $P \cap (\lambda + Z)$. Let $Z = \oplus_{n \geq 0} Z_{\lambda + n}$ and $Z_\lambda = U$. If $Z$ is not irreducible then $Z$ has a proper submodule $X = \sum_{n \geq 0} X_{\lambda + n_0 + n}$ for some $n_0 > 0$ with $X_{\lambda + n_0} \neq 0$ where $X_{\lambda + m} = X \cap Z_{\lambda + m}$. So $X_{\lambda + n_0}$ is an $A(W)$-module on which $\omega$ acts on $\lambda + n_0$. Since $\lambda + n_0 \in P \cap (\lambda + Z)$ is greater than $\lambda$ we have a contradiction. This shows that $Z$ must be irreducible.

It remains to prove the result with $\lambda = 0$ or $\lambda = 3/5$. If $\lambda = 0$, then $U \cong \mathbb{C}1$ and $Z$ is isomorphic to $W$ (see [26]). Now let $\lambda = 3/5$. By Theorem 5.9, $U$ can be either $W(1)_{3/5}$ or $W(2)_{3/5}$ (see Table 1). We can assume that $U = W(1)_{3/5}$ and the proof for $U = W(2)_{3/5}$ is similar. In this case $J(0)$ acts on $U$ as $2\sqrt{-3}$. Let $U = \mathbb{C}u$. Then $Z$ is spanned by

$$L(-m_1) \cdots L(-m_p)J(-n_1) \cdots J(-n_q)u$$

with $m_1 \geq \cdots \geq m_p \geq 1$, $n_1 \geq \cdots \geq n_q \geq 1$, $p = 0, 1, 2, \ldots$, and $q = 0, 1, 2, \ldots$ (see Remark 5.3). Since $8/5$ is the only number in $P \cap (3/5 + Z)$ greater than $3/5$, $Z$ is irreducible if and only if there is no nonzero vector $v \in Z_{8/5}$ such that $L(1)v = J(1)v = 0$.

Note that $Z_{8/5}$ is spanned by $L(-1)u$ and $J(-1)u$. By formulas (3.5)-(3.7) we see that

$$L(1)L(-1)u = \frac{6}{5}u,$$

$$L(1)J(-1)u = 6\sqrt{-3}u,$$

$$J(1)L(-1)u = 6\sqrt{-3}u,$$

$$J(1)J(-1)u = \left(\frac{237 \times 6}{5} - \frac{48 \times 39}{5}\right)u.$$

Now let $v = \alpha L(-1)u + \beta J(-1)u \in Z_{8/5}$ such that $L(1)v = J(1)v = 0$. Then we have a system of linear equations

$$\frac{6}{5}\alpha + 6\sqrt{-3}\beta = 0,$$

$$6\sqrt{-3}\alpha - 90\beta = 0.$$
We have to prove that $v = 0$. If $v$ is not zero, then $\mathbb{C}v$ is an irreducible module for $A(W)$ on which $\omega$ acts as $8/5$. Again by Table 1, $J$ must act on $v$ as 0. Using (3.6) and (3.7), we find out that $J(0)v = -120L(-1)u - 8\sqrt{-3}J(-1)u = -8\sqrt{-3}v$. This implies that $v = 0$. Clearly we have a contradiction. Thus $Z$ is an irreducible $W$-module. 

Combining Proposition 5.11, Theorem 5.12, and Lemma 5.13 together yields:

**Theorem 5.14.** The vertex operator algebra $W$ is rational.

It is proved in [1] that a rational and $C_2$-cofinite vertex operator algebra is regular in the sense that any weak module is a direct sum of irreducible admissible modules. Thus we, in fact, have proved that $W$ is also regular.

6. Characters of irreducible $M^r$-modules

We shall describe the characters of the 20 irreducible $M^r$-modules by the characters of irreducible modules for the Virasoro vertex operator algebras. Throughout this section $z$ denotes a complex number in the upper half plane $\mathcal{H}$ and $q = \exp(2\pi \sqrt{-1}z)$. First we recall the character of the irreducible module $L(c_m, h_{r,s}^{(m)})$ with highest weight $h_{r,s}^{(m)}$ for the Virasoro vertex operator algebra $L(c_m, 0)$ with central charge $c_m$, where

$$c_m = 1 - \frac{6}{(m+2)(m+3)}, \quad m = 1, 2, \ldots,$$

$$h_{r,s}^{(m)} = \frac{(m+3)r - (m+2)s}{4(m+2)(m+3)}, \quad 1 \leq s \leq r \leq m+1.$$

The character of $L(c_m, h_{r,s}^{(m)})$ is obtained in [32] as follows:

$$\chi L(c_m, h_{r,s}^{(m)}) = \sum_{k \in \mathbb{Z}} \left( q^{b(k)} - q^{a(k)} \right) \prod_{i=1}^{\infty} \left( 1 - q^i \right),$$

where

$$a(k) = \frac{(2(m+2)(m+3)k + (m+3)r + (m+2)s)^2 - 1}{4(m+2)(m+3)},$$

$$b(k) = \frac{(2(m+2)(m+3)k + (m+3)r - (m+2)s)^2 - 1}{4(m+2)(m+3)}.$$
Define $\Xi_{r,s}^{(m)}(z) = g^{-cmz/24} \chi \mathcal{L}(cm, h_{r,s}^{(m)})$. For $1 \leq s \leq r \leq m + 1$, the following transformation formula holds (cf. [18, Exercise 13.27]):

\begin{equation}
\Xi_{r,s}^{(m)} \left( \frac{-1}{z} \right) = \sqrt{\frac{8}{(m + 2)(m + 3)}} \cdot \sum_{1 \leq j \leq \leq m+1} (-1)^{(r+s)(i+j)} \sin \frac{\pi ri}{m+2} \sin \frac{\pi sj}{m+3} \Xi_{i,j}^{(m)}(z).
\end{equation}

Let $\eta(z) = q^{1/24} \prod_{i=1}^{\infty} (1 - q^i)$ be the Dedekind $\eta$-function. The following transformation formula is well-known (cf. [2]):

$$
\eta \left( \frac{-1}{z} \right) = (-\sqrt{-1}z)^{1/2} \eta(z),
$$

where we choose the branch of the square root function $x^{1/2}$ so that it is positive when $x > 0$.

We review notations and some properties of the trace function in [9]. Let $g, h \in \text{Aut}(M)$ be such that $gh = hg$. Let $\mathcal{C}_1(g, h)$ be the space of $(g, h)$ 1-point functions. Let $W$ be a $g$-twisted $h$-stable $M$-module with conformal weight $\lambda$. There is a linear isomorphism $\phi(h) : W \rightarrow W$ such that

$$
\phi(h)Y_W(u, z) = Y_W(hu, z)\phi(h).
$$

Define $T_W(u, (g, h), z) = \text{tr}_W u_{\text{wt}(u)-1} \phi(h)q^{L(0)-1/20}$ for homogeneous $u \in M$ and extend it for arbitrary $u \in M$ linearly. Note that the central charge of $M$ is 6/5. Then $T_W(\cdot, (g, h), z) \in \mathcal{C}_1(g, h)$ by [9, Theorem 8.1]. Let $F(\cdot, z) \in \mathcal{C}_1(g, h)$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. Define $F|_A$ by

$$
F|_A(u, z) = (cz + d)^{-k}F \left( u, \frac{az + b}{cz + d} \right)
$$

for $u \in M_{[k]}$ and extend it for arbitrary $u \in M$ linearly. Then $F|_A \in \mathcal{C}_1(g^a h^c, g^b h^d)$ by [9, Theorem 5.4]. We denote $T_W(1, (g, h), z)$ by $T_W((g, h), z)$ for simplicity. Recall that the character $\chi W$ of $W$ is defined to be $\text{tr}_W q^{L(0)}$.

We want to determine the characters of the 20 irreducible $M^\sigma$-modules $M(\epsilon), W(\epsilon), M_k^\epsilon, W_k^\epsilon, M_T(\tau^i)(\epsilon), M_T(\tau^i)(\epsilon)$ for $\epsilon = 0, 1, 2$ and $i = 1, 2$. We have shown in Theorem 2.1 that $\text{Aut}(M)$ is generated by $\sigma$ and $\tau$. We shall consider the cases where $g = 1$ and $h = \tau$ or $g = \tau$ and $h = 1$. We specify $\phi(h)$ as follows: If $h = 1$, we take $\phi(h) = 1$. We shall deal with the case $g = 1$ and $h = \tau$ for $W = M$ or $W_k^0$. In such a case we consider
the same $\phi(\tau)$ as in Section 4. Thus if $W = M$, we take $\phi(\tau)$ to be the automorphism $\tau$. If $W = W_k^0$, we take $\phi(\tau)$ to be the linear isomorphism which is naturally induced from the isometry $\tau$ of the lattice $(L, \langle \cdot, \cdot \rangle)$.

Note that $T_W((g, 1), z) = q^{-1/20} \text{ch} W$. Note also that the symmetry (4.9) induced by $\sigma$ implies $T_{M(1)}((1, 1), z) = T_{M(2)}((1, 1), z)$. A similar assertion holds for $W(1)$ and $W(2)$.

**Proposition 6.1.** For $i = 1, 2$,

\[
T_{M\tau^i}((\tau^i, 1), z) = \frac{\eta(z)}{\eta(z/3)} \left( -\Xi_{2,1}^{(3)} - \Xi_{3,1}^{(3)} + \Xi_{3,3}^{(3)} \right),
\]

\[
T_{W\tau^i}((\tau^i, 1), z) = \frac{\eta(z)}{\eta(z/3)} \left( \Xi_{1,1}^{(3)} + \Xi_{4,1}^{(3)} - \Xi_{4,3}^{(3)} \right).
\]

**Proof.** Since $\text{ch} V^{T_{x_j}}(\tau) = \text{ch} S(\tau)$ for $j = 0, 1, 2$, we have

\[
q^{-1/12} \text{ch} V^{T_{x_j}}(\tau) = \frac{\eta(z)}{\eta(z/3)}
\]

by (4.1). Then (4.7) and (4.8) imply that

\begin{equation}
\frac{\eta(z)}{\eta(z/3)} = T_{M\tau((\tau, 1), z)} \cdot \left( \Xi_{1,1}^{(3)} + \Xi_{4,1}^{(3)} \right)
\end{equation}

\[
+ T_{W\tau((\tau, 1), z)} \cdot \left( \Xi_{2,1}^{(3)} + \Xi_{3,1}^{(3)} \right),
\]

\begin{equation}
\frac{\eta(z)}{\eta(z/3)} = T_{M\tau((\tau, 1), z)} \cdot \Xi_{4,3}^{(3)} + T_{W\tau((\tau, 1), z)} \cdot \Xi_{3,3}^{(3)}.
\end{equation}

Now consider $\left( \Xi_{1,1}^{(3)} + \Xi_{4,1}^{(3)} \right) \Xi_{3,3}^{(3)} - \left( \Xi_{2,1}^{(3)} + \Xi_{3,1}^{(3)} \right) \Xi_{4,3}^{(3)}$. Using (6.2) we can verify that it is invariant under the action of $SL_2(\mathbb{Z})$. Moreover, its $q$-expansion is $1 + 0 \cdot q + \cdots$. Thus

\[
\left( \Xi_{1,1}^{(3)} + \Xi_{4,1}^{(3)} \right) \Xi_{3,3}^{(3)} - \left( \Xi_{2,1}^{(3)} + \Xi_{3,1}^{(3)} \right) \Xi_{4,3}^{(3)} = 1.
\]

Hence the assertions for $i = 1$ follow from (6.3) and (6.4). The assertions for $i = 2$ also hold by the symmetry (4.9).

**Theorem 6.2.** The characters of the 20 irreducible $M^\tau$-modules $M(\epsilon)$, $W(\epsilon)$, $M_k^\tau$, $W_k^\tau$, $M_T(\tau^i)(\epsilon)$, and $W_T(\tau^i)(\epsilon)$ for $\epsilon = 0, 1, 2$ and $i = 1, 2$ are given by the following formulas:
(1) For $\epsilon = 1, 2$ we have

\[
q^{-1/20} \text{ch} M(0) = \frac{1}{3} \left( e_{1,1}^{(1)} - e_{1,1}^{(2)} + 2 e_{2,1}^{(2)} - \frac{\eta(z)}{\eta(3z)} e_{3,3}^{(3)} \right),
\]

\[
q^{-1/20} \text{ch} M(\epsilon) = \frac{1}{3} \left( e_{1,1}^{(1)} - e_{1,1}^{(2)} + 2 e_{2,1}^{(2)} - \frac{\eta(z)}{\eta(3z)} e_{3,3}^{(3)} \right),
\]

\[
q^{-1/20} \text{ch} W(0) = \frac{1}{3} \left( e_{1,1}^{(1)} - e_{1,1}^{(2)} + 2 e_{2,1}^{(2)} - \frac{\eta(z)}{\eta(3z)} e_{3,3}^{(3)} \right),
\]

\[
q^{-1/20} \text{ch} W(\epsilon) = \frac{1}{3} \left( e_{1,1}^{(1)} - e_{1,1}^{(2)} + 2 e_{2,1}^{(2)} - \frac{\eta(z)}{\eta(3z)} e_{3,3}^{(3)} \right),
\]

\[
q^{-1/20} \text{ch} M^\xi_k = e_{22}^{(1)} e_{21}^{(2)},
\]

\[
q^{-1/20} \text{ch} W^\xi_k = e_{22}^{(1)} e_{22}^{(2)}.
\]

(2) For $i = 1, 2$ we have

\[
\begin{pmatrix}
q^{-1/20} \text{ch}(M_T(\tau^i)(0)) \\
q^{-1/20} \text{ch}(M_T(\tau^i)(1)) \\
q^{-1/20} \text{ch}(M_T(\tau^i)(2))
\end{pmatrix}
= \frac{1}{3} \begin{pmatrix}
1 & 1 & 1 \\
1 & \xi & \xi^2 \\
1 & \xi^2 & \xi
\end{pmatrix}
\begin{pmatrix}
T_{M_T(\tau^i)}((\tau^i, 1), z) \\
T_{M_T(\tau^i)}((\tau^i, 1), z + 1) \\
T_{M_T(\tau^i)}((\tau^i, 1), z + 2)
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 \\
1 & \xi & \xi^2 \\
1 & \xi^2 & \xi
\end{pmatrix}
\begin{pmatrix}
T_{W_T(\tau^i)}((\tau^i, 1), z) \\
T_{W_T(\tau^i)}((\tau^i, 1), z + 1) \\
T_{W_T(\tau^i)}((\tau^i, 1), z + 2)
\end{pmatrix},
\]

where $\xi = \exp(2\pi\sqrt{-1}/3)$.

Proof. Since $M_T(\tau^i) = \oplus_{\epsilon=0}^2 M_T(\tau^i)(\epsilon)$ for $i = 1, 2$, we have

\[
T_{M_T(\tau^i)}((\tau^i, 1), z) = \sum_{\epsilon=0}^2 T_{M_T(\tau^i)(\epsilon)}((1, 1), z).
\]

Replace $z$ with $z + k$, where $k = 0, 1, 2$. Then

\[
T_{M_T(\tau^i)(\epsilon)}((1, 1), z + k) = \text{tr}_{M_T(\tau^i)(\epsilon)} q^{L(0) - 1/20} \exp(2\pi\sqrt{-1}k) L(0) - 1/20.
\]

Note that $\exp(2\pi\sqrt{-1}k) L(0) - 1/20 = \exp(11\pi\sqrt{-1}k/90) \xi^{2\epsilon}$ on $M_T(\tau^i)(\epsilon)$, since the eigenvalues for $L(0)$ on $M_T(\tau^i)(\epsilon)$ are of the form $1/9 + 2\epsilon/3 + n$.
with \( n \in \mathbb{Z}_{\geq 0} \). Thus

\[
T_{M_T(\tau^i)}((\tau^i, 1), z + k) = \exp(11\pi \sqrt{-1} k/90) \sum_{\epsilon=0}^{2} \xi^{2k\epsilon} T_{M_T(\tau^i)(\epsilon)}((1, 1), z).
\]

We can solve these equations for \( k = 0, 1, 2 \) with respect to

\[
T_{M_T(\tau^i)(\epsilon)}((1, 1), z), \quad \epsilon = 0, 1, 2,
\]

and obtain the expressions of \( T_{M_T(\tau^i)(\epsilon)}((1, 1), z) = q^{-1/20} \chi(M_T(\tau^i)(\epsilon)) \) in the theorem.

Similarly, \( W_T(\tau^i) = \oplus_{\epsilon=0}^2 W_T(\tau^i)(\epsilon) \) and the eigenvalues for \( L(0) \) on \( W_T(\tau^i)(\epsilon) \) are of the form \( 2/45 + (2 - \epsilon)/3 + n, \ n \in \mathbb{Z}_{\geq 0} \). Hence \( \exp(2\pi \sqrt{-1} k L(0))^{-1/20} = \exp(-61\pi \sqrt{-1} k/90)\xi^{2k\epsilon} \) on \( W_T(\tau^i)(\epsilon) \) and we obtain the expressions of \( q^{-1/20} \chi(W_T(\tau^i)(\epsilon)), \ \epsilon = 0, 1, 2 \).

It is proved in [24] that \( M = M_k^0 \) is a rational vertex operator algebra. Moreover, there are exactly two inequivalent \( \tau \)-stable \( M \)-modules, namely, \( M \) and \( W_k^0 \) by Lemma 4.1. Since \( M_T(\tau) \) and \( W_T(\tau) \) are two inequivalent irreducible \( \tau \)-twisted \( M \)-modules, we have \( \dim C_1(\tau, 1) = \dim C_1(\tau, 2) = 2 \) and

\[
\{ T_{M_T(\tau)}(\cdot, (\tau, 1), z), T_{W_T(\tau)}(\cdot, (\tau, 1), z) \}
\]

is a basis of \( C_1(\tau, 1) \) by [9, Theorems 5.4 and 10.1]. Now \( T_M(\cdot, (1, \tau), z)|_S \in C_1(\tau, 1) \) for \( S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) by [9, Theorems 5.4 and 8.1]. Thus,

\[
T_M((1, \tau), z) = \alpha T_{M_T(\tau)}((\tau, 1), \frac{-1}{z}) + \beta T_{W_T(\tau)}((\tau, 1), \frac{-1}{z})
\]

for some \( \alpha, \beta \in \mathbb{C} \).

From (6.2) and Proposition 6.1 it follows that

\[
\frac{\eta(3z)}{\eta(z)} T_{M_T(\tau)}((\tau, 1), \frac{-1}{z}) = \frac{2 \sin(\frac{\pi}{5})}{\sqrt{5}} \Xi^{(3)}_{3,3} - \frac{2 \sin(\frac{2\pi}{5})}{\sqrt{5}} \Xi^{(3)}_{4,3},
\]

\[
\frac{\eta(3z)}{\eta(z)} T_{W_T(\tau)}((\tau, 1), \frac{-1}{z}) = \frac{2 \sin(\frac{2\pi}{5})}{\sqrt{5}} \Xi^{(3)}_{3,3} + \frac{2 \sin(\frac{\pi}{5})}{\sqrt{5}} \Xi^{(3)}_{4,3}.
\]

Thus

\[
T_M((1, \tau), z) = q^{-1/20} \left( \alpha \frac{2 \sin(\frac{\pi}{5})}{\sqrt{5}} + \beta \frac{2 \sin(\frac{2\pi}{5})}{\sqrt{5}} \right) + \left( - \alpha \frac{2 \sin(\frac{2\pi}{5})}{\sqrt{5}} + \beta \frac{2 \sin(\frac{\pi}{5})}{\sqrt{5}} \right) q^{3/5} + \cdots \right).
\]
Furthermore, we see that $T_M((1, 1), z) = q^{-1/20}(1 + q^{3/5} + \cdots)$ by a direct computation. Hence $\alpha = 2\sin\left(\frac{\pi}{5}\right)/\sqrt{5}$ and $\beta = 2\sin\left(\frac{2\pi}{5}\right)/\sqrt{5}$. Therefore,

$$T_M((1, \tau), z) = \frac{\eta(z)}{\eta(3z)} \Xi^{(3)}_{3,3}.$$

Note that

$$T_M((1, 1), z) = q^{-1/20} \text{ch } M = \Xi^{(1)}_{1,1} \Xi^{(2)}_{1,1} + \Xi^{(1)}_{2,1} \Xi^{(2)}_{3,3}.$$  

Now $M = M(0) \oplus M(1) \oplus M(2)$ and $T_M((1, 1), z) = T_{M(2)}((1, 1), z)$ by the symmetry (4.9). Then

$$T_M((1, 1), z) = T_{M(0)}((1, 1), z) + T_{M(1)}((1, 1), z) + T_{M(2)}((1, 1), z)$$

$$= T_{M(0)}((1, 1), z) + 2T_{M(1)}((1, 1), z)$$

and

$$T_M((1, \tau), z) = T_{M(0)}((1, 1), z) + \xi T_{M(1)}((1, 1), z) + \xi^2 T_{M(2)}((1, 1), z)$$

$$= T_{M(0)}((1, 1), z) - T_{M(1)}((1, 1), z)$$

by the definition of trace functions. Thus $q^{-1/20} \text{ch } M(\epsilon) = T_{M(\epsilon)}((1, 1), z)$ can be expressed as

$$q^{-1/20} \text{ch } M(0) = \frac{1}{3} \left( T_M((1, 1), z) + 2T_M((1, \tau), z) \right)$$

$$= \frac{1}{3} \left( \Xi^{(1)}_{1,1} \Xi^{(2)}_{1,1} + \Xi^{(1)}_{2,1} \Xi^{(2)}_{3,1} + 2 \frac{\eta(z)}{\eta(3z)} \Xi^{(3)}_{3,3} \right).$$

$$q^{-1/20} \text{ch } M(\epsilon) = \frac{1}{3} \left( T_M((1, 1), z) - T_M((1, \tau), z) \right)$$

$$= \frac{1}{3} \left( \Xi^{(1)}_{1,1} \Xi^{(2)}_{1,1} + \Xi^{(1)}_{2,1} \Xi^{(2)}_{3,1} - \frac{\eta(z)}{\eta(3z)} \Xi^{(3)}_{3,3} \right)$$

for $\epsilon = 1, 2$. The computations for $W(\epsilon), \epsilon = 0, 1, 2$ are similar.

Since $M^i_k, i = a, b, c$ are equivalent irreducible $M^r$-modules by Lemma 4.1, we have $q^{-1/20} \text{ch } M^a_k = q^{-1/20} \text{ch } M^a_k = \Xi^{(1)}_{2,2} \Xi^{(2)}_{2,1}$. Likewise, $q^{-1/20} \text{ch } W^a_k = q^{-1/20} \text{ch } W^a_k = \Xi^{(1)}_{2,2} \Xi^{(2)}_{2,1}$. The proof is complete. $\square$

We now discuss the relation between the characters computed here and those of modules for a $W$-algebra computed in [16]. We use the notation of [16] without any comments. We refer to their results in the case that $\mathfrak{g}$ is the simple finite dimensional Lie algebra over $\mathbb{C}$ of type $A_2$ and $(p, p') = (6, 5)$. 
In this case, we have

\[ P_+^{p-h'} = P_+^3 = \left\{ \sum_{i=0}^{2} a_i \Lambda_i \mid 0 \leq a_i \in \mathbb{Z} \text{ and } \sum_{i=0}^{2} a_i = 3 \right\}, \]

\[ P_+^{p-h} = P_+^{p-2} = \left\{ \sum_{i=0}^{2} b_i \Lambda_i \mid 0 \leq b_i \in \mathbb{Z} \text{ and } \sum_{i=0}^{2} b_i = 2 \right\}. \]

It can be easily shown that \( \widetilde{W}_+ = \langle g \rangle \) is the cyclic group of order 3 such that \( g(\Lambda_0) = \Lambda_1, g(\Lambda_1) = \Lambda_2, \) and \( g(\Lambda_2) = \Lambda_0. \) The cardinality of \( I_{p,p'} = (P_+^3 \times P_+^{p-2})/\widetilde{W}_+ \) is equal to 20.

For \( \lambda \in P_+^3, \mu \in P_+^{p-2}, \) define

\[ \varphi_{\lambda,\mu}(z) = \eta(z)^{-2} \sum_{w \in \widetilde{W}_+} \epsilon(w)q^{\frac{1}{2}p'w(\lambda + \rho) - p(\mu + \rho')}|z^2 w(\lambda + \rho) - p(\mu + \rho')|^2. \]

The vector space spanned by \( \varphi_{\lambda,\mu}(z), (\lambda, \mu) \in I_{p,p'} \) is invariant under the action of \( SL_2(\mathbb{Z}) \) and the transformation formula

\[ \varphi_{\lambda,\lambda'} \left( \frac{-1}{z} \right) = \sum_{(\mu, \mu') \in I_{p,p'}} S_{(\lambda, \lambda'), (\mu, \mu')} \varphi_{\mu, \mu'}(z) \]

is given by [16, (4.2.2)]. Define \( F_1 = \{ \varphi_{\lambda,\mu}(z) \mid (\lambda, \mu) \in I_{p,p'} \}. \) In [16, Section 3], it is shown that each \( \varphi_{\lambda,\mu}(z) \in F_1 \) is the character of a module for the \( \mathbb{Z} \)-algebra associated to \( \mathfrak{g} \) and \( (p, p') \) which is conjectured to be irreducible.

We denote by \( F_2 \) the set of characters of all irreducible \( M^r \)-modules computed in Theorem 6.2. For any \( m, \) there is a congruence subgroup \( \Gamma_m \) such that each \( \Xi^{(m)}_{(s)} \) is a modular form for \( \Gamma_m \) (cf. [33, (6.11)]). Then there is a congruence subgroup \( \Gamma \) such that each character in \( F_2 \) is invariant under the action of \( \Gamma. \) The following transformation formulas hold by the formula (6.2):

\[
\begin{pmatrix}
T_M((1,1), \frac{-1}{z}) \\
T_W^p(((1,1), \frac{-1}{z}) \\
T_{M_0}((1,1), \frac{-1}{z}) \\
T_{W_0}((1,1), \frac{-1}{z})
\end{pmatrix} =
\begin{pmatrix}
\sin(\frac{\pi}{5}) & \sin(\frac{2\pi}{5}) & 3\sin(\frac{\pi}{5}) & 3\sin(\frac{2\pi}{5}) \\
\sin(\frac{\pi}{5}) & \sin(\frac{2\pi}{5}) & 3\sin(\frac{\pi}{5}) & 3\sin(\frac{2\pi}{5}) \\
\sin(\frac{\pi}{5}) & \sin(\frac{2\pi}{5}) & 3\sin(\frac{\pi}{5}) & 3\sin(\frac{2\pi}{5}) \\
\sin(\frac{\pi}{5}) & \sin(\frac{2\pi}{5}) & 3\sin(\frac{\pi}{5}) & 3\sin(\frac{2\pi}{5})
\end{pmatrix}
\begin{pmatrix}
T_M((1,1), z) \\
T_W^p((1,1), z) \\
T_{M_0}((1,1), z) \\
T_{W_0}((1,1), z)
\end{pmatrix}
\]
and
\[
\begin{pmatrix}
T_M((1, \tau), \frac{-1}{z}) \\
T_W^0((1, \tau), \frac{-1}{z})
\end{pmatrix}
= \begin{pmatrix}
\frac{2 \sin(\frac{\pi}{5})}{\sqrt{5}} & \frac{2 \sin(\frac{2\pi}{5})}{\sqrt{5}} \\
\frac{2 \sin(\frac{3\pi}{5})}{\sqrt{5}} & -\frac{2 \sin(\frac{4\pi}{5})}{\sqrt{5}}
\end{pmatrix}
\begin{pmatrix}
T_M(\tau)((\tau, 1), z) \\
T_W(\tau)((\tau, 1), z)
\end{pmatrix}.
\]

Thus we have the transformation formulas for elements of $\mathcal{F}_2$. Comparing the $q$-expansions and the coefficients of transformation formulas of elements in $\mathcal{F}_1$ and $\mathcal{F}_2$, it can be shown that $\mathcal{F}_1 = \mathcal{F}_2$ using Lemma 1.7.1 in [20]. In particular, $\varphi_{3A_0,2A_0}(z) = q^{-1/20} \mathrm{ch} M^*$ holds.

**Appendix A.** $v^{12}$, $J(-1)v^{12}$, $J(-2)v^{12}$, and $J(-1)^2v^{12}$

\[
v^{12} = -(5877264800/3501) L(-12)1 + (3404702000/3501) L(-10) L(-2)1
+ (263990000/3501) L(-9) L(-3)1 - (2663768000/3501) L(-8) L(-4)1
+ (282988000/1167) L(-8) L(-2)1 L(-3)1 - (237448000/1167) L(-7) L(-5)1
+ (30820000/1167) L(-7) L(-3) L(-2)1 + (1242377600/1167) L(-6) L(-2)1
+ (6149200/3501) L(-6) L(-4) L(-2)1 - (1313806000/1167) L(-6) L(-3) L(-2)1
+ (45496000/1167) L(-6) L(-2)31 - (3047684000/3501) L(-5) L(-2) L(-2)1
+ (299428000/1167) L(-5) L(-4) L(-3)1 + (2347094000/3501) L(-5) L(-3) L(-2) L(-2)1
- (17280400/1167) L(-4) L(-2)31 - (2036373200/3501) L(-4) L(-2) L(-2)1
+ (82960000/3501) L(-4) L(-3) L(-2) L(-2)1 + (1074512000/3501) L(-4) L(-2) L(-2)1
+ (511628125/3501) L(-3) L(-2) L(-2)1 - (4185500000/3501) L(-3) L(-2) L(-2)1
+ (59680000/3501) L(-2) L(-2)1 - (505200/3501) L(-6) L(-3)21
+ (3380480/1167) L(-4) L(-2) L(-3)21 + (37886800/1167) L(-5) L(-4) J(-3)1
+ (8788400/3501) L(-3) L(-2) L(-2) J(-4) J(-3)1 - (12761440/3501) L(-4) L(-2) J(-5) J(-3)1
- (5727500/10503) L(-4) J(-4) L(-2) J(-5) J(-3)1
+ (5727500/10503) L(-2) J(-2) J(-4) 21 + (1593900/3501) L(-3) J(-6) J(-3)1
+ (12935800/10503) L(-3) J(-5) J(-4) 21 + (301086000/3501) L(-2) L(-2) J(-7) J(-3)1
+ (2811800/1167) L(-2) J(-6) J(-4) 21 - (3131600/10503) L(-2) L(-2) J(-5) J(-3)1
+ (14904160/3501) J(-9) J(-3) 21 + (326760000/10503) J(-8) J(-4) 21
+ (9232000/10503) J(-7) J(-5) 21 + (2432375/1167) J(-6) J(-3) 1
+ J(-3) J(-1).
\]

\[
J(-1) v^{12} = -(47528/389) L(-4) L(-2) J(-3) 21 - (53552200/1167) J(-7) J(-3) 21
- (14322122800/389) L(-10) J(-3) 1 - (7313862400/389) L(-8) L(-2) J(-3) 1
+ (2263268000/389) L(-7) L(-3) J(-3) 1
+ (714032840/1167) L(-6) L(-4) J(-3) 1
- (48700662400/389) L(-5) J(-3) 1 - (41271174880/3501) L(-9) J(-4) 1.
\]
\[ J(-2)v^{12} = -4272L(-5)J(-3)^31 - (21069744/389)L(-8)J(-3)^21 \\
- (1415043808/1167)L(-11)J(-3)1 \]
\begin{align*}
- (3639849600/389)L(-9)L(-2)J(-3)1 \\
- (9699222400/1167)L(-8)L(-3)J(-3)1 \\
+ (2157139840/1167)L(-7)L(-4)J(-3)1 \\
- (5925448960/1167)L(-6)L(-5)J(-3)1 \\
+ (3006435200/389)L(-10)J(-4)1 \\
+ (325064548960/389)J(-14)1 - (176823168000/389)L(-2)J(-12)1 \\
+ (10174691200/3501)L(-9)J(-5)1 + (38988751200/389)L(-3)J(-11)1 \\
- (4612321600/1167)L(-8)J(-6)1 - (33023056960/389)L(-4)J(-10)1 \\
- (1337157600/1167)L(-7)J(-7)1 + (54711326720/1167)L(-5)J(-9)1 \\
+ (4368409600/1167)L(-6)J(-8)1 - 960L(-3)L(-2)J(-3)^21 \\
+ (5080480/389)L(-2)J(-6)J(-3)^21 \\
- (1269523200/389)L(-7)L(-2)^2J(-3)1 \\
+ (1626342400/1167)L(-6)L(-3)L(-2)J(-3)1 \\
- (3954100480/1167)L(-5)L(-4)L(-2)J(-3)1 \\
+ (6597673600/1167)L(-8)L(-2)L(-2)J(-4)1 \\
+ (382495595200/3501)L(-2)^2J(-10)1 \\
+ (23370344000/3501)L(-7)L(-2)J(-5)1 \\
- (41865472000/1167)L(-3)L(-2)J(-9)1 \\
+ (5662851200/1167)L(-6)L(-2)L(-6)1 \\
+ (41335582200/1167)L(-4)L(-2)L(-8)1 \\
- (66974297600/3501)L(-5)L(-2)J(-7)1 + 7760L(-3)L(-5)J(-5)J(-3)^21 \\
- (13118000/1167)L(-6)L(-5)J(-3)1 \\
- (3036691200/389)L(-6)L(-3)L(-5)1 \\
+ (2541514240/1167)L(-5)L(-4)L(-5)1 \\
- (4489884400/1167)L(-5)L(-3)^2J(-3)1 \\
+ (48898000/389)L(-5)L(-3)L(-6)1 \\
+ (524720000/389)L(-4)^2L(-3)J(-3)1 \\
- (478727200/389)L(-4)^2J(-6)1 \\
- (315678400/3501)L(-7)L(-3)L(-4)1 \\
- (5656762000/1167)L(-3)^2J(-8)1 \\
+ (4388915200/1167)L(-4)L(-3)J(-7)1 \\
+ (5080480/1167)L(-4)L(-4)J(-3)^21 \\
+ (7809478400/1167)L(-6)L(-4)L(-4)1 \\
+ (117493120/3501)L(-7)L(-4)L(-3)1 \\
- (6924715520/3501)L(-5)^2J(-4)1 \\
+ (7972739200/3501)L(-5)L(-2)^2J(-5)1 \\
+ (726208000/3501)L(-4)L(-3)L(-2)L(-5)1 \\
+ (15160000/3501)L(-5)^2J(-4)1
\end{align*}
\[-(9229774400/1167)L(-4)L(-2)^2J(-6)1\]
\[-(1273060000/1167)L(-3)^2L(-2)J(-6)1\]
\[-(5021408000/3501)L(-6)L(-2)^2J(-4)1\]
\[+(4736835200/1167)L(-5)L(-3)L(-2)J(-4)1\]
\[-(4697646400/3501)L(-4)^2L(-2)J(-4)1\]
\[-(28325800/3501)J(-6)J(-4)^21\]
\[+(10330016000/1167)L(-3)L(-2)^2J(-7)1\]
\[+(6184910000/3501)L(-3)^3J(-5)1\]
\[-(2988476000/3501)L(-4)L(-3)^2J(-4)1\]
\[+(2298688000/1167)L(-4)L(-2)^3J(-4)1\]
\[-(5988676800/3501)L(-2)^3J(-8)1\]
\[+(1320284000/3501)L(-3)^2L(-2)^2J(-4)1\]
\[+(22910000/10503)L(-2)J(-4)^31\]
\[-(3979216000/3501)L(-3)L(-2)^3J(-5)1\]
\[-(122435200/389)L(-5)L(-2)^3J(-3)1\]
\[-(977670400/1167)L(-4)L(-3)L(-2)^2J(-3)1\]
\[-(22467200/3501)L(-2)J(-5)J(-4)J(-3)1\]
\[+(58888000/1167)L(-3)^3L(-2)J(-3)1\]
\[-(17575800/3501)L(-3)J(-4)^2J(-3)1\]
\[+(29504000/389)L(-3)L(-2)^4J(-3)1 + (2281792000/1167)L(-2)^4J(-6)1\]
\[-(368800/389)L(-2)^2J(-4)J(-3)^21 - (238720000/1167)L(-2)^5J(-4)1.\]

\[J(-1)^2v^{12} = (28587850894720/389)L(-14)1 + (4067943568000/1167)L(-12)L(-2)1\]
\[-(20370766707200/389)L(-11)L(-3)1\]
\[-(29040708661120/389)L(-10)L(-4)1\]
\[-(1357372140800/389)L(-10)L(-2)^21\]
\[-(120978369778240/1167)L(-9)L(-5)1\]
\[+(1504699864000/1167)L(-9)L(-3)L(-2)1\]
\[-(120139236131200/1167)L(-8)L(-6)1\]
\[+(7353135836800/1167)L(-8)L(-4)L(-2)1\]
\[+(5914869272000/389)L(-8)L(-3)^21\]
\[-(9027652192000/1167)L(-8)L(-2)^31 - (1975756187200/389)L(-7)^21\]
\[+(10357377908800/389)L(-7)L(-5)L(-2)1\]
\[+(6212435174400/389)L(-7)L(-4)L(-3)1\]
\[-(3066391744000/389)L(-7)J(-3)L(-2)^21\]
\[-(34866323814400/1167)L(-6)^2L(-2)1\]
\[-(360052761600/389)L(-6)L(-5)L(-3)1\]
\[-(3455809144320/389)L(-6)L(-4)^21\]
\[+(811406115200/1167)L(-6)L(-4)L(-2)^21\]
\[+(2356317080000/1167)L(-6)L(-3)^2L(-2)1\]
\[-(420030912000/1167)L(-6)L(-2)^41\]
\[+(2046779720960/389)L(-5)^2L(-4)1\]
\[+(5012264899200/389)L(-5)^2L(-2)^21\]
\[+(5606971697600/1167)L(-5)L(-4)L(-3)L(-2)1\]
\[+(4546296703000/1167)L(-5)L(-3)^31\]
\[-(398623128800/1167)L(-5)L(-3)L(-2)^31\]
\[-(824891421120/389)L(-4)^3L(-2)1\]
\[+(129922820000/389)L(-4)^2L(-3)^21\]
\[+(9190279446400/1167)L(-4)^2L(-2)^31\]
\[-(3417631724000/1167)L(-4)L(-3)^2L(-2)^21\]
\[-(1854416512000/1167)L(-4)L(-2)^31\]
\[-(339474200000/1167)L(-3)^4L(-2)1\]
\[+(472407520000/1167)L(-3)^2L(-2)^41 + (21565440000/389)L(-2)^71\]
\[-(33906046720/389)L(-8)L(-3)^21\]
\[-(38547928640/389)L(-6)L(-2)J(-3)^21\]
\[+(8889576280/389)L(-5)L(-3)J(-3)^21 - (52680368/389)L(-4)^2J(-3)^21\]
\[+(1681515680/389)L(-4)L(-2)^2J(-3)^21\]
\[-(4900781600/389)L(-3)^2L(-2)J(-3)^21\]
\[-(6806592000/389)L(-2)^4J(-3)^21\]
\[-(21316634560/1167)L(-7)L(-4)J(-3)1\]
\[+(15456968800/389)L(-5)L(-2)J(-4)J(-3)1\]
\[-(57407779520/1167)L(-4)L(-3)J(-4)J(-3)1\]
\[+(769371200/389)L(-3)L(-2)^2J(-4)J(-3)1\]
\[+(82018834560/389)L(-6)J(-5)J(-3)1\]
\[-(318755320000/3501)L(-6)J(-4)^21\]
\[-(62232722240/1167)L(-4)L(-2)J(-5)J(-3)1\]
\[+(59657182000/3501)L(-4)L(-2)J(-4)^21\]
\[+(4384283800/1167)L(-3)^2J(-5)J(-3)1\]
\[+(28313585300/1167)L(-3)^2J(-4)^21\]
\[+(14719931200/1167)L(-2)^3J(-5)J(-3)1\]
\[-(15017860000/3501)L(-2)^3J(-4)^21\]
\[-(102815580920/389)L(-5)J(-6)J(-3)1\]
+ (214806972640/3501) L(-5)J(-5)J(-4)1
+ (20784972000/389) L(-3)L(-2)J(-6)J(-3)1
− (133605586400/3501) L(-3)L(-2)J(-5)J(-4)1
+ (243575438080/1167) L(-4)J(-7)J(-3)1
+ (7292932400/389) L(-4)J(-6)J(-4)1
− (12891781760/389) L(-4)J(-5)^21
− (49983377600/389) L(-2)^2J(-7)J(-3)1
+ (10825750000/389) L(-2)^2J(-6)J(-4)1
− (13957486400/3501) L(-2)^2J(-5)^21
− (173848522640/1167) L(-3)J(-8)J(-3)1
− (65060216000/1167) L(-3)J(-7)J(-4)1
+ (25622862200/389) L(-3)J(-6)J(-5)1
+ (174271514560/389) L(-2)J(-9)J(-3)1
− (232573421600/3501) L(-2)J(-8)J(-4)1
+ (392430209600/3501) L(-2)J(-7)J(-5)1
− (31534947600/389) L(-2)J(-6)J(-6)1
− (5559840/389) L(-2)J(-3)^41 − (291151720080/389) J(-11)J(-3)1
+ (257458099600/1167) J(-10)J(-4)1 − (140099797760/389) J(-9)J(-5)1
+ (83988236280/389) J(-8)J(-6)1 − (44378890400/389) J(-7)^21
Appendix B. The images of four vectors in $A(W)$

For simplicity of notation we omit the symbol $*$ for multiplication in $A(W)$.

$[v^{12}] = -(5968000/3501)[\omega]^6 + (156040000/3501)[\omega]^5$
$- (115878400/3501)[\omega]^4$
$+ (- (184400/1167)[J]^2 + 32328400/3501)[\omega]^3$
$+ ((536500/1167)[J]^2 - 3155968/3501)[\omega]^2$
$+ ( - (87812/389)[J]^2 + 93184/3501)[\omega]$

$[J(-1)v^{12}] = -(89856000/389)[J][\omega]^5 + (228945600/389)[J][\omega]^4$
$- (555607520/1167)[J][\omega]^3$
$+ ( - (926640/389)[J]^3 + (57790304/389)[J])[\omega]^2$
$+ ((1637064/389)[J]^3 - (19542016/1167)[J])[\omega]$
$- (668408/389)[J]^3 + (186368/389)[J].$

$[J(-2)v^{12}] = (179712000/389)[J][\omega]^5 - (457891200/389)[J][\omega]^4$
$+ (1111215040/1167)[J][\omega]^3$
$+ ((1853280/389)[J]^3 - (115580608/389)[J])[\omega]^2$
$+ ( - (3274128/389)[J]^3 + (39084032/1167)[J])[\omega]$
$+ (1336816/389)[J]^3 - (372736/389)[J].$

$[J(-1)^2v^{12}] = (21565440000/389)[\omega]^7 + (513849856000/1167)[\omega]^6$
$- (552497504000/389)[\omega]^5$
$+ ( - (680659200/389)[J]^2 + 128551506304/1167)[\omega]^4$
$+ ((3994427840/389)[J]^2 - 121501591744/389)[\omega]^3$
$+ ( - (8220864912/389)[J]^2 + 36103315456/1167)[\omega]^2$
$+ ( - (5559840/389)[J]^4 + (3836073072/389)[J]^2$
$- (363417600/389)))[\omega]$

Acknowledgments. The authors would like to thank Toshiyuki Abe and Kiyokazu Nagatomo for helpful advice concerning $W$ algebras and Hiroshi
Yamauchi for a positive definite invariant Hermitian form on $V_L$. They also would like to thank Masahiko Miyamoto for valuable discussions. Chongying Dong was partially supported by NSF grant DMS-9987656 and a research grant from the Committee on Research, UC Santa Cruz, Ching Hung Lam was partially supported by NSC grant 91-2115-M-006-014 of Taiwan, R.O.C., Kenichiro Tanabe was partially supported by JSPS Grant-in-Aid for Scientific Research No. 14740061, Hiromichi Yamada was partially supported by JSPS Grant-in-Aid for Scientific Research No. 13640012.

References


Received October 21, 2003.

Department of Mathematics
University of California
Santa Cruz, CA 95064
E-mail address: dong@math.ucsc.edu

Department of Mathematics
National Cheng Kung University
Tainan, Taiwan 701
Republic of China
E-mail address: chlam@mail.ncku.edu.tw

Institute of Mathematics
University of Tsukuba
Tsukuba 305-8571
Japan
E-mail address: tanabe@math.tsukuba.ac.jp

Department of Mathematics
Hitotsubashi University
Kunitachi, Tokyo 186-8601
Japan
E-mail address: yamada@math.hit-u.ac.jp

Graduate School of Mathematics
Kyushu University
Fukuoka 812-8581
Japan
E-mail address: yokoyama@math.kyushu-u.ac.jp
We analyze in classical $L^q$($\mathbb{R}^n$)-spaces, $n = 2$ or $n = 3$, $1 < q < \infty$, a singular integral operator arising from the linearization of a hydrodynamical problem with a rotating obstacle. The corresponding system of partial differential equations of second order involves an angular derivative which is not subordinate to the Laplacian. The main tools are Littlewood–Paley theory and a decomposition of the singular kernel in Fourier space.

1. Introduction

Consider a three-dimensional rotating rigid body with angular velocity $\omega = (0, 0, 1)^T$ and assume that the complement, a time-dependent exterior domain $\Omega(t) \subset \mathbb{R}^3$, is filled with a viscous incompressible fluid modelled by the Navier–Stokes equations. By a simple coordinate transform we are led to the nonlinear system [6]

\[
\begin{align*}
    u_t - \nu \Delta u + u \cdot \nabla u - (\omega \wedge x) \cdot \nabla u + \omega \wedge u + \nabla p &= f \quad \text{in } \Omega \\
    \text{div } u &= 0 \quad \text{in } \Omega \\
    u &= \omega \wedge x \quad \text{on } \partial \Omega \\
    u &\to 0 \quad \text{at } \infty
\end{align*}
\]

(1.1)

for the unknown velocity $u$ and pressure function $p$ in a time-independent exterior domain $\Omega \subset \mathbb{R}^3$ where $\nu > 0$ is the coefficient of viscosity. Looking for stationary solutions of (1.1), i.e., for time-periodic solutions of the original problem, and ignoring the nonlinear term $u \cdot \nabla u$ we arrive at a linear stationary partial differential equation in $\Omega$.

The first step to analyzing this problem is the $L^q$-theory of the system

\[
\begin{align*}
    -\nu \Delta u - (\omega \wedge x) \cdot \nabla u + \omega \wedge u + \nabla p &= f \quad \text{in } \mathbb{R}^3 \\
    \text{div } u &= g \quad \text{in } \mathbb{R}^3
\end{align*}
\]

(1.2)

in the whole space. Here for later applications we allow div $u$ to equal an arbitrarily given function $g$. The Coriolis force $\omega \wedge u = (-u_2, u_1, 0)^T$ can be considered as a perturbation of the Laplacian. But the first order partial
differential operator \((\omega \wedge x) \cdot \nabla u\) is not subordinate to the Laplacian due to the increasing term \(\omega \wedge x = (-x_2, x_1, 0)^T\). Using cylindrical coordinates \((r, \theta, x_3) \in (0, \infty) \times [0, 2\pi) \times \mathbb{R}\) we get
\[
(\omega \wedge x) \cdot \nabla u = -x_2 \partial_1 u + x_1 \partial_2 u = \partial_\theta u
\]
showing that the crucial term \((\omega \wedge x) \cdot \nabla u\) is “just” an angular derivative of \(u\) w.r.t. \(\theta\). Since
\[
\text{div} \left( (\omega \wedge x) \cdot \nabla u - \omega \wedge u \right) = (\omega \wedge x) \cdot \nabla \text{div} u = \partial_\theta g,
\]
the pressure \(p\) will satisfy the equation
\[
\Delta p = \text{div} f + \nu \Delta g + \partial_\theta g \quad \text{in} \quad \mathbb{R}^3
\]
which can easily be solved in \(L^q\)-spaces. Given \(p\) and ignoring (1.2)2 we arrive at the system
\[
(1.3) \quad -\nu \Delta u - \partial_\theta u + \omega \wedge u = f \quad \text{in} \quad \mathbb{R}^3
\]
with another right-hand side \(f\). Note that (1.3) also makes sense for a two-dimensional vector field \(u\) on \(\mathbb{R}^2\); then \(\omega \wedge u = (-u_2, u_1)^T\) and \((r, \theta) \in (0, \infty) \times [0, 2\pi)\) denote polar coordinates in \(\mathbb{R}^2\).

**Theorem 1.1.**

1. Let \(f \in L^q(\mathbb{R}^n)^n, n = 2 \text{ or } n = 3, 1 < q < \infty\). Then (1.3) has a solution \(u \in L^1_{\text{loc}}(\mathbb{R}^n)^n\) satisfying the estimate
\[
(1.4) \quad \|\nu \nabla^2 u\|_q + \|\partial_\theta u - \omega \wedge u\|_q \leq c \|f\|_q.
\]
Its equivalence class in the homogeneous Sobolev space \(\dot{H}^{2,q}(\mathbb{R}^n)^n\) is unique.

2. Let \(f \in L^q(\mathbb{R}^3)^3 \cap L^q(\mathbb{R}^3)^3, 1 < q_1, q_2 < \infty\), and let \(u_1\) and \(u_2\) be solutions as given by (1) corresponding to \(q = q_1\) and \(q = q_2\), respectively. Then there are \(\alpha, \beta, \gamma, \delta \in \mathbb{R}\) such that \(u_1\) coincides with \(u_2\) up to an affine linear vector field \(\omega + \beta \omega \wedge x + (\gamma x_1, \gamma x_2, \delta x_3)^T\), and any solution remains a solution if one adds such a term. For \(n = 2\) the terms \(\omega\) and \((0, 0, \delta x_3)^T\) have to be omitted.

3. Let \(f \in L^q(\mathbb{R}^n)^n, n = 2 \text{ or } n = 3, \) and let \(g \in H^{1,q}_{\text{loc}}(\mathbb{R}^n)\) such that \((\omega \wedge x)g, \nabla g \in L^q(\mathbb{R}^n)^n, 1 < q < \infty\). Then (1.2) has a locally integrable solution \((u, p)\) satisfying the estimate
\[
\|\nu \nabla^2 u\|_q + \|\partial_\theta u - \omega \wedge u\|_q + \|\nabla p\|_q \leq c (\|f\|_q + \|\nu \nabla g + (\omega \wedge x)g\|_q)
\]
where (1.2)2 has to be understood in the sense \(\nabla \text{div} u = \nabla g\). Its equivalence class in \(\dot{H}^{2,q}(\mathbb{R}^n)^n \times \dot{H}^{1,q}(\mathbb{R}^n)\) is unique. Moreover, if \((u_1, p_1)\) and \((u_2, p_2)\) are two such solutions, then \(p_1\) equals \(p_2\) up to a constant and \(u_1\) equals \(u_2\) up to an affine linear vector field of the form
\[
\alpha \omega + \beta \omega \wedge x + (\gamma x_1, \gamma x_2, -2\gamma x_3)^T, \quad \alpha, \beta, \gamma \in \mathbb{R}, \text{ and any solution remains a solution if one adds such terms. For } n = 2, \text{ } u_1 \text{ equals } u_2 \text{ up to the linear term } \beta(-x_2, x_1)^T, \beta \in \mathbb{R}.
\]

The so-called *homogeneous* Sobolev spaces \( \dot{H}^{k,q}(\mathbb{R}^n) \) in Theorem 1.1 are defined as follows: Let \( \Pi_{k-1} \) denote the space of polynomials of degree \( \leq k - 1 \). Then, using multi-index notation,

\[
\dot{H}^{k,q}(\mathbb{R}^n) = \{ u \in L^1_{\text{loc}}(\mathbb{R}^n)/\Pi_{k-1} : \partial^\alpha u \in L^q(\mathbb{R}^n) \text{ for all } \alpha \in \mathbb{N}^n_0, |\alpha| = k \}
\]

is equipped with the norm \( \sum_{|\alpha| = k} \| \partial^\alpha u \|_q \). Note that elements in \( \dot{H}^{k,q}(\mathbb{R}^n) \) are equivalence classes of \( L^1_{\text{loc}} \)-functions being unique only up to polynomials from \( \Pi_{k-1} \). Since \( \dot{H}^{k,q}(\mathbb{R}^n) \) can be considered as a closed subspace of \( L^q(\mathbb{R}^n)^N \) for some \( N = N(k,n) \in \mathbb{N} \), it is reflexive for every \( q \in (1, \infty) \). For more details on these spaces see Chapter II in [3]. Notice, however, that the space \( \Pi_1^n \) is not completely contained in the kernel of the operator

\[
L = -\nu \Delta - \partial \theta + \omega \wedge
\]
arising in (1.3).

We note that separate \( L^q \)-estimates of the terms \( \omega \wedge u \) and \( \partial \theta u \) in Theorem 1.1 are not possible unless \( f \) satisfies an additional set of compatibility conditions, see Remark 2.3 and Proposition 2.4 below; in particular \( u \) or \( \omega \wedge u \) are not necessarily \( L^q \)-integrable. Furthermore Proposition 2.1 indicates that the main solution operator does not define a classical Calderón–Zygmund integral operator.

The underlying problem of the flow around a rotating obstacle has attracted much attention during the last years. Weak solutions have been considered in [1] and [2], whereas one of the present authors proved the existence of a unique instationary solution in an \( L^2 \)-setting using semigroup theory ([6] and [7]). It is a remarkable fact that the operator \( -\nu \Delta u - \partial \theta u + \omega \wedge u \) does not generate an analytic semigroup, but a contractive \( C^0 \)-semigroup. Several auxiliary linearized equations without the crucial term \( \partial \theta u \) have been considered in [8]. An \( L^2 \)- and an \( L^3/2 \)-theory of problem (1.2) have been established in [4], where the nonlinear problem is also solved for non-Newtonian, second-order fluids and rigid bodies moving due to gravity. Pointwise decay estimates for the linear and nonlinear case are obtained in [5]. For further references on moving bodies in fluids see [4] and [5].

### 2. Preliminaries

To find the fundamental solutions of (1.2) and of (1.3) (see also [6] and [7]), we use the Fourier transform \( \mathcal{F} = \wedge \), i.e.,

\[
\hat{u}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} u(x) dx.
\]
Note that in $S'(\mathbb{R}^n)$, the space of tempered distributions, $\widehat{\partial_j u} = i\xi_j \hat{u}$ and $\widehat{x_j u} = i\partial \hat{u} / \partial \xi_j$, $1 \leq j \leq n$. Hence (1.3) is related to the problem
\begin{equation}
\nu s^2 \hat{u} - \partial_\varphi \hat{u} + \omega \wedge \hat{u} = \hat{f}
\end{equation}
where $s = |\xi|$ and $\partial_\varphi = -\xi_2 \partial / \partial \xi_1 + \xi_1 \partial / \partial \xi_2 = (\omega \wedge \xi) \cdot \nabla_\xi$ is the angular derivative in Fourier space when using polar or cylindrical coordinates for $\xi \in \mathbb{R}^2$ or $\xi \in \mathbb{R}^3$, resp. Ignoring for a moment the term $\omega \wedge \hat{u}$ the ordinary differential equation $-\partial_\varphi \hat{u} + \nu s^2 \hat{u} = \hat{f}$ yields the solution
\begin{equation}
\hat{u}(\varphi) = e^{\nu s^2 \varphi} \hat{u}_0 - e^{\nu s^2 \varphi} \int_0^\varphi e^{-\nu s^2 t} \hat{f}(t) dt, \quad \hat{u}_0 \in \mathbb{R}^n,
\end{equation}
when omitting in $\hat{u}$, $\hat{f}$ the variables $s = |\xi|$ or $s' = (\xi_1^2 + \xi_2^2)^{1/2}$, $\xi_3$, resp. Due to the $2\pi$-periodicity of $\hat{u}$ w.r.t. $\varphi$ the unknown $\hat{u}_0$ is given by
$$
\hat{u}_0 = (1 - e^{-2\pi
u s^2})^{-1} \int_0^{2\pi} e^{-\nu s^2 t} \hat{f}(t) dt.
$$
Using for $s \neq 0$ the geometric series expansion of $(1 - e^{-2\pi
u s^2})^{-1}$ and the $2\pi$-periodicity of $\hat{f}$ w.r.t. $t$ we get $\hat{u}_0 = \int_0^\infty e^{-\nu s^2 t} \hat{f}(t) dt$. Then (2.2) yields
\begin{equation}
\hat{u}(\varphi) = \int_0^\infty e^{-\nu s^2 t} \hat{f}(\varphi + t) dt.
\end{equation}

Let $O(t)$ denote the orthogonal matrix
$$
O(t) = \begin{pmatrix}
\cos t & -\sin t & 0 \\
\sin t & \cos t & 0 \\
0 & 0 & 1
\end{pmatrix}
\quad \text{or} \quad
O(t) = \begin{pmatrix}
\cos t & -\sin t \\
\sin t & \cos t
\end{pmatrix}
$$
describing the rotation around the $\xi_3$-axis or in the plane by the angle $t$, resp. Thus, in the variable $\xi$,
$$
\hat{u}(\xi) = \int_0^\infty e^{-\nu s^2 t} \hat{f}(O(t)\xi) dt
$$
is the solution of (2.1) when $\omega \wedge u$ has been ignored. To deal with the term $\omega \wedge u$ note that $\partial_\varphi O(\varphi) = \omega \wedge O(\varphi)$ in the sense of linear maps. Applying $O(\varphi)^T$ to (2.1) the unknown $\hat{v}(\varphi) = O(\varphi)^T \hat{u}(\varphi)$ will satisfy the ordinary differential equation $\nu s^2 \hat{v}(\varphi) - \partial_\varphi \hat{v}(\varphi) = O(\varphi)^T \hat{f}(\varphi)$. Hence by (2.3) $\hat{v}(\varphi) = \int_0^\infty e^{-\nu s^2 t} O(\varphi + t)^T \hat{f}(\varphi) dt$ and consequently
\begin{equation}
\hat{u}(\xi) = \int_0^\infty e^{-\nu s^2 t} O(t)^T \hat{f}(O(t)\xi) dt.
\end{equation}

Since $e^{-\nu |\xi|^2 t}$ multiplied by $(2\pi)^{-n/2}$ is the Fourier transform of the heat kernel
$$
E_t(x) = \frac{1}{(4\pi \nu t)^{n/2}} e^{-\frac{|x|^2}{4\nu t}}
$$
and since \( f(\widetilde{O}(t)x) = \hat{f}(O(t)\xi) \), (2.4) yields the formal solution

\[
(2.5) \quad u(x) = \int_0^\infty O(t)^T E_t * f(O(t)\cdot)(x) dt
\]

of (1.3).

Note that for \( n = 3 \) and \( f \in \mathcal{S}(\mathbb{R}^3)^3 \), the integrals (2.4) and (2.5) do in fact converge absolutely and define a distributional solution \( u \in \mathcal{S}'(\mathbb{R}^3)^3 \) of (1.3).

However, if \( n = 2 \), then both integrals fail to converge in \( \mathcal{S}'(\mathbb{R}^2)^2 \), even when \( f \in \mathcal{S}(\mathbb{R}^2)^2 \). This is not surprising, in view of a similar phenomenon for the Poisson equation in dimension 2. In this case, we need to modify (2.4), by defining a solution \( u \in \mathcal{S}'(\mathbb{R}^2)^2 \) e.g., by means of the convergent integral

\[
\langle u, \varphi \rangle = \langle \hat{u}, \varphi \rangle = \int_0^\infty e^{-\nu s t} O(t)^T \hat{f}(O(t)\xi) \cdot \varphi(\xi) dt d\xi
\]

for all \( \varphi \in \mathcal{S}(\mathbb{R}^2)^2 \); here \( \cdot \) denotes the inverse Fourier transform.

Then, in both dimensions \( n = 2, 3 \), for \( f \in \mathcal{S}(\mathbb{R}^n)^n \), we have constructed a solution \( u \in \mathcal{S}'(\mathbb{R}^n)^n \) of (1.3). Moreover, in the next section we shall prove that \( u \) satisfies inequality (1.4) in Theorem 1.1(1). In particular, \( ||\nabla^2 u||_q \leq c||f||_q < \infty \) for \( 1 < q < \infty \), yielding \( u \in L^1_{\text{loc}}(\mathbb{R}^n)^n \). We will conclude that, for any \( f \in L^q(\mathbb{R}^n)^n \), there is a solution \( u \in L^1_{\text{loc}}(\mathbb{R}^n)^n \) of (1.3) satisfying (1.4).

To this end, consider the sequence of balls \( B_m(0) \subset \mathbb{R}^n \) and choose a sequence \( \{f_j\} \subset \mathcal{S}(\mathbb{R}^n)^n \) converging to \( f \) in \( L^q(\mathbb{R}^n)^n \). Let \( u_j \) be the solution of (1.3) corresponding to \( f_j \). The proof of completeness of \( \dot{H}^{2,q}(\mathbb{R}^n) \) in [3] reveals that we can find a sequence of polynomials \( \{r_j\} \subset \Pi_1^n \) and \( \tilde{u} \in L^1_{\text{loc}}(\mathbb{R}^n)^n \) such that for \( j \to \infty \)

\[
||\nabla^2 ((u_j + r_j) - \tilde{u})||_q \to 0
\]

and

\[
(2.6) \quad (u_j + r_j)|_{B_m} \to \tilde{u}|_{B_m} \text{ in } L^q(B_m)^n \quad \text{for all } m \in \mathbb{N}.
\]

Then (2.6) implies that \( Lu_j + Lr_j \to L\tilde{u} \) in the sense of distributions, which shows that \( Lr_j \to L\tilde{u} - f \) in \( \mathcal{D}'(\mathbb{R}^n)^n \). And, since \( L\Pi_1^n \) is closed, as a linear subspace of the finite-dimensional space \( \Pi_1^n \), we see that \( L\tilde{u} - f = Lr \), for some \( r \in \Pi_1^n \). Thus, if we put \( u = \tilde{u} - r \), then \( u \in L^1_{\text{loc}}(\mathbb{R}^n)^n \) and \( ||\nabla^2 u||_q \leq c||f||_q \), so that \( u \) satisfies (1.4).
Observe next that formula (2.5) may be rewritten by using
\[ E_t * f(O(t)x) = (E_t * f)(O(t)x), \]
the proof of which is based on the radial symmetry of \( E_t(\cdot) \).

For \( n = 3 \) we arrive at the identity
\[ u(x) = \int_{\mathbb{R}^3} \Gamma(x,y)f(y)dy \]
with the fundamental solution
\[ \Gamma(x,y) = \int_0^\infty O(t)^T E_t(O(t)x - y)dt. \]

Furthermore \( \Delta u(x) \) can be represented — as \( u(x) \) in (2.7) — with the help of the kernel
\[ K(x,y) = \Delta_x \Gamma(x,y) \]
\[ = \int_0^\infty \Delta_x O(t)^T E_t(O(t)x - y)dt \]
\[ = \int_0^\infty O(t)^T \frac{1}{(4\pi\nu t)^{n/2}} \left( - \frac{n}{2\nu t} + \frac{\|O(t)x - y\|^2}{(2\nu t)^2} \right) \exp \left( -\frac{\|O(t)x - y\|^2}{4\nu t} \right) dt, \]
for \( n = 2 \) or \( n = 3 \), cf. (3.4) below.

The following proposition indicates that \( K(x,y) = \Delta_x \Gamma(x,y) \) does not define a classical Calderón–Zygmund integral operator:

**Proposition 2.1.**

1. Let \( n = 3 \). Then, for \( |x|, |y| \to \infty \), the fundamental solution \( \Gamma(x,y) \) is not bounded by \( C|y|^{-1} \). Actually there exists an \( \alpha > 0 \) such that for suitable \( x, y \in \mathbb{R}^3 \) with \( |x|, |y| \to \infty \)
\[ |\Gamma(x,y)| \geq \alpha \frac{\log |x|}{|y|}. \]

2. Let \( n = 2 \) or \( n = 3 \). Then there exists an \( \alpha > 0 \) and suitable \( x, y \in \mathbb{R}^n \) with \( |x|, |y| \to \infty \) such that the kernel
\[ K_1(x,y) = \int_0^\infty t^{-n/2} \frac{1}{t} e^{-\|O(t)x - y\|^2/t} dt \]
satisfies the estimate
\[ K_1(x,y) \geq \frac{\alpha}{|y|}. \]
The same result holds for the kernel \( K_2(x,y) \) where the term \( \frac{1}{t} \) in the definition of \( K_1 \) is replaced by \( \|O(t)x - y\|^2/t^2 \), cf. (2.9).
Proof. (1) Considering only the component $\Gamma_{3,3}(x, y)$ and points $x, y \in \mathbb{R}^3$ with equal third component $x_3 = y_3$ and of equal norm $r = |x| = |y|$ we use complex notation. Thus we may omit the third component of $x, y$ and we restrict ourselves to complex numbers $x = r$ and $y = re^{i\theta}$, $0 < \theta < \pi$, yielding

$$|O(t)x - y| = r|e^{it} - e^{i\theta}| = 2r\left|\sin\frac{\theta - t}{2}\right|$$

and $|x - y| = 2r\sin\frac{\theta}{2}$. Now $\Gamma_{3,3}(x, y)$ is bounded from below by $N \sum_{k=0}^{N} I_k(r, \theta)$, where $N = \lfloor 2r^2\sin^2\frac{\theta}{2} \rfloor$ and

$$I_k(r, \theta) = \int_{\theta/2}^{\theta/2 + 2k\pi} \frac{1}{(4\pi\nu t)^{3/2}} \exp\left(-r^2\sin^2\left|\frac{\theta - t}{2}\right|/(\nu t)\right) dt.$$  

We find constants $\alpha_j > 0$ independent of $r, \theta$ and of $k$ such that for $k \geq 1$

$$I_k(r, \theta) \geq \frac{\alpha_1}{k^{3/2}} \int_{-\theta/2}^{\theta/2} \exp\left(-\alpha_2 r^2 t^2/k\right) dt$$

$$= \frac{2\alpha_1}{r^k} \int_0^{\sqrt{\theta}/(2\sqrt{k})} \exp\left(-\alpha_2 s^2\right) ds.$$  

For $1 \leq k \leq N \sim r^2\theta^2$ and $r\theta \gg 1$, we find $\alpha_3 > 0$ such that $I_k(r, \theta) \geq \frac{\alpha_4}{r^k}$. Summing up we are led to the inequality

$$\Gamma_{3,3}(x, y) \geq \sum_{k=1}^{N} I_k(r, \theta) \geq \alpha_3 \sum_{k=1}^{N} \frac{1}{r^k} \geq \frac{\alpha_4 \log(r\theta)}{r}$$

with a constant $\alpha_4 > 0$ independent of $r$ and of $\theta$ when $r\theta \gg 1$.

(2) Again we use complex notation and consider points $x = r$, $y = re^{i\theta}$, $0 < \theta < \pi$, where now $r^2\theta \gg 1$. Then $K_1(x, y)$ is bounded from below by

$$\int_{\theta - \sqrt{\theta}/r}^{\theta + \sqrt{\theta}/r} t^{-n/2} \exp\left(-4r^2\sin^2\left|\frac{\theta - t}{2}\right|/t\right) dt$$

$$\geq \frac{\alpha_1}{\theta^{1+n/2}} \int_0^{\sqrt{\theta}/r} \exp\left(-\alpha_2 r^2 t^2/\theta\right) dt$$

$$\geq \frac{\alpha_1}{\theta^{1+n/2}} \int_0^1 e^{-\alpha_2 s^2} ds.$$  

Hence $K_1(x, y) \geq \frac{\alpha_3}{\theta^{n/2 - 1/2}|x - y|}$. The kernel $K_2(x, y)$ can be estimated analogously. \hfill $\square$

Before proving Theorem 1.1 in Section 3 below we consider the much simpler case $q = 2$, the question of separate estimates for $u_\theta$ and $\omega \wedge u$ and a variation of (2.10) when the integrals w.r.t. $t$ extend from $2\pi$ to $\infty$. 


Proposition 2.2. Given $f \in L^2(\mathbb{R}^n)$, $n = 2$ or $n = 3$, the solution $u$ of (1.3) given by (2.5) satisfies the estimate
\begin{equation}
\|\nabla^2 u\|_2 + \| (\omega \wedge x) \cdot \nabla u - \omega \wedge u \|_2 \leq c\|f\|_2.
\end{equation}

\textbf{Proof.} By Plancherel’s theorem, Fubini’s theorem and the inequality of Cauchy–Schwarz (with $s = |\xi|$)
\begin{align*}
\|\Delta u\|_2^2 &= \int_{\mathbb{R}^n} s^4 \left( \int_0^\infty e^{-|s|t^2} O(t)^T \hat{f}(O(t)\xi) \, dt \right)^2 \, d\xi \\
&\leq \int_{\mathbb{R}^n} \left( \int_0^\infty s^2 e^{-|s|t^2} \, dt \right) \cdot \left( \int_0^\infty s^2 e^{-|s|t^2} |\hat{f}(O(t)\xi)|^2 \, dt \right) \, d\xi \\
&= \frac{1}{\nu} \int_0^\infty \left( \int_{\mathbb{R}^n} s^2 e^{-|s|t^2} |\hat{f}(O(t)\xi)|^2 \, d\xi \right) \, dt \\
&= \frac{1}{\nu^2} \|f\|_2^2.
\end{align*}
Furthermore, for any second order partial derivative
\[ \| \partial_j \partial_k u \|_2 = \| \xi_j \xi_k \hat{u} \|_2 \leq \| |\xi|^2 \hat{u} \|_2 = \| \Delta u \|_2 \leq \frac{1}{\nu} \|f\|_2. \]

\hfill \Box

\textbf{Remark 2.3.} Inequality (2.10) cannot be improved in the sense that both $\|\omega \wedge u\|_2$ and $\| (\omega \wedge x) \cdot \nabla u \|_2$ are finite or can even be estimated by $\|f\|_2$.

In the two-dimensional case let
\[ u(x) = u(r, \theta) = a(r) \frac{1}{r} \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix} = a(r) \frac{1}{r} x^\perp \]
where $x^\perp$ is obtained from $x$ by rotation with the angle $\frac{\pi}{4}$ and $a \in C^\infty(\mathbb{R}_+)$ satisfies $a = 1$ for large $r$ and $a = 0$ for $r \in [0, 1)$. Obviously $u \in C^\infty(\mathbb{R}^2)^2$ is solenoidal, $|\nabla^2 u(x)| \sim \frac{1}{r}$ for large $r$ yielding $\nabla^2 u \in L^2(\mathbb{R}^2)^4$, supp $\Delta u \subset \text{supp } a$ and $\omega \wedge u = \frac{a(r)}{r} \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix} = u_0$. Consequently $\omega \wedge u - u_0 \equiv 0$ and the right-hand side $f = -\nu \Delta u \in L^2(\mathbb{R}^2)^2$, but $|\omega \wedge u| \sim \frac{1}{r} \not\in L^2(\mathbb{R}^2)$. An analogous result holds in $L^q$-spaces, $q \neq 2$, when choosing $u(x) = a(r)r^{-\lambda} x^\perp$ for suitable $\lambda > 0$.

Proposition 2.4. Let $f \in L^q(\mathbb{R}^2)^2$ satisfy the compatibility conditions
\begin{equation}
f_m(r) := \frac{1}{2\pi} \int_0^{2\pi} O(\theta)^T f(r, \theta) \, d\theta = 0 \quad \text{for a.a. } r > 0.
\end{equation}

Then one can find a suitable representative $u$ of the unique solution in $H^2(\mathbb{R}^2)^2$ of (1.3) given by Theorem 1.1, satisfying the estimate
\[ \|\nabla^2 u\|_q + \|\partial_\theta u\|_q + \|u\|_q \leq c\|f\|_q. \]
An analogous result holds for \( n = 3 \) where (2.11) is replaced by the assumption \( \frac{1}{2\pi} \int_0^{2\pi} O(\theta)^T f(r, \theta, x_3) \, d\theta = 0 \) for a.a. \( r = \sqrt{x_1^2 + x_2^2} > 0, \ x_3 \in \mathbb{R}. \)

**Proof.** The main idea is to show that the integral mean

\[
\mathcal{u}_m(r) = \frac{1}{2\pi} \int_0^{2\pi} O(\theta)^T \mathcal{u}(r, \theta) \, d\theta
\]

vanishes for a.a. \( r > 0 \), for a suitable representative \( \mathcal{u} \); for \( n = 3 \) the integral mean \( \mathcal{u}_m(r, x_3) \) is defined analogously. Then the identity \( O(\theta)\partial_\theta(O(\theta)^T \mathcal{u}) = \partial_\theta \mathcal{u} - \omega \wedge \mathcal{u} \) and Wirtinger's inequality will imply that

\[
\|u\|^q_q = \int_0^\infty r \int_0^{2\pi} |O(\theta)^T \mathcal{u}(r, \theta)|^q \, d\theta \, dr
\leq c\|\partial_\theta (O(\theta)^T \mathcal{u})\|^q_q \leq c\|\partial_\theta \mathcal{u} - \omega \wedge \mathcal{u}\|^q_q,
\]

and Theorem 1.1(1) will complete the proof for \( n = 2 \) and also for \( n = 3 \).

In order to prove that \( \mathcal{u}_m(r) \equiv 0 \) notice that, for \( n = 2, \ \tilde{u}(x) = O(\theta)u_m(r) \) satisfies (1.3) with \( f \) replaced by \( f = 0 \) since

\[
\mathcal{L}(\tilde{u}) = \mathcal{L}(O(\theta)u_m(r)) = O(\theta)(\mathcal{L}u)_m(r) = O(\theta)f_m(r) = 0.
\]

Furthermore, since \( \tilde{u} \in S'(\mathbb{R}^2)^2 \), the proof of Theorem 1.1(2), see Section 3 below, implies that \( \tilde{u} \in \Pi^2_1 \). Replacing \( u \) by \( u - \tilde{u} \), we may then assume that \( \mathcal{u}_m = 0 \). This argument easily extends to the case \( n = 3 \). \( \square \)

**Remark 2.5.** The difficulties in the proof of Theorem 1.1 when estimating \( \Delta \mathcal{u} \) with \( \mathcal{u} \) given by (2.5) arise from the corresponding integrals on \( (0, \varepsilon), \ \varepsilon > 0 \). Actually, consider the operator \( \mathcal{S} \) on \( L^q(\mathbb{R}^n) \) given by

\[
\mathcal{S}f(x) = \int_{\mathbb{R}^n} (-\Delta)O(t)^T \mathcal{E}_t * f(O(t)\cdot)(x) \, dt,
\]

i.e., in Fourier space

\[
\mathcal{S}\hat{f}(\xi) = \int_{\mathbb{R}^n} s^2 e^{-\nu s^2 t}O(t)^T \hat{f}(O(t)\xi) \, dt, \quad s = |\xi|.
\]

Since \( O(t) \) is 2\( \pi \)-periodic and \( s^2 \sum_{k=1}^\infty e^{-2k\pi\nu s^2} = s^2 e^{-2\pi\nu s^2} (1 - e^{-2\pi\nu s^2})^{-1} =: m(\xi) \), we get that

\[
\mathcal{S}\hat{f}(\xi) = m(\xi) \int_0^{2\pi} e^{-\nu s^2 t}O(t)^T \hat{f}(O(t)\xi) \, dt
= m(\xi) \mathcal{F} \left( \int_0^{2\pi} O(t)^T \mathcal{E}_t * f(O(t)\cdot)(x) \, dt \right).
\]

Obviously \( m(\xi) \) satisfies the classical Michlin–Hörmander multiplier condition, cf. [9], and due to properties of the heat kernel

\[
\left\| \int_0^{2\pi} O(t)^T \mathcal{E}_t * f(O(t)\cdot)(x) \, dt \right\|_q \leq \int_0^{2\pi} \| f(O(t)\cdot) \|_q \, dt = 2\pi \| f \|_q.
\]
Then multiplier theory yields the estimate $\|Sf\|_q \leq c\|f\|_q$ for every $q \in (1, \infty)$ with a constant $c = c(m, q)$.

3. Proof of Theorem 1.1

Due to the well-known estimate $\|\partial_j \partial_k u\|_q \leq c\|\Delta u\|_q$, $1 < q < \infty$, $1 \leq j, k \leq n$, cf. [9], it suffices to consider only $\Delta u$. The main ideas are Littlewood–Paley theory and a decomposition of the integral operator

$$Tf(x) = \int_0^\infty (-\Delta)O(t)^T(E_t * f)(O(t)x)dt = \int_{\mathbb{R}^n} K(x, y)f(y)dy$$

in Fourier space where each integral kernel has compact support. Since

$$\mathcal{F}(\Delta O(t)^T(E_t * f)(O(t) \cdot))(\xi) = O(t)^T|\xi|^2 e^{-\nu|\xi|^2} \hat{f}(O(t)\xi)$$

define $\psi \in \mathcal{S}(\mathbb{R}^n)$ by

$$\hat{\psi}(\xi) = (2\pi)^{-n/2}|\xi|^2 e^{-\nu|\xi|^2} = (-\Delta)^1$$

and

$$\psi_t(x) = t^{-n/2}\psi\left(\frac{x}{\sqrt{t}}\right), \quad \hat{\psi}_t(\xi) = \hat{\psi}(\sqrt{t}\xi) = (2\pi)^{-n/2}t|\xi|^2 e^{-\nu t|\xi|^2}.$$  

Thus the kernel $K(x, y)$ may be written in the form

$$K(x, y) = \int_0^\infty O(t)^T \psi_t(O(t)x - y) \frac{dt}{t}.$$  

To decompose $\hat{\psi}_t$ choose $\tilde{\varphi}, \tilde{\chi} \in C_0^\infty(1/2, 2)$ such that $0 \leq \tilde{\varphi}, \tilde{\chi} \leq 1$ and

$$\sum_{j=-\infty}^{\infty} \tilde{\chi}(2^{-j}r) = 1, \quad \sum_{j=-\infty}^{\infty} \tilde{\varphi}(sr)^2 \frac{ds}{s} = \frac{1}{2} \text{ for all } r > 0.$$  

Then define for $\xi \in \mathbb{R}^n$ and for $j \in \mathbb{Z}$, $s > 0$

$$\hat{\chi}_j(\xi) = \tilde{\chi}(2^{-j}|\xi|), \quad \hat{\varphi}_s(\xi) = \tilde{\varphi}(\sqrt{s}|\xi|)$$

yielding

$$\text{supp } \hat{\chi}_j \subset A(2^{j-1}, 2^{j+1}) := \{ \xi \in \mathbb{R}^n : 2^{j-1} < |\xi| < 2^{j+1} \},$$

$$\text{supp } \hat{\varphi}_s \subset A\left(\frac{1}{2\sqrt{s}}, \frac{2}{\sqrt{s}}\right);$$

moreover $\int_{\mathbb{R}^n} \varphi_s(x)dx = 0$ and

$$\sum_{j=-\infty}^{\infty} \hat{\chi}_j(\xi) = 1, \quad \int_0^\infty \hat{\varphi}_s(\xi)^2 \frac{ds}{s} = 1 \quad (\xi \neq 0).$$
The family of functions \( \{ \varphi_s : s > 0 \} \) will be used in Littlewood–Paley theory, see I§8.23 in [10], yielding the inequalities
\[
\| f \|_q \leq \left\| \left( \int_0^\infty |\varphi_s * f(\cdot)|^2 \frac{ds}{s} \right)^{1/2} \right\|_q \leq c_2 \| f \|_q
\]
with constants \( c_1, c_2 > 0 \) depending on \( q \in (1, \infty) \), but independent of \( f \in L^q(\mathbb{R}^n)^n \). Furthermore we decompose \( K \) by defining \( \psi_j \in \mathcal{S}(\mathbb{R}^n) \) by
\[
\psi_j = (2\pi)^{-n/2} \chi_j \ast \psi \quad \text{or equivalently} \quad \hat{\psi}_j = \hat{\chi}_j \cdot \hat{\psi}, \quad j \in \mathbb{Z},
\]
yielding \( \psi = \sum_{j=-\infty}^{\infty} \psi_j \) and, cf. (3.4),
\[
K_j(x, y) = \int_0^\infty O(t) T \psi_j(t)(O(t)x - y) \frac{dt}{t}, \quad j \in \mathbb{Z}.
\]
Given \( K_j \) we define the operator
\[
T_j f(x) = \int_{\mathbb{R}^n} K_j(x, y) f(y) dy = \int_0^\infty O(t)^T (\psi_t^j * f)(O(t)x) \frac{dt}{t}
\]
such that formally and even w.r.t to the operator norm topology \( T = \sum_{j=-\infty}^{\infty} T_j \), see the proof below.

**Lemma 3.1.** The functions \( \psi_t^j \) have the following properties:

1. For \( j \in \mathbb{Z} \) and \( t > 0 \)
   \[
   \text{supp} \hat{\psi}_t^j \subset A \left( \frac{2^j-1}{\sqrt{t}}, \frac{2^j+1}{\sqrt{t}} \right)
   \]

2. For \( m > \frac{n}{2} \) let \( h(x) = (1+|x|^2)^{-m} \) and, cf. (3.3), \( h_t(x) = t^{-n/2} h \left( \frac{x}{\sqrt{t}} \right) \).
   Then there exists a constant \( c > 0 \) independent of \( j \in \mathbb{Z} \) such that
   \[
   |\psi^j(x)| \leq c 2^{-2j} |h_{2-2j}(x)| \quad \text{for all } x \in \mathbb{R}^n.
   \]
   In particular
   \[
   \| \psi^j \|_1 \leq c 2^{-2|j|}.
   \]

**Proof.** (1) is obvious due to (3.3), (3.5) and (3.8). To prove (2) we show first of all the pointwise estimate
\[
|2^j|^{\alpha} \partial_\alpha \hat{\psi}_t^j(\xi)| \leq c_\alpha 2^{-2|j|} \eta(2^{-j}|\xi|)
\]
for all \( \xi \in \mathbb{R}^n, j \in \mathbb{Z}, \) for all multi-indices \( \alpha \in \mathbb{N}_0^n \) and with a function \( \eta \in C_0^\infty \left( \frac{1}{4}, 4 \right) \), \( 0 \leq \eta \leq 1 \). By the definition of \( \hat{\chi}_j, (3.5) \) and the pointwise estimates
\[
|\partial_\beta \hat{\psi}(\xi)| \leq c_{\beta,N} \begin{cases} 
|\xi|^{\max(0, 2-|\beta|)}, & |\xi| < 1 \\
|\xi|^{-N}, & |\xi| \geq 1
\end{cases}, \quad \beta \in \mathbb{N}_0^n,
\]
for every $N \in \mathbb{N}$, cf. (3.2), Leibniz’s formula yields the estimate

$$|2^{|a|} \partial^\alpha \hat{\psi}_j(\xi)| \leq c \sum_{0 \leq \beta \leq \alpha} 2^{|a|} |\partial^{\alpha-\beta} \chi(2^{-j}|\xi|)| |\partial^{\beta} \hat{\psi}(\xi)|$$

$$\leq c \sum_{0 \leq \beta \leq \alpha} 2^{|a|} |2^{|a|} \eta(2^{-j}|\xi|)| |\partial^{\beta} \hat{\psi}(\xi)|.$$  

For $j \geq 0$ where only $|\xi| \sim 2^j$ has to be considered, we get (3.11) immediately, even with $2^{-N-1}$ replacing $2^{-2j}$. For $j < 0$ and $|\xi| \sim 2^j$, the right-hand side of the last inequality is bounded by

$$c \sum_{0 \leq \beta \leq \alpha} |\eta(2^{-j}|\xi|)| 2^j \lambda_{\eta}|2^{-j}| \leq c 2^{-2|j|} |\eta(2^{-j}|\xi|)|.$$

Now (3.11) is proved.

To estimate $\psi^j(x)$ we use for $m > \frac{n}{2}$ the identity

$$(1 + |2^j x|^2)^m \psi^j(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} (1 - 2^j \Delta)^m \hat{\psi}_j(\xi) e^{ix \cdot \xi} \, d\xi.$$

By (3.11)

$$|(1 - 2^j \Delta)^m \hat{\psi}_j(\xi)| \leq C_{m,N} 2^{-2|j|} |\eta(2^{-j}|\xi|)|$$

for all $j \in \mathbb{Z}$ and $\xi \in \mathbb{R}^n$. Hence

$$\|1 - 2^j \Delta|^{m} \hat{\psi}_j\|_1 \leq C_m 2^{nj - 2|j|}$$

and consequently $|(1 + |2^j x|^2)^m \psi^j(x)| \leq c 2^{nj - 2|j|}$ proving Part (2). \qed

Lemma 3.2. For $j \in \mathbb{Z}$ let $\mathcal{M}^j$ denote the maximal operator

$$\mathcal{M}^j g(x) = \sup_{r > 0} \int_{A_r} (|\psi^j| * |g|)(O(t)x) \frac{dt}{t}$$

where $A_r = \left[ \frac{r}{16}, 16r \right]$. Then for $q \in (2, \infty)$ the operator $T_j$ satisfies the estimate

$$\|T_j f\|_q \leq c \|\psi^j\|_1^{1/2} \|\mathcal{M}^j\|_{(q/2)'}^{1/2} \|f\|_q$$

with a constant $c > 0$ independent of $j \in \mathbb{Z}$. The term $\|\mathcal{M}^j\|_{(q/2)'}$ denotes the operator norm of the sublinear operator $\mathcal{M}^j$ on $L^{(q/2)'}(\mathbb{R}^n)$, where $1 + \frac{1}{(q/2)'} = 1$.

Proof. To estimate $\|T_j f\|_q$ we use the Littlewood–Paley decomposition (3.7) of $T_j f$ and find a function $0 \leq g \leq L^{(q/2)'}(\mathbb{R}^n)$ with $\|g\|_{(q/2)'} = 1$ (note that $q > 2$) such that

$$\|T_j f\|_q^2 \leq \frac{1}{c_1^2} \|\int_0^\infty |\varphi_s * T_j f(\cdot)|^2 \frac{ds}{s}\|_{q/2}$$

$$= \frac{1}{c_1^2} \int_0^\infty \int_{\mathbb{R}^n} |\varphi_s * T_j f|^2 g dx \frac{ds}{s}. $$
By (3.9), (3.10)
\[
\varphi_s \ast T_j f(x) = \int_0^\infty O(t)^T (\varphi_s \ast \psi_t^j) (O(t)x) \frac{dt}{t},
\]
where due to (3.5) \(\varphi_s \ast \psi_t^j = 0\) unless \(t \in A(s, j) := [2^{2j-4}s, 2^{2j+4}s]\). Since
\[
\int_{t \in A(s, j)} \frac{dt}{t} = \log 2^8
\]
for every \(j \in \mathbb{Z}, s > 0\), the inequality of Cauchy–Schwarz and the associativity of convolutions yield
\[
|\varphi_s \ast T_j f(x)|^2 \leq c \int_{A(s, j)} |(\psi_t^j \ast (\varphi_s \ast f))(O(t)x)|^2 \frac{dt}{t}
\]
\[
\leq c \|\psi_t^j\|_1 \int_{A(s, j)} (|\psi_t^j| \ast |\varphi_s \ast f|^2)(O(t)x) \frac{dt}{t}.
\]
Here we used the inequality
\[
||\psi_t^j \ast (\varphi_s \ast f))|y|^2 \leq \|\psi_t^j\|_1 (|\psi_t^j| \ast |\varphi_s \ast f|^2)(y)
\]
and that \(\|\psi_t^j\|_1 = \|\psi_t^j\|\) for all \(t > 0\). Thus
\[
\|T_j f\|_q^2 \leq c \|\psi_t^j\|_1 \int_{\mathbb{R}^n} \int_0^\infty \int_{A(s, j)} (|\psi_t^j| \ast |\varphi_s \ast f|^2)(x) g(O(-t)x) dx \frac{dt}{t} ds.
\]
In the inner integral on \(\mathbb{R}^n\) note that \(\phi = |\psi_t^j|\) is radially symmetric; thus for arbitrary functions \(f\) and \(h\) we get \(\int (\phi \ast f)h \, dx = \int f \phi \ast h \, dx\). Then the elementary identity \(\phi \ast [g(O(-t) \cdot)] = (\phi \ast g)(O(-t) \cdot)\) implies that
\[
\|T_j f\|_q^2 \leq c \|\psi_t^j\|_1 \int_{\mathbb{R}^n} \int_0^\infty |\varphi_s \ast f|^2(x) \int_{A(s, j)} (|\psi_t^j| \ast g)(O(-t)x) \frac{dt}{t} ds \, dx.
\]
Here the inner integral on \(A(s, j)\) is bounded by \(\mathcal{M}jg(x)\) uniformly in \(s > 0\). Now Hölder’s inequality and (3.7) show that
\[
\|T_j f\|_q^2 \leq c \|\psi_t^j\|_1 \left( \int_{\mathbb{R}^n} \left( \int_0 |\varphi_s \ast f|^2 \frac{ds}{s} \right)^{q/2} dx \right)^{2/q} \|\mathcal{M}jg\|_{(q/2)'}
\]
\[
\leq c c_2 \|\psi_t^j\|_1 \|f\|_2^{q/2} \|\mathcal{M}j\|_{(q/2)'} \|g\|_{(q/2)'}.
\]
Since \(\|g\|_{(q/2)'} = 1\), the proof is complete. \(\square\)

**Lemma 3.3.** Let \(\mathcal{M}\) denote the classical Hardy–Littlewood maximal operator on \(\mathbb{R}^n\), i.e.,
\[
\mathcal{M}g(x) = \sup_{r > 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |g(y)| \, dy,
\]
and let \(\tilde{\mathcal{M}}_\theta g\) denote the “angular” maximal operator
\[
\tilde{\mathcal{M}}_\theta g(x) = \sup_{r > 0} \int_{A_r} |g(O(t)^T x)| \frac{dt}{t},
\]
where \( A_r = [\frac{r}{16}, 16r] \). Then \( \mathcal{M}^j \) in Lemma 3.2 satisfies the estimates

\[
\mathcal{M}^j g(x) \leq c 2^{-2|j|} \mathcal{M}(\widetilde{M}_\theta g)(x) \quad \text{for a.a. } x \in \mathbb{R}^n,
\]

\[
\|\mathcal{M}^j g\|_q \leq c 2^{-2|j|} \|g\|_q \quad \text{for } 1 < q < \infty.
\]

**Proof.** By Lemma 3.1 (2) \(|\psi^j_t(x)| \leq c 2^{-2|j|} h_{t_2-2j}(x)\) and consequently

\[
\mathcal{M}^j g(x) \leq c 2^{-2|j|} \sup_{r>0} \int_{A_r} (h_{t_2-2j} * |g|)(O(t)^T x) \frac{dt}{t}.
\]

There exists a constant \( c > 0 \) independent of \( r, j \) such that \( h_{t_2-2j} \leq ch_{r_2-2j} \) for all \( t \in A_r \). Hence

\[
\mathcal{M}^j g(x) \leq c 2^{-2|j|} \sup_{r>0} h_{r_2-2j} * \int_{A_r} |g|(O(t)^T x) \frac{dt}{t}
\]

\[
\leq c 2^{-2|j|} \sup_{t>0} h_t * \mathcal{M}_\theta g(x).
\]

Note that \( h \) is a nonnegative, radially decreasing function and that \( \int h_t \, dx = c_0 > 0 \) for all \( t > 0 \). Therefore we conclude by II§2.1 in [10] that

\[
\sup_{t>0} h_t * \mathcal{M}_\theta g(x) \leq c_0 \mathcal{M}(\mathcal{M}_\theta g)(x)
\]

proving the first assertion.

For \( q \in (1, \infty) \) the maximal operator \( \mathcal{M} \) is bounded on \( L^q(\mathbb{R}^n) \). Concerning \( \mathcal{M}_\theta \) we consider for given \( g \in L^q(\mathbb{R}^n) \) its restriction

\[
g_r(\theta) = g(r, \theta) \quad \text{or} \quad g_{r,x_3}(\theta) = g(r, \theta, x_3)
\]

for \( n = 2 \) or \( n = 3 \), resp., when using polar or cylindrical coordinates. For \( n = 2 \) \( g_r(\theta) \in L^q(0, 2\pi) \) for a.a. \( r > 0 \) by Fubini’s theorem, and with the classical one-dimensional Hardy–Littlewood maximal operator \( \mathcal{M}_1 \) on \( L^q(0, 2\pi) \)

\[
|\mathcal{M}_\theta g(r, \theta)| \leq c (\mathcal{M}_1 g_r)(\theta) \quad \text{for a.a. } r > 0.
\]

Thus

\[
\|\mathcal{M}_\theta g\|_q \leq c \int_0^\infty r \|\mathcal{M}_1 g_r\|_{L^q(0, 2\pi)}^q dr \leq c \int_0^\infty r \|g_r\|_{L^q(0, 2\pi)}^q dr = c \|g\|_q^q
\]

due to the \( L^q \)-boundedness of \( \mathcal{M}_1 \). For \( n = 3 \) the proof is analogous. \( \square \)

**End of the proof of Theorem 1.1 (1).** Let \( q \in (2, \infty) \). Then by Lemmata 3.1-3.3

\[
\|T_j f\|_q \leq c 2^{-|j|} \cdot 2^{-|j|} \|f\|_q.
\]

Thus \( \sum_{j \in \mathbb{Z}} T_j \) converges in the \( L^q \)-operator norm and \( T = \sum_{j \in \mathbb{Z}} T_j \) is bounded on \( L^q(\mathbb{R}^n) \) for \( q > 2 \).
Closely related to $T$ is the operator $T^* f(x) = \int K^*(x,y) f(y) dy$ with kernel
\[ K^*(x,y) = \int_0^\infty \psi(t) (O(t) y - x) O(t) \frac{dt}{t}. \]

Analogous arguments as before show that $T^*$ is bounded on $L^q(\mathbb{R}^n)^n$ for every $q > 2$. Now let $q \in (1,2)$. Then for $f \in L^q(\mathbb{R}^n)^n$, $g \in L^q(\mathbb{R}^n)^n$
\[ |\langle T f, g \rangle| = |\langle f, T^* g \rangle| \leq \|f\|_q \|g\|_q' \]
implying the $L^q$-boundedness of $T$. The case $q = 2$ had been considered in Proposition 2.2.

**Proof of Theorem 1.1(2).** It suffices to prove that every solution $u \in S'(\mathbb{R}^3)^3$ of (1.3) when $f = 0$ and $\nabla^2 u \in L^q(\mathbb{R}^3)$ equals a polynomial of the form $\alpha \omega + \beta \omega \wedge x + (\gamma x_1, \gamma x_2, \delta x_3) T$. Given $u$ define $\hat{v}(s', \varphi, \xi_3) = O(\varphi)^T \hat{u}(s', \varphi, \xi_3) \in S'(\mathbb{R}^3)^3$ using cylindrical coordinates for $\xi \in \mathbb{R}^3$ and $s' = \sqrt{\xi_1^2 + \xi_2^2}$. Then, cf. Section 2,
\[ \nu |\xi|^2 \hat{v} - \partial_\varphi \hat{v} = 0 \quad \text{in } S'(\mathbb{R}^3)^3. \]
Let us show that $\langle \hat{v}, \psi \rangle = 0$ for all $\psi \in C^\infty_c(\mathbb{R}^3 \setminus \{0\})^3$. Given $\psi$ define
\[ \psi_0(s', \varphi, \xi_3) = e^{-\nu |\xi|^2} \int_{-\infty}^\nu e^{|\xi|^2} \psi(s', \varphi, \xi_3) d\varphi. \]
Obviously $\psi_0 \in C^\infty_c(\mathbb{R}^3 \setminus \{0\})^3$ and $(\nu |\xi|^2 + \partial_\varphi) \psi_0 = \psi$. Consequently
\[ \langle \hat{v}, \psi \rangle = \langle \hat{v}, (\nu |\xi|^2 + \partial_\varphi) \psi_0 \rangle = \langle (\nu |\xi|^2 - \partial_\varphi) \hat{v}, \psi_0 \rangle = 0 \]
proving that $\text{supp } \hat{v} \subset \{0\}$ and also $\text{supp } \hat{u} \subset \{0\}$. Hence $u$ is a polynomial.
Since $\nabla^2 u \in L^q(\mathbb{R}^3)$, $u$ is even affine linear, $u(x) = a + Bx$ for $a \in \mathbb{R}^3$, $B \in \mathbb{R}^{3 \times 3}$. Then (1.3) with $f = 0$, i.e., $(\omega \wedge x) \cdot \nabla u = \omega \wedge u$, shows that $\omega \wedge a = 0$ or equivalently $a = \alpha \omega$, $\alpha \in \mathbb{R}$. Furthermore $Bx$ must be of the form $Bx = \beta \omega \wedge x + (\gamma x_1, \gamma x_2, \delta x_3) T$ with constants $\beta, \gamma, \delta \in \mathbb{R}$. For $n = 2$ one easily obtains that $a = 0$ and $Bx = \beta \omega \wedge x + \gamma x$.

**Proof of Theorem 1.1(3).** As explained in Section 1 problem (1.2) may be reduced to (1.3) by solving the equation
\[ \Delta p = \text{div } f + \nu \Delta g + \partial_\theta g = \text{div } F \quad \text{in } \mathbb{R}^n \]
where $F = f + \nu \nabla g + (\omega \wedge x) g$ satisfies the estimate $\|F\|_q \leq c(\|f\|_q + \|\nu \nabla g + (\omega \wedge x) g\|_q)$. Thus $\text{div } F$ may be considered as a continuous linear functional on $H^{1,q}_0(\mathbb{R}^n)$. Since the operator $\Delta$ is easily seen to be an isomorphism from $H^{1,q}_0(\mathbb{R}^n)$ to its dual $H^{1,q}_0(\mathbb{R}^n)^*$ there exists a unique $p \in H^{1,q}_0(\mathbb{R}^n)$ solving $\Delta p = \text{div } F$ and satisfying $\|\nabla p\|_q \leq c \|F\|_q$. Then Part (1) yields a $u \in H^{2,q}_0(\mathbb{R}^n)^n$ satisfying $-\nu \Delta u - \partial_\theta u + \omega \wedge u = f - \nabla p$ and the estimate $\|\nabla^2 u\|_q + \|\partial_\theta u - \omega \wedge u\|_q \leq c(\|f\|_q + \|\nabla p\|_q)$. In particular $(-\nu \Delta - \partial_\theta)(\text{div } u = \text{div } f - \Delta p$ and consequently $(-\nu \Delta - \partial_\theta)(\text{div } u - g) = 0$. By the reasoning of Part (2) we may conclude that $\text{div } u - g$ is a polynomial and due to the
integrability assumptions even a constant. Replacing $u$ by $u - \gamma(x_1, x_2, 0)^T$, if necessary, we get a solution $(u, p)$ of (1.2) satisfying also $\text{div} \, u = g$. The uniqueness assertion is proved as in Part (2).

\[\square\]

**References**


Received July 1, 2003. The second author was supported in part by an Alexander von Humboldt research fellowship, Germany.

**Department of Mathematics**  
Darmstadt University of Technology  
Schloßgartenstr. 7  
D-64289 Darmstadt  
Germany  
E-mail address: farwig@mathematik.tu-darmstadt.de

**Faculty of Engineering**  
Niigata University  
Niigata 950–2181  
Japan
E-mail address: hishida@eng.niigata-u.ac.jp

Mathematisches Seminar
Universität Kiel
D-24118 Kiel
Germany
E-mail address: mueller@math.uni-kiel.de
FIBRATIONS ON BANACH MANIFOLDS

OLIVIA GUTÚ AND JESÚS A. JARAMILLO

Let \( f \) be a split submersion between paracompact Banach manifolds. We obtain here various conditions for \( f \) to be a fiber bundle. First, we give general conditions in terms of path-liftings. As a consequence, we deduce several criteria: For example, \( f \) is a fiber bundle provided it satisfies either some topological requirements (such as being a proper or a closed map) or, in the case of Finsler manifolds, some metric requirements (such as Hadamard integral condition).

1. Introduction

A classical theorem of Ehresmann [5] asserts that, if \( M \) and \( N \) are finite-dimensional manifolds with \( M \) paracompact and \( N \) connected, every proper submersion \( f : M \to N \) is a fiber bundle. This result was extended by Earle and Eells [4] to the case where \( M \) and \( N \) are Finsler manifolds modelled on Banach spaces, \( M \) complete, and \( f : M \to N \) is a proper surjective submersion with split kernels. In fact, Earle and Eells obtained a more general result (see [4, Theorem 3C]) in which the fiber bundle structure depends on the existence of a certain kind of right inverse of the differential \( df \). More recently, Rabier [14] extends the theorem of Ehresmann by proving that, if \( M \) and \( N \) are Finsler manifolds modelled on Banach spaces with \( M \) complete and \( N \) connected, and \( f : M \to N \) is a “strong submersion with uniformly split kernels”, then \( f \) is a fiber bundle (see Section 4 for this terminology). A large number of applications and ramifications of this result are also discussed in [14]. On the other hand, Plastock [13] obtained conditions for a function \( f \) to be globally equivalent to a projection, in the particular case that \( f \) is a nonlinear Fredholm map between Banach spaces. Plastock used a powerful continuation method (the “method of line lifting”), which had proved to be also quite useful in problems of global inversion of functions (see e.g., [12], [15] and [7]).

In this paper we are concerned with the problem of giving conditions for a submersion to be a fibre bundle. Our purpose is to make a direct connection between the fiber bundle structure and suitable path-lifting properties. In this way, we obtain some fairly general results, formulated in terms of path-liftings, and from which all the above mentioned theorems can be derived as corollaries. The contents of the paper are as follows: In Section 2 we consider
a split submersion \( f : M \to N \) between paracompact Banach manifolds, \( N \) connected. Note that no Finsler structure is needed here. We define the continuation property for \( f \), and we prove that it implies that \( f \) is a fiber bundle. Then, some consequences and variants of this result are given. In particular, we obtain that \( f \) is a fiber bundle provided it is either a proper or a closed map (conditions which are in turn equivalent in this setting). In Section 3 we assume, in addition, that \( M \) and \( N \) are Finsler manifolds, \( M \) complete. We introduce the bounded path-lifting property for \( f \), proving that it is a sufficient condition for \( f \) to be a fiber bundle. Then we describe several instances where \( f \) has the bounded path-lifting property. For example, this is the case (for connected manifolds) when \( f \) satisfies a Hadamard integral condition. This integral condition was first used by Hadamard \([8]\) in problems of global inversion of functions, and it has been widely used in this context (see e.g., \([12]\), \([13]\), \([11]\), \([7]\) and references therein). On the other hand, the case where \( f \) is a local diffeomorphism is also considered, and conditions are given here for \( f \) to be a covering projection or a global diffeomorphism. Finally, Section 4 is devoted to submersions with uniformly split kernels in the sense of Rabier. We introduce a path-lifting condition adapted to this case, which allows us to apply our previous results.

2. Fiber bundles via continuation property

We start with some notations and definitions. Throughout this paper, \( M \) and \( N \) will denote \( C^2 \) paracompact Banach manifolds without boundary, modelled on Banach spaces \( E \) and \( F \), respectively. Following the terminology of \([10]\), by \( C^k \) we mean “\( C^{k-1} \) with \((k-1)\)-th derivative locally Lipschitz”. In this way, a cross section \( s : M \to TM \) is said to be of class \( C^1 \) if for each \( x \in M \) there is a chart \( \phi : V \to E \) at \( x \), such that the map \( d\phi(\phi^{-1}(\cdot)) \circ s(\phi^{-1}(\cdot)) : \phi(V) \to E \) is locally Lipschitz. As it might be suspected, every map of class \( C^2 \) is of class \( C^1 \) and the composition of \( C^2 \) maps is of class \( C^2 \). As usual, \( f \) is said to be a split submersion if, for each \( x \in M \), \( df(x) \in \mathcal{L} (T_xM; T_{f(x)}N) \) is surjective and its kernel splits. It will be quite useful for us the possibility of “gluing together” continuous linear sections of each \( df(x) \) in a locally Lipschitz way. More precisely, we will say that \( s(\cdot) \) is a \( C^1 \) right inverse of \( df(\cdot) \) if the following two conditions are satisfied:

(i) For every \( x \in M \), \( s(x) \in \mathcal{L} (T_{f(x)}N; T_xM) \) and \( df(x) \circ s(x) = id \).
(ii) For every chart \( \phi : V \to E \) in \( M \) and every chart \( \psi : W \to F \) in \( N \) with \( f(V) \subset W \), the map
\[
d\phi(\phi^{-1}(\cdot)) \circ s(\phi^{-1}(\cdot)) \circ [d\psi(f(\phi^{-1}(\cdot)))]^{-1}
\]
is locally Lipschitz in \( \phi(V) \).

The next well-known lemma provides us the existence of such a kind of right inverse for \( df(\cdot) \). It depends on the fact that every paracompact Banach manifold admits partitions of unity of class \( C^{1-} \) (see [10]). A direct proof for the case of a Fredholm submersion between Banach spaces is given in [13, Lemma 2.6].

Lemma 2.1 ([4, Lemma 3(B)]). Let \( M \) and \( N \) be \( C^2 \) paracompact Banach manifolds and let \( f : M \to N \) be a split submersion of class \( C^{2-} \). Then, there exists a \( C^{1-} \) right inverse of \( df(\cdot) \).

Remark 2.2. In the special case that \( M \) and \( N \) are Hilbert manifolds, there is a canonical \( C^{1-} \) right inverse of \( df(\cdot) \), which is given by an explicit formula. Namely
\[
s_0(x) = (df(x)[|\ker df(x)|^{-1}])^{-1} = df(x)^*[df(x)df(x)^*]^{-1},
\]
where \( df(x)^* \in \mathcal{L}(T_f(x)N; T_xM) \) denotes the Hilbert space adjoint of \( df(x) \) (see [13, Lemma 2.5]).

Next we give the central result of this section. First, we need the following definition: We shall say that \( f : M \to N \) has the continuation property if, for every \( C^1 \) path \( p : [0,1] \to N \) and every \( C^1 \) path \( q : [0,b) \to M \) with \( f \circ q = p \), where \( 0 < b \leq 1 \), there exists an increasing sequence \( t_n \to b \) such that the sequence \( \{q(t_n)\} \) converges in \( M \).

Theorem 2.3. Let \( M \) and \( N \) be \( C^2 \) paracompact Banach manifolds, \( N \) connected. Let \( f : M \to N \) be a split submersion of class \( C^{2-} \). If \( f \) has the continuation property then \( f \) is a fiber bundle.

Proof. We are much inspired by the proof of Theorem 4.1 in [14]. First of all, consider a \( C^{1-} \) right inverse \( s(\cdot) \) of \( df(\cdot) \), which always exists by Lemma 2.1. Let \( \{(W_\kappa, \psi_\kappa) : \kappa \in \Lambda \} \) be an atlas of \( N \). Without loss of generality, we can suppose that for every \( \kappa \in \Lambda \), \( \psi_\kappa(W_\kappa) = \tilde{W}_\kappa \) is a ball centered at the origin in the Banach space \( E \). For each \( \kappa \in \Lambda \), denote \( V_\kappa := f^{-1}(W_\kappa) \subset M \).

Now, let us fix \( \kappa \in \Lambda \) such that \( V_\kappa \neq \emptyset \). Note that \( V_\kappa \) is an open submanifold of \( M \) and, for each \( y \in F \), the cross section \( s_y(\cdot) := s(\cdot)[d\psi_\kappa(f(\cdot))]|^{-1}y : V_\kappa \to TV_\kappa \) is of class \( C^{1-} \). For each \( x \in V_\kappa \) and \( y \in F \), if we consider the initial value problem:

\[
\begin{align*}
\dot{q}(t) &= s_y(q(t)) \\
q(0) &= x
\end{align*}
\]
we obtain that Equation (2.1) has a unique solution \( q(t, x, y) \) in \( V_\kappa \) over an open maximal interval \( I_{x,y} = (a(x, y), b(x, y)) \), with \( 0 \in I_{x,y} \). Furthermore the set

\[
\Omega = \bigcup_{(x,y) \in V_\kappa \times F} I_{x,y} \times \{(x,y)\}
\]

is open in \( \mathbb{R} \times V_\kappa \times F \) and \( q : \Omega \to V_\kappa \) is continuous (see [10, §1]). Now let \( (x, y) \in V_\kappa \times F \) be fixed. From Equation (2.1), it follows that \( \dot{q}(t) = s(q(t)) \cdot [d\psi_\kappa(f(q(t)))]^{-1}y \) for every \( t \in I_{x,y} \). If we denote \( \tilde{f} := \psi_\kappa \circ f |_{V_\kappa} : V_\kappa \to W_\kappa \), we obtain that \( \tilde{f}(q(0)) = \tilde{f}(x) \) and \( d\tilde{f}(q(t))(\dot{q}(t)) = y \) for every \( t \in I_{x,y} \). As a consequence,

\[
(2.2) \quad \tilde{f}(q(t, x, y)) = \tilde{f}(x) + ty, \quad \forall t \in I_{x,y}.
\]

Before going further, we will prove the following:

**Claim.** The continuation property implies that:

(i) If \( x \in V_\kappa \), then \([-1, 0] \subset I_{x,\tilde{f}(x)} \) and

(ii) if \( (x, y) \in \tilde{f}^{-1}(0) \times \overline{W_\kappa} \), then \([0, 1] \subset I_{x,y} \).

Indeed, for (i) let \( x \in V_\kappa \), let \( a := a(x, \tilde{f}(x)) \) and suppose that \( a \in [-1, 0) \). Consider the line that joins 0 with \( \tilde{f}(x) \) in \( W_\kappa \), that is, \( l(t) = (1 - t)\tilde{f}(x) \) for \( t \in [0, 1] \), and define the path \( p = \psi_\kappa^{-1} \circ l : [0, 1] \to N \). There is a path \( \overline{q}(t) := q(-t, x, \tilde{f}(x)) \) on \([0, -a]\) such that \( \tilde{f}(\overline{q}(t)) = l(t) \), for all \( t \in [0, -a] \). Therefore \( f \circ \overline{q} = p \) on \([0, -a]\), and by the continuation property, there is an increasing sequence \( t_n \to -a \) such that \( \overline{x} = \lim_{n} q(t_n) \) exists in \( M \).

In fact, \( \overline{x} \in V_\kappa \), since by continuity \( \tilde{f}(x) = l(-a) \in \overline{W_\kappa} \). Then, by the unique solvability of (2.1), \( q \) can be extended outside \( I_{x,\tilde{f}(x)} \), contradicting its maximality. In consequence, \( a < -1 \) and in particular \( q(-1, x, \tilde{f}(x)) \in \tilde{f}^{-1}(0) \) and \( \tilde{f}^{-1}(0) \neq \emptyset \). For (ii) let \( x \in \tilde{f}^{-1}(0) \) and \( y \in \overline{W_\kappa} \). Consider the line in \( \overline{W_\kappa} \) given by \( l(t) = \tilde{f}(x) + ty = ty \), for \( t \in [0, 1] \). If \( b(x, y) \leq 1 \), by the same argument as before \( q \) can be extended outside \( I_{x,y} \), which is not possible. Therefore \( q(1, x, y) \) is defined for every \( x \in \tilde{f}^{-1}(0) \) and \( y \in \overline{W_\kappa} \).

Therefore we can define \( \theta : \tilde{f}^{-1}(0) \times \overline{W_\kappa} \to V_\kappa \) by \( \theta(x, y) = q(1, x, y) \). We know that \( \theta \) is continuous, and we are going to prove that it is a homeomorphism. Let \( (x_1, y_1) \neq (x_2, y_2) \) in \( \tilde{f}^{-1}(0) \times \overline{W_\kappa} \) such that \( q(1, x_1, y_1) = q(1, x_2, y_2) \). By (2.2), \( \tilde{f}(x_1) + y_1 = \tilde{f}(x_2) + y_2 \) and therefore \( y_1 = y_2 = y \). In other words, \( q(1, x_1, y) \) and \( q(1, x_2, y) \) are values at time 1 of the differential equation (2.1) with initial values \( x_1 \) and \( x_2 \) respectively, and then \( x_1 = x_2 \). In conclusion, \( \theta \) is injective. On the other hand, let \( x \in V_\kappa \). By the above arguments, we can choose \( x' = q(-1, x, \tilde{f}(x)) \in \tilde{f}^{-1}(0) \) and we obtain that \( \theta(x', \tilde{f}(x)) = x \). Then \( \theta \) is surjective. Furthermore, the map
\( x \mapsto (q(-1,x,\tilde{f}(x)),\tilde{f}(x)) \) is the continuous inverse of \( \theta \). Note that, for every \( y \in \tilde{W}_\kappa \), the map \( \theta(\cdot,y) \) is a homeomorphism from \( \tilde{f}^{-1}(0) \) onto \( \tilde{f}^{-1}(y) \). Thus for each \( w \in W_\kappa \) we have that \( f^{-1}(w) = \tilde{f}^{-1}(\psi_\kappa(w)) \) is homeomorphic to \( \mathcal{F}_\kappa := \tilde{f}^{-1}(0) \). Now the map

\[
\Theta_\kappa := \theta \circ (id \times \psi_\kappa) : \mathcal{F}_\kappa \times W_\kappa \to V_\kappa = f^{-1}(W_\kappa)
\]

is a homeomorphism and if \( (x,w) \in (\mathcal{F}_\kappa,W_\kappa) \), then \( f(\Theta_\kappa(x,w)) = w \), that is, \( f \circ \Theta_\kappa \) is the natural projection.

Next we will show that all the fibers of \( f \) are homeomorphic. Note that this implies that \( f \) is onto, and \( V_\kappa \neq \emptyset \) for every \( \kappa \in \Lambda \). In this way, we complete the proof that \( f : M \to N \) is a fiber bundle. Let \( \kappa_0 \in \Lambda \) be fixed such that \( V_{\kappa_0} \neq \emptyset \) and consider \( N_0 = \{ w \in N : f^{-1}(w) \approx \mathcal{F}_{\kappa_0} \} \), where \( \approx \) denotes “is homeomorphic to”. As we noticed before, \( W_{\kappa_0} \subset N_0 \). Given \( w \in N_0 \), there exists \( \kappa \in \Lambda \) such that \( w \in W_\kappa \), and then \( f^{-1}(v) \approx f^{-1}(w) \approx \mathcal{F}_{\kappa_0} \) for all \( v \in W_\kappa \). Therefore \( N_0 \) is open in \( N \). In the same way, it can be seen that the complement of \( N_0 \) is open in \( N \). By connectedness, we get that \( N_0 = N \). \( \square \)

A particular case of split submersion is obtained when \( f : M \to N \) is a local diffeomorphism. In this case, if \( f \) is a fiber bundle then the fibre is discrete, and therefore \( f \) is a covering projection. On the other hand, every covering projection has the unique path lifting property, and as a consequence it also has the continuation property. Thus by Theorem 2.3 we have the following:

**Corollary 2.4.** Let \( M \) and \( N \) be \( C^2 \) paracompact Banach manifolds, \( N \) connected. Let \( f : M \to N \) be a \( C^2 \) map such that

\[
df(x) \in \text{Isom} \left( T_x M; T_{f(x)} N \right),
\]

for every \( x \in M \). Then \( f \) has the continuation property if and only if it is a covering projection.

If \( f : M \to N \) is a fiber bundle and \( N \) is contractible, it is well-known that \( f \) is a trivial fiber bundle (see, e.g., [1, Theorem 3.4.35]). So, we have the following immediate consequence of Theorem 2.3, which can be seen as a global implicit function theorem:

**Corollary 2.5.** Let \( M \) and \( N \) be \( C^2 \) paracompact Banach manifolds, \( N \) contractible. Let \( f : M \to N \) be a split submersion of class \( C^2 \) if \( f \) has the continuation property, then \( f \) is a trivial fiber bundle with trivialization \( \mathcal{F} \times N \approx M \).

**Remark 2.6.** Suppose that \( f : E \to F \) is a split submersion of class \( C^2 \) between Banach spaces, which satisfies the continuation property. In this situation, the proof of Theorem 2.3 can be considerably simplified, and it gives
directly a global trivialization. Following the proof without using charts, we search for a right inverse \( s(x) \in \mathcal{L}(F; E) \) of \( df(x) \in \mathcal{L}(E; F) \) and we consider Equation (2.1) with \( s_y(\cdot) = s(\cdot)y \), for \( y \) fixed. The continuation property gives us the existence of the flow in the appropriate interval. In fact, it is enough here to lift lines, as in the method of line lifting used by Plastock in [13] for the case of a \( C^2 \)–Fredholm map. In conclusion, there exist a map \( \Theta \) and a fiber \( \mathfrak{F} \) such that: \( \Theta : \mathfrak{F} \times \mathfrak{F} \rightarrow E \) is a homeomorphism and the composition \( f \circ \Theta : \mathfrak{F} \times \mathfrak{F} \rightarrow \mathfrak{F} \) is the natural projection.

Recall that a continuous map \( f : M \rightarrow N \) is said to be proper if, for every compact subset \( K \) of \( N \), the set \( f^{-1}(K) \) is compact in \( M \). More generally, we say that \( f \) is weakly proper if, for every compact subset \( K \) of \( N \), each connected component of \( f^{-1}(K) \) is compact in \( M \). Now we can deduce, as a direct consequence of Theorem 2.3, the following extension of classical Ehresmann Theorem [5] to the infinite dimensional setting. In [14] Rabier gives an analogous result for proper maps in the case of Finsler manifolds.

**Theorem 2.7.** Let \( M \) and \( N \) be \( C^2 \) paracompact Banach manifolds, \( N \) connected. If \( f : M \rightarrow N \) is a weakly proper split submersion of class \( C^2 \), then \( f \) is a fiber bundle.

**Proof.** It is easy to check that if \( f \) is weakly proper then it has the continuation property. Indeed, let \( p : [0, 1] \rightarrow N \) be a \( C^1 \) path, consider \( 0 < b \leq 1 \), and suppose that \( q : [0, b) \rightarrow M \) satisfies \( f \circ q = p \) over \([0, b)\). Then we have that \( \text{Im} q \) is relatively compact in \( M \), since it is contained in a connected component of set \( f^{-1}(\text{Im} p) \). If we now choose an increasing sequence \( t_n \rightarrow b \), there exists a subsequence \( (t_{n_k}) \) such that \( \{q(t_{n_k})\} \) is convergent in \( M \). \( \square \)

A slight variation in the proof of Theorem 2.3 gives the following result (compare with [3]):

**Theorem 2.8.** Let \( M \) and \( N \) be \( C^2 \) paracompact Banach manifolds, \( N \) connected. If \( f : M \rightarrow N \) is a closed split submersion of class \( C^2 \), then \( f \) is a fiber bundle.

**Proof.** If we follow the proof of Theorem 2.3, we see that the continuation property is only used in the Claim. Therefore, in order to reach the same conclusion, it is enough to prove the following (where we maintain the notation of Theorem 2.3):

**Claim.** The closedness of \( f \) implies that

(i) if \( x \in V_\kappa \), then \([-1, 0] \in I_{x, \bar{f}(x)} \)

(ii) if \((x, y) \in \bar{f}^{-1}(0) \times \bar{W}_\kappa \), then \([0, 1] \in I_{x, y} \).

Indeed, for (i) let \( x \in V_\kappa \), let \( a := a(x, \bar{f}(x)) \) and suppose that \( a \in [-1, 0) \). Consider the path \( p = \psi_\kappa^{-1} \circ l : [0, 1] \rightarrow N \), where \( l(t) = (1 - t)\bar{f}(x) \). There
is a path \( q(t) := q(-t, x, \tilde{f}(x)) \) on \([0, -a)\) such that \( f \circ q = p \) on \([0, -a)\). Let 
\( C \) be the closure of \( \text{Im} \tilde{q} \) in \( M \). Since \( f(C) \) is closed in \( N \) and it contains \( p(t) \) for every \( t \in [0, -a) \), there exists \( \tilde{x} \in C \) such that \( f(\tilde{x}) = p(-a) \). Therefore, there is a sequence \( (t_n) \) in \([0, -a)\) such that \( q(t_n) \to p(-a) \), and we can also assume that \( (t_n) \) is convergent to some \( \ell \in [0, -a] \). By continuity, \( p(\ell) = p(-a) \), and since \( p \) is one-to-one we have that \( \ell = -a \). Then, by the unique solvability of (2.1), \( q \) can be extended outside \( I_x, e_{\tilde{f}(x)} \), contradicting its maximality. As a consequence, \( a < -1 \). For (ii), the argument is similar. □

From the above theorem it follows that properness and closedness are in fact equivalent in our setting. More precisely, we have:

**Corollary 2.9.** Let \( M \) and \( N \) be \( C^2 \) paracompact Banach manifolds, with \( N \) connected, and let \( f : M \to N \) be a split submersion of class \( C^{2-} \). Then \( f \) is proper if, and only if, \( f \) is closed.

**Proof.** In general, every proper map \( f : M \to N \) is closed. Conversely, suppose that \( f \) is closed. By Theorem 2.8 we have that \( f \) is a fiber bundle. In order to prove that \( f \) is proper, it is sufficient to show that the fiber is compact (see, e.g., [6, Theorem 3.7.2]). Indeed, if this is not the case, the fiber contains a sequence with no convergent subsequence. Using this and considering a convergent sequence \( (y_n) \) in \( N \) with pairwise different terms, it is easy to construct a sequence \( (x_n) \) in \( M \) with no convergent subsequence and such that \( f(x_n) = y_n \). This contradicts the closedness of \( f \), since the set \( C = \{ x_n : n \in \mathbb{N} \} \) is closed but \( f(C) \) is not. □

On the other hand, we note that properness (or closedness) is in fact a quite restrictive condition in this situation. For example, Berger and Plastock [2] proved that, in the case of functions between Banach spaces, there is no \( C^{2-} \) proper Fredholm submersion with index \( \geq 1 \). More generally we obtain that, in the case of a \( C^{2-} \) split submersion between contractible manifolds, properness (or closedness) is in fact equivalent to global homeomorphism.

**Corollary 2.10.** Let \( M \) and \( N \) be \( C^2 \) paracompact, contractible Banach manifolds. If \( f : M \to N \) is a split submersion of class \( C^{2-} \), the following are equivalent:

1. \( f \) is a homeomorphism,
2. \( f \) is a closed map,
3. \( f \) is a proper map.

**Proof.** By Corollary 2.9, it only remains to prove that (3) implies (1). Suppose that \( f \) is a proper map. By Theorem 2.7 we get that \( f \) is a fiber bundle. Again by [1, Theorem 3.4.35], \( f \) is actually a trivial fiber bundle with trivialization \( M \approx \mathfrak{F} \times N \). Since \( M \) is contractible, so is the fiber \( \mathfrak{F} \). On the other hand, the properness of \( f \) implies that the fiber \( \mathfrak{F} \) is a compact submanifold
of $M$ without boundary. Since there is no contractible compact $C^1$ manifold with positive dimension (see e.g., [9, Theorem 5.1.6]), we obtain that $\mathfrak{F}$ is a singleton. Therefore, $f$ is a homeomorphism. □

We conclude this section with a technical improvement of Theorem 2.3 which will be quite useful in the sequel. We need the following definition: Let $f : M \to N$ be a $C^2$- split submersion and $s(\cdot)$ a fixed $C^1$ - right inverse of $df(\cdot)$. Let $p : [0, 1] \to N$ be a $C^1$ path, and $0 < b \leq 1$. We shall say that a $C^1$ path $q : [0, b) \to M$ is an horizontal lifting of $p$ if $f \circ q = p$ on $[0, b)$ and:

\begin{equation}
\dot{q}(t) = s(q(t)) \dot{p}(t), \quad \forall t \in [0, b).
\end{equation}

**Theorem 2.11.** Let $M$ and $N$ be $C^2$ paracompact Banach manifolds, $N$ connected. Let $f : M \to N$ be a split submersion of class $C^2$, and $s(\cdot)$ a fixed $C^1$ - right inverse of $df(\cdot)$. If $f$ has the continuation property for horizontal liftings, then $f$ is a fiber bundle.

**Proof.** We note as before that, in the proof of Theorem 2.3, the continuation property is only used in the Claim. Therefore, in order to obtain the same conclusion, it is enough to have the continuation property to hold, not for arbitrary paths, but only for solutions $q(t, x, y)$ of the initial value problem (2.1) where $x$ and $y$ are conveniently chosen. More precisely, with $x \in V_\kappa$ and $y = \tilde{f}(x)$ in Case (i) and with $x \in \tilde{f}^{-1}(0)$ and $y \in \tilde{W}_\kappa$ in Case (ii).

It is easy to see that, in both cases, these solutions are horizontal lifts. Indeed, for the first case, let $p(t) := \psi_{\kappa}^{-1}((1 - t)\tilde{f}(x))$ defined on $[0, 1]$, and let $\bar{q}(t) = q(-t, x, \tilde{f}(x))$ defined on the maximal half-interval $[0, -a)$. As we saw in the proof of Theorem 2.3, $f \circ \bar{q} = p$ over $[0, -a)$ and then $\dot{\bar{q}}(t) = -[d\psi_{\kappa}(f(q(-t))))^{-1}\dot{\tilde{f}}(x)$, for $t \in [0, -a)$. Therefore, $\dot{\bar{q}}(t) = \dot{q}(-t) = s(q(-t)) \dot{p}(t) = s(\bar{q}(t)) \dot{p}(t)$ over $[0, -a)$. Case (ii) is analogous. □

### 3. Bounded path-lifting property

In this section we will work in the framework of Finsler manifolds. Recall from [10] that a *Finsler manifold* $M$ is a Banach manifold that admits for its tangent bundle a functional $\| \cdot \| : TM \to \mathbb{R}$ satisfying the following two conditions:

(i) For every $x \in M$, the map $v \mapsto \|(x, v)\|$ is an admissible norm for $T_xM$, which we denote by $\| \cdot \|_x$.

(ii) Given $x_0 \in M$, a chart $\phi : V \to E$ at $x_0$ in $M$ and $k > 1$, there exists an open neighborhood $U_{x_0}$ of $x_0$ in $V$ such that:

\[ k^{-1} \|d\phi^{-1}(\phi(x_0))v\|_{x_0} \leq \|d\phi^{-1}(\phi(x))v\|_x \leq k \|d\phi^{-1}(\phi(x_0))v\|_{x_0} \]

for all $x \in U_{x_0}$ and $v \in E$.

In this case, the *length* of a piecewise $C^1$ path $p : [a, b] \to M$ is defined by $\ell_F(p) := \int_a^b \|\dot{p}(t)\|_x dt$. If the domain of $p$ is a half-open interval $[a, b)$,
then \( \ell_F(p) := \lim_{t \to b^-} \ell_F(p|_{[0,t]}) \). On each connected component of \( M \), the Finsler metric is defined by

\[
d_F(x,y) = \inf \{ \ell_F(p) : p \text{ is a piecewise } C^1 \text{ path from } x \text{ to } y \}.
\]

This metric is compatible with the topology of the corresponding component of \( M \), see [10, Theorem 3.3]. Note in particular that, since each connected component of \( M \) is open, \( M \) is metrizable and therefore it is paracompact. Now \( M \) is said to be complete if each connected component is complete with respect to its Finsler metric. For technical reasons, we are interested in defining a boundedness condition for path-liftings in terms of a weight. As we will see, this is useful in order to obtain a type of Hadamard integral condition. The relevance of such weights has been also noted in [14, Remark 4.4]. By a weight we mean an increasing monotonic map \( \omega : [0, \infty) \to (0, \infty) \) such that:

\[
\int_0^\infty \frac{dt}{\omega(t)} = \infty.
\]

**Lemma 3.1.** Let \( M \) and \( N \) be \( C^2 \) Finsler manifolds, \( M \) complete and \( N \) connected. Let \( f : M \to N \) be a \( C^2 \) split submersion and \( s(\cdot) \) a fixed \( C^1 \) right inverse of \( df(\cdot) \). Consider a \( C^1 \) path \( p : [0, 1] \to N \), a \( C^1 \) horizontal lifting \( q : [0, b) \to M \) of \( p \), where \( 0 < b \leq 1 \), and let \( M_0 \) be the connected component of \( M \) containing \( \text{Im} q \). Then the following conditions are equivalent:

1. There exists \( C > 0 \) such that \( \sup \{ \|s(x)\| : x \in \text{Im} q \} \leq C \).
2. There exist \( C > 0 \), some weight \( \omega \), and some \( x_0 \in M_0 \) such that
   \[
   \sup \left\{ \frac{\|s(x)\|}{\omega(d_F(x, x_0))} : x \in \text{Im} q \right\} \leq C.
   \]
3. \( \ell_F(q) < \infty \).
4. The path \( q \) can be extended continuously to \( [0, b] \).
5. \( \text{Im} q \subset M \) is compact.

**Proof.** (1 \( \Rightarrow \) 2) Is obvious, by taking \( \omega(t) \equiv 1 \).

(2 \( \Rightarrow \) 3) We can suppose, with no loss of generality, that \( x_0 = q(0) \) (since otherwise we could consider the alternative weight \( \omega(t) := \omega(d_F(x_0, q(t)) + t) \)). Let \( 0 < \delta < b \) be fixed, and define the map \( \xi : [0, \delta] \to \mathbb{R} \) by

\[
\xi(\tau) = \max_{t \in [0, \tau]} d_F(q(t), x_0).
\]

It is clear that \( \xi \) is continuous and non-decreasing. Before going further, we are going to show that:

\[
(3.1) \quad \xi(\tau) - \xi(\tau') \leq C \cdot \omega(\xi(\tau)) \cdot \ell_F(p|_{[\tau', \tau]}), \quad \forall \tau' \leq \tau.
\]
Indeed, let \( \tau' < \tau \) in \([0, \delta]\). From the equality (2.3) we get that for every \( t \in [\tau', \tau] \):
\[
\| \dot{q}(t) \| \leq \| s(q(t)) \| \cdot \| \dot{p}(t) \|.
\]
Therefore,
\[
\ell_F \left( p_{|[\tau', \tau]} \right) = \int_{\tau'}^\tau \| \dot{p}(t) \| \, dt \geq \inf \left\{ \| s(x) \|^{-1} : x \in \text{Im} q_{|[\tau', \tau]} \right\} \cdot d_F(q(\tau'), q(\tau)).
\]
Since for every \( x \in \text{Im} q_{|[\tau', \tau]} \) we have that
\[
\frac{\| s(x) \|}{\omega(\xi(\tau))} \leq \frac{\| s(x) \|}{\omega(d_F(x, x_0))} \leq C,
\]
we obtain that
\[
\ell_F \left( p_{|[\tau', \tau]} \right) \geq \frac{d_F(q(\tau'), q(\tau))}{C \cdot \omega(\xi(\tau))}.
\]
As a consequence we have that, for every \( \tau' < \tau \) in \([0, \delta]\):
\[
d_F(q(\tau), x_0) \leq d_F(q(\tau'), x_0) + C \cdot \omega(\xi(\tau)) \cdot \ell_F \left( p_{|[\tau', \tau]} \right)
\]
\[
\leq \xi(\tau') + C \cdot \omega(\xi(\tau)) \cdot \ell_F \left( p_{|[\tau', \tau]} \right).
\]
In order to establish (3.1), note that the inequality is clear if \( \xi(\tau') = \xi(\tau) \).
On the other hand, if \( \xi(\tau') < \xi(\tau) \), there exists \( \tau^* \in (\tau', \tau] \) such that \( \xi(\tau) = d_F(x_0, q(\tau^*)) \).
In this case, by applying the above argument to \([\tau', \tau^*] \), we obtain that
\[
\xi(\tau) = d_F(x_0, q(\tau^*)) \leq \xi(\tau') + C \cdot \omega(\xi(\tau^*)) \cdot \ell_F \left( p_{|[\tau', \tau^*]} \right)
\]
\[
\leq \xi(\tau') + C \cdot \omega(\xi(\tau)) \cdot \ell_F \left( p_{|[\tau', \tau]} \right).
\]
Now, given a partition \( 0 = t_0 < t_1 < \cdots < t_n = \xi(\delta) \) of \([0, \xi(\delta)] \), since \( \xi \) is continuous and non-decreasing we can find \( 0 = \tau_0 < \tau_1 < \cdots < \tau_n = \delta \) such that \( t_i = \xi(\tau_i) \), for \( i = 0, \ldots, n \). Then, by inequality (3.1), we have:
\[
\sum_{i=0}^{n-1} \omega(t_{i+1})(t_{i+1} - t_i) \leq C \cdot \sum_{i=0}^{n-1} \ell_F \left( p_{|[\tau_i, \tau_{i+1}]} \right) = C \cdot \ell_F \left( p_{|[0, \delta]} \right) \leq C \cdot \ell_F(p).
\]
Therefore, for every \( \delta \in [0, b) \) we obtain that
\[
\int_0^{\xi(\delta)} \frac{dt}{\omega(t)} \leq C \cdot \ell_F(p) < \infty.
\]
Since \( \omega \) is a weight, we conclude that there exists some \( r > 0 \) such that \( \xi(\delta) \leq r \), for every \( \delta \in [0, b) \). Therefore, for all \( x \in \text{Im} q \) we have:
\[
\frac{\| s(x) \|}{\omega(r)} \leq \frac{\| s(x) \|}{\omega(d_F(x, x_0))} \leq C.
\]
Using equality (2.3) we get that, for every \( t \in [0, b) \),
\[
\| \dot{q}(t) \| \leq \| s(q(t)) \| \cdot \| \dot{p}(t) \| \leq C \cdot \omega(r) \cdot \| \dot{p}(t) \|.
\]
Then for each $\delta \in [0, b)$ we have
\[
\ell_F(q|[0,\delta]) = \int_0^\delta \|\dot{q}(t)\|dt \leq C \cdot \omega(r) \cdot \ell_F(p|[0,\delta]) \\
\leq C \cdot \omega(r) \cdot \ell_F(p) < \infty.
\]

In conclusion, $\ell_F(q) < \infty$.

(3 $\Rightarrow$ 4) Suppose that $\ell_F(q) < \infty$. Let $\{t_n\} \subset [0, b)$ be any sequence such that $t_n \to b$. Since $\int_0^{t_n} \|\dot{q}(t)\|dt$ converges to $\ell_F(q)$, we obtain that $\{q(t_n)\}$ is a Cauchy sequence for the Finsler metric in the connected component $M_0$. By completeness, $\{q(t_n)\}$ is then convergent.

(4 $\Rightarrow$ 5) It is clear.

(5 $\Rightarrow$ 1) It follows directly from the fact that $\|s(x)\| > 0$ for all $x \in M$ and the mapping $x \mapsto \|s(x)\|$ is continuous on $M$ (see [10, §2]).

Let $f : M \to N$ be a split submersion of class $C^{2-}$ and let $s(\cdot)$ be a right inverse of $df(\cdot)$ of class $C^{1-}$. We shall say that $f$ has the bounded path-lifting property with respect to $s(\cdot)$ if, for each $C^1$ path $p : [0, 1] \to N$ and each $0 < b \leq 1$, we have that every $C^1$ horizontal lifting $q : [0, b) \to M$ of $p$ satisfies the equivalent conditions of Lemma 3.1. With this terminology Lemma 3.1(4) gives that the bounded path-lifting property implies the continuation property for horizontal liftings. Thus combining with Theorem 2.11 we get at once the following:

**Theorem 3.2.** Let $M$ and $N$ be $C^2$ Finsler manifolds, $M$ complete and $N$ connected, and let $f : M \to N$ be a $C^{2-}$ split submersion. Suppose that $f$ has the bounded path-lifting property with respect to some $C^{1-}$ right inverse $s(\cdot)$ of $df(\cdot)$. Then $f$ is a fiber bundle.

Now from Theorem 3.2 we can easily deduce the following corollary, related to a result by Earle and Eells [4, Theorem 3C]. There, Earle and Eells do not assume that $N$ is connected, but they assume instead that $f$ is surjective.

**Corollary 3.3.** Let $M$ and $N$ be $C^2$ Finsler manifolds, $M$ complete and $N$ connected. Let $f : M \to N$ be a $C^{2-}$ split submersion and let $s(\cdot)$ be a $C^{1-}$ right inverse of $df(\cdot)$. Suppose that for every $y_0 \in N$, there exist a neighborhood $V_0$ of $y_0$ in $N$ and $C_0 > 0$ such that, for every $x \in f^{-1}(V_0)$, $\|s(x)\| \leq C_0$. Then, $f$ has the bounded path-lifting property with respect to $s(\cdot)$, and therefore it is a fiber bundle.

**Proof.** Let $p : [0, 1] \to N$ be a $C^1$ path. By compactness, we can find a finite open covering $\text{Im} p \subset V_1 \cup \ldots \cup V_n$ and some $C > 0$ such that $\|s(x)\| \leq C$, for every $x \in f^{-1}(V_i)$, for all $i = 1, \ldots, n$. If $q : [0, b) \to M$ is such that $f \circ q = p$ over $[0, b)$, we obtain that $\sup\{\|s(x)\| : x \in \text{Im} q\} \leq C$. Thus every lifting of $p$ satisfies Condition (1) of Lemma 3.1. \qed
Next we give a result for connected manifolds, in terms of the Hadamard integral condition. Here we denote by $B_t(x_0)$ the closed ball with center $x_0$ and radius $t$, with respect to the Finsler metric $d_F$ of $M$. We note that Corollary 3.4 below was obtained by Plastock [13, Theorem 3.2] for the special case of Fredholm submersions with positive index between Banach spaces.

**Corollary 3.4.** Let $M$ and $N$ be $C^2$ connected Finsler manifolds, $M$ complete. Suppose that $f$ is a $C^{2-}$ split submersion that satisfies

$$\int_0^\infty \inf_{x \in B_t(x_0)} \|s(x)\|^{-1} \, dt = \infty$$

for some $C^1$ right inverse $s(\cdot)$ of $df(\cdot)$ and some $x_0 \in M$. Then $f$ has the bounded path-lifting property with respect to $s(\cdot)$, and therefore it is a fiber bundle.

**Proof.** We can consider the weight $\omega(t) := \inf\{\|s(x)\|^{-1} : x \in B_t(x_0)\}^{-1}$, defined for $t \geq 0$. It is not difficult to check that $\|s(x)\| \cdot \omega(d_F(x,x_0))^{-1} \leq 1$, for each $x \in M$. In particular for every path $p : [0,1] \to N$, every lifting satisfies Condition (2) of Lemma 3.1.

**Remark 3.5.** In the special case that $M$ and $N$ are Riemannian manifolds (hence modelled on Hilbert spaces) we can use the right inverse of $df(\cdot)$ considered in Remark 2.2, that is, $s_0(x) = \left( df(x) |_{\ker df(x)} \right)^{-1} = df(x)^* df(x)^* \cdot df(x)$. In this case the norm $\|s_0(x)\|$ can be computed explicitly, by means of the following formula (see Remark 4.1 below):

$$\|s_0(x)\|^{-1} = \inf\{\|df(x)(u)\| : u \in \ker df(x)^* ; \|u\| = 1\}$$

$$= \inf\{\|df(x)^*(v)\| : v \in T_{f(x)}N ; \|v\| = 1\}.$$

To finish this section, we specialize to the case where $f : M \to N$ is a local diffeomorphism. Of course, in this case the only choice for a right inverse of $df(\cdot)$ is $s(x) = df(x)^{-1}$, and every lifting is horizontal. Thus we obtain:

**Theorem 3.6.** Let $M$ and $N$ be $C^2$ Finsler manifolds, $M$ complete and $N$ connected. Let $f : M \to N$ be a map of class $C^{2-}$ and suppose that, for every $x \in M$, $df(x) \in \text{Isom}(T_xM;T_{f(x)}N)$. The following statements are equivalent:

1. $f$ is a covering projection.
2. $f$ has the continuation property.
3. $f$ has the bounded path-lifting property with respect to $s(x) = df(x)^{-1}$.

If in addition, we assume that either $N$ is simply connected or $\pi_1(M) = \pi_1(N)$ is finite, then the previous conditions are also equivalent to the following:

4. For every compact subset $K \subset N$, there is a constant $C > 0$ such that $\|df(x)^{-1}\| \leq C$, for every $x \in f^{-1}(K)$. 
(5) $f$ is a proper map.
(6) $f$ is a closed map.
(7) $f$ is a homeomorphism.

Proof. The equivalences (1)–(3) can be deduced by combining Corollary 2.4 and Lemma 3.1. For the equivalences (4)–(7), consider Corollary 2.9, and use for example [16, Corollary 2.4.7].

4. Submersions with uniformly split kernels

A map $f : M \to N$ of class $C^1$ between $C^1$ Finsler manifolds is said to have uniformly split kernels if there is a constant $k > 0$ such that for each $x \in M$, there exists a projection $P_x \in \mathcal{L}(T_x M, T_x M)$ with $\ker P_x = \ker df(x)$ and $\|P_x\|_x \leq k$, where $\|\cdot\|_x$ denotes the norm induced by the Finsler structure $\|\cdot\|$ on $TM$. This concept was introduced by Rabier in [14]. As shown in [14, Proposition 3.1], the map $f$ has uniformly split kernels, for example, in the following cases:

(i) When $M$ is a Riemannian manifold (hence modelled on a Hilbert space);
(ii) when $N$ is finite-dimensional;
(iii) when $f$ is a Fredholm submersion of nonnegative index.

On the other hand, for a linear operator $T \in \mathcal{L}(E, F)$ between Banach spaces, Rabier [14] sets the quantity $\nu(T) = \inf_{\|y^*\|=1} \|T^* y^*\|$, where $T^* \in \mathcal{L}(F^*, E^*)$ denotes the transpose of $T$. It is easy to see that if $T \in \text{Isom}(E, F)$, then $\nu(T) = \|T^{-1}\|^{-1}$. More generally, we have the following:

Remark 4.1. Let $T : E \to F$ be a linear onto map between Banach spaces, and consider the canonical isomorphism $\hat{T} : E/\ker T \to F$. Then $\nu(T) = \|\hat{T}^{-1}\|^{-1}$. Indeed, if $\pi : E \to E/\ker T$ denotes the quotient map, we have that $T = \hat{T} \circ \pi$ and $T^* = \pi^* \circ \hat{T}^*$. Now consider $(\ker T)^0 = \{x^* \in E^* : x^*(\ker T) = 0\}$. We know that $\pi^* : (E/\ker T)^* \to (\ker T)^0$ is an isometry and $T^* : F^* \to (\ker T)^0$ is an isomorphism. Therefore,

$$\nu(T) = \inf_{\|y^*\|=1} \|T^* y^*\| = \inf_{\|y^*\|=1} \|\hat{T}^* y^*\| = \|\hat{T}^*\|^{-1} = \|\hat{T}^{-1}\|^{-1}.$$ 

Next we are going to see that, when $f : M \to N$ is a $C^{2-}$ submersion with uniformly split kernels, it is possible to construct a $C^1$ right inverse $s(\cdot)$ of $df(\cdot)$ satisfying a special boundedness condition. We know that, for every $x \in M$, there exists a projection $P_x \in \mathcal{L}(T_x M, T_x M)$ with $\ker P_x = \ker df(x)$ and $\|P_x\|_x \leq k$, for some uniform constant $k > 0$. For each $x \in M$, consider the quotient map $\pi_x : T_x M \to T_x M/\ker df(x)$ and the canonical isomorphism
If we define \( \sigma_x(u + \ker df(x)) = P_x(u) \), we obtain that \( \sigma_x : T_x M / \ker df(x) \to T_f(x) N \) is a right inverse of \( \pi_x \) with \( \| \sigma_x \| \leq k \), and

\[
s_x := \sigma_x \circ \hat{df}(x)^{-1} : T_f(x) N \to T_x M
\]

is a right inverse of \( df(x) \) with \( \| s_x \| \leq k \cdot \| \hat{df}(x)^{-1} \| = k \cdot \nu(df(x))^{-1} \). Now using a partition of unity of class \( C^{1-} \), it is possible to obtain the following result, due to Rabier (it is essentially [14, Proposition 3.1] with minor modifications):

**Lemma 4.2.** Let \( M \) and \( N \) be \( C^2 \) Finsler manifolds and let \( f : M \to N \) be a \( C^{2-} \) submersion with uniformly split kernels. Then there exist a \( C^{1-} \) right inverse \( s(\cdot) \) of \( df(\cdot) \) and a constant \( k > 0 \) such that, for every \( x \in M \):

\[
\| s(x) \| \leq k \cdot \nu(df(x))^{-1}.
\]

Now we introduce a path-lifting property whose definition does not involve explicitly a \( C^{1-} \) right inverse. We shall say that \( f : M \to N \) has the \( \nu \)-bounded path-lifting property if for every \( C^1 \) path \( p : [0, 1] \to N \), every \( b \in (0, 1] \) and every \( C^1 \) path \( q : [0, b) \to M \) with \( f \circ q = p \) over \([0, b)\), there exist a weight \( \omega \), some \( C > 0 \) and some \( x_0 \) in the connected component of \( M \) containing \( \im q \), such that:

\[
\sup \left\{ \frac{\nu(df(x))^{-1}}{\omega(d_F(x, x_0))} : x \in \im q \right\} \leq C.
\]

By Lemma 4.2, if \( f : M \to N \) is a \( C^{2-} \) submersion with uniformly split kernels satisfying the \( \nu \)-bounded path-lifting property, then \( f \) has the bounded path-lifting property for some \( C^{1-} \) right inverse of \( df(\cdot) \). Thus from Theorem 3.2 it follows at once:

**Theorem 4.3.** Let \( M \) and \( N \) be \( C^2 \) Finsler manifolds, \( M \) complete and \( N \) connected. Let \( f : M \to N \) be a \( C^{2-} \) submersion with uniformly split kernels. If \( f \) has the \( \nu \)-bounded path-lifting property, then \( f \) is a fiber bundle.
and extra assumption over \( f \); the same can be done here to obtain the desired result.)

Next we give a result for submersions with uniformly split kernels between connected manifolds, using a Hadamard integral condition in terms of \( \nu(df(\cdot)) \). The proof is similar to that of Corollary 3.4.

**Corollary 4.4.** Let \( M \) and \( N \) be \( C^2 \) connected Finsler manifolds, \( M \) complete. Suppose that \( f \) is a \( C^2^- \) submersion with uniformly split kernels, such that

\[
\int_0^\infty \inf_{x \in B_r(x_0)} \nu(df(x)) \, dt = \infty,
\]

for some \( x_0 \in M \). Then \( f \) is a fiber bundle.

In what follows, we are going to show that the different conditions that we have considered along this section are, in fact, nonequivalent. This can be seen by means of quite simple examples.

**Example 4.5.** Consider a \( C^\infty \)-diffeomorphism \( \psi : \mathbb{R} \to \mathbb{R} \) such that \( \inf \{ |\psi'(t)| : t \in \mathbb{R} \} = 0 \), and define the \( C^\infty \) submersion \( f : \mathbb{R} \to S^1 \) by \( f(t) = (\cos \psi(t), \sin \psi(t)) \). In this case \( \nu(df(t)) = |\psi'(t)| \) for every \( t \in \mathbb{R} \), and therefore \( f \) is not a strong submersion. On the other hand, since \( f \) is a covering, \( f \) has the \( \nu \)-bounded path-lifting property.

In the next two examples, we consider a \( C^\infty \)-diffeomorphism \( \phi : (0, \infty) \to (0, \infty) \), and we define \( f : \mathbb{R} \times (0, \infty) \to \mathbb{R} \) by \( f(x,y) = x \cdot \phi(y) \). Then \( f \) is a \( C^\infty \) submersion, and it is not difficult to check that \( \nu(df(x,y)) = [\phi(y)^2 + \phi'(y)^2 x^2]^{\frac{1}{2}} \), for every \( (x,y) \in \mathbb{R} \times (0, \infty) \).

**Example 4.6.** Consider here \( \phi(y) = y^{-1} \) and define \( f : \mathbb{R} \times (0, \infty) \to \mathbb{R} \) by \( f(x,y) = xy^{-1} \). Since in this case \( \nu(df(x,y)) = [y^{-2} + y^{-4} x^2]^{\frac{1}{2}} \), we have that \( f \) is not a strong submersion. On the other hand, we are going to see that \( f \) satisfies the Hadamard integral condition of Corollary 4.4. Indeed, for \( z_0 = (0,1) \in \mathbb{R} \times (0, \infty) \) we have that \( d_F((x,y), z_0) = [x^2 + (y-1)^2]^{\frac{1}{2}} \). Note that if \( d_F((x,y), z_0) \leq r \), then \( y \leq 1 + r \). Therefore for every \( (x,y) \in B_r(z_0) \) we have that \( \nu(df(x,y)) \geq \frac{1}{y} \geq \frac{1}{1+r} \), and as a consequence:

\[
\int_0^\infty \inf_{(x,y) \in B_r(z_0)} \nu(df(x,y)) \, dr \geq \int_0^\infty \frac{dr}{1+r} = \infty.
\]

In our final example we exhibit a submersion \( f \) which does not satisfy the \( \nu \)-bounded path-lifting property (and in particular \( f \) is not a strong submersion), and nevertheless \( f \) satisfies the bounded path-lifting property for some right inverse \( s(\cdot) \) of \( df(\cdot) \).

**Example 4.7.** Consider here \( \phi(y) = y^{-2} \) and define \( f : \mathbb{R} \times (0, \infty) \to \mathbb{R} \) by \( f(x,y) = xy^{-2} \). In this case \( \nu(df(x,y)) = [y^{-4} + 4y^{-6} x^2]^{\frac{1}{2}} \), and for \( z_0 =
(0, 1) \in \mathbb{R} \times (0, \infty) \) we have that \( d_F((x, y), z_0) = [x^2 + (y - 1)^2]^{1/2} \). Consider the paths \( p : [0, 1] \to \mathbb{R} \) given by \( p(t) = t \) and \( q : (0, 1] \to \mathbb{R} \times (0, +\infty) \) given by \( q(t) = (1, t^{-1/2}) \). It is clear that \( f(q(t)) = p(t) \), for every \( t \in (0, 1] \). Furthermore, it is easy to check that \( \nu(df(q(t))) = \sqrt{5} t \), and \( d_F(q(t), z_0) \leq \sqrt{2} t^{-1/2} \), for every \( t \in (0, 1] \). Suppose now that \( f \) satisfies the \( \nu \)-bounded path-lifting property. Then there exist a constant \( C > 0 \) and a weight \( \omega \) such that \( C \cdot \nu(df(q(t))) \geq \omega(d_F(q(t), z_0))^{-1} \), for every \( t \in (0, 1] \). Taking \( r = \sqrt{2} t^{-2} \) we obtain that \( \omega(r)^{-1} \leq 2\sqrt{5} C r^{-2} \), for every \( r \in [\sqrt{2}, \infty) \). As a consequence,

\[
\int_\sqrt{2}^\infty \frac{dr}{\omega(r)} \leq \int_\sqrt{2}^\infty \frac{C}{r^2} dr < \infty,
\]

which is a contradiction with the fact that \( \omega \) is a weight. On the other hand, consider the right inverse \( s(\cdot) \) of \( df(\cdot) \) defined, for every \( (x, y) \in \mathbb{R} \times (0, \infty) \) by \( s(x, y) : \mathbb{R} \to \mathbb{R} \times \mathbb{R} \), where \( s(x, y)(r) = (ry^2, 0) \). Let \( p : [0, 1] \to \mathbb{R} \) be a \( C^1 \)-path, let \( 0 < b \leq 1 \), and consider an horizontal lifting \( q : [0, b] \to \mathbb{R} \times (0, \infty) \). If we denote \( q(t) = (q_1(t), q_2(t)) \), it is easy to see that \( q_1(t) = q_2(t)^2 \dot{p}(t) \) and \( q_2(t) = 0 \), for every \( t \in [0, b] \). Therefore \( q_2 \) is constant, say \( q_2(t) = c \), and \( q_1(t) = c^2 \dot{p}(t) \), for every \( t \in [0, b] \). Then it is clear that \( q \) can be continuously extended to \([0, b] \). In this way we see that \( f \) has the bounded path-lifting property with respect to \( s(\cdot) \).

Acknowledgements. It is a great pleasure to thank Professors Javier Gómez Gil and María Isabel Garrido for some valuable conversations concerning this paper.

References


Received July 9, 2003 and revised October 24, 2003. The first author was supported by CONACyT (México) Grant 119933. The second author was supported in part by DGES (Spain) BFM2000-0609.

**Departamento de Análisis Matemático**
**Universidad Complutense de Madrid**
**28040 Madrid**
**Spain**

**Centro de Investigación en Matemáticas**
**Universidad Autónoma del Estado de Hidalgo**
**42074 Pachuca Hidalgo**
**México**

E-mail address: olivia@uaeh.reduaeh.mx

**Departamento de Análisis Matemático**
**Universidad Complutense de Madrid**
**28040 Madrid**
**Spain**

E-mail address: jaramil@mat.ucm.es
REGULARITY OF THE HEAT OPERATOR ON A MANIFOLD WITH CYLINDRICAL ENDS

THALIA D. JEFFRES AND PAUL LOYA

We study mapping properties of the heat operator $e^{tA}$ of an $m$-th order elliptic $b$-differential operator in appropriately defined spaces of whole and fractional (Hölder) derivatives. An application is made to short time existence of solutions to certain semilinear parabolic equations.

1. Introduction

Let $(M,g)$ be a compact manifold with boundary. We say that $M$ has a cylindrical end if there is a neighborhood in which there is a local diffeomorphism to a product $[0, \varepsilon)_x \times N_z$ in which the metric may be approximated by

$$\left( \frac{dx}{x} \right)^2 + h_N,$$

where $h_N$ is a Riemannian metric on the cross section $N$. Note that in this metric, the boundary is infinitely far away. $M$ is called a manifold with cylindrical ends if it has at least one cylindrical end and no other type of boundary.

The purpose of this paper is to study differentiability properties of solutions to the heat equation

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u &= 0 \\ u|_{t=0} &= f \end{cases}$$

(1)

on such a manifold $M$. The Laplacian in such a metric $g$ will be a $b$-differential operator, that is to say, an operator composed of vector fields tangent to the boundary. In the local coordinates given above, the set of such vector fields is generated by

$$\left\{ x \frac{\partial}{\partial x}, \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_{n-1}} \right\}.$$

These are called the boundary vector fields. In fact, the Laplacian could be replaced by $A$, any elliptic combination of the boundary vector fields. See [Ma] or [Me] for more on these elliptic operators. Elliptic $b$-operators are essentially self-adjoint.
Example 1.1. If $M = [0, \infty)$, the heat equation is

$$\frac{\partial u}{\partial t} - \left( x \frac{\partial}{\partial x} \right)^2 u = 0.$$ 

The relevant analytic feature of this operator is the vanishing coefficient $x$ in the elliptic part, which prevents it from being uniformly elliptic in the traditional sense. It was for the purpose of studying such operators that Melrose developed the $b$-calculus; see his book [Me] and also the paper of Mazzeo [Ma] and the many others referenced in their work. This is exactly what is needed here, for in this point of view the vector field $x\partial/\partial x$ and $\partial/\partial z_1$ bounded. We also make use of the Hölder space $C^\gamma_b(M)$ consisting of continuous functions for which $\|u\|_\gamma = \sup_M |u| + \sup_{p \neq q} \frac{|u(p) - u(q)|}{(d_g(p,q))^\gamma} < \infty$, where $d_g(p,q)$ represents the distance between the points $p$ and $q$ in the cylindrical end metric as long as $p$ and $q$ are ‘close’. For a precise definition, see Section 4.

The main theorem is:

**Theorem 1.2.** Let $e^{tA}$ be the heat operator of an $m$-th order $b$-differential operator $A$ which is elliptic with respect to a sector whose complement is negative (see Theorem 2.4). Then for any $\alpha \in \mathbb{R}$, $e^{tA}$ is bounded for $t$ within any fixed bounded interval of $(0, \infty)$, between the following weighted spaces and with the corresponding estimates:

$$e^{tA} : x^\alpha C^0_b(M) \longrightarrow x^\alpha C^k_b(M); \quad \|u\|_k \leq Ct^{-k/m}\|f\|_0,$$

and

$$e^{tA} : x^\alpha C^\gamma_b(M) \longrightarrow x^\alpha C^m_b(M); \quad \|u\|_m \leq Ct^{-1+\gamma/m}\|f\|_\gamma.$$

We note two immediate features of these estimates. One is that the estimates blow up as $t$ approaches zero. This occurs because of the well-known smoothing phenomenon associated to heat transfer. The second observation is that although our spaces are defined in a way that accounts for the singular metric, the powers of $t$ are the same as in the smooth case.

The method of proof is roughly the same also. The solution to the heat equation may be expressed as the integral of the product of the heat kernel with the initial data. For the heat equation of the usual Laplacian in $\mathbb{R}^n$, this solution is

$$u(x, t) = \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}} e^{-\|x-y\|^2/4t} f(y) \, dy.$$
To estimate derivatives, one passes the derivative inside the integral and checks that the integrals converge. One may calculate directly that each successive differentiation produces a factor of $t^{-1/2}$. See for example a standard text in PDE such as that of Evans [E] or Folland [F]. On a smooth compact manifold, or in the interior of our manifold with cylindrical ends, the heat kernel of the metric Laplacian is approximated by that of $\mathbb{R}^n$, as one discovers in Aubin [A]. Following a similar method, we explain in Section Two the form of the heat kernel on a manifold with cylindrical ends, and in Sections Three and Four we carry out the calculations.

Boundedness properties of the heat operator $e^{tA}$ in appropriately defined $L^p$ and Sobolev spaces have been studied extensively by several authors; see for example [Ma], [Me], and [L]. Our purpose in investigating the differentiability and Hölder properties is to allow applications to nonlinear problems. An existence theorem for certain semilinear equations will be given in the last section.

This paper is continuation of our earlier paper [JL] in which we studied the heat equation on a cone manifold, and further description of related work of other authors may be found there.

Finally, we thank the referee for helpful comments in improving the paper.

2. The heat space and heat kernel

We wish to express the heat operator as an integral operator, and a detailed understanding of the singular structure of the integral kernel will enable us to determine the mapping properties. Such a description is given in [L] and we will quote that result in this section after establishing some concepts and notations.

A central idea of Melrose is that when $M$ is singular or has boundary, these integral kernels are more naturally described as functions or distributions not on $M \times M \times [0, \infty)_t$, but rather on a more complicated manifold with corners, whose interior is identified with the interior of $M \times M \times [0, \infty)_t$. The singularities of the kernel are then described by the asymptotic behavior of the kernel on approach to the various corners, edges, and faces. It is explained in Melrose' book [Me] that the correct space is obtained by an ordinary blow up of $Z \times [0, \infty) \subset M^2 \times [0, \infty)$, where $Z = (\partial M)^2 \subset M^2$ is the corner, followed by what he calls a parabolic blow up at the diagonal at $t = 0$. This space is denoted here by $M^2_H$ and is a well-defined object with a differentiable structure and so on; all these details may be found in [Me].

We remark that Melrose focused primarily on Laplace-type operators. We wish to include $m$-th order operators. Because the blow up space is not essentially different, we will call this last step of its construction an "$m$-th order parabolic blow up," and we will also use the same notation, $M^2_H$, to denote it.
For our immediate purpose here, we only need to know that locally these blow ups are simply introduction of singular coördinates, and these coördinates are very natural because they reflect the familiar radial symmetry of $b$-differential operators and the parabolic symmetry of the heat operator. As a manifold with corners, there is no global coördinate chart for all of $M^2_b$. We now describe the local charts.

Again let $(x, z_1, \ldots, z_{n-1})$ be coördinates in a neighborhood of the boundary where $M$ is locally diffeomorphic to $[0, \varepsilon)_x \times N_z$. Let $(y, z_1', \ldots, z_{n-1}')$ be local coördinates for the second copy of $M$ in $M^2 \times [0, \infty)$. To compress notation we will often write $z = (z_1, \ldots, z_{n-1})$ and $z' = (z_1', \ldots, z_{n-1}')$. The first blow up is accomplished by the introduction of projective (really polar, reflecting the radial symmetry of the elliptic part — see $[G]$) coördinates near $Z \times [0, \infty) \subset M^2 \times [0, \infty)$. At this stage we have two neighborhoods.

I. Near the left-hand corner

$$\{ t = t, \xi = x/y, \eta = y, z = z, z' = z' \} .$$

Three boundary hypersurfaces meet at the left-hand corner. Using notation consistent with $[L]$, we define them as follows:

**Definition 2.1.** The left boundary, $lb$, is the lift of $\{ x = 0 \}$ and is given in these coördinates by $\{ \xi = 0 \}$. The front face, $ff$, is the blow up of the corner and is given by $\{ \eta = 0 \}$. The temporal boundary, $tb$, is the lift of $\{ t = 0 \}$ and is described in by $\{ t = 0 \}$. This submanifold is also denoted by $M^2_b$ and is often referred to as the ‘$b$-stretched product’.

II. Near the right-hand corner

$$\{ t = t, \xi' = x, \eta' = y/x, z = z, z' = z' \} .$$

**Definition 2.2.** The right boundary, $rb$, is given by $\{ \eta' = 0 \}$.

Note however that approach to the right boundary can also be described in the coördinates I. by $\xi \to \infty$.

Next we do the $m$-th order parabolic blow up of the lifted diagonal $\{ x = y, z = z' \} \cap \{ t = 0 \}$. This set belongs to either coördinate patch, so choosing those near the left-hand corner, it is described locally by $\{ \xi = 1, z = z' \} \cap \{ t = 0 \}$. Parabolic blow up is introduction of a singular system of coördinates in which three data are used to locate a point: Its position $\eta$ along the diagonal, its location $(\omega_0, \omega')$ on a unit reference heat sphere $\mathbb{S}_H^0$ centered at $(\xi, z - z', \eta, t) = (1, 0, \eta, 0)$, and its parabolic distance from the center. The upper hemisphere of the unit heat sphere with center at the origin is defined to be

$$\mathbb{S}_H^0 = \{ \tilde{\omega} = (\omega_0, \omega') \in \mathbb{R}^{n+1} \mid \omega_0 \geq 0, \omega_0^2 + |\omega'|^{2m} = 1 \} ,$$
and the parabolic distance of \((\xi, z - z', \eta, t)\) to \((1, 0, \eta, 0)\) is
\[
d = \frac{2m}{m} \sqrt{t^2 + |(\xi - 1, z - z')|^{2m}}.
\]

Global “coördinates” are
\[
\{\omega_0 = t/d, \omega' = (\xi - 1)/d, (z - z')/d, \eta, z', d\}.
\]

There appears to be an extra coördinate because \(\omega_0\) and \(\omega'\) are not independent. In the regions where \(\xi - 1\) and \(z - z'\) are small compared to \(t\), i.e., near the top of the blown-up diagonal, local coördinates are
\[
\left\{ \rho = t^{1/m}, w = \frac{\xi - 1}{t^{1/m}} = \frac{x - y}{yt^{1/m}}, \zeta = \frac{z - z'}{t^{1/m}}, \eta = y, z' = z' \right\}.
\]

Near the top where the blown-up diagonal meets the front face, there are two boundary hypersurfaces, the front face and a new one.

**Definition 2.3.** The blown-up diagonal is called the temporal face and is denoted by \(tf\). It is locally given by \(\{\rho = 0\}\).

We could also define a separate coördinate system valid near \(tb\) where the diagonal hits the front face, but it won’t be necessary because \(tb\) is reached as \(|(w, \zeta)| \to \infty\).

We now recall an ellipticity condition that guarantees the existence of the heat operator for a \(b\)-differential operator. Let \(A\) be an \(m\)-th order \(b\)-differential operator on \(M\). Then on a neighborhood \([0, \varepsilon) \times N\) near the boundary, we can write
\[
A = \sum_{k=0}^{m} A_{m-k}(x) \cdot \left( x \frac{\partial}{\partial x} \right)^k,
\]
where the \(A_{m-k}\)'s are differential operators of order \(m - k\) on \(N\) depending smoothly in \(x\). The \(b\)-principal symbol of \(A\), \(ab\), is just the usual principal symbol of \(A\) in the interior of \(M\), and in the decomposition above is defined by
\[
ab(\xi) = \sum_{k=0}^{m} a_{m-k}(x, \xi') \cdot (i\xi_1)^k, \quad \xi = (\xi_1, \xi') \in \mathbb{R} \times T^*(N),
\]
where \(a_{m-k}(x, \xi')\) is the principal symbol of the operator \(A_{m-k}(x)\) as a differential operator on \(N\). The \(b\)-principal symbol \(ab\) is a function on the \(b\)-cotangent bundle (see [Me]). \(A\) is said to be elliptic with respect to a sector \(\Lambda \subset \mathbb{C}\) if \(ab(\xi) - \lambda \neq 0\) for all \(\lambda \in \Lambda \cup \{0\}\) and \(b\)-cotangent vectors \(\xi \neq 0\). Finally, a sector \(\Lambda\) is said to be negative if it is of the form \(\Lambda = \{\lambda \in \mathbb{C} | \pi - \delta < \arg \lambda < \pi + \delta\}\) for some \(0 < \delta < \pi/2\).

For instance, the principal symbol of the one-dimensional Laplacian \(\Delta = (x\partial_x)^2\) is \(-\xi^2\). Since \(-\xi^2 - \lambda \neq 0\) for \(\xi \neq 0\) and for \(\lambda\) not on the negative
real axis, $\Delta$ is elliptic with respect to any sector whose complement is negative. In general, the Laplacian on a manifold with cylindrical ends is always elliptic with respect to any sector whose complement is negative.

**Theorem 2.4** ([L]). Let $A$ be an $m$-th order $b$-differential operator that is elliptic with respect to a sector whose complement is negative. Then the heat operator $e^{tA}$ exists and its Schwartz kernel, when written in the singular coördinates of $M^2_H$, defines a smooth function on the interior of $M^2_H$ that vanishes to infinite order at $lb$, $rb$, and $tb$, is smooth up to $ff$, and finally, has an expansion up to $tf$ with leading power given by $-n$.

The primary example of an operator satisfying the conditions of this theorem is the Laplacian with respect to a cylindrical end metric (cf. Theorem 7.24 of [Me]).

In Theorem 2.4, we choose to treat the kernel $\kappa_H$ of the heat operator $H = e^{tA}$ as a function. In other words, the action of the operator on the initial data $f$, also a function, will be to multiply $f$ by $\kappa_H$ and integrate against the Riemannian volume form:

$$Hf(x,z,t) = \int \kappa_H(x,y,z,z')f(y,z')y^{-1}dydz'. $$

To illustrate the above theorem, consider the coördinates $\rho = t^{1/m}$, $w = (\xi - 1)/t^{1/m} = (x - y)/yt^{1/m}$, $\zeta = (z - z')/t^{1/m}$, $\eta = y$, and $z' = z'$ near the blown-up diagonal. Here, $\rho$ defines $tf$, $\eta$ defines $ff$, and $tb$ corresponds to the face in the limit $|(w,\zeta)| \to \infty$. Then the above theorem states that we can write

$$\kappa_H = H(\rho, w, \zeta, \eta, z'),$$

where $H$ is a function that vanishes to infinite order as $|(w,\zeta)| \to \infty$, is smooth in $\eta$ and $z'$, and finally, has an expansion as $\rho \to 0$ with leading power $-n$. Thus, we have

$$H \sim \sum_{j=0}^{\infty} \rho^{-n+j}a_j(w,\zeta,\eta, z') \text{ as } \rho \to 0,$$

where the $a_j$’s are smooth functions vanishing to infinite order as $|(w,\zeta)| \to \infty$ and smooth in $\eta$ and $z'$.

### 3. Local calculations of sup and whole derivative bounds

This section and the next comprise the proof of Theorem 1.2.

**Definition 3.1.** The space $C^k_b(M)$ consists of continuous functions on $M$ such that

$$\sup |u| + \sup |V_1 u| + \cdots + \sup |V_1 \cdots V_k u| < \infty$$

for every choice of $k$ boundary vector fields.
Recall that the boundary vector fields are locally generated by vector fields $x\partial/\partial x$ and $\partial/\partial z$.

Henceforth, we fix an $m$-th order $b$-differential operator $A$ that is elliptic with respect to a sector whose complement is negative. We begin by reducing Theorem 1.2 to the case of zero weights. Indeed, observe that for any $\alpha \in \mathbb{R}$, we have

$$e^{tA} : x^{\alpha}C^0(M) \longrightarrow x^{\alpha}C^b_k(M) \iff x^{-\alpha}e^{tA}x^{\alpha} : C^0(M) \longrightarrow C^b_k(M).$$

If $\kappa_H$ denotes the Schwartz kernel of $e^{tA}$, then $x^{-\alpha}\kappa_H y^{\alpha} = \xi^{-\alpha}\kappa_H$ is the kernel of $x^{-\alpha}e^{tA}x^{\alpha}$. By Theorem 2.4, $\kappa_H$ vanishes to infinite order at $lb = \{\xi = 0\}$ and $rb = \{\xi = \infty\}$. Since the multiplier $\xi^{-\alpha}$ can only affect the asymptotics of $\kappa_H$ at the left and right boundaries, it follows that the kernels of $e^{tA}$ and $x^{-\alpha}e^{tA}x^{\alpha}$ have the same asymptotics as functions on $M^2_H$. We prove Theorem 1.2 using only the asymptotic properties of the heat kernel, and thus without loss of generality we assume that $\alpha = 0$. We examine the heat kernel in each of the coordinate patches on $M^2_H$ separately, multiplying by a cut-off function so that we can assume each time that the heat kernel is supported in a box within that coordinate patch.

**Estimates near the left and right-hand corners**

Recalling the local coordinates from Section Two, we write the heat operator out, for simplicity omitting the $z$ and $z'$ variables corresponding to the compact cross section and which play no role yet. Here $H$ stands for both the operator and its integral kernel:

$$Hf(x,t) = \int H\left(t,\frac{x}{y},y\right)f(y)\,y^{-1}dy.$$

According to Theorem 2.1 above, $H(t,\xi,y)$ vanishes to infinite order as $t \to 0$ or as $\xi = x/y \to 0$; $H$ is also vanishing to infinite order as $\xi \to \infty$, for this is approach to the right boundary. We use this infinite vanishing to absorb the nonintegrable factor of $y^{-1}$ appearing in the measure, by introducing a change of variables, $u = y/x$. Then

$$Hf(x,t) = \int H\left(t,\frac{1}{u},xu\right)f(xu)\,u^{-1}du.$$

Absorbing $u^{-1}$ into a new kernel by defining

$$G\left(t,\frac{1}{u},xu\right) = u^{-1}H\left(t,\frac{1}{u},xu\right),$$

the new kernel $G$ is still vanishing to infinite order at both ends, $u \to 0$ and $u \to \infty$. Therefore,

$$Hf(x,t) = \int G\left(t,\frac{1}{u},xu\right)f(xu)\,du,$$
and estimating $f$ by its maximum,

$$|Hf(x,t)| \leq C\|f\|_0.$$ 

For the derivatives, we pass the derivative inside the integral,

$$x \frac{\partial}{\partial x} (Hf)(x,t) = \int x \frac{\partial H}{\partial \xi} \frac{\partial \xi}{\partial x} f(y) y^{-1} dy = \int \frac{\partial H}{\partial \xi} y f(y) y^{-1} dy = \int \xi \frac{\partial H}{\partial \xi} f(y) y^{-1} dy.$$ 

Now note that because the kernel $H$ was vanishing to infinite order in $\xi$ at both zero and infinity, the very same is true of $\xi \partial H/\partial \xi$. In fact, any finite number of derivatives can be applied and the resulting factors of $\xi^k$ can be absorbed without changing the nature of the asymptotics. Therefore, we find ourselves with a kernel of the same description as $H$ itself, and may apply the same argument as used in the sup estimate. There is never any problem with the $z_i$ derivatives, since these represent compact directions, so we obtain

$$\left| x \frac{\partial}{\partial x} H f(x,z,t) \right| \leq C_{k+l} \|f\|_0.$$ 

Where the diagonal hits the front face

For the first time the coordinates in the direction of the cross section appear, and we write the action of the heat operator on $f$ in these local coordinates as

$$Hf(x,z,t) = \int H \left( \frac{t^{1/m}}{y^{1/m}} \frac{x-y}{y^{1/m}}, \frac{z-z'}{t^{1/m}}, y, z' \right) f(y,z') y^{-1} dy dz',$$

with leading asymptotic $H \sim \rho^{-n} = t^{-n/m}$ and vanishing to infinite order as $|(w,\zeta)| \to \infty$. Upon first inspection it looks as though the sup bound will produce an unexpectedly bad factor of $t^{-n/m}$, but once again we can use the variables in which $H$ vanishes to infinite order to absorb some of this. Letting $w$ and $\zeta$ be the variables of integration, we have

$$Hf(x,z,t) = \int H \left( \frac{t^{1/m}}{wt^{1/m} + 1}, \frac{x}{wt^{1/m} + 1}, \frac{z-t^{1/m} \zeta}{n-1/m} \right)$$

$$\cdot \frac{t^{1/m}}{wt^{1/m} + 1} t^{(n-1)/m} dw d\zeta.$$
The total factor of \( t \) coming from the measure is now \( t^{n/m} \) and we cancel this with the \( t^{-n/m} \) in the kernel and call the new kernel \( G \) to get
\[
Hf(x, z, t) = \int G(t^{1/m}, w, \zeta, y, z') f \left( \frac{x}{wt^{1/m} + 1}, z - t^{-1/m} \zeta \right) \cdot \frac{1}{wt^{1/m} + 1} \, dwd\zeta.
\]
Under the assumption that the heat kernel is supported in a small box around the diagonal, the factor \( 1/(wt^{1/m} + 1) \) may be bounded above, because the denominator equals \( \xi \), which is one at the diagonal. The new kernel \( G \) is bounded in \( \rho = t^{1/m}, y, \) and \( z' \), plus is rapidly decreasing in \( w \) and \( \zeta \), so that
\[
|Hf(x, z, t)| \leq C\|f\|_0.
\]
For the derivative in the transversal direction, we calculate
\[
\frac{x}{\partial x} (Hf)(x, z, t) = \int \left( \frac{\partial H}{\partial w} + t^{-1/m} \frac{\partial H}{\partial w} \right) \cdot f(y, z') y^{-1} \, dydz'.
\]
The factor of \( t^{-1/m} \) comes outside the integral, and the remaining terms inside the integral have the same asymptotics as the kernel \( H \) itself, so we apply to the integral the same argument as above in the sup bound and obtain the estimate
\[
\left| \frac{x}{\partial x} (Hf)(x, z, t) \right| \leq t^{-1/m} C\|f\|_0.
\]
It is easily seen that derivatives by \( \partial / \partial z_i \) also produce a factor of \( t^{-1/m} \) and do not otherwise change the asymptotics. Each successive differentiation by a boundary vector field produces a new factor of \( t^{-1/m} \) so that in the end we have
\[
\left| \left( \frac{x}{\partial x} \right)^k \frac{\partial}{\partial z_i}, \ldots, \frac{\partial}{\partial z_i} (Hf)(x, z, t) \right| \leq C_{k+l} t^{-(k+l)/m} \|f\|_0.
\]

4. The Hölder estimates

We define \( d_g(p, q) \) as the geometric distance between \( p \) and \( q \) if this distance is less than or equal to 1, and 1 if the geometric distance is greater than 1.

**Definition 4.1.** For \( 0 < \gamma < 1 \), the Hölder space \( C^\gamma_b(M) \) contains those functions which are continuous and for which
\[
\|u\|_\gamma = \sup_{M} |u| + \sup_{p \neq q} \frac{|u(p) - u(q)|}{(d_g(p, q))^{\gamma}} < \infty.
\]
There is a simple description of the distance $d_g$ in local coördinates. Near
the boundary, the cylindrical end metric is of the form

$$
\left( \frac{dx}{x} \right)^2 + h = d(\log x)^2 + h,
$$

where $h$ is a metric on $N$. Let $(x, z)$ be coördinates in a neighborhood of the
boundary in $M$ and let $(y, z')$ be a second copy of the same coördinates. In
these coördinates, the square of the geometric distance between $(x, z)$ and
$(y, z')$ has the same order of magnitude as

$$
| \log x - \log y |^2 + | z - z' |^2 = u^2 + | z - z' |^2, \tag{2}
$$

where $u = \log(x/y)$ takes values in $(-\infty, \infty)$. If $v = (x - y)/(x + y)$, then $v$
takes values in $[-1, 1]$, and a short computation shows that

$$
v = \frac{1 - e^{-u}}{1 + e^{-u}} \iff u = \log \frac{v + 1}{1 - v}.
$$

Thus, each $u$ corresponds to a unique $v$ and each $v$ corresponds to a unique
$u$. Moreover, $u = -\infty$ corresponds to $v = -1$, $u = 0$ corresponds to $v = 0$, and
$u = +\infty$ corresponds to $v = +1$, and hence the squared distance (2) is
has the same order of magnitude as

$$
v^2 + | z - z' |^2 = \left( \frac{x - y}{x + y} \right)^2 + | z - z' |^2,
$$
as long as $\log(x/y)$ is bounded by some fixed constant.

In view of this discussion, the Hölder part of the norm has the same order
of magnitude as

$$
\frac{| u(x, z) - u(y, z') |}{(x - y)/(x + y)^\gamma + \sum_{i=1}^{n-1} | z_i - z'_i |^\gamma},
$$

which after multiplication by $| x + y |^\gamma$ can be written as

$$
\frac{| x + y |^\gamma | u(x, z) - u(y, z') |}{| x - y |^\gamma + | x + y |^\gamma \sum_{i=1}^{n-1} | z_i - z'_i |^\gamma}.
$$

We now derive Hölder estimates on the heat operator. To do so, we apply
$m$ derivatives to $Hf$ and pass the derivatives inside to get

$$
\left( x \frac{\partial}{\partial x} \right)^m \left( x \frac{\partial}{\partial x} \right)^m \int \left( x \frac{\partial}{\partial x} \right)^m H \cdot f(y, z') y^{-1} dy dz' \tag{3}
$$

$$
= \int \left( x \frac{\partial}{\partial x} \right)^m H \cdot [f(y, z') - f(x, z)] y^{-1} dy dz' + f(x, z) \int \left( x \frac{\partial}{\partial x} \right)^m H \cdot y^{-1} dy dz'.$$
The first integral is estimated using the fact that the initial data function $f$ is bounded in the Hölder norm:

$$
\left| \int \left( x \frac{\partial}{\partial x} \right)^m H \cdot [f(y, z') - f(x, z)] y^{-1} dy dz \right|
\leq \int \left| \left( x \frac{\partial}{\partial x} \right)^m H \right| \cdot \left| f(y, z') - f(x, z) \right| y^{-1} dy dz'
\leq \|f\|_\gamma \int \left| \left( x \frac{\partial}{\partial x} \right)^m H \right| \frac{|x - y|^{\gamma} + |x + y|^{\gamma} \sum_{i=1}^{n-1} |z_i - z_i'|^{\gamma}}{|x + y|^{\gamma}} y^{-1} dy dz'
= \|f\|_\gamma \int \left| \left( x \frac{\partial}{\partial x} \right)^m H \right| \frac{|x - y|^{\gamma}}{|x + y|^{\gamma}} y^{-1} dy dz'
+ \|f\|_\gamma \sum_{i=1}^{n-1} \int \left| \left( x \frac{\partial}{\partial x} \right)^m H \right| |z_i - z_i'|^{\gamma} y^{-1} dy dz'.
$$

In the first term, notice that $x$ and $y$ are both positive, so that $|x + y|^{\gamma} > |y|^{\gamma}$, and therefore

$$
\frac{|x - y|^{\gamma}}{|x + y|^{\gamma}} \leq \frac{|x - y|^{\gamma}}{y^{\gamma}} = \frac{|x - y|^{\gamma}}{y^{\gamma}} t^{\gamma/m} = |w|^{\gamma} t^{\gamma/m}.
$$

In this way, the first term can be rewritten as

$$
\int \left| \left( x \frac{\partial}{\partial x} \right)^m H \right| |w|^{\gamma} t^{\gamma/m} y^{-1} dy dz'.
$$

Because $H$ is bounded in compact regions for $w$ and vanishing to infinite order as $|w| \to \infty$, the $|w|^{\gamma}$ factor can be absorbed into the kernel without changing the asymptotics. The $m$-th derivative of the kernel produces a factor of $t^{-1}$. Therefore, by the sup and whole derivative estimates done above, the first integral can be bounded by $C t^{-1+\gamma/m} \|f\|_\gamma$.

Similarly in the second term, in the $i$-th summand, we can write

$$
|z_i - z_i'|^{\gamma} = \frac{|z_i - z_i'|^{\gamma}}{t^{\gamma/m}} t^{\gamma/m} = |\xi_i|^{\gamma} t^{\gamma/m},
$$

and absorb $|\xi_i|^{\gamma}$ into the kernel without altering its asymptotics. Therefore the second term and also the first integral is bounded in terms of

$$
t^{-1+\gamma/m} \|f\|_\gamma.
$$

To estimate the second integral in (3), we show that

$$
(4) \quad \left| \int \left( x \frac{\partial}{\partial x} \right)^m H \cdot y^{-1} dy dz' \right| \leq C t^{-1+1/m}.
$$
This estimate implies that for $t \leq 1$,
\[
\left| f(x,z) \int \left( x \frac{\partial}{\partial x} \right)^m H \cdot y^{-1} dydz' \right| \leq \| f \|_0 \cdot C t^{-1+1/m} \\
\leq C t^{-1+\gamma/m} \| f \|_{\gamma}.
\]
Together with the estimate on the first integral in (3) we have
\[
\left| \left( x \frac{\partial}{\partial x} \right)^m Hf(x,z,t) \right| \leq C t^{-1+\gamma/m} \| f \|_{\gamma}.
\]
Similar arguments give analogous estimates for derivatives by the vector fields $\partial/\partial z_i$. This concludes the proof of Theorem 1.2.

To prove the estimate (4), we consider the heat kernel in coordinates where the diagonal hits the front face:
\[
\int \left( x \frac{\partial}{\partial x} \right)^m H \cdot y^{-1} dydz' = \int \left( x \frac{\partial}{\partial x} \right)^m H \left( t^{1/m}, \frac{\log(x/y)}{t^{1/m}}, \frac{z-z'}{t^{1/m}}, y, z' \right) y^{-1} dydz',
\]
where we use $\log(x/y)$ in place of the coordinate $x/y - 1$ that we have been using. Here, we used the fact that $\log 1 = 0$ and the logarithm is a diffeomorphism of $\mathbb{R}^+$ onto $\mathbb{R}$ to justify this substitution. Setting $v = \log(x/y)/t^{1/m}$ and $\zeta = (z - z')/t^{1/m}$, the kernel $H(t^{1/m}, v, \zeta, y, z')$ has the leading asymptotics $H \sim t^{-n/m}$ and vanishes to infinite order as $|(v, \zeta)| \to \infty$.

Since $y = xe^{-t^{1/m}v}$, $z' = z - t^{1/m}\zeta$, and $dy/y dz' = t^{n/m} dv d\zeta$, we can write
\[
(5) \quad \int \left( x \frac{\partial}{\partial x} \right)^m H \cdot y^{-1} dydz' = t^{-1+n/m} \int \left( \frac{\partial^m H}{\partial v^{m-1}} \right) \left( t^{1/m}, v, \zeta, xe^{-t^{1/m}v}, z - t^{1/m}\zeta \right) dv d\zeta,
\]
where the partial derivatives in $v$ apply only to the second variable in $H(t^{1/m}, v, \zeta, y, z')$. With $v = \log(x/y)/t^{1/m}$ and $\zeta = (z - z')/t^{1/m}$, we have
\[
\frac{\partial}{\partial v} \left( \frac{\partial^{m-1} H}{\partial v^{m-1}} \right) = \left( \frac{\partial^m H}{\partial v^m} \right) - t^{1/m}xe^{-t^{1/m}v} \left( \frac{\partial}{\partial y} \cdot \frac{\partial^{m-1} H}{\partial v^{m-1}} \right) \\
= \left( \frac{\partial^m H}{\partial v^m} \right) - t^{1/m} \left( y \frac{\partial}{\partial y} \cdot \frac{\partial^{m-1} H}{\partial v^{m-1}} \right).
\]
Now integrating by parts and using the fact that $H$ vanishes to infinite order as $|(v, \zeta)| \to \infty$ shows that Equation (5) is equal to
\[
t^{-1+n/m} \cdot t^{1/m} \int \left( y \frac{\partial}{\partial y} \cdot \frac{\partial^{m-1} H}{\partial v^{m-1}} \right) \left( t^{1/m}, v, \zeta, xe^{-t^{1/m}v}, z - t^{1/m}\zeta \right) dv d\zeta,
\]
where we recall that $y = xe^{-t^{1/m}v}$. Since $H(t^{1/m}, v, \zeta, y, z') \sim t^{-n/m}$ and vanishes to infinite order as $|(v, \zeta)| \to \infty$, this integral is bounded by a constant times $t^{-1+1/m}$. The estimate (4) is now proved.

5. Applications to semilinear problems

In this section we apply the above results to obtain solutions to some semilinear equations. The reaction-diffusion equations are famous examples of such equations with wide applicability to problems in physics and engineering. Other equations, applications, and theory may be found in the book of Cazenave and Haraux, [CH], dedicated to this subject. Taylor’s volume on nonlinear PDE’s, [T], also provides a thorough discussion of nonlinear parabolic equations and includes many interesting physical examples.

The book of Taylor, [T], explains how to reformulate a nonlinear problem as a contraction mapping. Suppose in addition to the linear part $Au$, our equation also involves quadratic and higher order terms in $u$ and its first derivatives by boundary vector fields. We denote these higher order terms by $Q(u, Vu)$, separate them out, and rewrite the equation as

$$\frac{\partial u}{\partial t} - Au - Q(u, Vu) = 0, \quad u(0) = f.$$  \hspace{1cm} (6)

Treating $Q$ as an inhomogeneous term and applying Duhamel’s principle, Taylor proves the following result:

**Theorem 5.1** ([T], p. 273). Suppose that $X$ and $Y$ are Banach spaces such that the following four conditions are satisfied:

1. $e^{tA} : X \to X$ is a strongly continuous semigroup, for $t \geq 0$.
2. $Q : X \to Y$ is locally Lipschitz.
3. $e^{tA} : Y \to X$, for $t > 0$.
4. For some $-\gamma > -1$,

$$\|e^{tA}\|_{L(Y; X)} \leq Ct^{-\gamma}, \quad \text{for} \; 0 < t \leq 1.$$  

Then the initial value problem (6) with $f \in X$ has a unique solution $u \in C([0, T], X)$ where $T > 0$ and is estimable from below in terms of $\|f\|_X$.

Interpreting this in the context of a manifold with cylindrical ends, we have the following:

**Theorem 5.2.** On a manifold with cylindrical ends, let either $Y = C^0(M)$ and $X = C^{m-1}_b(M)$ or else $Y = C^1_b(M)$ and $X = C^m_b(M)$, and let $Q : X \to Y$ be locally Lipschitz. Then given any initial data in $X$, the semilinear initial value problem (6) has a solution in $X$ existing on some time interval $[0, T]$ for some $T > 0$.  

Proof. The third and fourth conditions are exactly the content of Theorem 1.2. Therefore the only hypothesis that remains to check is the semigroup property. Namely, we need to show that
\[ \| H f(\cdot, t) - f \|_X \to 0 \text{ as } t \to 0. \]

This result is very similar to the proof given in Section Six of the earlier paper, [JL]. Let us simply note here that local calculations similar to the ones we gave in Sections Three and Four show that away from the diagonal, the heat operator becomes more like the zero map as \( t \to 0 \), while at the diagonal it becomes more and more like the identity operator. For details we refer the reader to [JL].

As a concluding remark, we observe that long time existence of solutions involves completely different considerations. The reaction-diffusion equation, for example, models a fluid or gas that is diffusing while simultaneously undergoing a chemical reaction. The nonlinear term representing the reaction may be a growth term which is in competition with the linear diffusion term, and which is stronger is highly dependent on the individual problem. Likewise, in geometric applications, global aspects such as curvature and topology influence the long time behavior of solutions.

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Received September 24, 2003. The second author was supported in part by a Ford Foundation fellowship.

**Universidad Michoacana**
Morelia
Mexico
*E-mail address:* thalia@ginette.ifm.umich.mx

**Department of Mathematics**
SUNY at Binghamton
Binghamton, NY 13902
*E-mail address:* paul@math.binghamton.edu
THE HOPF ALGEBRA OF A UNISERIAL GROUP

ALAN KOCH

We give a Dieudonné module description for the finite commutative infinitesimal unipotent group schemes over a perfect field of prime characteristic. Witt vector addition is used to describe the algebra and coalgebra structures of the representing Hopf algebra of a uniserial group.

1. Introduction

Let $k$ be an algebraically closed field of characteristic $p > 0$. The purpose of this paper is to describe the Hopf algebra of a finite commutative infinitesimal unipotent $k$-group scheme which is uniserial, i.e., which has a unique composition series. As there is only one simple finite commutative infinitesimal unipotent group scheme (namely $\alpha_p := \ker \{ F : \mathbb{G}_a \to \mathbb{G}_a \}$, with $\mathbb{G}_a$ being the additive group scheme and $F$ the Frobenius map), composition series on this class of group schemes are a bit easier to study than for arbitrary group schemes. A certain class of uniserial groups, namely the $V$-uniserial groups, are important in studying representation theory: A finite connected $k$-group scheme $G$ has finite representation type if and only if the quotient $G/M(G)$, where $M(G)$ is the multiplicative center of $G$, is a semidirect product of a $V$-uniserial unipotent group $U$ together with a group of type $\mu_p^n$ [FV, 2.7].

In [FRV], the authors introduce Dieudonné modules to classify the $V$-uniserial unipotent groups in an effort to describe all groups of finite representation type. It is shown that the (isomorphism classes of) uniserial groups follow one of six different “types”, three of which are dual to the other three.

Here, we will also use classical Dieudonné module theory to describe uniserial groups, but we will reduce the number of types needed in the description. Surprisingly, there is an easy way to write the isomorphism classes of uniserial groups in terms of Dieudonné modules: They all fit into one of two types, and furthermore the types are dual to each other. Since any representation-finite local algebra is of the form $k[t]/(t^m)$, by [FV, 2.7 and 3.1] we have that $G = \text{Spec}(H)$ is $V$-uniserial if and only if $H^* = \text{Hom}(H, k)$ is monogenic (i.e., generated as a $k$-algebra by a single element), and by duality $G$ is $F$-uniserial if and only if $H$ is monogenic. Using the classification
found in [K2], this enables us to quickly describe all uniserial groups in terms of their Dieudonné modules.

In [FRV, 5.1], most of the attention is focused on the classification of $V$-uniserial groups due to their relationship with representation theory. A group is $V$-uniserial if and only if the dual to the representing algebra is monogenic. We will look primarily at $F$-uniserial groups due to the simple structure of their representing Hopf algebras. Duality will give us the results for $V$-uniserial groups. We will explicitly show how our classification is equivalent to the Farnsteiner–Röhrle–Voigt classification in the third section.

While the representation-theoretic applications appear to be most common when $k$ is algebraically closed, in the final section we extend our results to the case where $k$ is a perfect (not necessarily algebraically closed) field of characteristic $p$. In this case, it is still true that $G = \text{Spec}(H)$ is uniserial if and only if either $H$ or $H^*$ is monogenic, however there are usually many more isomorphism classes of monogenic Hopf algebras than when $k$ is algebraically closed. For the case where $k$ is finite we provide a formula for the number of (isomorphism classes of) uniserial groups of order $p^n$ for some $n$.

Throughout this paper, all groups are commutative, finite, connected, and unipotent. Until the final section, $k$ is an algebraically closed field of positive characteristic $p$.

2. Uniserial groups and their Dieudonné modules

Let $W = W(k)$ be the ring of Witt vectors with coefficients in $k$. Following the notation in [O] we let $E$ be the non-commutative polynomial ring $E = W[F, V]$ with relations $FV = VF = p$, $Fw = w^\sigma F$, and $wV = Vw^\sigma$, where $\sigma$ is the Frobenius map on $W$. There is an antiequivalence $D^*$ between finite connected unipotent commutative group schemes and finite length $E$-modules killed by a power of $F$ and $V$. The correspondence is given by

$$D^*(G) = \text{Hom}(G, C)$$

where $C$ is the ring of Witt covectors as described in [F1, p. 1273]. We will use the term Dieudonné module to indicate a finite length $E$-module killed by a power of $F$ and $V$. There are other definitions of Dieudonné module which encompass a larger variety of group schemes, for example in [F1] and [F2] a Dieudonné module theory is constructed that can be used to describe formal groups as well as non-connected, non-unipotent group schemes. However, our definition is all that is needed for the results to follow.

The Cartier dual of an infinitesimal unipotent group scheme $G$, given by $G^* = \text{Hom}(G, \mathbb{G}_m)$ (where $\mathbb{G}_m$ is the multiplicative group scheme), is also infinitesimal and unipotent. In terms of Dieudonné modules, $D^*(G^*)$ is simply the module $D^*(G)$ with the roles of $F$ and $V$ interchanged [DG, V, 4, 5.6].
Of particular interest to us will be Dieudonné modules that are cyclic, i.e., are of the form \(E/I\) for some ideal \(I \subset E\). Each of the uniserial groups will be what is called a Witt subgroup, that is, a subgroup of a group \(W_n\) of Witt vectors of length \(n\) for some \(n\). The antiequivalence \(D^*\) restricts to a correspondence between Witt subgroups and cyclic Dieudonné modules [K1, 1.2].

Recall that a group scheme \(G\) is uniserial if it has a unique composition series. Additionally, there are the notions of \(F\)-uniserial and \(V\)-uniserial. The group scheme \(G\) is \(F\)-uniserial if the kernel of the Frobenius map \(F : G \to G^{(p)}\) is simple, i.e., if \(\ker F \cong \alpha_p\). Likewise, \(G\) is \(V\)-uniserial if the cokernel of the Verschiebung \(V : G^{(p)} \to G\) is simple. A group scheme \(G\) is uniserial if and only if it is either \(F\)-uniserial or \(V\)-uniserial [FRV, Sec. 1.2].

A Dieudonné module \(M\) is called \(F\)-uniserial if \(\text{coker } F\) is simple as a Dieudonné module, i.e., if \(\text{coker } F \cong E/E(F,V)\). Letting \(n\) be the smallest positive integer so that \(F^nM = 0\), this gives rise to a composition series

\[
0 = F^nM \subset F^{n-1}M \subset \cdots \subset F^2M \subset FM \subset M
\]

which is the only composition series for \(M\), hence an \(F\)-uniserial module is necessarily a uniserial module (that is it has a unique composition series). Indeed, suppose \(M\) is a nonsimple uniserial module with composition series

\[
0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_{n-1} \subset M_n = M.
\]

Observe that \(F\) acts trivially on \(M_{i+1}/M_i \cong k\) and hence \(FM_{i+1} \subset M_i\). Then we have

\[
FM_{i+1} \subset M_i \subset M_{i+1}
\]

with \(M_i \neq M_{i+1}\). Since \(F\) induces a composition series on \(M_{i+1}\) it follows that \(FM_{i+1} = M_i\).

Similarly, \(M\) is \(V\)-uniserial if \(\ker V \cong E/E(F,V)\), that is we have that

\[
0 \subset V^{n-1}M \cdots \subset V^2M \subset VM \subset M
\]

is a composition series, where \(V^{n-1}M \neq 0\) and \(V^nM = 0\). By an argument similar to the one above a \(V\)-uniserial module is also uniserial. Conversely, it is straightforward to show that any uniserial Dieudonné module is either \(F\)-uniserial or \(V\)-uniserial.

Using the definitions of uniserial above it follows that \(G\) is \(F\)-uniserial (resp. \(V\)-uniserial) if and only if \(D^*(G)\) is \(F\)-uniserial (resp. \(V\)-uniserial) [FRV, 2.5].

We will now give a simple description for the Dieudonné module structure of an \(F\)-uniserial group.
Proposition 2.1. A commutative finite infinitesimal unipotent group scheme $G$ is $F$-uniserial if and only if
\[ D^r(G) \cong E/E(F^n, F^r - V) \]
for some $1 \leq r \leq n$.

Proof. Of course, $G = \text{Spec} \, H$ is $F$-uniserial if and only if $H$ is monogenic. The result follows from \[ K2, 2.2 \] taking $\eta = 1$, which can be done since $E/E(F^n, F^r - V) \cong E/E(F^n, F^r - \eta V)$ for all $\eta \in k^\times$ as $k$ is algebraically closed (see \[ K2, \text{p. 199} \]).

Dualizing switches the roles of $F$ and $V$, and consequently changes $F$-uniserial modules to $V$-uniserial modules, giving us:

Corollary 2.2. A commutative finite infinitesimal unipotent group scheme $G$ is $V$-uniserial if and only if
\[ D^r(G) \cong E/E(V^n, V^r - F) \]
for some $1 \leq r \leq n$.

3. Equivalence to the Farnsteiner–Röhrle–Voigt classification

As stated above, the classification of (finite commutative infinitesimal unipotent) uniserial group schemes has also been done by R. Farnsteiner, G. Röhrle and D. Voigt in \[ FRV \] using Dieudonné modules. Here we will show how the two classifications are equivalent.

We start by recalling the Farnsteiner–Röhrle–Voigt classification. For each pair of positive integers $d$ and $n$, let $M_{n,d}$ be a free $W_n(k)$-module with basis $\{e_1, \ldots, e_d\}$. For $d \geq 2$, let $E_{n,d}$ be the Dieudonné module defined as follows: $E_{n,d} \cong M_{n,d}$ as $W_n(k)$-modules, with $E$-module structure given by
\[
V(e_i) = e_{i+1} \text{ for } 1 \leq i \leq d - 1, \quad V(e_d) = pe_1
\]
\[
F(e_1) = e_d, \quad F(e_i) = pe_{i-1} \text{ for } 2 \leq i \leq d.
\]

For $d, n \geq 2$ and $1 \leq j \leq d - 1$ let
\[
E^j_{n,d} = E_{n,d}/V^{(n-1)d+j}E_{n,d}.
\]

For $d \geq 1$ let $E_d$ be the Dieudonné module defined as follows: $E_d \cong M_{1,d}$ as $W_1(k)$-modules, with $E$-module structure given by
\[
V(e_i) = e_{i+1} \text{ for } 1 \leq i \leq d - 1, \quad V(e_d) = 0
\]
\[
F(e_i) = 0 \text{ for } 1 \leq i \leq d.
\]

Then every $V$-uniserial Dieudonné module is of one of these three forms: $E_{n,d}$, $E^j_{n,d}$, or $E_d$; furthermore every module of each of these forms is $V$-uniserial. Duality, of course, gives us the $F$-uniserial modules.
Proposition 3.1. With the notation above we have:

1) \( E_{n,d} \cong E/E(V^{nd}, V^{d-1} - F) \)
2) \( E_{n,d}^j \cong E/E(V^{(n-1)d+j}, V^{d-1} - F) \)
3) \( E_d \cong E/E(V^d, F) = E/E(V^d, V^d - F) \).

Thus every \( V \)-uniserial group is of the form \( E/E(V^s, V^r - F) \). Furthermore, any \( M = E/E(V^s, F^r - V) \) is isomorphic to either \( E_{n,d} \), \( E_{n,d}^j \), or \( E_d \) for some choice of \( j, n, \) and \( d \).

Proof. We will consider each of these three forms, and show that they are isomorphic to \( E/E(V^s, V^r - F) \) for some choice of \( r \) and \( s \). Define \( \phi : E_{n,d} \to E/E(V^{nd}, V^{d-1} - F) \) by \( \phi(e_i) = F^{i-1}e \) for \( 1 \leq i \leq d \), where \( e \) is the projection of the identity 1 under the canonical projection \( E \to E/E(V^{nd}, V^{d-1} - F) \). As \( E/E(V^{nd}, V^{d-1} - F) \) has \( W_n(k) \)-basis \( \{e, V^1e, \ldots, V^{d-1}e\} \), this is clearly an isomorphism of \( W_n(k) \)-modules. In fact, it is quite easy to show that this map also preserves the \( F \) and \( V \) actions on each module, hence \( E_{n,d} \cong E/E(V^{nd}, V^{d-1} - F) \) as Dieudonné modules.

For the second form, the module \( E_{n,d}^j \) is given by

\[ E_{n,d}^j = E_{n,d}/V^{(n-1)d+j}E_{n,d} \]

where \( n, d \geq 2 \) and \( 1 \leq j \leq d - 1 \). Identifying \( E_{n,d} \) with the isomorphism above gives us

\[ E_{n,d}^j \cong E/E(V^{nd}, V^{d-1} - F, V^{(n-1)d+j}) = E/E(V^{(n-1)d+j}, V^{d-1} - F). \]

For the third form, \( E_d \) is the \( k \)-module with basis \( \{e_1, e_2, \ldots, e_d\} \) with \( F \) acting trivially and \( V(e_i) = e_{i+1} \) for \( 1 \leq i \leq d - 1 \) and \( V(e_d) = 0 \). Here we define \( \phi : E_d \to E/E(V^d, F) \) by \( \phi(e_i) = V^{i-1}e \) for \( 1 \leq i \leq d \). This is an isomorphism of Dieudonné modules.

If we let \( s = nd \) and \( r = d - 1 \) with, then we see that the first form is \( E/E(V^s, V^r - F) \). The substitutions of \( s \) for \( (n-1)d + j \) and \( r \) for \( d - 1 \) shows that the second class is also \( E/E(V^s, V^r - F) \). Finally, setting \( s = nd \) and observing \( E/E(F^n, V) = E/E(F^n, F^n - V) \) gives the result for the third form. Thus the Dieudonné module for any \( F \)-uniserial group is of the form \( E/E(F^s, F^r - V) \) for some \( 1 \leq r \leq s \).

Conversely, suppose \( M = E/E(V^s, V^r - F) \). If \( s = r \) we get

\[ M = E/E(V^s, V^s - F) = E/E(V^s, F) \cong E_s. \]

If \( r + 1 \) divides \( s \) then letting \( t = s/(r + 1) \) gives

\[ M = E/E(V^{t(r+1)}, V^{(r+1)-1} - F) \cong E_{t,r+1}. \]

If \( r + 1 \) does not divide \( s \), then writing \( s = t(r + 1) + j \) gives

\[ M = E/E(V^{t(r+1)+j}, V^{(r+1)-1} - F) \cong E_{t,r+1}^j. \]
Thus every Dieudonné module in our classification also appears somewhere in the Farnsteiner–Röhrle–Voigt classification.

Our classification of uniserial Dieudonné modules translates easily to the corresponding uniserial group schemes. The proofs of the statements about the heights will be obvious by the explicit algebra structure for the representing algebras given in the next section.

**Corollary 3.2.**

1) Suppose $D^* (G) = E/E(V^s, V^r - F)$. Then $G = \ker \{ V^r - F : W_s \to W_s \}$. Explicitly, for any $k$-algebra $A$ we have

$$G(A) = \left\{ (a_0, \ldots, a_{s-1}) \in W_s(A) \mid a_j^{[s/r]} = 0 \text{ for } j < r, \right. a_i = a_{i-r}^p \text{ for all } r \leq i \leq s - 1 \right\} \leq W_s(A)$$

when $s > r$, and

$$G(A) = \{ (a_0, \ldots, a_{s-1}) \in W_s(A) \mid a_i^p = 0 \text{ for } 0 \leq i \leq s - 1 \} \leq W_s(A).$$

This group scheme has height $[s/r]$, where $[\ ]$ is the ceiling function.

2) If $D^*(G) = E/E(F^s, F^r - V)$, then $G = \ker \{ F^r - V : W_m \to W_m \}$ where $m = [s/r]$. For any $k$-algebra $A$ we have

$$G(A) = \left\{ (a_0, \ldots, a_{s-1}) \in W_m(A) \mid a_i^{r^\alpha} = 0 \text{ for all } i, a_i^{r^\alpha} = a_{i-1} \text{ for } i \neq 0 \right\} \leq W_m(A).$$

This group scheme has height $s$.

**Remark.** While the use of cyclic Dieudonné modules unifies the description of the $F$-uniserial (resp. $V$-uniserial) modules in a nice way, there are differences in these classes that may be important for various applications. For example, the modules $E_{n,d}$ correspond to the $F$-uniserial group schemes $G_{n,d}$ that can be lifted to $W(k)$, i.e., for which there is a $W(k)$-group scheme $\hat{G}$ with $\hat{G} \times_{\text{Spec } W(k)} \text{Spec}(k) \cong G$. The other two classes do not lift to $W(k)$. See [K2, 4.2], where $\alpha = 1$.

4. The Hopf algebra structure

We now use the simplified Dieudonné module structure for uniserial groups to compute their representing Hopf algebras. By insisting on the form used in Proposition 2.1 we can determine the underlying Hopf algebra structure.

Given a finite abelian (that is, commutative and cocommutative) local Hopf algebra $H$ with local dual, we can associate to it a Dieudonné module $M$ by setting $M = D^*(\text{Spec}(H))$. Conversely, given a Dieudonné module $M$, we can associate to it a $k$-Hopf algebra $\mathcal{H}(M)$. We will show how this
is done after introducing the notation of Witt polynomials. For any $n > 0$
define a polynomial $w_n(Z_0, Z_1, \ldots, Z_n)$ by

$$w_n(Z_0, Z_1, \ldots, Z_n) = p^n Z_n + p^{n-1} Z_{n-1}^p + \cdots + Z_0^p.$$  

The $w_n$’s are used to define polynomials $S_0, S_1, \ldots, P_0, P_1, \ldots$ via

$$w_n(S_0, \ldots, S_n) = w_n(X_0, \ldots, X_n) + w_n(Y_0, \ldots, Y_n)$$

$$w_n(P_0, \ldots, P_n) = w_n(X_0, \ldots, X_n)w_n(Y_0, \ldots, Y_n).$$

For example,

$$S_0(X_0, Y_0) = X_0 + Y_0$$

$$S_1((X_0, X_1), (Y_0, Y_1)) = X_1 + Y_1 - \frac{(X_0 + Y_0)^p - X_0^p - Y_0^p}{p}$$

$$P_0(X_0, Y_0) = X_0Y_0$$

$$P_1((X_0, X_1), (Y_0, Y_1)) = X_0^p Y_1 + X_1 Y_0^p + pX_1 Y_1.$$  

Further properties of these polynomials can be found in [J].

We are now ready to describe $H(M)$. As a $k$-algebra, $H(M) = k[T_x \mid x \in M]$ with the following relations:

$$T_{Fx} = (T_x)^p$$

$$T_{x+y} = S_N((T_{V^N x}, T_{V^N-1 x}, \ldots, T_x); (T_{V^N y}, T_{V^N-1 y}, \ldots, T_y))$$

$$T_{wx} = P_N((w_0^{p^{-N}}, w_1^{p^{-N}}, \ldots, w_N^{p^{-N}}); (T_{V^N x}, T_{V^N-1 x}, \ldots, T_x))$$

where $x, y \in M$, $w = (w_0, w_1, \ldots) \in W(k)$, and $N$ is any nonnegative integer so that $V^{N+1} M = 0$. The comultiplication is given by

$$\Delta(T_x) = S_N \left( (T_{V^N x} \otimes 1, T_{V^N-1 x} \otimes 1, \ldots, T_x \otimes 1); \right.$$

$$\left. (1 \otimes T_{V^N x}, 1 \otimes T_{V^N-1 x}, \ldots, 1 \otimes T_x) \right).$$

These operations make $H(M)$ into a Hopf algebra, giving a 1-1 correspondence between finite local Hopf algebras with local dual and Dieudonné modules [G, II, Sec. 5].

We can now give an explicit description of the Hopf algebra based on the Dieudonné module. If $M = E/E(F^s, F^r - V)$ then $H(M) \cong k[t]/(t^{p^r})$ (under this isomorphism $t \in H$ corresponds to $T_e$ where $e$ is the image of 1 in $E$ under the canonical map $E \to M$) with

$$\Delta(t) = S_N \left( \left( t^{p^r} \otimes 1, \ldots, t^{p^r} \otimes 1, t \otimes 1 \right); \left( 1 \otimes t^{p^r}, \ldots, 1 \otimes t^{p^r}, 1 \otimes t \right) \right)$$

where $N = \lfloor s/r \rfloor - 1$. Clearly $H(E/E(F, V))$ is the representing algebra of $\alpha_p$. In general the generator $t$ is primitive if and only if $s = r$.  

If, on the other hand, \( M \cong E/E(V^s, V^r - F) \) we let \( \varepsilon_i = [(s + 1 - i)/r] \) for \( 1 \leq i \leq r - 1 \), which gives us
\[
\mathcal{H}(M) \cong k \langle t_1, t_2, \ldots, t_{r-1} \rangle \bigg/ \left( t_1^{p^s_1}, t_2^{p^s_2}, \ldots, t_{r-1}^{p^s_{r-1}} \right)
\]
with a straightforward but usually messy coalgebra structure: In this case \( t_i \in H \) corresponds to \( T_{V_{i+1}} \). As a small example, if \( s = 3 \) and \( r = 2 \) we have
\[
\mathcal{H}(M) \cong k[t_1, t_2]/(t_1^2, t_2^2)
\]
\[
\Delta(t_1) = S_2 ((t_1^0 \otimes 1, t_2^1 \otimes 1, t_1 \otimes 1); (1 \otimes t_1^0, 1 \otimes t_2, 1 \otimes t_1))
\]
\[
\Delta(t_2) = S_2 ((0, t_1^0 \otimes 1, t_2 \otimes 1); (0, 1 \otimes t_1^0, 1 \otimes t_2)) = S_1 ((t_1^0 \otimes 1, t_2 \otimes 1); (1 \otimes t_1^0, 1 \otimes t_2)).
\]

5. Uniserial groups over perfect fields of characteristic \( p \)

Finally, we can quickly extend these results to the case where \( k \) is any perfect (but not necessarily algebraically closed) field of characteristic \( p \). The definitions of uniserial, \( F \)-uniserial, and \( V \)-uniserial are the same as in the algebraically closed case, both when speaking about group schemes as well as Dieudonné modules. In fact, it is still true that \( G = \text{Spec} H \) is uniserial if and only if \( H \) or \( H^* \) is monogenic. By [K2, Th. 2.2], we obtain:

**Proposition 5.1.** Let \( M \) be an \( F \)-uniserial Dieudonné module. Then
\[
M \cong E/E(F^n, F^r - \eta V)
\]
where \( \eta \) is a nonzero element of \( k \).

**Caveat.** The \( F \)-uniserial Dieudonné modules are not parameterized by triples \((n, r, \eta)\) since different choices of \( \eta \) can give isomorphic modules. Adopting the notation \( M_{n, r, \eta} = E/E(F^n, F^r - \eta V) \) gives that if \( M_{n, r, \eta} \cong M_{n', r', \eta'} \) then \( n = n' \) and \( r = r' \); [K1, 3.1]. Furthermore, \( M_{n, r, \eta} \cong M_{n, r, \eta'} \) if and only if there is an invertible element \( a \in k \) such that
\[
\left( \frac{\eta}{\eta'} \right)^{p^n} = a^{p^i + m - 1}
\]
[K1, 3.2]. In the case where \( k \) is a finite field, it is possible to give a complete list of non-isomorphic \( F \)-uniserial Dieudonné modules. The proof follows immediately from the above proposition together with [K2, 3.1].

**Corollary 5.2.** Let \( k = \mathbb{F}_{p^d} \), and let \( M \) be an \( F \)-uniserial Dieudonné module. Let \( k_0 = \mathbb{F}_{p^d} \), where \( d = \gcd(t, r + 1) \). Fix \( \alpha \in k \) such that \( k = k_0[\alpha] \).
Then
\[
M \cong E/E(F^n, F^r - \alpha^z V) \quad \text{or} \quad M \cong E/E(F^n, V)
\]
where \( 1 \leq r \leq n - 1 \) and \( 0 \leq z \leq p^d - 2 \).
For an explicit description of $H(M)$ in the former case, as well as a description of the corresponding $V$-uniserial group scheme, see [K3, Sec. 4].

Finally, the above corollary enables us to provide a formula for the number of uniserial groups of order $p^n$ for some $n$ when $k$ is a finite field.

**Corollary 5.3.** The number of isomorphism classes of uniserial group schemes over a field $k = F_{p^l}$ of order $p^n$ for $n > 1$ is given by

$$2 \left( 2 - n - \sum_{r=1}^{n-1} p^{\gcd(l,r+1)} \right).$$

Of course, if $n = 1$ then the only isomorphism class is the one containing $\alpha_p$.

**References**


Received January 13, 2003.
DEPARTMENT OF MATHEMATICS
AGNES SCOTT COLLEGE
DECATURE, GA 30030
E-mail address: akoch@agnesscott.edu
POISSON BRACKETS ASSOCIATED TO INVARIANT EVOLUTIONS OF RIEMANNIAN CURVES

G. Marí Beffa

In this paper we show that Poisson brackets linked to geometric flows of curves on flat Riemannian manifolds are Poisson reductions of the Kac–Moody bracket of $SO(n)$. The bracket is reduced to submanifolds defined by either the Riemannian or the natural curvatures of the curves. We show that these two cases are (formally) Poisson equivalent and we give explicit conditions on the coefficients of the geometric flow guaranteeing that the induced flow on the curvatures is Hamiltonian.

1. Introduction

The study of infinite dimensional Poisson geometry has traditionally been an important component in the study of completely integrable systems. In fact, the majority of completely integrable systems of PDEs are Hamiltonian with respect to two different but compatible infinite dimensional Hamiltonian structures, that is, they are biHamiltonian. This property allows the generation of a recursion operator that produces an infinite sequence of preserved quantities, effectively integrating the system. Infinite dimensional Poisson geometry is also linked to some analytic problems, in particular to the classification of normal forms of differential operators. These are found using versal deformations of the symplectic foliation (see [LPa] and [M1]).

The connection between finite dimensional differential geometry and completely integrable PDEs dates back to Liouville, Bianchi and Darboux ([Li], [Bi] and [Da]), but it was after Hasimoto’s work in the vortex filament flow evolution that the close relation between integrable PDEs and the evolution of curvature and torsion (rather than the curve flow itself) was clear. In fact, Hasimoto ([Ha]) proved that the vortex filament flow induces a completely integrable evolution on the curvature and torsion of the flow. In particular, the evolution of curvature and torsion were biHamiltonian.

Langer and Perline pointed out in their papers on the subject (see [LP1] and [LP2]) that the Hamiltonian structures that were used to integrate some of these systems were defined directly from the Euclidean geometry of spatial curves. Indeed, the structures found in [MSW] for the evolution of Riemannian curves in three dimensions were all defined geometrically with
the use of Frenet frames. So were the ones in [DSa] and others. The so-called natural frames were also used to integrate systems in [LP2]. A direct relationship between the evolution of differential invariants (curvatures) of the evolving curves and infinite dimensional Poisson structures exists not only in Riemannian geometry but also in projective geometry. In fact, KdV and its generalizations can be viewed as the evolution of projective differential invariants of a certain invariant flow of curves in $\mathbb{RP}^n$ ([DS]). The Hamiltonian structure used to integrate them can be defined directly from the projective geometry of the curves ([M2]).

In this paper we describe the evolution induced on the Riemannian curvatures of curves evolving invariantly on a Riemannian manifold with constant curvature. Our approach using Cartan connections allows us to establish directly the connection between these evolutions and Hamiltonian structures on the dual of the algebra of loops on $o(n)$, $Lo(n)^*$. In fact, we prove that there exists a Poisson structure on the quotient $Lo(n)^*/LSO(n - 1)$ obtained through a standard Poisson reduction procedure as described in [MR]. The Poisson reduction procedure links directly the geometry of the curves and the quotient $Lo(n)^*/LSO(n - 1)$. The reduced structure on $Lo(n)^*/LSO(n - 1)$ can be found in the literature (see [TU] and references within), although defined from a different point of view. We prove that both Frenet frames and natural frames can be viewed as transverse sections of the foliation induced on $Lo(n)^*$ by the coadjoint action of $LSO(n - 1)$. Therefore, there exist two natural Hamiltonian structures defined on the spaces of Frenet curvatures and natural curvatures. They are given through the identification of $Lo(n)^*/LSO(n - 1)$ with its sections. The Poisson map (a gauge) that takes one structure to the other is a generalization of the Hasimoto transformation found in [Ha] for $n = 3$.

The emphasis of this paper is on the geometric description of these Poisson structures and their precise relationship with invariant evolutions of Riemannian curves, not on the classification and study of integrable systems. For more information about completely integrable systems appearing in this setting, please see [LP2] and [TT].

In Chapter 2 we have included some background definitions and information on Cartan connections, Riemannian geometry and moving frames, since they are used and some of the readers might not be familiar with it. We have also included basic information about Poisson geometry in Chapter 4. Chapter 3 obtains a general formula for the evolution of Frenet and natural curvatures, evolutions induced by an arc-length preserving invariant flow of curves on a Riemannian manifold with constant curvature. In Chapter 4 we define the relevant Poisson structures via the Poisson reduction method. Using the Poisson reduction method allows us to define explicitly these brackets in both the Frenet and natural cases. We show that both Frenet and natural cases are simply different choices of transverse sections.
in \( Lo(n)^*/LSO(n-1) \). Chapter 5 establishes the relation, in the flat case, between these reduced brackets and the evolution of curvatures that were found in Chapter 3. In the natural case the relation is far simpler than in the Frenet case. That might explain why the latter is a preferred choice in the treatment of the associated integrable systems.

2. A short introduction to Riemannian manifolds, Cartan connections and moving frames

In this section we will provide the background definitions and results in Differential Geometry that will be used along this paper. Much of it is stated as in [Sh] and [K].

**Definition 1.** Let \( G \) be a Lie group and let \( H \subset G \) be a closed subgroup such that \( G/H \) is connected. The pair \( (G,H) \) is called a **Klein Geometry**.

Assume that a group \( G \) acts transitively on a manifold \( M \), and let \( H_p = \{ g \in G, \text{such that } g.p = p \} \). The manifold \( M = G/H_p \) is called a **homogeneous space**. Examples of homogeneous spaces are Euclidean and projective space and the Möbius sphere.

A **Cartan Geometry** \((P,\omega)\) on a manifold \( M \) modeled on the Klein geometry \((G,H)\) consists of the following data:

1. A smooth manifold \( M \);
2. a principal \( H \)-bundle \( P \) over \( M \);
3. a \( \mathfrak{g} \)-value 1-form \( \omega \) on \( P \) satisfying the following conditions:
   1. For each point \( p \in P \), the linear map \( \omega_p : T_p P \rightarrow \mathfrak{g} \) is an isomorphism;
   2. \( (R_h)^* \omega = \text{Ad}(h^{-1}) \omega \) for all \( h \in H \);
   3. \( \omega(0,X) = X \) for all \( X \in \mathfrak{h} \)

where, as usual, \( R_h \) denotes the right multiplication map, \( h \in H \), \( \text{Ad} \) represents the Adjoint action of the group, and \( (0,X) \) is a trivialization of the element in \( P \) associated to \( X \in \mathfrak{h} \). The form \( \omega \) is usually called the **Cartan connection**.

If \((M,P,\omega)\) is a Cartan geometry, the \( \mathfrak{g} \)-valued 2-form on \( P \) given by

\[
\Omega = d\omega + \frac{1}{2}[\omega,\omega]
\]

is called **the curvature**. Equation (2.1) is called the **structural equation**.

Several interesting facts are known about Cartan connections and Cartan curvature forms. The most relevant to us can be found in [Sh] pp. 187-188:

1. The curvature form \( \Omega \) can be regarded as a 2-form on the pullback of the tangent bundle of \( M \) to the principal bundle \( P \).
2. The restriction of the Cartan connection \( \omega \) to each fiber of the principal bundle coincides with the Maurer–Cartan form on \( H \) (if one identifies each fiber with \( H \)). This is a direct consequence of Property (iii) above.
(3) Let \( \pi : P \to M \) be the projection from \( P \) to \( M \). For \( x \in M \) and \( p \in P \) with \( \pi(p) = x \) there exists a canonical isomorphism \( \omega_p \)
\[
\omega_p : T_x M \to g/h
\]
such that \( \omega_p h = \text{Ad}(h^{-1}) \omega_p \) for any \( h \in h \).

If \( \rho : g \to g/h \) is the canonical projection, then \( \rho(\Omega) \) is called the torsion.

**Definition 2.** The \( n \)-dimensional Euclidean space is a homogeneous space given by \( \mathbb{R}^n = \text{Euc}_n(\mathbb{R})/\text{SO}_n(\mathbb{R}) \). By \( \text{Euc}_n(\mathbb{R}) \) we denote the Euclidean group defined as
\[
\text{Euc}_n(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & 0 \\ v & \vartheta \end{pmatrix} \right\},
\]
where we are identifying \( \text{SO}_n(\mathbb{R}) \) with its copy inside \( \text{Euc}_n(\mathbb{R}) \) (that is, \( v = 0 \)). \( \text{Euc}_n(\mathbb{R}) \) acts on \( \mathbb{R}^n \) by multiplication of matrices if we identify \( x \in \mathbb{R}^n \) with \( \begin{pmatrix} 1 \\ x \end{pmatrix} \). The subgroup \( \text{SO}_n(\mathbb{R}) \) leaves the origin fixed.

Let \( M \) be a smooth manifold. A Euclidean geometry on (oriented) \( M \) is a Cartan geometry on \( M \) modeled on Euclidean space \( (\text{Euc}_n(\mathbb{R}),\text{SO}_n(\mathbb{R})) \). A Riemannian geometry on \( M \) is a Euclidean geometry with torsion equals zero. We will say \( M \) is a Riemannian manifold.

**Definition 3.** Let
\[
p = \left\{ \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix} \right\} \subset \text{euc}_n(\mathbb{R}).
\]

Comment (3) above shows that there exists an isomorphism between \( T_x M \) and \( p \) given by \( \omega_p \), depending on a point \( p \in P \) with \( \pi(p) = x \). Therefore, if \( v \in p \) then \( \omega_p^{-1}(v) \in T_x M \), or rather it belongs to the pullback of \( T_x M \) to the principal bundle \( P \). We define the curvature function \( K : P \to \text{Hom}(\bigwedge^2 p, h) \) on \( P \) as
\[
K(p)(v_1, v_2) = \Omega_p(\omega_p^{-1}(v_1), \omega_p^{-1}(v_2)),
\]
for any \( v_1, v_2 \in p \).

We say that a Riemannian manifold has constant curvature \( \kappa \) if, whenever \( \{e_i\} \) are basis of \( \mathbb{R}^n \) and \( e_{ij} \) are generators of \( so(n) \) (same as \( so_n(\mathbb{R}) \)) such that \( \text{ad}(e_{ij})e_k = \delta_{jk}e_i - \delta_{ik}e_j \), then
\[
K(p) = \kappa \sum_{i<j} e_i^* \wedge e_j^* \otimes e_{ij}.
\]

In the case of a Riemannian manifold there are two kinds of invariant frames that will be relevant to this paper. These are the Frenet frame and the natural frame. While the Frenet frame is better known, the natural frame description can be found in [B] and can be summarized as follows: Assume we have a curve in \( \mathbb{R}^n \). We say that a normal vector field along the
curve is relatively parallel if its derivative is tangential to the curve. The following theorem can be found in [B] for the case $n = 3$ although the result yields identically for the general case, as the authors point out at the end of the paper.

**Theorem 1.** Let $\gamma$ be a regular $C^2$ curve in $\mathbb{R}^n$. Then, for any vector $V_0$ at $\gamma(t_0)$ there is a unique $C^1$ relatively parallel field $V$ along $\gamma$ such that $V(t_0) = V_0$. The scalar product of two relatively parallel fields is constant.

Given a regular Euclidean curve $\gamma$, the tangent space along $\gamma$ is divided into an oriented tangential component and the normal subspace. One can thus choose an orthonormal basis in the normal subspace formed by relatively parallel vector fields. This basis is determined up to a constant matrix in $O(n-1)$ and, together with the tangential vector field, formed a so-called natural frame. See [B] for more details. The evolution of this frame is given by the equation $F_x = FN$, where $F$ contains the natural frame as columns, and where

$$N = \begin{pmatrix} 0 & -u^T \\ u & 0 \end{pmatrix}.$$ 

The vector $u^T = (u_1, \ldots, u_{n-1})$ is formed by what are known as natural curvatures, which are differential invariants for the curve. Notice that $u$ is unique only up to the action of $O(n-1)$.

If $F_N$ is a natural frame and $F$ is the Frenet frame, then

$$g = F^{-1}F_N = \begin{pmatrix} 1 & 0 \\ 0 & \theta \end{pmatrix}$$

satisfies

$$-g'g^{-1} + gNg^{-1} = K$$

where $K$ is as in (3.4) below. From here, if $K_2$ is such that

$$K = \begin{pmatrix} 0 & -\kappa_1 e_1^T \\ \kappa_1 e_1 & K_2 \end{pmatrix},$$

then $\theta$ as in (2.3) satisfies

$$u = \kappa_1 \theta^T e_1, \quad \theta' = K_2 \theta$$

which determines $u$ from $\kappa$ up to a constant matrix in $O(n-1)$. Also, if $u$ is known, (2.5) gives $\kappa_1 = \|u\|$, and $K_2$ can be obtained from the first row of $\theta$ (that is $1/\kappa_1 u$) using a process analogous to the construction of a Frenet frame.

**Example 1.** In the case $n = 3$, let $\{T, N, B\}$ be the Frenet frame, $\kappa$ and $\tau$ the curvature and torsion of the curve, respectively, and let $\{T, M_1, M_2\}$ be a natural frame. Then

$$M_i = \cos \alpha_i N + \sin \alpha_i B$$
where the relatively parallel condition determines \( \alpha'_i = -\tau \). That is, up to a rotation, \( M_1 \) and \( M_2 \) are determined. If we further ask \( M_1 \) and \( M_2 \) to be orthogonal, we can choose \( \alpha_2 = \frac{\pi}{2} - \alpha_1 \) and the natural curvatures are then given by \( u_1 = -\kappa \cos \alpha_1 \), \( u_2 = -\kappa \sin \alpha_1 \).

3. Invariant evolutions of Riemannian curves and the evolution of their differential invariants

Let \( \phi \) be a curve parametrized by the arc-length. Assume we have an evolution of the form

\[
\phi_t = F h = h_1 T + \sum_{k=1}^{n-1} h_k N_{k-1}
\]

where \( F = (T, N_1, \ldots, N_{n-1}) \) is the matrix having in columns an invariant frame (for example the Frenet or natural frames) along the curve, and where \( h = (h_k) \) is a vector whose entries are functions of the Riemannian invariants associated to the invariant frame, \( k_i, i = 1, \ldots, n - 1 \) and their derivatives with respect to arc-length (for example, the usual Riemannian curvatures in the case of a Frenet frame, or the natural curvatures in the natural case).

The following theorem describes the evolution induced on the invariants by evolution (3.1):

**Theorem 2.** Let \( M \) be a Riemannian manifold, and let \( P \) be its associated principal bundle. Let \( \omega \) and \( \Omega \) be its associated Cartan connection and curvature tensor, respectively. Let

\[
\phi_t = F h
\]

be an evolution of curves on \( M \) as in (3.1). Assume the evolution is arc-length preserving. Then, the evolution induced on the Riemannian curvatures of \( \phi \) by (3.2) can be found by evaluating the structural Equation (2.1) on the vector fields \( (\phi_x, F_x) \) \( (\phi_t, F_t) \) tangential to \( P \) along the family of curves \( (\phi(t, x), F(t, x)) \). Indeed, in the Frenet case,

\[
\frac{d}{dx} \left( \begin{array}{cc} 0 & 0 \\ h & F^T F_t \end{array} \right) + \left[ \left( \begin{array}{ccc} 0 & 0 & \end{array} \right), \left( \begin{array}{ccc} 0 & 0 & \end{array} \right) \right] + \Omega(\phi_x, \phi_t)
\]

where

\[
K = \begin{pmatrix}
0 & -\kappa_1 & 0 & \cdots & 0 \\
\kappa_1 & 0 & -\kappa_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \kappa_{n-2} & 0 & -\kappa_{n-1} \\
0 & \cdots & 0 & \kappa_{n-1} & 0
\end{pmatrix}
\]
The matrix $K$ is substituted by

$$
N = \begin{pmatrix}
0 & -u_1 & -u_2 & \ldots & -u_{n-1} \\
u_1 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
u_{n-2} & 0 & \ldots & 0 & 0 \\
u_{n-1} & 0 & \ldots & 0 & 0
\end{pmatrix}
$$

(3.5)

in the natural frame case.

Furthermore, in the Frenet case, $F^T F_t$ can be found directly from Equation (3.3) itself using simple algebraic manipulations. In the natural case $F^T F_t$ is also determined by the equation but only up to a constant element in $o(n-1)$.

Before proving this theorem we will prove a convenient lemma.

**Lemma 1.** Let $B(x,t)$ be in $\text{so}(n)^*$ and assume that its first column is fixed, that is $B e_1 = \begin{pmatrix} 0 \\ f \end{pmatrix}$, where $f$ is given. Then, equation

$$
\frac{d}{dt} K = B' + [K, B] + \hat{\Omega} (\phi_x, \phi_t)
$$

(3.6)

completely determines $B$, where $\Omega = \begin{pmatrix} 0 & 0 \\ 0 & \hat{\Omega} \end{pmatrix}$ and $K$ is as in (3.4).

**Proof.** Assume $B e_1$ is fixed.

Let $\text{so}(n) = \sum_{i=1}^{n-1} [g_i \oplus g_{-i}]$ be the usual decomposition of $\text{so}(n)$ given by the standard gradation of $gl(n)$. That is, $g_i$ is given by matrices whose only nonzero entries are in place $(r,r+i)$, $r = 1, 2, \ldots, n - i$. We will assume $g_i = 0$ whenever $i > n - 1$ or $i < -n + 1$. For $x$ and $t$ fixed we have that $K = K_1 + K_{-1} \in g_1 \oplus g_{-1}$. Decompose $B = \sum_{i=1}^{n-1} (B_i + B_{-i})$ in its components with respect to the gradation, so that $B_i^T = -B_{-i}$. Decompose also $\Omega(\phi_x, \phi_t) = \sum_{i=1}^{n-1} \hat{\Omega}_i + \hat{\Omega}_{-1}$ in its components.

Since $\frac{d}{dt} K \in g_1 \oplus g_{-1}$, from equality (3.6) we get

$$
B' + [K_1, B_{i-1}] + [K_{-1}, B_{i+1}] + \hat{\Omega}_i = 0
$$

(3.7)

for $i \neq 1, -1$, where by definition $B_i = 0$ for $i \geq n$ or $i \leq -n$. If the first row of $B$ is determined, clearly $B_{n-1}$ and $B_{-n+1}$ are determined. Now, (3.7) for $i = n - 1$ will determine $B_{n-2}$. Indeed, for $i = n - 1$ we have

$$
B'_{n-1} + [K_1, B_{n-2}] + \hat{\Omega}_{n-1} = 0
$$

(3.8)

and $[K_1, B_{n-2}] = (\kappa_{n-1} b_{1,n-1} - \kappa_1 b_{2,n}) E_{1,n}$, where $B = (b_{i,j})$ and $E_{i,j}$ is the matrix having a 1 in the $(i,j)$ entry and zeroes elsewhere. Thus, if $b_{1,n-1}$ is known, $b_{2,n}$ can be found from (3.8).
A simple induction shows that, if \( B_r \) is known \( r = n - 1, \ldots, s \) and \( b_1, s \) is known, then

\[
B'_s + [K_1, B_{s-1}] + [K_{-1}, B_{s+1}] + \hat{\Omega}_s = 0
\]

determines \( B_{s-1} \).

The last equation to be used from the group of Equations (3.7) is the case \( i = 2 \)

\[
B'_2 + [K_1, B_1] + [K_{-1}, B_3] + \hat{\Omega}_2 = 0
\]

which solves for \( B_1 \). Hence, we can solve for \( B \) using (3.6).

\[\square\]

From now on we will denote by \( B(f) \) the matrix determined by \( f \) via Equation (3.6) with \( B(f)e_1 = (0_0 e_1 K) \).

**Proof of Theorem 2.** The first part of the proof is quite simple. Indeed, if evolution (3.2) is arc-length preserving, differentiation with respect to \( x \) and \( t \) commute and so \([(\phi_x, F_x), (\phi_t, F_t)] = 0\) as vector fields on the tangent to the bundle along the curve. We use now the formula

\[
d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y])
\]

(3.10)

for vector fields \( X, Y \in TP \). Recall that, as vector fields, the application of the vectors fields \((\phi_x, F_x)\) and \((\phi_t, F_t)\) on a function amounts to \( x \) and \( t \)-differentiation, respectively, of the function evaluated along the curve. Therefore, the evaluation of the structural equation on the vector fields \((\phi_x, F_x)\), \((\phi_t, F_t)\) along the curve \((\phi, F)\) results in equation

\[
\frac{d}{dt}\omega(\phi_x, F_x) = \frac{d}{dx}\omega(\phi_t, F_t) + [\omega(\phi_x, F_x), \omega(\phi_t, F_t)] + \Omega((\phi_x, F_x), (\phi_t, F_t)).
\]

(3.11)

Thus, we need to show that along the curve

\[
\omega(\phi_x, F_x) = \begin{pmatrix} 0 & 0 \\ e_1 & K \end{pmatrix}, \quad \omega(\phi_t, F_t) = \begin{pmatrix} 0 & 0 \\ h & F^TF_t \end{pmatrix}.
\]

It is known (see [Sh]) that

\[
\omega = \begin{pmatrix} 0 & 0 \\ \theta & \omega_H \end{pmatrix}
\]

where \( \theta^T = (\theta_1, \ldots, \theta_n) \) are coframe fields dual to the frame under consideration, along the curve, and where \( \omega_H \) is the Maurer–Cartan of \( SO_n(\mathbb{R}) \). Therefore

\[
\omega(\phi_x, F_x) = \begin{pmatrix} 0 & 0 \\ \theta(\phi_x) & F^TF_x \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ e_1 & K \end{pmatrix}
\]
and

\[ \omega(\phi_t, F_t) = \begin{pmatrix} 0 & 0 \\ \theta(\phi_t) & F^T F_t \end{pmatrix}. \]

We just need to apply that \( \phi_t = Fh \) and so \( \theta(\phi_t) = h \).

The last part of the theorem is to show that one can find \( F^T F_t \) directly from Equation (3.11) using algebraic computations. The Frenet case is a direct consequence of Lemma 1. Indeed, the first row of \( F^T F_t \) is determined by the first column of (3.3) which reads \( 0 = h' + Kh - F^T F_t e_1 \), so that \( F^T F_t e_1 = h' + Kh \). Therefore, \( F^T F_t = B(\hat{\pi}(h' + Kh)) \), where \( \hat{\pi} : \mathbb{R}^n \to \mathbb{R}^{n-1} \) is the projection on the last \( n - 1 \) components. Notice that, since \( B(\hat{\pi}(h' + Kh)) \) is in \( \mathfrak{o}(n) \), the first entry of \( h' + Kh \) needs to be zero. Indeed, this implies \( h_2 = \frac{h_1'}{\kappa_1} \), which is known to be the arc-length preserving condition on evolution (3.2).

The natural frame case is simpler. If \( N \) is given as in (3.5) and we denote \( F^T F_t \) by \( S \), Equation (3.11) in the natural case can be rewritten as

\[ Se_1 = h' + Nh, \quad N_t = S' + [N,S]. \]

Therefore, \( Se_1 = \begin{pmatrix} 0 \\ r \end{pmatrix} = h' + Nh \) is determined. Furthermore, if

\[ S = \begin{pmatrix} 0 & -r^T \\ r & \hat{S} \end{pmatrix} \]

then \( N_t = S' + [N,S] \) becomes

\[ N_t = \begin{pmatrix} 0 & -u^T \hat{S} \\ -\hat{S}u & \hat{S}' + ru^T - ur^T \end{pmatrix}. \]

Therefore, condition \( \hat{S}' = ur^T - ru^T \) determines \( \hat{S} \) in terms of \( r \) up to a constant matrix in \( o(n-1) \). \( \square \)

The next step is to determine the value of \( \Omega((\phi_x, F_x), (\phi_t, F_t)) \). Of course, that will depend on the tensor \( \Omega \) itself. The next lemma gives us an answer in the special case of manifolds with constant curvature tensor.

**Lemma 2.** Let \( M \) be a Riemannian manifold with constant curvature and let \( \Omega \) be its curvature 2-form. Let \( \phi(x,t) \) be a family of curves evolving according to (3.1). Then, along the family of curves \( (\phi(x,t), F(x,t)) \) on the principal bundle \( P \), we have

\[ \Omega((\phi_x, F_x), (\phi_t, F_t)) = \kappa \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & h_2 & \ldots & h_{n-1} \\ 0 & -h_2 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -h_{n-1} & 0 & \ldots & 0 \end{pmatrix} \]

(3.13)
where \( \kappa \) is the curvature of the manifold, and where \( h = (h_i) \) is as in (3.1).

Proof. Again, the proof of this lemma can be found solely based on known descriptions of Riemannian manifolds with constant curvatures. Indeed, if \( \theta = (\theta_1, \ldots, \theta_n) \) are the dual coframe fields, it is known that a Riemannian manifold has constant curvature whenever

\[
\Omega = \kappa \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \theta \wedge \theta \end{pmatrix}.
\]

Therefore, \( \Omega((\phi_x, F_x), (\phi_t, F_t)) \) along \((\phi, F)\) is determined by the application of \( \theta \wedge \theta \) to \((\phi_x, \phi_t) = (T, Fh)\). When applied we obtain directly the result of the lemma. \(\square\)

To finish this section, and to illustrate the simplicity of this method, we will apply the procedure described above to the special case of a 3-dimensional Riemannian manifold with constant curvature. Compare this procedure to the more traditional one used in [MSW].

**Example 2.** Let \( M \) be a 3-dimensional Riemanian manifold with constant curvature \( \kappa \). Let \( \phi(x, t) \) be a family of curves on \( M \), parametrized by arc-length, with associated curvatures and torsion given by \( \kappa \) and \( \tau \) (\( \kappa_1 \) and \( \kappa_2 \) in the theorems above). Assume \( \phi \) is solution of an evolution of the form

\[
\phi_t = h_1 T + h_2 N + h_3 B
\]

where \( T \) is the tangent to \( \phi \), \( N \) the normal, and \( B \) the binormal. Assume that the evolution is arc-length preserving.

From Theorem 2 we have the following equation to hold true:

\[
(3.14) \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -\kappa & 0 \\ 0 & \kappa & 0 & \tau \\ 0 & 0 & -\tau & 0 \end{pmatrix}_t = \begin{pmatrix} 0 & 0 & 0 & 0 \\ h_1 & 0 & \alpha & \beta \\ h_2 & -\alpha & 0 & \gamma \\ h_3 & -\beta & -\gamma & 0 \end{pmatrix}_{x} + \kappa \begin{pmatrix} h_2 & h_3 \\ h_3 & -\beta & -\gamma & 0 \end{pmatrix}
\]

where

\[
F^T F_t = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix}
\]
is to be found from the equation. We can rewrite (3.14) as

\begin{equation}
(3.15)
\begin{pmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & -\kappa & 0 \\
0 & \kappa & 0 & -\tau \\
0 & 0 & \tau & 0
\end{pmatrix}_t =
\begin{pmatrix}
0 & 0 & 0 & 0 \\
h'_1 - \kappa h_2 & 0 & \alpha' + \tau \beta + \kappa h_2 & \beta' + \tau \alpha - \kappa \gamma + \kappa h_3 \\
h'_2 + \kappa h_1 - \tau h_3 + \alpha & * & 0 & \gamma' + \kappa \beta \\
h'_3 + \tau h_2 + \beta & * & * & 0
\end{pmatrix}
\end{equation}

where * represent entries that are determined by the matrix being an element of the Euclidean algebra, and where we denote \( \frac{d}{dx} \) by \( ' \). Comparison of entries in the first column in both sides of the equality leads to the condition

\[ h_2 = \frac{h_1'}{\kappa} \]

which is known to be the arc-length preserving condition for evolution (3.1). It also leads to the determination of \( \alpha \) and \( \beta \) in terms of \( h, \kappa \) and \( \tau \), namely

\[ \alpha = -h'_2 - \kappa h_1 + \tau h_3, \quad \beta = -h'_3 - \tau h_2. \]

Comparison of the (2,4)-entries in the equation determines \( \gamma \) also in terms of \( h, \kappa \) and \( \tau \), namely

\[ \gamma = \frac{1}{\kappa} (\beta' + \tau \alpha + \kappa h_3). \]

Substituting the values of \( \alpha, \beta, \gamma \) and \( h_2 \) into the entries (3,2) and (4,3) yields to the following evolution for \( \kappa \) and \( \tau \):

\begin{equation}
(3.16)
\begin{align*}
\kappa_t &= \left( \frac{h_1'}{\kappa} \right)'' + (\kappa h_1)' + \tau^2 \frac{h'_1}{\kappa} - \tau' h_3 + \kappa \frac{h_1'}{\kappa} \\
\tau_t &= \left( \frac{\tau}{\kappa} \left( \frac{h_1'}{\kappa} \right)' \right)' + \left( \frac{1}{\kappa} \left( \frac{\tau}{\kappa} h_1' \right)' \right)' + \tau h'_1 + (\tau h_1)' \\
&+ \left( \frac{1}{\kappa} h''_3 \right)' - \left( \frac{\tau^2}{\kappa} h_3 \right)' + \kappa h'_3 + \kappa \frac{h_3'}{\kappa}
\end{align*}
\end{equation}

evolution that can be found, for example, in [MSW].

In the natural frame case, assume the curve \( \phi \) is evolving following an evolution of the form:

\[ \phi_t = h_1 T + h_2 M_1 + h_3 M_2 \]
where \( \{ T, M_1, M_2 \} \) is a natural frame along the curve. Assume the evolution is arc-length preserving. Then the evolution of the natural curvatures is

\[
N_t = S' + [N, S] + \kappa \begin{pmatrix}
0 & h_2 & h_3 \\
-h_2 & 0 & 0 \\
-h_3 & 0 & 0
\end{pmatrix}
\]

where

\[
S = \begin{pmatrix}
0 & r^T \\
-r & \hat{S}
\end{pmatrix}
= \begin{pmatrix}
0 & \alpha & \beta \\
-\alpha & 0 & \gamma \\
-\beta & -\gamma & 0
\end{pmatrix}
\]

and \( \begin{pmatrix}
0 \\
r
\end{pmatrix} = h' + Nh \). From here we have that \( h'_1 = u_1h_2 + u_2h_3 \) (the arc-length preserving condition) and \( \alpha = -h'_2 - u_1h_1, \beta = -h'_3 - u_2h_1 \). Furthermore, (3.17) implies \( \hat{S}' = ur^T - ru^T \) and so \( \gamma' = \alpha u_2 - \beta u_1 \). Putting all this information together yields the evolution

\[
(u_1)_t = h''_2 + [u_1D^{-1}(u_1h_2 + u_2h_3)]' - u_2D^{-1}(u_1h'_3 - u_2h'_2) - \kappa h_2
\]

\[
(u_2)_t = h''_3 + [u_2D^{-1}(u_1h_2 + u_2h_3)]' + u_1D^{-1}(u_1h'_3 - u_2h'_2) - \kappa h_3
\]

where \( D^{-1} \) represents formally the integral, determined up to a constant.

The advantages of using this formulation are not only calculational. Indeed this way of writing the evolutions of the curvatures give us the inside view of where the associated Poisson brackets come from, as we will readily see in the next section.

4. A family of Poisson structures

In this section we will first give a very brief description of Poisson manifolds in finite dimensions, since most readers will be more familiar with this case, and we will describe how the picture translates into infinite dimensions. We will then define the Poisson structure that seems to generate all the interesting known Poisson structures associated to geometric evolutions.

**Definition 4.** A Hamiltonian structure or Poisson bracket on a finite dimensional manifold is a bilinear and skew-symmetric map

\[
\{,\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)
\]

such that the following two additional properties hold:

1. \( \{ FG, H \} = F\{ G, H \} + G\{ F, H \} \) (Leibniz’s property),
2. \( \{ F, \{ G, H \} \} + \{ G, \{ H, F \} \} + \{ H, \{ F, G \} \} = 0 \) (Jacobi’s property),

for any \( F, G, H \in C^\infty(M) \).

If \( H \in C^\infty(M) \), the vector field \( \{ H, \cdot \} = \xi_H : C^\infty(M) \to C^\infty(M) \) is called the Hamiltonian vector field associated to the Hamiltonian \( H \).
If \( u : \mathbb{R} \to M \), the evolution
\[
(4.1) \quad u_t = \xi_H(u(t))
\]
is called a Hamiltonian evolution associated to the Hamiltonian \( H \).

The flow of Hamiltonian evolutions remains always on a certain submanifold for all times. These submanifolds foliate the original manifold and they are called the symplectic leaves of the Poisson structure.

In the special case where \( M = g^* \) there is a very natural Poisson bracket defined by the Lie algebra structure of \( g \). Denote by \([,]\) the Lie bracket on \( g \) and let \( \langle,\rangle \) be the pairing between \( g \) and \( g^* \).

**Definition 5.** Let \( F,G \in C^\infty(g^*) \) be two real functions defined on the dual of a Lie algebra \( g \). The total derivatives of these functions at a point \( L \in g^* \) can be naturally identified with elements in the Lie algebra, say \( f \) and \( g \), respectively. Define
\[
\{F,G\}(L) = \langle [f,g],L \rangle.
\]
The bracket \( \{,\} \) is clearly Poisson and it is called the Lie–Poisson bracket on \( g^* \).

It is well-known, and it is crucial in the description that follows, that the symplectic leaves of the Lie–Poisson bracket coincide with the coadjoint orbits of \( g^* \) under the action of the Lie group.

In the case of \( M \) being an infinite dimensional manifold a general definition is technically complicated to give, so I will limit myself to the definition of the bracket that interests us. Let \( G \) be a semisimple Lie group, and \( g \) its Lie algebra. Let \( LG = C^\infty(S^1,G) \) be the group of loops on \( G \) and let \( Lg = C^\infty(S^1,g) \) be its Lie algebra. Let \( Lg^* = C^\infty(S^1,g^*) \) be its dual (it is not really its dual but what is called the regular part of the dual, dense in the dual of the algebra of loops). The space of loops could be replaced by functions from \( \mathbb{R} \) to \( G \) vanishing at infinity, or any condition that ensures that no boundary terms will appear when we integrate by parts. Let \( Q \) be the Killing form associated to \( g \). Define the following cocycle of the algebra \( Lg \):
\[
w(L,M) = \int_{S^1} Q\left( L, \frac{dM}{dx} \right) dx
\]
for any \( L,M \in Lg \). The form \( w \) is called a cocycle because it has the properties necessary to guarantee that \( Lg \oplus \mathbb{R} \) is a Lie algebra with Lie bracket given by
\[
[(L,s),(M,r)] = ([L,M],w(L,M)).
\]
This algebra is called the central extension of \( Lg \), also known as a Kac–Moody algebra on the circle associated to \( g \) (we will denote it by \( \text{kac}(g) \)).
The Poisson bracket we are interested in is the Lie–Poisson bracket on the dual of the Kac–Moody algebra. It is defined as follows: Let $G : Lg^* \rightarrow \mathbb{R}$ be a functional and let $\frac{\delta G}{\delta L} \in Lg$ be its variational derivative. The Lie–Poisson bracket on $Lg^*$ is defined as

$$\{H, G\}(L, s) = \int_{S^1} \text{tr} \left( \frac{\delta G}{\delta L} \left( -s \left( \frac{\delta H}{\delta L} \right)' + \left[ L, \frac{\delta H}{\delta L} \right] \right) \right) dx$$

where $\text{tr}$ denotes $-\frac{1}{2}$ times the trace. From this expression we readily see that the Hamiltonian vector field associated to the Hamiltonian $H$ is given by

$$\xi_H = -s \left( \frac{\delta H}{\delta L} \right)' + \left[ L, \frac{\delta H}{\delta L} \right].$$

It is known ([Ki]) that the coadjoint action of the Kac–Moody group on $kac^*(g)$ reduces to the following action of the group of loops:

$$\text{Ad}^*(g)(L, s) = (-sg^{-1}g' + g^{-1}Lg, s).$$

We can see from this formula that $kac^*(g)$ foliates into Poisson manifolds corresponding to a fix value of $s$. Also, it is customary to identify an element $(L, s) \in kac^*(g)$ with the differential operator

$$s \frac{d}{dx} + L$$

so that the coadjoint action corresponds to conjugation of such operator by $g \in LG$. This conjugation (or gauge) by $g$ corresponds to the change of variable $X = Yg$ on the solutions of the system $sX' = XL$.

Compare Equation (4.3) to (3.3). The similarity between these equations suggests that geometric evolutions (3.3) for both Frenet and natural cases might be Hamiltonian evolutions with respect to the Lie–Poisson bracket on the dual of the Kac–Moody algebra associated to $so(n)$ (with $s = -1$), after being reduced to the submanifolds $K$ of Frenet matrices of the form (3.4), or to the submanifold $N$ of natural matrices of the form (3.5), respectively.

The following definition of Poisson reduction can be found in [MR]:

**Definition 6.** Let $(M, \{\cdot, \cdot\})$ be a Poisson manifold and $P$ a submanifold of $M$, $i : P \rightarrow M$ the inclusion. Let $E$ be a subbundle of $TM$ along $P$.

Assume $E \cap TP$ is an integral subbundle of $TP$ defining a foliation $\Phi$ on $P$. We say that $(P, \{\cdot, \cdot\}, E)$ is Poisson reducible if $P/\Phi$ has a Poisson structure $\{\cdot, \cdot\}_R$ defined the following way: For any (locally defined) smooth functions $f, g$ on $P/\Phi$ and any smooth extensions $F, G$ of $f \circ \pi, g \circ \pi$ with differentials vanishing on $E$, we have

$$\{f, g\}_R \circ \pi = \{F, G\} \circ i,$$

where $\pi : P \rightarrow P/\Phi$ is the projection.
Notice that I have not commented on the nature of $P/\Phi$ (whether or not it is a manifold and what kind of manifold) and its projection (its smooth character, etc.). These conditions are, of course, included in the original definition found in [MR]. They are difficult questions in the infinite dimensional case and I rather perform these reductions formally (the definition above can be taken as a formal definition of reduction). Later on, one can look at both the quotient $P/\Phi$ and these formal definitions of the brackets and wonder if they are well-defined. This way we also preserve the simplicity and beauty of the geometric picture behind the reductions.

**Theorem 3.** Let $\{,\}$ be the Lie–Poisson bracket on $\mathfrak{kac}^*(\mathfrak{o}(n))$ with central parameter $s = -1$. Let $E$ be the subbundle of $T(\text{Lo}(n)^*)$ generated by the Hamiltonian vector fields of Hamiltonian functionals $H$ such that

$$
\frac{\delta H}{\delta L} = \left( \begin{array}{cc}
0 & 0 \\
0 & \Gamma
\end{array} \right)
$$

with $\Gamma \in \text{Lo}(n-1)$. Let $\Phi$ be the foliation defined by the orbits of $\text{LSO}(n-1)$, the foliation known to be associated to $E$.

Then, the submanifold $K$ given by the set of all Frenet matrices of the form (3.4) with $k_i \in C^\infty(S^1, \mathbb{R})$, $k_i > 0$, $i = 1, \ldots, n-1$, is a section transverse to the orbits of $\text{LSO}(n-1)$ under the coadjoint action on an open set of $\text{Lo}(n)^*$, say $U$. That is, we can identify $U/\Phi$ with $K$. Furthermore, $(U, \{,\}, E)$ is Poisson reducible, and, hence, there exists a Poisson bracket $\{,\}_R$ defined on $K$.

**Proof.** Consider $P = M = \text{Lo}(n)^*$ with the Lie–Poisson Kac–Moody bracket for $s = -1$. Let $E$ be given as in the statement of the theorem. Given that $E \cap T\text{Lo}(n)^* = E$, the foliation $\Phi$ associated to this intersection is simply the coadjoint orbits under the action (4.4) with

$$
g = \left( \begin{array}{cc}
1 & 0 \\
0 & \theta
\end{array} \right)
$$

$\theta \in \text{LSO}(n-1)$.

We will first show that, for some open set $U \subset \text{Lo}(n)^*$, $U/\Phi$ can be identified with $K$. The description of such an identification is very simple and pretty. Let’s identify each element in $\text{Lo}(n)^*$ with its differential operator as in (4.5), $s = -1$. We need to show that the set $U$ of coadjoint orbits intersecting $K$ is an open set of $\text{Lo}(n)^*$ and that such an orbit intersects $K$ at only one point.

Indeed, let $-\frac{d}{dx} + L$ be an element in an orbit $\Theta(L)$ intersecting $K$. Let $X$ be a fundamental matrix solution for equation $X' = XL$, with $X(0) = I$, the identity matrix. Let $T$ be the first column of $X$. Clearly the Riemannian length of $T$ is constant and equal to one. Besides, since $L$ has periodic coefficient, there exists a matrix $M(L) \in O(n)$, the monodromy of $L$, such
that

\[ T(x + 2\pi) = M(L)T(x). \]

Define \( \phi \) to be the curve whose tangent is given by \( T \). Clearly \( \phi \) is determined up to a translation and it is uniquely determined if we ask \( \phi \) to have the same monodromy property as \( T \) does. If \( \Theta(L) \) intersects \( K \), there exists \( g \in LSO(n - 1) \) such that \( Xg = Y \) satisfies \( Y' = YK, \) \( K \) as in (3.4). Indeed, \( g \in LSO(n - 1) \) since the tangent to the curve (the first column of \( X \)) is also the first vector in the Frenet frame. Clearly, \( Y \) must be the Frenet frame and \( K \) the Frenet matrix associated to \( \phi \). \( K \) is unique by the uniqueness of the Frenet matrix and must have periodic entries since \( \phi \) has a monodromy. Summarizing, moving along the orbit corresponds to changing the normal components of the frame and intersecting \( K \) corresponds to fixing the normal vectors to be the normal Frenet vectors.

It is clear now that the set of all these orbits is an open set in \( Lo(n)^* \). Each orbit can be identified up to translations with a curve \( \phi \) with a certain monodromy matrix. A nearby orbit will yield a nearby tangent vector and a nearby tangent vector will produce a nearby curve. This curve can be chosen to have a monodromy matrix because of the periodicity of the equation that the tangent satisfies. But if a curve is nondegenerate in the sense of having a well-defined Frenet frame, any nearby curve will also be nondegenerate. Thus, its orbit will intersect \( K \).

We will finally show that \((U, \{ \}, E)\) is Poisson reducible. Let \( f, g : U/\Phi \to \mathbb{R} \) be two Fréchet differentiable functionals and define

\[ \{ f, g \}_R \circ \pi = \{ F, G \}|_U \]

for any two extensions of \( f \circ \pi \) and \( g \circ \pi \), \( F \) and \( G \) respectively, such that \( \frac{\delta F}{\delta L} \) and \( \frac{\delta G}{\delta L} \) vanish on \( E \). The definition does not depend on \( F \) and \( G \) since any two different extensions will coincide on \( U \). Also, (4.8) above is well-defined. Indeed, since \( F \) and \( G \) are constant on the leaves of \( \Phi \), \( \{ F, G \} \) will also be constant on the leaves, by Jacobi’s identity. If the variational derivative of \( H \) is as in (4.7), then

\[ \{ \{ F, G \}, H \} = -\{ \{ G, H \}, F \} - \{ \{ H, F \}, G \} = 0. \]

Thus, \( \{ F, G \} \) defines a functional on \( U/\Phi \). The bracket \( \{ , \}_R \) is Poisson since it inherits its properties from \( \{ , \} \). We only need to point out that, if \( F \) and \( G \) are extensions of \( f \) and \( g \), respectively, both of them constant on the leaves of \( \Phi \), then clearly \( \{ F, G \} \) is an extension of \( \{ f, g \}_R \), constant on the leaves of \( \Phi \). With this in mind the verification of the properties is straightforward. \( \square \)

Unfortunately, the natural case can be carried out only formally. Indeed, from (2.5) we see that neither \( \theta \) nor \( u \) will be, in general, periodic.
Theorem 4. The set $\mathcal{N}/O(n-1)$, where $\mathcal{N}$ is the manifold of matrices of the form (3.5), can also be formally identified with $\text{Lo}(n)^*/\Phi$, so there exist an additional reduced bracket denoted by $\{,\}_{NR}$, on $\mathcal{N}/O(n-1)$. Both reduced brackets $\{,\}_R$ and $\{,\}_{NR}$ are formally Poisson equivalent (formally since the gauge that takes one to the other will not be periodic in general).

Proof. Notice that since moving along the orbit corresponds to changing the normal components of the frame along the curve, we will intersect $\mathcal{N}$ when a natural frame is reached (we are allowing ourselves the use of non-periodic elements $g(x) \in O(n-1)$). The only condition needed to do that is for the curve to be regular ($T \neq 0$). Thus, any orbit intersects $\mathcal{N}$. But a natural frame is determined only up to the action of $O(n-1)$ (see [B]). Therefore, $\text{Lo}(n)^*/\Phi$ can be identified with $\mathcal{N}/O(n-1)$. □

This simple picture gives us also a clear description of how the reduced bracket can be found explicitly. Indeed, the definition of $\{,\}_R$ is given by formula (4.6). Now, if $f$ is defined on $K$, there exists a unique local extension which is constant on the leaves of the foliation $\Phi$, namely $F = f \circ \pi$. Since $F$ is an extension for $f$, its variational derivative in the $K$-direction, along $K$, coincides with that of $f$. This is reflected in the following algebraic fact:

Proposition 1. Let $A \in \text{Lo}(n)^*$ and assume all entries of $A$ in the $K$-direction $A_{i,i+1} = f_i$, $i = 1, \ldots, n-1$ are fixed. Then, there exists a unique choice of $A$ such that $A' + [K,A]$ is of the form (3.5) (that is, $A$ vanishes on $E$).

The proof of this proposition is very similar to that of Lemma 1 for $\kappa = 0$ and we will not include it. We will denote by $A(f)$ the matrix determined by $f = (f_i)$ via Proposition 1.

Now the explicit formula of the reduced bracket can be readily given. Let $f, g : K \to \mathbb{R}$ be two Fréchet differentiable functionals. Let $f_i = \frac{\delta f}{\delta \kappa_i}$ and $g_i = \frac{\delta g}{\delta \kappa_i}$, $i = 1, \ldots, n-1$. If we place $f_i$ in the entry $(i,i+1)$ of a matrix $A(f)$ so that $A(f) = \frac{\delta (f \circ \pi)}{\delta L}(K)$, $L \in U, K \in K$, the rest of the entries of $A(f)$ are uniquely determined by $A(f)' + [K,A(f)]$ being of the form (3.5). We would then have

$$\{f,g\}_R(K) = \int_{S^1} \text{tr} \left( (A(f)' + [K,A(f)]) A(g) \right) dx.$$  \hspace{1cm} (4.9)

The same description can be given for the natural case if, instead of $A(f)$ we use a matrix $S(f)$ with $S(f) e_1 = \begin{pmatrix} 0 \\ f \end{pmatrix}$, with $f = (f_i)$, and such that $S(f)' + [N,S(f)]$ is of the form (3.5). If $f, g : \mathcal{N} \to \mathbb{R}$ are two functionals which are constant on the $E$-orbits, then the reduced bracket on $\mathcal{N}$ is given
by

\[
(f, g)_{NR}(N) = \int_{S^1} \text{tr} \left( (S(f)' + [N, S(f)])S(g) \right) dx,
\]

which, from the theorem, will be invariant under the action of \(O(n-1)\).

**Example 3.** In the case \(n = 3\), if \(\delta f = f_i, \delta g = g_i\), and if \(A(f) = \begin{pmatrix} 0 & f_1 & \alpha \\ -f_1 & 0 & f_2 \\ -\alpha & -f_2 & 0 \end{pmatrix}\), then \((f')' + [K, A(f)]\) being tangent to \(N\) implies \(\alpha = -\frac{f_2'}{\kappa}\). A short computation yields

\[
A(f)' + [K, A(f)] = \begin{pmatrix} 0 & f_1' + \frac{\tau}{\kappa} f_2' - \left(\frac{f_2'}{\kappa}\right)' - \kappa f_2 + \tau f_1 \\ * & 0 \\ * & 0 \end{pmatrix}.
\]

With this information, (4.9) above can be written as

\[
(f, g)_R(K) = \int_{S^1} g_1 \begin{pmatrix} f_1' + \frac{\tau}{\kappa} f_2' - \left(\frac{f_2'}{\kappa}\right)' - \kappa f_2 + \tau f_1 \\ 0 \\ 0 \end{pmatrix}^T \begin{pmatrix} D & \frac{\tau}{\kappa} D & 0 \\ D^\top & -D - D\frac{1}{\kappa} D \frac{1}{\kappa} D & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \delta f \\ \delta u \\ \delta u \end{pmatrix} dx.
\]

This Poisson bracket has appeared in the literature ([MSW]) in connection to the study of integrable systems associated to invariant evolutions of Riemannian differential invariants.

In the natural case, the matrix \(S(f)\) is given as in (3.12) with \(r = f\). From here

\[
S(f)' + [N, S(f)] = \begin{pmatrix} 0 & 0 \\ f' + D^{-1}(uf^T - fu^T)u & -f' + D^{-1}(uf^T - fu^T) \\ 0 & 0 \end{pmatrix}
\]

and so bracket (4.10) is defined as

\[
(f, g)_{NR}(N) = \int_{S^1} g_1 u_2 D^{-1}(u_2 f_1 - f_2 u_1) + g_2 u_1 D^{-1}(u_1 f_2 - f_1 u_2) + g_1 f_1' + g_2 f_2' dx
\]

which can be rewritten as

\[
(f, g)_{NR}(N) = \int_{S^1} \begin{pmatrix} \delta g \\ \delta u \end{pmatrix}^T \begin{pmatrix} D + u_2 D^{-1} u_2 & -u_2 D^{-1} u_1 \\ -u_1 D^{-1} u_2 & D + u_1 D^{-1} u_1 \end{pmatrix} \begin{pmatrix} \delta f \\ \delta u \end{pmatrix}.
\]

This bracket has appeared in association to the integrability of modified KdV equations.
The precise connection between these two brackets and the curve evolutions is described in the next section.

5. Relationship between the reduced brackets and the evolution of the Riemannian curvatures

In this section we will give, in the flat case, necessary conditions so that the evolutions induced on the Riemannian curvatures by arc-length preserving flows of the form (3.1) are Hamiltonian with respect to the reduced brackets.

First of all, notice that, if \( A(f) \) is defined as in Proposition 1, it is immediate that all other entries in \( A(f) \) are defined as differential polynomials in \( f_i, i = 1, \ldots, n - 1 \) whose coefficients are rational functions of \( \kappa_i \) and their derivatives. This is a simple consequence of the fact that the entries are obtained algebraically from \( f \) using that \( A(f)' + [K, A(f)] \) belongs to \( TN' \).

Let's denote by \( A \) the differential operator holding

\[
A(f)' + [K, A(f)] = \begin{pmatrix} 0 & (Af)^T \\ -A f & 0 \end{pmatrix}.
\]

Likewise, if \( B(g) \) is the matrix given in Lemma 1 (with \( \varkappa = 0 \)), the entries of \( B(g) \) are also differential polynomials in \( g_i, i = 1, \ldots, n - 1 \) since they are obtained algebraically from \( g = (g_i) \) using the fact that \( B(g)' + [K, B(g)] \) lies in \( TK \). Their coefficients will also be rational functions on \( \kappa_i \) and their derivatives. Let's denote by \( B \) and \( C \) the differential operators that associates to \( g \) the nonzero entries in the 1-graded component of \( B(g)' + [K, B(g)] + \varkappa \hat{\Omega} \), that is, \((Bg)_i + \varkappa (Cg)_i\) is the \((i, i + 1)\) entry in \( B(g)' + [K, B(g)] \).

Also, directly from Equation (3.3) we see that the entries of the right-hand side of (3.3) can be written as differential polynomials on \( h \) with coefficients depending on \( \kappa_i \) and their derivatives with respect to \( x \). That is, if \( \varkappa = 0 \), evolution (3.3) can be rewritten as

\[
(5.1) \quad \kappa_t = \mathcal{P} \hat{h}
\]

for some matrix of differential operators \( \mathcal{P} \), and where \( \kappa = (\kappa_i) \) and \( \hat{h} = (h_1, h_3, \ldots, h_n) \). Notice that \( h_2 \) can be eliminated from the equation using the arc-length preserving condition \( h_2 = \frac{h_1'}{\kappa} \).

Lemma 3.

\[
(5.2) \quad B^* = -A.
\]
Proof. The proof of this lemma is a consequence of the following basic calculation:

\begin{align}
\langle A(f)' + [K, A(f)], B(g) \rangle &= \int_{S^1} A f \cdot g dx \\
&= -\langle A(f), B(g)' + [K, B(g)] \rangle \\
&= - \int_{S^1} B g \cdot f dx = - \int_{S^1} B^* f \cdot g dx
\end{align}

for any \( f, g \).

\[ \square \]

**Theorem 5.** Let

\[ \phi_t = Fh \]

be an arc-length preserving evolution of curves \( \phi(x, t) \) parametrized by the arc-length, where \( F(x, t) \) contains in columns the Frenet frame along \( \phi(x, t) \).

Assume \( h \) is a vector which depends on the Riemannian curvatures \( \kappa_i, i = 1, \ldots, n - 1 \) and their derivatives with respect to \( x \). Furthermore, assume \( \hat{\pi} : \mathbb{R}^n \to \mathbb{R}^{n-1} \) to be the projection on the last \( n-1 \) components and assume that there exists a functional \( g : \mathbb{K} \to \mathbb{R} \) such that

\[ \hat{\pi}(h' + Kh) = C^* \left( \frac{\delta g}{\delta \kappa} \right), \]

where \( C \) is the matrix of differential operators associated to the constant curvature of the manifold (and whose value is independent of the value of the curvature \( \kappa \)). Then, in the flat case \( \kappa = 0 \), the evolution induced by (5.4) on the Riemannian curvatures \( \kappa_i \) via Theorem 2, is Hamiltonian with respect to the reduced bracket \( \{ , \}_R \) and its associated Hamiltonian functional is \( g \).

Proof. Let’s analyze the reduced evolutions a little further. Assume that \( A(f) \) is given as in Proposition 1, and assume we can write it as

\[ A(f) = \begin{pmatrix} 0 & (\mathcal{R}f)^T \\ -\mathcal{R}f & \ast \end{pmatrix} \]

for some matrix of differential operators \( \mathcal{R} \). In order to rewrite the reduced evolution in terms of matrices of differential operators, notice that if \( f \) and \( g \) are two functionals

\[ \{ f, g \}_R(K) = \langle A(f)' + [K, A(f)], A(g) \rangle = \int_{S^1} A f \cdot \mathcal{R} g dx = \int_{S^1} \mathcal{R}^* A f \cdot g dx. \]

From here it is clear that the Hamiltonian vector field associated to \( f \) is given by \( \xi_f = \mathcal{R}^* A f \), where \( f = \frac{df}{\delta \kappa} \).
Assume $\phi_t = Fh$ is arc-length preserving and that $\dot{\phi}$ is parametrized by arc-length. We see from (3.3) that the evolution induced on the Riemannian curvatures is given by

$$K_t = (F^T F_t)' + [K, F^T F_t]$$

where, in the notation of Lemma 1, $F^T F_t = B(\hat{\pi}(h' + K h))$. Therefore, the evolution induced upon the Riemannian curvatures can be written as

$$\kappa_t = -B(\hat{\pi}(h' + K h)).$$

On the other hand, the reduced Hamiltonian vector field is given by

$$\xi_f = R^*A(f) = -A^*R(f)$$

because of the skew-symmetry of the bracket. The result of Theorem 5 now follows from Lemma 3 and the following lemma:

**Lemma 4.**

$$\mathcal{C} = R^*.$$

**Proof.** First of all, notice that $\mathcal{C}$ is defined the following way: Given an element of $TN$, for example

$$\begin{pmatrix} 0 & f^T \\ f & 0 \end{pmatrix}$$

there exists a unique element of $TN^0$ such that

$$\begin{pmatrix} 0 & f^T \\ -f & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix}' + \begin{bmatrix} K, \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix} \end{bmatrix}$$

is tangent to $K$. That is, the $(i, i + 1)$ diagonal of (5.10) defines $\mathcal{C}\hat{\pi}(h)$, whenever $f = \hat{\pi}(h)$. The uniqueness of $\mathcal{C}$ is clear since (5.10) tangent to $K$ implies that $\hat{\pi}(C e_1) = \hat{\pi}(\frac{1}{\kappa_1} h)$ and that condition together with $C' + [K_2, C]$ having zeroes off the $(i, i + 1)$ and $(i, i - 1)$ diagonals completely determines $C$, as it was shown in Lemma 1 ($K_2$ is given as in (2.4)). On the other hand,

$$\int_{S^1} \mathcal{C} \mathbf{f} \cdot \mathbf{g} dx = \int_{S^1} \left\langle \begin{pmatrix} 0 & f^T - \kappa_1 e_1^T C \\ -f + \kappa_1 C e_1 & C' + [K_2, C] \end{pmatrix}, A(g) \right\rangle dx,$$

since $A(g)$ has $\mathbf{g}$ in its +1 component with respect to the standard gradation. But

$$\left\langle \begin{pmatrix} 0 & f^T - \kappa_1 e_1^T C \\ -f + \kappa_1 C e_1 & C' + [K_2, C] \end{pmatrix}, A(g) \right\rangle dx = \left\langle \begin{pmatrix} 0 & f^T \\ -f & 0 \end{pmatrix}, A(g) \right\rangle dx$$

$$= \int_{S^1} \mathbf{f} \cdot R \mathbf{g} dx$$
where we use that $A(g)' + [K, A(g)]$ is tangent to $\mathcal{N}$ and so
\[
\left\langle \left( \begin{array}{cc} 0 & 0 \\ 0 & C \end{array} \right)' + \left[ K, \left( \begin{array}{cc} 0 & 0 \\ 0 & C \end{array} \right) \right], A(g) \right\rangle
= -\left\langle \left( \begin{array}{cc} 0 & 0 \\ 0 & C \end{array} \right), A(g)' + [K, A(g)] \right\rangle = 0.
\]

If we use natural frames, the connection is even simpler.

**Theorem 6.** Let
\begin{equation}
\phi_t = Nh
\end{equation}
be an arc-length preserving evolution of curves $\phi(x, t)$ parametrized by the arc-length, where $N(x, t)$ contains in columns the natural frame along $\phi(x, t)$. Assume $h$ is a vector which depends on the natural curvatures $u_i$, $i = 1, \ldots, n - 1$ and their derivatives with respect to $x$. Furthermore, assume that there exists a functional $g : \mathcal{N} \to \mathbb{R}$ such that
\begin{equation}
\hat{\pi}((h)' + Nh) = \frac{\delta g}{\delta u}.
\end{equation}
Then, in the flat case $\kappa = 0$, the evolution induced by (5.12) on the natural curvatures $u_i$ via Theorem 2, is Hamiltonian with respect to the reduced bracket $\{,\}_{NR}$, and its associated Hamiltonian functional is $g$.

**Proof.** The proof of this theorem is analogous to the proof of the previous theorem. One has only to notice that the evolution induced upon $u$ by (5.12) equals
\begin{equation}
u_t = S(\hat{\pi}(h' + Nh))
\end{equation}
where $S$ is the operator defining the natural reduced bracket. That is, if $S(f)$ is the matrix such that $S(f)e_1 = \left( \begin{array}{c} 0 \\ f \end{array} \right)$ and such that $S(f)' + [N, S(f)]$ lies on $T\mathcal{N}$, then
\[
S(f)' + [N, S(f)] = \left( \begin{array}{ccc} 0 & Sf^T \\ -Sf & 0 \end{array} \right).
\]

Condition (5.13) is far simpler than condition (5.5). This, I believe, explains why the natural frame has been favored in the study of integrable systems associated to Riemannian geometry. Geometrically, though, there is hardly any difference, except the fact that the Frenet case is uniquely and well-defined (from a Poisson reduction point of view), unlike the natural case which requires the choice of a section in $\mathcal{N}/O(n - 1)$ and a formal approach. In fact, once the section is fixed, both brackets are formally equivalent and
the gauge that takes the Frenet frame to the natural frame of our choice is a Poisson map between both Poisson manifolds. As we saw before, any such a map would be a generalization of the well-known Hasimoto transformation.

Finally, the Lie–Poisson bracket on \( \mathfrak{g}^\ast \) is known to have compatible Poisson brackets given by the following formula:

\[
\{ F, G \}^1(L) = - \left\langle \frac{\delta F}{\delta L}, \frac{\delta G}{\delta L} \right\rangle, H_0
\]

for a fixed element \( H_0 \in \mathfrak{g}^\ast \) with some nondegeneracy conditions. One can easily check that these brackets are also reducible using the foliation \( \Phi \). Since \( \{ , \} \) and \( \{ , \}^1 \) are compatible on \( \mathfrak{kac}^\ast (\mathfrak{o}(n)) \), when reduced they will still be compatible. Different choices of \( H_0 \) will produce different companions that can be used to integrate PDE’s.

Acknowledgements. The author wishes to express her gratitude to Prof. Jan Sanders and J.P. Wang for extensive discussions on the subject.

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Received December 2, 2002 and revised October 27, 2003.

MATHEMATICS DEPARTMENT
UNIVERSITY OF WISCONSIN
MADISON, WISCONSIN 53706
E-mail address: maribeff@math.wisc.edu
ON THE MINIMAL NUMBER OF RAMIFIED PRIMES IN SOME SOLVABLE EXTENSIONS OF $\mathbb{Q}$

Bernat Plans

For each finite solvable group $G$, there is a minimal positive integer $\text{ram}(G)$ (resp. $\text{ram}^t(G)$) such that $G$ appears as the Galois group of an extension of $\mathbb{Q}$ (resp. a tamely ramified extension of $\mathbb{Q}$) ramified at only $\text{ram}(G)$ (resp. $\text{ram}^t(G)$) finite primes. We obtain bounds for $\text{ram}(G)$ and $\text{ram}^t(G)$, where $G$ is either a nilpotent group of odd order or a generalized dihedral group.

1. Introduction

Given a finite group $G$, let $\text{ram}(G)$ (resp. $\text{ram}^t(G)$) denote the minimal positive integer such that $G$ can be realized as the Galois group of an extension of $\mathbb{Q}$ (resp. a tamely ramified extension of $\mathbb{Q}$) ramified only at $\text{ram}(G)$ (resp. $\text{ram}^t(G)$) finite primes. The present paper is devoted to study $\text{ram}(G)$ and $\text{ram}^t(G)$, for some solvable groups $G$. More precisely, we consider the case where $G$ is either a finite nilpotent group of odd order or a generalized dihedral group.

Let $l$ be an odd prime number. The Scholz-Reichardt’s Theorem establishes that every $l$-group $G$ can be realized as the Galois group of some extension of $\mathbb{Q}$ [Re]. By Burnside’s Basis Theorem and Kronecker–Weber’s Theorem, $\text{ram}(G)$ must be greater than or equal to the minimal number of generators of $G$. At present, it is not known whether this lower bound coincides with the exact value of $\text{ram}(G)$ and $\text{ram}^t(G)$, although this is claimed in [Cu-He] (see Remark 2.10). A Galois extension of $\mathbb{Q}$ with Galois group $G$ arises by proper resolution of a chain of central embedding problems, starting with the trivial epimorphism $G_{\mathbb{Q}} \to \{1\}$. Moreover, if one restricts himself to embedding problems with kernel of order $l$, then this process can be made adding only one new ramified prime at each step. Hence, $\text{ram}^t(G) \leq n$, where $l^n$ is the order of $G$ [Se, Chap. 2] (see also the generalization in [Ge-Ja], where the ground field $\mathbb{Q}$ is replaced by a general global field). We prove a better upper bound for $\text{ram}^t(G)$, less than or equal to the sum of the minimal number of generators of the factors in the lower central series of $G$. In order to obtain this improvement, we allow arbitrary cyclic kernels and we show that it still suffices to admit just one new ramified prime (for Frattini or split embedding problems). In addition,
we obtain the best possible generalization of this bound to the case of finite nilpotent groups of odd order.

Now let $G$ be a generalized dihedral group. From the theory of ring class fields of quadratic fields, we prove that $\text{ram}^l(G)$ can be upper bounded by the minimal number of generators of $G$. We also consider the question of which of these groups can have $\text{ram}^l(G) = 1$ (resp. $\text{ram}(G) = 1$). Assuming the validity of Hypothesis (H) of Schinzel, we give the exact value of $\text{ram}^l(D_{2n})$ and $\text{ram}(D_{2n})$, where $D_{2n}$ denotes the dihedral group of order $2n$.

2. Finite nilpotent groups of odd order

For a finite group $G$, let $d(G)$ denote the minimal number of generators of $G$. Given a prime number $p$, we always assume that a prime $\mathfrak{p}$ of $\mathbb{Q}$ over $p$ has been fixed. We denote by $D_p$ (resp. $I_p$) its corresponding decomposition (resp. inertia) subgroup in $G_{\mathbb{Q}}$.

Let us first recall the so-called Scholz’s condition for an $l$-extension of $\mathbb{Q}$.

**Definition 2.1.** Let $l$ be an odd prime number and let $G$ be an $l$-group. Given a positive integer $N$, an epimorphism $\varphi : G_{\mathbb{Q}} \to G$ is said to be of type $(S_N)$ if, for every prime number $p$ ramified by $\varphi$, the following conditions hold:

- $p \equiv 1 \pmod{l^N}$,
- $\varphi(I_p) = \varphi(D_p)$.

One also says that the extension of $\mathbb{Q}$ given by (the fixed field of the kernel of) $\varphi$ is of type $(S_N)$.

This condition is introduced in order to ensure ($N$ large) the local solvability, at all ramified primes, of central embedding problems for $\varphi$ with, say, kernel $\mathbb{Z}/l\mathbb{Z}$. Hence, one also obtains globally solvable embedding problems of the same type.

**Theorem 2.2** (cf. [Se, Chap. 2]). Let $1 \to C \to G \overset{\pi}{\to} H \to 1$ be a central extension of $l$-groups ($l \neq 2$) and let $\varphi : G_{\mathbb{Q}} \to H$ be an epimorphism of type $(S_N)$. If the exponent of $G$ is at most $l^N$, then the embedding problem given by $(\pi, \varphi)$ is solvable.

We want to twist an arbitrary solution to the above embedding problem in order to obtain a (proper) solution of type $(S_N)$. Moreover, we want to increase as few as possible the ramification set when carrying over this process. This amounts to finding suitable elements in $H^1(G_{\mathbb{Q}}, C) = \text{Hom}(G_{\mathbb{Q}}, C)$, whose existence must first be proved.

Let $\text{Ram}(K/\mathbb{Q})$ (resp. $\text{Ram}(\varphi)$) denote the set of ramified prime numbers in an extension $K/\mathbb{Q}$ (resp. an epimorphism $\varphi : G_{\mathbb{Q}} \to G$).

**Proposition 2.3.** Let $l$ be an odd prime number, $K/\mathbb{Q}$ an $l$-extension of type $(S_N)$ and $C = \langle c \rangle$ a cyclic group of order $l^r$ such that $r \leq N$. Let
us define \( S = \text{Ram}(K/Q) \setminus \{ p_0 \} \), where \( p_0 \) is a fixed prime number which ramifies in \( K/Q \). Let \( \{ \nu_p \}_{p \in S} \) be arbitrary integers. Then:

(i) For every positive integer \( k < r \), there exist infinitely many prime numbers \( q \) such that:

- \( q \) splits completely in \( K \left( \zeta_l^N, \{ \sqrt[l^{k}] {p_0^\nu p} \}_{p \in S}, \sqrt[l^{k}] {p_0} \right) \),
- \( q \) does not split completely in \( Q \left( \zeta_l^N, \sqrt[l^{k+1}] {p_0} \right) \).

(ii) For every integer \( \nu_0 \) such that \( v_l(\nu_0) < r \) and every prime \( q \) which satisfies Statement (i) for \( k = v_l(\nu_0) \), there exists an epimorphism \( \chi : (\mathbb{Z}/q\mathbb{Z})^* \rightarrow C \) such that:

- \( \chi(p_0) = c^{\nu_0} \),
- \( \chi(p) = c^{\nu_0 \cdot \nu_p} \), for every \( p \in S \).

**Proof.** By Txebotarev’s density theorem, Statement (i) is reduced to showing that

\[
\sqrt[l^{k+1}] {p_0} \notin K \left( \zeta_l^N, \{ \sqrt[l^{k}] {p_0^\nu p} \}_{p \in S}, \sqrt[l^{k}] {p_0} \right).
\]

From Kummer’s theory, it suffices to prove that

\[
\sqrt[l^{k}]{p_0} \notin K \left( \zeta_l^N, \{ \sqrt[l^{k-1}] {p_0^\nu p} \}_{p \in S} \right)
\]

or, equivalently,

\[
K \left( \zeta_l^N, \{ \sqrt[l^{k-1}] {p_0^\nu p} \}_{p \in S}, \sqrt[l^{k}] {p_0} \right) \subseteq K \left( \zeta_l^N, \{ \sqrt[l^{k-1}] {p_0^\nu p} \}_{p \in S} \right).
\]

In order to see that this is always true, we first note that the following isomorphism holds (see [Se, Lemma 2.1.9]):

\[
\text{Gal} \left( K \left( \zeta_l^N, \{ \sqrt[l^{k}]{p_0^\nu p} \}_{p \in S}, \sqrt[l^{k}] {p_0} \right) / K (\zeta_l^N) \right) \cong \mathbb{Z}/l\mathbb{Z} \times \mathbb{Z}/l\mathbb{Z},
\]

where \( s \) denotes the cardinality of \( S \). This is all we need, since the Galois group

\[
\text{Gal} \left( K \left( \zeta_l^N, \{ \sqrt[l^{k-1}] {p_0^\nu p} \}_{p \in S} \right) / K (\zeta_l^N) \right)
\]

can be generated with \( s \) elements.

Finally, Statement (ii) is clear once we observe that, if \( q \) satisfies the conditions in Statement (i), then:

- \( p_0 \) is an \( l^k \)-th power residue modulo \( q \),
- \( \sqrt[l^{k}] {p_0^\nu p} \) is an \( l^r \)-th power residue modulo \( q \), for each \( p \in S \),
- \( p_0 \) is not an \( l^{k+1} \)-th power residue modulo \( q \).

\( \Box \)
We can now prove the following generalization of [Se, Thm. 2.1.3]:

**Proposition 2.4.** Let \( l \) be an odd prime number. Suppose given a cyclic central extension of \( l \)-groups \( 1 \rightarrow C \rightarrow G \xrightarrow{\varphi} H \rightarrow 1 \) and an epimorphism \( \varphi : G_Q \rightarrow H \) of type \((S_N)\), for some positive integer \( N \) such that the exponent of \( G \) is at most \( l^N \). If the embedding problem \((\pi, \varphi)\) is Frattini or split, then \((\pi, \varphi)\) admits a proper solution \( \tilde{\varphi} \) of type \((S_N)\) such that:

\[
\sharp \text{Ram}(\tilde{\varphi}) \leq 1 + \sharp \text{Ram}(\varphi).
\]

**Proof.** For each prime number \( p \), let us choose a preimage \( \sigma_p \in D_p \) of the Frobenius automorphism \( \text{Frob}_p \in D_p/I_p \cong G_{\hat{F}_p} \cong \hat{\mathbb{Z}} \). Hypothesis \((S_N)\) over \( \varphi \) allows us to assume that \( \varphi(\sigma_p) = 1 \), for every \( p \in \text{Ram}(\varphi) \).

In order to deal with the Frattini case, let us first consider a (necessarily proper) solution \( \psi \) to the embedding problem \((\pi, \varphi)\), such that \( \text{Ram}(\psi) = \text{Ram}(\varphi) \). Such a \( \psi \) always exists [Se, Cor. 2.1.8]. Let \( p_0 \in \text{Ram}(\varphi) \) be such that all the elements \( \{\psi(\sigma_p)\}_{p \in \text{Ram}(\varphi)} \) belong to \( \langle \psi(\sigma_{p_0}) \rangle \subseteq C \). From Proposition 2.3, it follows that there exists a prime number \( q \equiv 1 \pmod{l^N} \) such that \( \varphi(D_q) = \{1\} \), and an epimorphism

\[
\chi : G_Q \rightarrow \text{Gal}(\mathbb{Q}(\zeta_q)/\mathbb{Q}) \cong (\mathbb{Z}/q\mathbb{Z})^* \rightarrow C
\]

such that \( \chi(\sigma_p) = \psi(\sigma_p) \), for every \( p \in \text{Ram}(\varphi) \). Clearly, \( \tilde{\varphi} := \psi \chi^{-1} \) is a proper solution to the embedding problem \((\pi, \varphi)\) such that \( \text{Ram}(\tilde{\varphi}) = \text{Ram}(\varphi) \cup \{q\} \). Furthermore, it is of type \((S_N)\), since \( \tilde{\varphi}(D_q) \subseteq C = \tilde{\varphi}(I_q) \) and \( \tilde{\varphi}(D_p) = \langle \tilde{\varphi}(\sigma_p), \tilde{\varphi}(I_p) \rangle = \tilde{\varphi}(I_p) \), for every \( p \in \text{Ram}(\varphi) \).

In the split case, it suffices to argue as in [Se, p. 11]. More precisely, let \( K/\mathbb{Q} \) denote the \( H \)-extension obtained from \( \varphi \) and let \( q \) be a prime number which splits completely in \( K \left( \zeta_{l^N}, \left\{ \sqrt[l]{p} \right\}_{p \in \text{Ram}(\varphi)} \right)/\mathbb{Q} \). Then, for every epimorphism \( \chi : G_Q \rightarrow \text{Gal}(\mathbb{Q}(\zeta_q)/\mathbb{Q}) \cong (\mathbb{Z}/q\mathbb{Z})^* \rightarrow C \), we can take

\[
\tilde{\varphi} : G_Q \xrightarrow{(\varphi, \chi)} H \times C \cong G.
\]

\( \square \)

Let \( C_i(G) \) be the \( i \)-th higher commutator subgroup of a finite group \( G \). This is defined inductively by \( C_1(G) = G \) and \( C_{i+1}(G) = [C_i(G), G] \). Recall that \( G \) is nilpotent if there exists a positive integer \( n \) such that \( C_{n+1}(G) = \{1\} \). The smallest such \( n \) is the nilpotency class of \( G \). Let us denote \( d_i(G) = d(C_i(G)/C_{i+1}(G)) \).

**Proposition 2.5.** Let \( l \) be an odd prime number and let \( n \) be the nilpotency class of an \( l \)-group \( G \). Then:

\[
d(G) \leq \text{ram}(G) \leq \text{ram}^l(G) \leq d(G) + \sum_{2 \leq i \leq n-1} d_i(G),
\]

where the above sum is assumed to be 0 in case \( n \leq 2 \).
Proof. The first inequality is a direct consequence of Burnside’s basis theorem and Kronecker–Weber’s theorem.

In order to obtain a $G$-extension of $\mathbb{Q}$ it suffices to properly solve a chain of $n$ central embedding problems given by the natural central extensions

$$1 \to C_i(G)/C_{i+1}(G) \to G/C_{i+1}(G) \to G/C_i(G) \to 1,$$

and starting from the trivial epimorphism $G_\mathbb{Q} \to G/C_1(G) = \{1\}$. Since $C_1(G) = G$ and $C_2(G)$ is nothing but the derived subgroup of $G$, our first ($i = 1$) embedding problem can be decomposed into $d(G) = d_1(G)$ cyclic split ones. For every $i > 1$, the abelianizations of $G/C_{i+1}(G)$ and $G/C_i(G)$ are the same (namely, $G/C_2(G)$). In this case, we thus have to consider a Frattini embedding problem which gives rise to $d_i(G)$ cyclic (Frattini) ones. The stated result then follows from Proposition 2.4. It should be noted that, since the final $G$-extension of $\mathbb{Q}$ is not required to be of type $(S_N)$, the (last) Frattini central embedding problem with kernel $C_n(G)$ can be (properly) solved without adding new ramification. □

In the above process, we certainly have to add a new ramified prime in order to properly solve each of the $d(G)$ first cyclic split embedding problems. For the remaining Frattini embedding problems, it may happen that we are not forced to increase the number of ramified primes. However, this would be restricted (if possible) by the following fact:

**Proposition 2.6.** Let $l$ be an odd prime number and let $G$ be an $l$-group of exponent at most $l^N$. Let $1 \to \mathbb{Z}/l\mathbb{Z} \to G \xrightarrow{\pi} H \to 1$ be a central Frattini extension and let $\varphi : G_\mathbb{Q} \to H$ be an epimorphism of type $(S_N)$ ramified at $d(H)$ finite primes. Then, the solutions to the embedding problem $(\pi, \varphi)$ ramified at $d(H)$ finite primes are, either all of them or none, of type $(S_N)$.

Proof. We will show that all solutions to $(\pi, \varphi)$ ramified at $d(H)$ finite primes define the same extension of $\mathbb{Q}$.

Let $\tilde{\varphi}_1, \tilde{\varphi}_2$ be different solutions to $(\pi, \varphi)$ ramified at $d(H)$ finite primes. Hence, $\tilde{\varphi}_1 = \tilde{\varphi}_2 \cdot \chi$, for some epimorphism $\chi \in \text{Hom}(G_\mathbb{Q}, \mathbb{Z}/l\mathbb{Z})$ ramified only at finite primes in $\text{Ram}(\varphi)$. We are assuming that $\varphi$ defines an $H$-extension of $\mathbb{Q}$ ramified at $d(H) = d(H^{ab})$ finite primes, where $H^{ab}$ denotes the abelianization of $H$. Hence, every abelian extension of $\mathbb{Q}$ of exponent $l$ ramified only at finite primes in $\text{Ram}(\varphi)$ must be a subextension of (the maximal abelian subextension of) $\mathbb{Q}^{\text{Ker} \varphi}/\mathbb{Q}$.

Thus, $\mathbb{Q}^{\text{Ker} \chi} \subseteq \mathbb{Q}^{\text{Ker} \varphi}$ and we have that

$$\mathbb{Q}^{\text{Ker} \tilde{\varphi}_1} = \mathbb{Q}^{\text{Ker} \tilde{\varphi}_2}.$$

The following one is the main result of this section. It is the best possible generalization of Proposition 2.5 for nilpotent groups of odd order.
Theorem 2.7. Let $G$ be a finite nilpotent group of odd order and let $\{G_1, \ldots, G_s\}$ be their Sylow subgroups, each of nilpotency class $n_j$. Then:

$$d(G) \leq \text{ram}(G) \leq \text{ram}^t(G) \leq \max_{1 \leq j \leq s} \left\{ d(G_j) + \sum_{2 \leq i \leq n_j-1} d_i(G_j) \right\}.$$  

What must be proved is that the bounds for each $G_i$ obtained from Proposition 2.5 can be reached in a compatible way. This is reduced to show the following generalization of Proposition 2.4:

Proposition 2.8. Let $L = \{l_1, \ldots, l_s\}$ be a finite set of odd prime numbers. For each $j \in L$, let us consider a cyclic central extension of $l_j$-groups

$$1 \to C_j \to G_j \xrightarrow{\pi_j} H_j \to 1$$

and an epimorphism $\varphi_j : G_Q \to H_j$. Let $N$ be a positive integer. Let us assume that, for every $1 \leq j \leq s$, the exponent of $G_j$ is at most $l_j^N$, the epimorphism $\varphi_j$ is of type $(S_N)$, the set $\text{Ram}(\varphi_j)$ does not contain any prime of $L$ and the embedding problem $(\pi_j, \varphi_j)$ is Frattini or split. Then, there exist proper solutions $\{\widehat{\varphi}_j\}_j$ to the embedding problems $\{(\pi_j, \varphi_j)\}_j$, each of them of type $(S_N)$, such that

$$\# \left( \bigcup_j \text{Ram}(\widehat{\varphi}_j) \right) \leq 1 + \# \left( \bigcup_j \text{Ram}(\varphi_j) \right).$$

Proof. It suffices to show the appropriate generalization of Proposition 2.3 (i).

Let $j \in \{1, \ldots, s\}$ be momentarily fixed. Let $K/Q$ be the $H_j$-extension obtained from $\varphi_j$ and let us denote $l = l_j$, $C = C_j$ and $S = \text{Ram}(\varphi_j)$. Given $p_0$, $k$ for which the hypothesis of Prop. 2.3 hold, let us define $M_j = Q(\zeta_{lN}, \sqrt[l^k]{p_0})$ and

$$L_j = K\left(\zeta_{lN}, \left\{ \sqrt[l^k]{p_0} \right\}_{p \in S}, \sqrt[l^k]{p_0} \right).$$

We want to prove that there exists a prime number $q$ such that, for every $j \in \{1, \ldots, s\}$, $q$ splits completely in $L_j/Q$ and does not in $M_j/Q$.

From Prop. 2.3 (i) we know that $[L_j : M_j] = l_j$, for every $j$. Let us denote $a = (l_1 \cdots l_s)^N$ and $a(j) = \frac{a}{l_j^N}$. Since the extensions $L_j, M_j/Q$ and $\mathbb{Q}(\zeta_{a(j)})/Q$ have no common ramified finite primes, it must be $[L_j : M_j(\zeta_a) : L_j(\zeta_a)] = l_j$. As a consequence, we have that $[L_1 \cdots L_s : M_1 \cdots L_s] = l_j$. Hence, there exists some $\sigma \in \text{Gal}(L_1 \cdots L_s, M_1 \cdots M_s/L_1 \cdots L_s)$ such that:

$$\sigma|_{M_j} \neq \text{id}, \quad \text{for every } j \in \{1, \ldots, s\}.$$  

It only remains to invoke Tchebotarev’s density theorem. \qed
Remark 2.9. From Theorem 2.7, we obtain that \( \text{ram}(G) = \text{ram}^t(G) = d(G) \), for every finite nilpotent group \( G \) of odd order such that

\[
 d(G) = \max_{1 \leq j \leq s} \left\{ d(G_j) + \sum_{2 \leq i \leq n_j - 1} d(C_i(G_j)/C_{i+1}(G_j)) \right\}.
\]

This equality holds for more groups than just the easy ones of nilpotency class \( n = \max_j \{ n_j \} \leq 2 \).

Remark 2.10. In [Cu-He, Thm. 5] it is claimed that the equality \( \text{ram}(G) = \text{ram}^t(G) = d(G) \) holds for every finite nilpotent group \( G \) of odd order. However, there is an error in the proof of this result (p. 308, “Therefore, \( q_1, \ldots, q_{b+1} \) are fleissig in \( K_1^n/Q \)...”). Moreover, Proposition 2.6 contradicts the argument followed there.

3. Generalized dihedral groups

Given a finite abelian group \( A \), consider the \( \mathbb{Z}/2\mathbb{Z} \)-action on \( A \) which sends \( 1 \in \mathbb{Z}/2\mathbb{Z} \) to \( \{ \sigma \mapsto \sigma^{-1} \} \in \text{Aut}(A) \). One says that the corresponding semidirect product \( A \rtimes \mathbb{Z}/2\mathbb{Z} \) is a generalized dihedral group and it will be denoted by \( D_{2,A} \). Note that, if \( B \) is a quotient of \( A \), then \( D_{2,B} \) is a quotient of \( D_{2,A} \). The abelianization of \( D_{2,A} \) will be denoted by \( D_{2,A}^{ab} \). One easily checks that \( D_{2,A}^{ab} \cong D_{2,A}/A^2 \cong D_{2,A/A^2} \).

Let \( K \) be a quadratic field. Given a positive integer \( f \geq 2 \), \( K(\tilde{f}) \) will denote the ring class field of \( K \) of conductor \( \tilde{f} := (f)O_K \cdot \infty_K \) (see, for example, [Co]). Let us just recall that \( K(\tilde{f})/K \) is a finite abelian extension, unramified away from \( \tilde{f} \), which contains the narrow Hilbert class field of \( K \). Every intermediate field between \( K \) and some \( K(\tilde{f}) \) will be called a ring class field of \( K \).

Generalized dihedral groups and ring class fields are intimately related by the following known result:

**Theorem 3.1** (cf. [Bru, Satz 8]). The following conditions on a number field \( L \) are equivalent:

(i) \( L/Q \) is a generalized dihedral extension, that is, it is a Galois extension with group isomorphic to a generalized dihedral group \( (D_{2,A}) \).

(ii) \( L \) is a ring class field of some quadratic field \( (K = L^A) \).

Our main result in this section is the following one:

**Theorem 3.2.** Let \( A \) be a finite abelian group. Then:

\[
 d(D_{2,A}^{ab}) \leq \text{ram}^t(D_{2,A}) \leq d(D_{2,A})
\]

**Proof.** The first inequality follows from Kronecker–Weber’s theorem.
Let us denote \( r = d(A) \) and let \((m_1, \ldots, m_r)\) be the invariant factors of \( A \). Certainly, \( d(D_2, A) = r + 1 \).

In order to obtain the second inequality, we will prove the existence of a ring class field \( L \) (of some quadratic field \( K \)) such that:

(a) At most \( r + 1 \) finite primes ramify in the extension \( L/\mathbb{Q} \), all of them being tamely ramified,

(b) \( A \) is isomorphic to a quotient of \( \text{Gal}(L/K) \).

The stated result then follows from Theorem 3.1 (and the remarks previous to it).

Let \( q \) be an arbitrary fixed odd prime number. Take \( K := \mathbb{Q}(\sqrt{q^*}) \), the unique quadratic field unramified away from \( \{q, \infty\} \). Let \( H_+ \) denote the narrow Hilbert class field of \( K \). The degree of the extension \( K(\tilde{f})/K \) is known to be \([K(\tilde{f}) : K] = [H_+ : K] \prod_{p | f} \left( 1 - \frac{d}{p} \right) \cdot \frac{1}{E_\tilde{f}} \),

where \( d \) denotes the discriminant of \( K \) and \( E_\tilde{f} \) is a suitable positive integer which, in case \( d < 0 \), depends only on \( K \) (not on \( f \)).

One can always find prime numbers \( p_1, \ldots, p_r \) such that:

\[ p_i \equiv 1 \pmod{m_i E_{p_i} q}, \]

for every \( i \in \{1, \ldots, r\} \). This is clear for imaginary \( K \). For real \( K \), it suffices to take \( p_i \) being completely split in \( K(\zeta_{m_i q}, \zeta_{m_i q^*}) \), where \( \epsilon_+ \) is the generator of the totally positive units in the ring of integers of \( K \) (see the proof of [Je-Yui, Thm. I.2.1]).

Let us fix a set of \( r \) prime numbers \( \{p_1, \ldots, p_r\} \) as above and let us define \( L = K(\overline{p_1}) \ldots K(\overline{p_r}) \). Since \( L \) is a subfield of \( K(p_1 \cdots p_r) \), it is a ring class field of \( K \). We are going to check Conditions (a) and (b) for such a choice of \( L \) (and \( K \)).

By assumption, each prime \( p_i \) splits completely in \( K \), \( p_i \mathcal{O}_K = p_i, p_i' \). The inertia subgroups in \( \text{Gal}(K(\overline{p_i})/K) \) at the primes \( p_i \) and \( p_i' \) are conjugate one from another in (the generalized dihedral group) \( \text{Gal}(K(\overline{p_i})/\mathbb{Q}) \), hence they are equal. Moreover, they must be equal to the inertia subgroup \( I_i \subset \text{Gal}(K(\overline{p_i})/\mathbb{Q}) \) at \( p_i \). Since \( p_i \) and \( p_i' \) are the only ramified prime ideals in the extension \( K(\overline{p_i})/K \), it must be \( I_i = \text{Gal}(K(\overline{p_i})/H_+) \). Let \( n_i \) denote the order of \( I_i \). From the above formula (with \( f = p_i \)), we obtain that \( n_i = \frac{p_i - 1}{E_{p_i}} \).

Hence, it must be

\[ \text{Gal}(K(\overline{p_i})/H_+) \cong \mathbb{Z}/n_i \mathbb{Z}, \]

the extension \( K(\overline{p_i})/H_+ \) being totally and tamely ramified at all primes of \( H_+ \) over \( p_i \). In addition, for \( i \neq j \), the extensions \( K(\overline{p_i})/H_+ \) and \( K(\overline{p_j})/H_+ \)
have no common ramified finite primes. Thus, we obtain an isomorphism
\[ \text{Gal}(L/H_+) \cong \mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_r\mathbb{Z}. \]

Our choice of \( p_i \) ensures \( n_i \) being divisible by \( m_i \). So, \( A \) is isomorphic to a quotient of \( \text{Gal}(L/K) \) (Condition (b)).

Finally, the only finite ramified primes in \( L/\mathbb{Q} \) are \( q, p_1, \ldots, p_r \) and their ramification indices are, respectively, \( 2, n_1, \ldots, n_r \). Hence, Condition (a) also holds.

\[ \square \]

**Remark 3.3.** We proved more than just the inequality \( \text{ram}^t(D_{2,A}) \leq d(D_{2,A}) \). The given proof shows that, for every odd prime number \( q \), there exist infinitely many ring class fields \( L \) of \( \mathbb{Q}(\sqrt{q^*}) \) such that the extension \( L/\mathbb{Q} \) ramifies at most at \( d(D_{2,A}) \) finite primes and has Galois group isomorphic to \( D_{2,A} \).

**Corollary 3.4.** Let \( A \) be a finite abelian group such that \( d(A) = d(A_2) \), where \( A_2 \) denotes the 2-primary component of \( A \). Then, \( \text{ram}^t(D_{2,A}) = d(D_{2,A}) \).

In the general case, it certainly happens that \( \text{ram}^t(D_{2,A}) < d(D_{2,A}) \).

**Proposition 3.5.** Let \( A \) be a finite abelian group. Then, the following conditions are equivalent:

(i) \( \text{ram}^t(D_{2,A}) = 1 \).

(ii) \( A \) is isomorphic to a subgroup of the narrow class group \( \text{Cl}_+(\mathbb{Q}(\sqrt{p^*})) \) of \( \mathbb{Q}(\sqrt{p^*}) \), for some odd prime number \( p \).

**Proof.** On the one hand, the Galois group over \( \mathbb{Q} \) of the narrow Hilbert class field \( H_+ \) of a quadratic field \( K \) is isomorphic to a generalized dihedral group. On the other hand, if \( K = \mathbb{Q}(\sqrt{p^*}) \) for an odd \( p \), then \( H_+ / K \) is the maximal tamely ramified subextension of \( K((p^m))^r / K \), for every \( m \geq 1 \). \( \square \)

Given a finite abelian group \( A \) of even order, it must be \( \text{ram}^t(D_{2,A}) > 1 \). However, it may be \( \text{ram}(D_{2,A}) = 1 \).

**Proposition 3.6.** Let \( A \) be a finite abelian group of even order and let \( p \) be a prime number. Let \( L/\mathbb{Q} \) be a Galois extension unramified away from \( \{p, \infty\} \), with Galois group \( \text{Gal}(L/\mathbb{Q}) \cong D_{2,A} \). Then, \( p = 2 \) and \( A \) is a cyclic 2-group.

**Proof.** From Theorem 3.1, \( L \) must be a ring class field of the quadratic field \( K = L^A \). Since \( D_{2,A}^{ab} \) is isomorphic to a quotient of \( (\mathbb{Z}/p^n\mathbb{Z})^* \) and \( d(D_{2,A}^{ab}) = 1 + d(A_2) \geq 2 \), it must be \( p = 2 \) and \( d(A_2) = 1 \). Hence, \( K \) is one of the fields \( \mathbb{Q}((\sqrt{2})) \), \( \mathbb{Q}((\sqrt{-2})) \) or \( \mathbb{Q}((i)) \). All of them have trivial narrow class group, so \( A \) must be a cyclic 2-group. \( \square \)

**Corollary 3.7.** Let \( A \) be a finite abelian group of even order. Then, the following conditions are equivalent:
(i) \( \text{ram}(D_{2,A}) = 1 \),
(ii) \( A \) is a cyclic 2-group.

Proof. (ii) \( \Rightarrow \) (i) follows from the fact that a 2-group \( G \) appears as the Galois group of an extension of \( \mathbb{Q} \) unramified away from \( \{2, \infty\} \) if and only if \( G \) can be generated by two elements, one of them of order 2 (cf. [Ma] or [Ha, p. 59]). \( \square \)

For the usual dihedral group \( D_{2n} \) of order \( 2n \), we have the following:

**Corollary 3.8.** Let \( n \) be an even positive integer. Then:
- \( \text{ram}(D_{2n}) = 1 \) and \( \text{ram}^t(D_{2n}) = 2 \), if \( n \) is a power of 2,
- \( \text{ram}(D_{2n}) = \text{ram}^t(D_{2n}) = 2 \), otherwise.

We next prove a conditional result about \( \text{ram}(D_{2n}) \) and \( \text{ram}^t(D_{2n}) \), for odd \( n \). Let us first recall the:

**Hypothesis (H) of Schinzel [Sc-Si]:** Let \( p_1(T), \ldots, p_r(T) \) be irreducible polynomials in \( \mathbb{Z}[T] \), all of them having positive leading coefficient. Assume also that, for every prime number \( p \), there exists an integer \( n_p \) such that \( p \) does not divide \( p_1(n_p) \cdots p_r(n_p) \). Then, there exist infinitely many positive integers \( n \) such that \( p_1(n), \ldots, p_r(n) \) are all prime numbers.

**Proposition 3.9.** Under the Hypothesis (H) of Schinzel, every dihedral group \( D_{2n} \) satisfies \( \text{ram}^t(D_{2n}) = d(D_{2n}^0) \).

Proof. We must prove the equality \( \text{ram}^t(D_{2n}) = 1 \), for every odd \( n \).

Let \( l \equiv 1 \pmod{n} \) be an odd prime number and let \( x \) be an odd integer which generates \( \left( \mathbb{Z}/l\mathbb{Z} \right)^* \). A result of Yamamoto [Ya, Prop. 1] establishes that, if \( t \in \mathbb{Z} \) is coprime with \( x \) and \( x^2 - 4^t l^m n < 0 \), then the class group of \( \mathbb{Q}(\sqrt{x^2 - 4^t l^m n}) \) has an element of order \( n \).

On the other hand, \( p(T) = 4^t l^m T^2 - x^2 \in \mathbb{Z}[T] \) is an irreducible polynomial in \( \mathbb{Q}[T] \) such that \( (p(0), p(1)) = 1 \). Then, Hypothesis (H) of Schinzel claims the existence of infinitely many integers \( t \in \mathbb{N} \) such that \( p(t) \) is a prime number \( q \), necessarily \( q \equiv 3 \pmod{4} \). Hence, \( q \) is the only ramified prime number in the extension \( \mathbb{Q}(\sqrt{-q})/\mathbb{Q} \). \( \square \)

**Remark 3.10.** Similar results can also be obtained for other finite groups. For instance, assuming Hypothesis (H), one can always find monic trinomials in \( \mathbb{Z}[X] \) of degree \( n \) (every \( n \)) whose Galois group over \( \mathbb{Q} \) is isomorphic to the symmetric group \( S_n \) and whose discriminant is a prime number greater than \( n \). Hence, under the Hypothesis (H) of Schinzel, the symmetric group satisfies \( \text{ram}(S_n) = \text{ram}^t(S_n) = 1 \), for every \( n \). It should be mentioned that the analogous argument for other groups may not work. For example, if \( n \equiv 2, 6 \pmod{8} \), then every realization of the alternating group \( A_n \) as the Galois group over \( \mathbb{Q} \) of a degree \( n \) trinomial must be ramified at all prime numbers \( p \equiv 3 \pmod{4} \) which divide \( n \) [Pl-Vi].
Acknowledgements. This work is part of my Ph.D. Thesis. I am very much indebted to my thesis advisor, Núria Vila, for many valuable suggestions concerning material in this paper.

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Received June 30, 2003 and revised September 25, 2003. Research partially supported by MCYT grant BFM2000-0794-C02-01.

DEPT. DE MATEMÀTIQUA APPLICADA I
UNIVERSITAT POLITÈCNICA DE CATALUNYA
AV. DIAGONAL, 647
08028 BARCELONA
SPAIN
E-mail address: bernat.plans@upc.es
TWO APPLICATIONS OF PREQUANTIZATION IN LAGRANGIAN TOPOLOGY

Jon Wolfson

The main theorem characterizes the Lagrangian homology classes of a compact symplectic 2n-manifold with integral symplectic form $\omega$. An integral homology $n$-class $\alpha$ is Lagrangian (i.e., can be represented by a Lagrangian $n$-cycle) if and only if $\alpha \cap [\omega] = 0$.

1. Introduction

In Euclidean space $\mathbb{R}^{2n}$ equipped with the standard symplectic structure $\omega = \sum dx_i \wedge dy_i$, a Lagrangian submanifold $\Sigma$ is called exact if the integral of the Liouville form $\lambda = 1/2 \sum (x_i \wedge dy_i - y_i \wedge dx_i)$ along any closed path on $\Sigma$ is zero. It is well-known that a Lagrangian submanifold is exact if and only if it admits a lift to a Legendrian submanifold of the contact manifold $\mathbb{R}^{2n+1}$. It is interesting that the notion of “exact Lagrangian” can be generalized to symplectic manifolds for which the symplectic form is not an exact form. To do this we use a construction sometimes called prequantization. Suppose that $(N, \omega)$ is a compact symplectic manifold with integral symplectic form $\omega$. There is a complex line bundle $L$ over $N$ with unitary connection $\eta$ and curvature $\omega[K]$. Denote the associated principal $S^1$-bundle by $P$. Then $(P, \eta)$ is a contact manifold. The line bundle $L$ is flat over a Lagrangian submanifold $\Sigma$ and therefore, locally, $\Sigma$ admits a Legendrian lift to $(P, \eta)$. When $\Sigma$ admits a global lift we call it exact. In this note we utilize the contact manifold $(P, \eta)$ and the Legendrian lift to prove two results on the topology of Lagrangian submanifolds. Before we describe these results we remark on the integrality assumption. The Chern class of the line bundle $L$ is $[\omega]$ and therefore it is necessary that the symplectic form be integral. However if $\omega$ is rational then after multiplying by a suitable scalar the form is integral. Since the property of being Lagrangian is independent of taking scalar multiples we can weaken the integrality assumption on $\omega$ to a rationality assumption.

Consider an integral homology class $\alpha \in H_n(N, \mathbb{Z})$ in the symplectic manifold $(N, \omega)$. An integral $n$-cycle is called a Lagrangian cycle if the $n$-simplices are Lagrangian (more precisely, if the $n$-simplices are the images of piecewise $C^1$ Lagrangian maps $\Delta^n \to N$). The class $\alpha$ is a Lagrangian class if $\alpha$ can be represented by a Lagrangian cycle. Note that if $C$ is a Lagrangian
n-cycle and $\phi$ is a closed $(n-2)$-form on $N$ then $\int_C \omega \wedge \phi = 0$. Therefore, by Poincare duality, a Lagrangian class $\alpha$ satisfies the condition:
(1.1) $\alpha \cap [\omega] = 0$,
where $[\omega]$ denotes the cohomology class represented by $\omega$. Is this necessary condition also sufficient? That is, if an integral class $\alpha$ satisfies $\alpha \cap [\omega] = 0$ is it a Lagrangian class? We prove the following answer:

**Theorem 1.1.** Let $N$ be a compact symplectic manifold of dimension $2n$ with an integral symplectic form $\omega$. An integral homology class $\alpha \in H_n(N, \mathbb{Z})$ is a Lagrangian class if and only if $\alpha \cap [\omega] = 0$.

When $n = 2$ the condition (1.1) becomes $[\omega](\alpha) = 0$. In [S-W] it is shown that this condition implies that $\alpha$ is a Lagrangian class. When $n = 3$ and $N$ is simply connected (1.1) is satisfied for any class $\alpha$. In [W2] it is shown that in this case every class is Lagrangian. In fact, every class can be represented by a Lagrangian immersion.

Next let $\Sigma$ be a minimal submanifold of $S^{2n-1}$ that is Legendrian for the standard contact structure on $S^{2n-1}$. It is easy to see that the image of $\Sigma$ under the Hopf map is a minimal, Lagrangian submanifold $\Sigma$ of $\mathbb{C}P^{n-1}$. Conversely, we show:

**Theorem 1.2.** Let $\Sigma$ be a minimal, Lagrangian submanifold in $\mathbb{C}P^{n-1}$. Then $\Sigma$ lifts to a minimal Legendrian submanifold $\Sigma \subset S^{2n-1}$.

The local version of the theorem is elementary. The global version follows from a consideration of the holonomy of the complex line bundle $L$ described above. We remark that the tangent cone of an isolated special Lagrangian singularity is the cone on a minimal Legendrian submanifold of $S^{2n-1}$ and therefore this theorem has some application to such singularities.

### 2. Lagrangian homology

Let $N$ be a compact symplectic manifold of dimension $2n$ equipped with an integral symplectic form $\omega$. Adopting the notation of the introduction we let $(P, \eta)$ be the contact manifold such that $d\eta = \omega$. In particular $\pi : P \to N$ is an $S^1$ bundle with Chern class $[\omega]$.

**Proposition 2.1.** Let $\alpha \in H_n(N, \mathbb{Z})$. There is a class $\tilde{\alpha} \in H_n(P, \mathbb{Z})$ such that $\pi_* (\tilde{\alpha}) = \alpha$ if and only if $\alpha \cap [\omega] = 0$.

**Proof.** If $\pi_* (\tilde{\alpha}) = \alpha$ then
$$\alpha \cap [\omega] = \pi_* (\tilde{\alpha}) \cap [\omega] = \pi_* (\tilde{\alpha} \cap \pi^* [\omega]) = \pi_* (\tilde{\alpha} \cap [d\eta]) = 0.$$

Conversely, consider the Thom–Gysin sequence $[S]$ of the $S^1$-bundle $\pi : P \to N$:
$$\cdots \to H_n(P; \mathbb{Z}) \xrightarrow{\pi^*} H_n(N; \mathbb{Z}) \xrightarrow{\Phi} H_{n-2}(N; \mathbb{Z}) \to \cdots,$$
where for \( z \in H_n(N; \mathbb{Z}) \), \( \Phi(z) = z \cap [\omega] \). By assumption \( \Phi(\alpha) = 0 \) and therefore there exists \( \tilde{\alpha} \in H_n(P; \mathbb{Z}) \) such that \( \pi_*(\tilde{\alpha}) = \alpha \). \( \square \)

The next proposition uses successive applications of a horizontal extension lemma due to Gromov [G, 3.5]. Let \( V \) be a compact contact \( 2n + 1 \)-manifold and \( S \) be an open \( k \)-simplex. A piecewise \( C^1 \) map \( h : S \to V \) is called horizontal if the image of the tangent space of \( S \) lies in the contact plane at each point. Suppose \( W \) is a simplicial \( k \)-complex with \( k \leq n \) and \( W_0 \subset W \) is a subcomplex. Let \( f_0 : W_0 \to V \) be a piecewise \( C^1 \) horizontal map and suppose that there is continuous extension \( g : W \to V \) of \( f_0 \). Then there is a piecewise \( C^1 \) horizontal extension \( f : W \to V \) of \( f_0 \) that is homotopic to \( g \).

**Proposition 2.2.** Any homology class in \( H_n(P; \mathbb{Z}) \) can be represented by a horizontal \( n \)-cycle.

**Proof.** Let \( b \in H_n(P; \mathbb{Z}) \) and suppose \( B \) is a simplicial cycle that represents \( b \). Suppose that \( B \) is sufficiently fine so that the 0 and 1 simplices have the following property: If \( v^0_1 \) and \( v^0_2 \) are vertices such that there is a 1-simplex \( s^1 \) in \( B \) joining them then there is a horizontal 1-simplex \( \sigma^1 \) joining \( v^0_1 \) and \( v^0_2 \) that is \( C^0 \) close to \( s^1 \). In particular there is a homotopy joining \( s^1 \) and \( \sigma^1 \). Use this property to construct a horizontal 1-cycle approximating the 1-simplices of \( B \) (and with the same 0-simplices as \( B \)). Consider a 2-simplex \( t^2 \) of \( B \) with 1-simplices \( s^1_i \), \( i = 1, 2, 3 \) such that \( \partial t^2 = s^1_1 - s^1_2 + s^1_3 \). To each simplex \( s^1_i \) there is a corresponding horizontal simplex \( \sigma^1_i \). The homotopies between \( s^1_i \) and \( \sigma^1_i \) and the 2-simplex \( t^2 \) show that the cycle \( \sigma^1_1 - \sigma^1_2 + \sigma^1_3 \) has a continuous extension and therefore by the horizontal extension lemma there is a horizontal 2-simplex \( \tau^2 \) with \( \partial \tau^2 = \sigma^1_1 - \sigma^1_2 + \sigma^1_3 \). Moreover \( \tau^2 \) is homotopic to \( t^2 \) and the homotopy can be taken to extend with the homotopies between \( s^1_i \) and \( \sigma^1_i \). Carrying out this construction for each 2-simplex of \( B \) there is a horizontal 2-cycle approximating the 2-simplices of \( B \). Successively apply this construction to the 3-simplices, the 4-simplices, \( \ldots \), the \( n \)-simplices to construct a horizontal \( n \)-cycle \( \Lambda \) that is homotopic to \( B \). The image of the homotopy determines an \((n + 1)\)-chain \( C \) with \( B - \Lambda = \partial C \), proving the proposition. \( \square \)

The following proposition and Proposition 2.1 together prove Theorem 1.1.

**Proposition 2.3.** Suppose that \((N, \omega)\) is a compact symplectic \( 2n \)-manifold with integral symplectic form \( \omega \). If \( \alpha \in H_n(N; \mathbb{Z}) \) has a lift \( \tilde{\alpha} \in H_n(P; \mathbb{Z}) \) then \( \alpha \) is a Lagrangian class.

**Proof.** By Proposition 2.2 there is a Legendrian cycle \( \Lambda \) that represents \( \tilde{\alpha} \). Then \( \pi(\Lambda) \) is a Lagrangian cycle representing \( \alpha \). \( \square \)
Remark. The above proof shows that under the hypothesis of Theorem 1.1 a homology class $\alpha$ satisfying $\alpha \cap [\omega] = 0$ can be represented by a Lagrangian cycle that lifts to a Legendrian cycle in $(P, \eta)$.

As an application of Theorem 1.1 we prove:

**Theorem 2.4.** Let $(N, \omega)$ be a compact symplectic $2n$-manifold with integral symplectic form $\omega$. Suppose that $\alpha \in H_n(N; \mathbb{Z})$ satisfies $\alpha \cap [\omega] = 0$. Then $\alpha$ can be represented by a Lagrangian integral current $T$ that minimizes mass (volume) among all Lagrangian $n$-cycles representing $\alpha$.

**Proof.** By Theorem 1.1 the set of Lagrangian integral currents $S$ with $\partial S = \emptyset$ representing $\alpha$ is nonempty. A mass minimizing sequence of such currents converges in the weak topology to a current $T$ with $\partial T = \emptyset$. It follows easily that $T$ is Lagrangian. Since convergence in the weak topology is equivalent to convergence in the flat norm topology it follows from $\mathbb{F} \cdot \mathbb{F}$ that $T$ is an integral current that represents $[\alpha]$. Thus

$$\|T\| = \inf_{\{S \in \mathcal{IL} : [S] = \alpha\}} \|S\|,$$

where $\mathcal{IL}$ denotes the set of Lagrangian integral currents. \qed

### 3. Lagrangian cones

Suppose that $M$ is a special Lagrangian submanifold with an isolated singularity at $p \in M$ in the Calabi–Yau manifold $N$. Let $C_p$ denote the tangent cone of $p$. Then $C_p$ is a special Lagrangian cone in $\mathbb{C}^n$. Denote the link $C_p \cap S^{2n-1}$ by $\Sigma$. Then $\Sigma$ is an imbedded Legendrian $n-1$-submanifold for the standard contact structure on $S^{2n-1}$ and, in addition, $\Sigma$ is a minimal submanifold of $S^{2n-1}$. Let $h : S^{2n-1} \to \mathbb{CP}^{n-1}$ be the Hopf map. Equivalently, let $L \to \mathbb{CP}^{n-1}$ be the tautological bundle so that the total space of the associated $S^1$ bundle is $S^{2n-1}$. Then the bundle projection is the Hopf map. Denote the image of $\Sigma$ under the bundle projection by $\Sigma$. It is easy to verify that $\Sigma$ is a minimal, Lagrangian submanifold in $\mathbb{CP}^{n-1}$. This (well-known) construction thus associates a minimal Lagrangian submanifold of $\mathbb{CP}^{n-1}$ to an isolated singularity in a special Lagrangian submanifold. The converse is given in the statement of Theorem 1.2. It follows from Theorem 1.2 that isolated special Lagrangian singularities are classified by minimal, Lagrangian submanifolds of $\mathbb{CP}^{n-1}$.

The proof of the theorem proceeds as follows: Consider $\mathbb{CP}^{n-1}$ equipped with the Fubini-Study metric and Kähler form $\omega$. Note that $\omega$ is integral. The tautological line bundle $L$ over $\mathbb{CP}^{n-1}$ has a unitary connection $\eta$ with curvature $\omega$. The total space of the associated $S^1$ principal bundle is $S^{2n-1}$ and $(S^{2n-1}, \eta)$ is a contact manifold over $(\mathbb{CP}^{n-1}, \omega)$. Consider a smooth Lagrangian immersion $i : \Lambda \to \mathbb{CP}^{n-1}$. Locally $i$ has a Legendrian lift to $S^{2n-1}$. To determine when this lift is global note that the pulled-back
connection $i^*\eta$ of $i^*L$ is flat. Thus there is a homomorphism (the holonomy homomorphism):

$$\text{Hol}(i^*\eta) : H_1(\Sigma, \mathbb{Z}) \to S^1.$$ 

**Proposition 3.1.** $i$ lifts to a Legendrian immersion if and only if the values of $\text{Hol}(i^*\eta)$ are integral.

Suppose next instead of the line bundle $L$, we consider the canonical line bundle $K$ with connection $\tau$ induced by the metric and with curvature $\text{Ric} = R\omega$. As above if $i : \Lambda \to \mathbb{C}\mathbb{P}^{n-1}$ is a smooth Lagrangian immersion (not necessarily minimal) then the pulled-back connection $i^*\tau$ of $i^*K$ is flat and $i$ lifts to a Legendrian immersion if and only if the values of the holonomy homomorphism of $i^*\tau$ are integral. On $i(\Lambda)$ the mean curvature 1-form $\sigma_H = H_\perp \omega$ is closed since $\mathbb{C}\mathbb{P}^{n-1}$ is Kähler–Einstein.

**Proposition 3.2.** Let $\gamma$ be a closed curve on $\Lambda$. Then

$$\text{Hol}_\gamma(i^*\tau) = \exp \left( -2\pi i \int_\gamma \sigma_H \right).$$

**Proof.** See [W1].

**Proof of Theorem 1.2.** Note that $\Sigma$ lifts locally to a minimal Legendrian submanifold of $S^{2n-1}$. The issue is the existence of a global lift. Because $\Sigma$ is minimal the periods of $\sigma_H$ vanish. Since $K = \mathbb{L}^n$ and $\tau = n\eta$, it follows from Proposition 3.2 that the holonomy of $\eta$ is integral. Therefore by Proposition 3.1 there is a global Legendrian lift.

**References**


Received August 22, 2003. The author was partially supported by NSF grant DMS-0304587.

DEPARTMENT OF MATHEMATICS
MICHIGAN STATE UNIVERSITY
EAST LANSING, MI 48824
E-mail address: wolfson@math.msu.edu
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