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L^q-THEORY OF A SINGULAR "WINDING" INTEGRAL OPERATOR ARISING FROM FLUID DYNAMICS

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We analyze in classical $L^q(\mathbb{R}^n)$-spaces, $n = 2$ or $n = 3$, $1 < q < \infty$, a singular integral operator arising from the linearization of a hydrodynamical problem with a rotating obstacle. The corresponding system of partial differential equations of second order involves an angular derivative which is not subordinate to the Laplacian. The main tools are Littlewood–Paley theory and a decomposition of the singular kernel in Fourier space.

1. Introduction

Consider a three-dimensional rotating rigid body with angular velocity $\omega = (0,0,1)^T$ and assume that the complement, a time-dependent exterior domain $\Omega(t) \subset \mathbb{R}^3$, is filled with a viscous incompressible fluid modelled by the Navier–Stokes equations. By a simple coordinate transform we are led to the nonlinear system [**6**]

(1.1)
$$\begin{aligned} u_t - \nu\Delta u + u\cdot\nabla u - (\omega\wedge x)\cdot\nabla u + \omega\wedge u + \nabla p &= f && \text{in } \Omega \\ \operatorname{div} u &= 0 && \text{in } \Omega \\ u &= \omega\wedge x && \text{on } \partial\Omega \\ u &\to 0 && \text{at } \infty \end{aligned}$$

for the unknown velocity u and pressure function p in a time-independent exterior domain $\Omega \subset \mathbb{R}^3$ where $\nu > 0$ is the coefficient of viscosity. Looking for stationary solutions of (1.1), i.e., for time-periodic solutions of the original problem, and ignoring the nonlinear term $u\cdot\nabla u$ we arrive at a linear stationary partial differential equation in Ω.

The first step to analyzing this problem is the L^q-theory of the system

$$(1.2)\qquad \begin{aligned} -\nu\Delta u - (\omega\wedge x)\cdot\nabla u + \omega\wedge u + \nabla p &= f && \text{in } \mathbb{R}^3 \\ \operatorname{div} u &= g && \text{in } \mathbb{R}^3 \end{aligned}$$

in the whole space. Here for later applications we allow $\operatorname{div} u$ to equal an arbitrarily given function g. The Coriolis force $\omega\wedge u = (-u_2, u_1, 0)^T$ can be considered as a perturbation of the Laplacian. But the first order partial differential operator $(\omega \wedge x) \cdot \nabla u$ is *not* subordinate to the Laplacian due to the increasing term $\omega \wedge x = (-x_2, x_1, 0)^T$. Using cylindrical coordinates $(r, \theta, x_3) \in (0, \infty) \times [0, 2\pi) \times \mathbb{R}$ we get

$$(\omega \wedge x) \cdot \nabla u = -x_2 \partial_1 u + x_1 \, \partial_2 u = \partial_\theta u$$

showing that the crucial term $(\omega \wedge x) \cdot \nabla u$ is "just" an angular derivative of u w.r.t. θ. Since

$$\operatorname{div}\big((\omega \wedge x) \cdot \nabla u - \omega \wedge u\big) = (\omega \wedge x) \cdot \nabla \operatorname{div} u = \partial_\theta \, g,$$

the pressure p will satisfy the equation

$$\Delta p = \operatorname{div} f + \nu \Delta g + \partial_\theta g \quad \text{in } \mathbb{R}^3$$

which can easily be solved in L^q-spaces. Given p and ignoring $(1.2)_2$ we arrive at the system

$$-\nu \Delta u - \partial_\theta u + \omega \wedge u = f \quad \text{in } \mathbb{R}^3 \tag{1.3}$$

with another right-hand side f. Note that (1.3) also makes sense for a two-dimensional vector field u on $\mathbb{R}^2$; then $\omega \wedge u = (-u_2, u_1)^T$ and $(r, \theta) \in (0, \infty) \times [0, 2\pi)$ denote polar coordinates in $\mathbb{R}^2$.

Theorem 1.1.

(1) *Let $f \in L^q(\mathbb{R}^n)^n$, $n = 2$ or $n = 3$, $1 < q < \infty$. Then* (1.3) *has a solution $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)^n$ satisfying the estimate*

$$\|\nu \nabla^2 u\|_q + \|\partial_\theta u - \omega \wedge u\|_q \le c \, \|f\|_q. \tag{1.4}$$

Its equivalence class in the homogeneous Sobolev space $\hat{H}^{2,q}(\mathbb{R}^n)^n$ is unique.

(2) *Let $f \in L^{q_1}(\mathbb{R}^3)^3 \cap L^{q_2}(\mathbb{R}^3)^3$, $1 < q_1, q_2 < \infty$, and let u_1 and u_2 be solutions as given by (1) corresponding to $q = q_1$ and $q = q_2$, respectively. Then there are $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that u_1 coincides with u_2 up to an affine linear vector field $\alpha\omega + \beta\omega \wedge x + (\gamma x_1, \gamma x_2, \delta x_3)^T$, and any solution remains a solution if one adds such a term. For $n = 2$ the terms $\alpha\omega$ and $(0, 0, \delta x_3)^T$ have to be omitted.*

(3) *Let $f \in L^q(\mathbb{R}^n)^n$, $n = 2$ or $n = 3$, and let $g \in H^{1,q}_{\mathrm{loc}}(\mathbb{R}^n)$ such that $(\omega \wedge x) g, \nabla g \in L^q(\mathbb{R}^n)^n$, $1 < q < \infty$. Then* (1.2) *has a locally integrable solution (u, p) satisfying the estimate*

$$\|\nu \nabla^2 u\|_q + \|\partial_\theta u - \omega \wedge u\|_q + \|\nabla p\|_q \le c \left(\|f\|_q + \|\nu \nabla g + (\omega \wedge x) g\|_q\right)$$

where $(1.2)_2$ has to be understood in the sense $\nabla \operatorname{div} u = \nabla g$. Its equivalence class in $\hat{H}^{2,q}(\mathbb{R}^n)^n \times \hat{H}^{1,q}(\mathbb{R}^n)$ is unique. Moreover, if (u_1, p_1) and (u_2, p_2) are two such solutions, then p_1 equals p_2 up to a constant and u_1 equals u_2 up to an affine linear vector field of the form

$\alpha\omega + \beta\omega \wedge x + (\gamma x_1, \gamma x_2, -2\gamma x_3)^T$, $\alpha, \beta, \gamma \in \mathbb{R}$, and any solution remains a solution if one adds such terms. For $n = 2$, u_1 equals u_2 up to the linear term $\beta(-x_2, x_1)^T$, $\beta \in \mathbb{R}$.

The so-called *homogeneous* Sobolev spaces $\hat{H}^{k,q}(\mathbb{R}^n)$ in Theorem 1.1 are defined as follows: Let Π_{k-1} denote the space of polynomials of degree $\leq k-1$. Then, using multi-index notation,

$$\hat{H}^{k,q}(\mathbb{R}^n) = \left\{ u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)/\Pi_{k-1} : \partial^\alpha u \in L^q(\mathbb{R}^n) \quad \text{for all } \alpha \in \mathbb{N}_0^n,\ |\alpha| = k \right\}$$

is equipped with the norm $\sum_{|\alpha|=k} \|\partial^\alpha u\|_q$. Note that elements in $\hat{H}^{k,q}(\mathbb{R}^n)$ are equivalence classes of L^1_{loc}-functions being unique only up to polynomials from Π_{k-1}. Since $\hat{H}^{k,q}(\mathbb{R}^n)$ can be considered as a closed subspace of $L^q(\mathbb{R}^n)^N$ for some $N = N(k,n) \in \mathbb{N}$, it is reflexive for every $q \in (1, \infty)$. For more details on these spaces see Chapter II in [**3**]. Notice, however, that the space Π_1^n is not completely contained in the kernel of the operator

$$L = -\nu\Delta - \partial_\theta + \omega\wedge$$

arising in (1.3).

We note that separate L^q-estimates of the terms $\omega \wedge u$ and $\partial_\theta u$ in Theorem 1.1 are not possible unless f satisfies an additional set of compatibility conditions, see Remark 2.3 and Proposition 2.4 below; in particular u or $\omega \wedge u$ are not necessarily L^q-integrable. Furthermore Proposition 2.1 indicates that the main solution operator does not define a classical Calderón–Zygmund integral operator.

The underlying problem of the flow around a rotating obstacle has attracted much attention during the last years. Weak solutions have been considered in [**1**] and [**2**], whereas one of the present authors proved the existence of a unique instationary solution in an L^2-setting using semigroup theory ([**6**] and [**7**]). It is a remarkable fact that the operator $-\nu\Delta u - \partial_\theta u + \omega \wedge u$ does *not* generate an analytic semigroup, but a contractive C^0-semigroup. Several auxiliary linearized equations without the crucial term $\partial_\theta u$ have been considered in [**8**]. An L^2- and an $L^{3/2}$-theory of problem (1.2) have been established in [**4**], where the nonlinear problem is also solved for non-Newtonian, second-order fluids and rigid bodies moving due to gravity. Pointwise decay estimates for the linear and nonlinear case are obtained in [**5**]. For further references on moving bodies in fluids see [**4**] and [**5**].

2. Preliminaries

To find the fundamental solutions of (1.2) and of (1.3) (see also [**6**] and [**7**]), we use the Fourier transform $\mathcal{F} = ^\wedge$, i.e.,

$$\hat{u}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\, u(x)dx.$$

Note that in $\mathcal{S}'(\mathbb{R}^n)$, the space of tempered distributions, $\widehat{\partial_j u} = i\xi_j \hat{u}$ and $\widehat{x_j u} = i\partial\hat{u}/\partial\xi_j$, $1 \le j \le n$. Hence (1.3) is related to the problem

$$\nu s^2 \hat{u} - \partial_\varphi \hat{u} + \omega \wedge \hat{u} = \hat{f} \tag{2.1}$$

where $s = |\xi|$ and $\partial_\varphi = -\xi_2\,\partial/\partial\xi_1 + \xi_1\partial/\partial\xi_2 = (\omega\wedge\xi)\cdot\nabla_\xi$ is the angular derivative in Fourier space when using polar or cylindrical coordinates for $\xi \in \mathbb{R}^2$ or $\xi \in \mathbb{R}^3$, resp. Ignoring for a moment the term $\omega \wedge \hat{u}$ the ordinary differential equation $-\partial_\varphi \hat{u} + \nu s^2 \hat{u} = \hat{f}$ yields the solution

$$\hat{u}(\varphi) = e^{\nu s^2 \varphi}\hat{u}_0 - e^{\nu s^2\varphi}\int_0^\varphi e^{-\nu s^2 t}\hat{f}(t)dt, \quad \hat{u}_0 \in \mathbb{R}^n, \tag{2.2}$$

when omitting in $\hat{u}$, $\hat{f}$ the variables $s = |\xi|$ or $s' = (\xi_1^2 + \xi_2^2)^{1/2}, \xi_3$, resp. Due to the 2π-periodicity of $\hat{u}$ w.r.t. φ the unknown $\hat{u}_0$ is given by

$$\hat{u}_0 = \left(1 - e^{-2\pi\nu s^2}\right)^{-1}\int_0^{2\pi} e^{-\nu s^2 t}\hat{f}(t)dt.$$

Using for $s \neq 0$ the geometric series expansion of $\left(1 - e^{-2\pi\nu s^2}\right)^{-1}$ and the 2π-periodicity of $\hat{f}$ w.r.t. t we get $\hat{u}_0 = \int_0^\infty e^{-\nu s^2 t}\hat{f}(t)dt$. Then (2.2) yields

$$\hat{u}(\varphi) = \int_0^\infty e^{-\nu s^2 t}\hat{f}(\varphi + t)dt. \tag{2.3}$$

Let $O(t)$ denote the orthogonal matrix

$$O(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad O(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

describing the rotation around the ξ_3-axis or in the plane by the angle t, resp. Thus, in the variable ξ,

$$\hat{u}(\xi) = \int_0^\infty e^{-\nu s^2 t}\hat{f}\big(O(t)\xi\big)dt$$

is the solution of (2.1) when $\omega \wedge u$ has been ignored. To deal with the term $\omega \wedge u$ note that $\partial_\varphi O(\varphi) = \omega \wedge O(\varphi)$ in the sense of linear maps. Applying $O(\varphi)^T$ to (2.1) the unknown $\hat{v}(\varphi) = O(\varphi)^T \hat{u}(\varphi)$ will satisfy the ordinary differential equation $\nu s^2 \hat{v}(\varphi) - \partial_\varphi \hat{v}(\varphi) = O(\varphi)^T\,\hat{f}(\varphi)$. Hence by (2.3) $\hat{v}(\varphi) = \int_0^\infty e^{-\nu s^2 t}\,O(\varphi + t)^T\,\hat{f}(\varphi + t)dt$ and consequently

$$\hat{u}(\xi) = \int_0^\infty e^{-\nu s^2 t}\,O(t)^T \hat{f}(O(t)\xi)dt. \tag{2.4}$$

Since $e^{-\nu|\xi|^2 t}$ multiplied by $(2\pi)^{-n/2}$ is the Fourier transform of the heat kernel

$$E_t(x) = \frac{1}{(4\pi\nu t)^{n/2}}\,e^{-\frac{|x|^2}{4\nu t}}$$

and since $\widehat{f(O(t)x)} = \hat{f}(O(t)\xi)$, (2.4) yields the formal solution

$$u(x) = \int_0^\infty O(t)^T E_t * f(O(t)\cdot)(x)dt \tag{2.5}$$

of (1.3).

Note that for $n = 3$ and $f \in \mathcal{S}(\mathbb{R}^3)^3$, the integrals (2.4) and (2.5) do in fact converge absolutely and define a distributional solution $u \in \mathcal{S}'(\mathbb{R}^3)^3$ of (1.3).

However, if $n = 2$, then both integrals fail to converge in $\mathcal{S}'(\mathbb{R}^2)^2$, even when $f \in \mathcal{S}(\mathbb{R}^2)^2$. This is not surprising, in view of a similar phenomenon for the Poisson equation in dimension 2. In this case, we need to modify (2.4), by defining a solution $u \in \mathcal{S}'(\mathbb{R}^2)^2$ e.g., by means of the convergent integral

$$\begin{aligned}\langle u, \varphi\rangle &= \langle \hat{u}, \check{\varphi}\rangle \\ &= \int_{|\xi|\geq 1}\int_0^\infty e^{-\nu s^2 t}\, O(t)^T \hat{f}(O(t)\xi)\cdot\check{\varphi}(\xi)\, dt\, d\xi \\ &\quad + \int_{|\xi|<1}\int_0^\infty e^{-\nu s^2 t}\, O(t)^T \hat{f}(O(t)\xi)\cdot(\check{\varphi}(\xi)-\check{\varphi}(0))\, dt\, d\xi\end{aligned}$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^2)^2$; here $\check{}$ denotes the inverse Fourier transform.

Then, in both dimensions $n = 2, 3$, for $f \in \mathcal{S}(\mathbb{R}^n)^n$, we have constructed a solution $u \in \mathcal{S}'(\mathbb{R}^n)^n$ of (1.3). Moreover, in the next section we shall prove that u satisfies inequality (1.4) in Theorem 1.1(1). In particular, $||\nabla^2 u||_q \leq c||f||_q < \infty$ for $1 < q < \infty$, yielding $u \in L^1_{\text{loc}}(\mathbb{R}^n)^n$. We will conclude that, for any $f \in L^q(\mathbb{R}^n)^n$, there is a solution $u \in L^1_{\text{loc}}(\mathbb{R}^n)^n$ of (1.3) satisfying (1.4).

To this end, consider the sequence of balls $B_m(0) \subset \mathbb{R}^n$ and choose a sequence $\{f_j\} \subset \mathcal{S}(\mathbb{R}^n)^n$ converging to f in $L^q(\mathbb{R}^n)^n$. Let u_j be the solution of (1.3) corresponding to f_j. The proof of completeness of $\hat{H}^{2,q}(\mathbb{R}^n)$ in [**3**] reveals that we can find a sequence of polynomials $\{r_j\} \subset \Pi_1^n$ and $\widetilde{u} \in L^1_{\text{loc}}(\mathbb{R}^n)^n$ such that for $j \to \infty$

$$||\nabla^2\left((u_j + r_j) - \widetilde{u}\right)||_q \to 0$$

and

$$(u_j + r_j)|_{B_m} \to \widetilde{u}|_{B_m} \text{ in } L^q(B_m)^n \quad \text{for all} \quad m \in \mathbb{N}. \tag{2.6}$$

Then (2.6) implies that $Lu_j + Lr_j \to L\widetilde{u}$ in the sense of distributions, which shows that $Lr_j \to L\widetilde{u} - f$ in $\mathcal{D}'(\mathbb{R}^n)^n$. And, since $L\,\Pi_1^n$ is closed, as a linear subspace of the finite-dimensional space Π_1^n, we see that $L\widetilde{u} - f = Lr$, for some $r \in \Pi_1^n$. Thus, if we put $u = \widetilde{u} - r$, then $u \in L^1_{\text{loc}}(\mathbb{R}^n)^n$ and $||\nabla^2 u||_q \leq c||f||_q$, so that u satisfies (1.4).

Observe next that formula (2.5) may be rewritten by using

$$E_t * f(O(t)\cdot)(x) = (E_t * f)(O(t)x),$$

the proof of which is based on the radial symmetry of $E_t(\cdot)$.

For $n = 3$ we arrive at the identity

$$u(x) = \int_{\mathbb{R}^3} \Gamma(x,y) f(y) dy \tag{2.7}$$

with the fundamental solution

$$\Gamma(x,y) = \int_0^\infty O(t)^T E_t(O(t)x - y) dt. \tag{2.8}$$

Furthermore $\Delta u(x)$ can be represented — as $u(x)$ in (2.7) — with the help of the kernel

(2.9)

$$\begin{aligned} &K(x,y) \\ &= \Delta_x \Gamma(x,y) \\ &= \int_0^\infty \Delta_x O(t)^T E_t(O(t)x - y) dt \\ &= \int_0^\infty O(t)^T \frac{1}{(4\pi\nu t)^{n/2}} \left(-\frac{n}{2\nu t} + \frac{|O(t)x - y|^2}{(2\nu t)^2} \right) \exp\left(\frac{-|O(t)x - y|^2}{4\nu t} \right) dt, \end{aligned}$$

for $n = 2$ or $n = 3$, cf. (3.4) below.

The following proposition indicates that $K(x,y) = \Delta_x \Gamma(x,y)$ does *not* define a classical Calderón–Zygmund integral operator:

Proposition 2.1.

(1) *Let $n = 3$. Then, for $|x|$, $|y| \to \infty$, the fundamental solution $\Gamma(x,y)$ is not bounded by $C|x - y|^{-1}$. Actually there exists an $\alpha > 0$ such that for suitable x, $y \in \mathbb{R}^3$ with $|x|$, $|y| \to \infty$*

$$|\Gamma(x,y)| \geq \alpha \, \frac{\log|x - y|}{|x - y|}.$$

(2) *Let $n = 2$ or $n = 3$. Then there exists an $\alpha > 0$ and suitable x, $y \in \mathbb{R}^n$ with $|x|$, $|y| \to \infty$ such that the kernel*

$$K_1(x,y) = \int_0^\infty t^{-n/2} \, \frac{1}{t} \, e^{-|O(t)x - y|^2/t} dt$$

satisfies the estimate

$$K_1(x,y) \geq \frac{\alpha}{|x - y|}.$$

The same result holds for the kernel $K_2(x,y)$ where the term $\frac{1}{t}$ in the definition of K_1 is replaced by $|O(t)x - y|^2/t^2$, cf. (2.9).

Proof. (1) Considering only the component $\Gamma_{3,3}(x,y)$ and points $x,y \in \mathbb{R}^3$ with equal third component $x_3 = y_3$ and of equal norm $r = |x| = |y|$ we use complex notation. Thus we may omit the third component of x, y and we restrict ourselves to complex numbers $x = r$ and $y = re^{i\theta}$, $0 < \theta < \pi$, yielding

$$|O(t)x - y| = r|e^{it} - e^{i\theta}| = 2r\left|\sin\frac{\theta - t}{2}\right|$$

and $|x-y| = 2r|\sin\frac{\theta}{2}|$. Now $\Gamma_{3,3}(x,y)$ is bounded from below by $\sum_{k=0}^{N} I_k(r,\theta)$, where $N = [2r^2\sin^2\frac{\theta}{2}]$ and

$$I_k(r,\theta) = \int_{\theta/2+2k\pi}^{3\theta/2+2k\pi} \frac{1}{(4\pi\nu t)^{3/2}} \exp\left(-r^2\sin^2\left|\frac{\theta-t}{2}\right| \Big/ (\nu t)\right) dt.$$

We find constants $\alpha_j > 0$ independent of r, θ and of k such that for $k \geq 1$

$$\begin{aligned} I_k(r,\theta) &\geq \frac{\alpha_1}{k^{3/2}} \int_{-\theta/2}^{\theta/2} \exp\big(-\alpha_2 r^2 t^2/k\big)dt \\ &= \frac{2\alpha_1}{rk} \int_0^{r\theta/(2\sqrt{k})} \exp\big(-\alpha_2 s^2\big)ds. \end{aligned}$$

For $1 \leq k \leq N \sim r^2\theta^2$ and $r\theta \gg 1$, we find $\alpha_3 > 0$ such that $I_k(r,\theta) \geq \frac{\alpha_3}{rk}$. Summing up we are led to the inequality

$$\Gamma_{3,3}(x,y) \geq \sum_{k=1}^{N} I_k(r,\theta) \geq \alpha_3 \sum_{k=1}^{N} \frac{1}{rk} \geq \alpha_4 \frac{\log(r\theta)}{r}$$

with a constant $\alpha_4 > 0$ independent of r and of θ when $r\theta \gg 1$.

(2) Again we use complex notation and consider points $x = r$, $y = re^{i\theta}$, $0 < \theta < \pi$, where now $r^2\theta \gg 1$. Then $K_1(x,y)$ is bounded from below by

$$\begin{aligned} &\int_{\theta-\sqrt{\theta}/r}^{\theta+\sqrt{\theta}/r} t^{-n/2} \exp\left(-4r^2\sin^2\left|\frac{\theta-t}{2}\right| \Big/ t\right) \frac{dt}{t} \\ &\geq \frac{\alpha_1}{\theta^{1+n/2}} \int_0^{\sqrt{\theta}/r} \exp\big(-\alpha_2 r^2 t^2/\theta\big)dt \\ &\geq \frac{\alpha_1}{r\theta^{1/2+n/2}} \int_0^1 e^{-\alpha_2 s^2} ds. \end{aligned}$$

Hence $K_1(x,y) \geq \frac{\alpha_3}{\theta^{n/2-1/2}|x-y|}$. The kernel $K_2(x,y)$ can be estimated analogously. □

Before proving Theorem 1.1 in Section 3 below we consider the much simpler case $q = 2$, the question of separate estimates for u_θ and $\omega \wedge u$ and a variation of (2.10) when the integrals w.r.t. t extend from 2π to ∞.

Proposition 2.2. *Given* $f \in L^2(\mathbb{R}^n)^n$, $n = 2$ *or* $n = 3$, *the solution* u *of* (1.3) *given by* (2.5) *satisfies the estimate*

$$\|\nabla^2 u\|_2 + \|(\omega \wedge x) \cdot \nabla u - \omega \wedge u\|_2 \le c\|f\|_2. \tag{2.10}$$

Proof. By Plancherel's theorem, Fubini's theorem and the inequality of Cauchy–Schwarz (with $s = |\xi|$)

$$\begin{aligned}
\|\Delta u\|_2^2 &= \int_{\mathbb{R}^n} s^4 \left| \int_0^\infty e^{-\nu s^2 t}\, O(t)^T \hat{f}(O(t)\xi)\, dt \right|^2 d\xi \\
&\le \int_{\mathbb{R}^n} \left(\int_0^\infty s^2 e^{-\nu s^2 t}\, dt \right) \cdot \left(\int_0^\infty s^2 e^{-\nu s^2 t} |\hat{f}(O(t)\xi)|^2 dt \right) d\xi \\
&= \frac{1}{\nu} \int_0^\infty \left(\int_{\mathbb{R}^n} s^2 e^{-\nu s^2 t} |\hat{f}(O(t)\xi)|^2\, d\xi \right) dt \\
&= \frac{1}{\nu} \int_0^\infty \left(\int_{\mathbb{R}^n} s^2 e^{-\nu s^2 t} |\hat{f}(\xi)|^2\, d\xi \right) dt \\
&= \frac{1}{\nu^2}\, \|f\|_2^2.
\end{aligned}$$

Furthermore, for any second order partial derivative

$$\|\partial_j \partial_k u\|_2 = \|\xi_j \xi_k \hat{u}\|_2 \le \||\xi|^2 \hat{u}\|_2 = \|\Delta u\|_2 \le \frac{1}{\nu}\|f\|_2.$$

□

Remark 2.3. Inequality (2.10) cannot be improved in the sense that both $\|\omega \wedge u\|_2$ and $\|(\omega \wedge x) \cdot \nabla u\|_2$ are finite or can even be estimated by $\|f\|_2$. In the two-dimensional case let

$$u(x) = u(r, \theta) = a(r)\frac{1}{r}\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = a(r)\frac{1}{r^2}x^\perp$$

where $x^\perp$ is obtained from x by rotation with the angle $\frac{\pi}{2}$ and $a \in C^\infty(\overline{\mathbb{R}_+})$ satisfies $a = 1$ for large r and $a = 0$ for $r \in [0, 1)$. Obviously $u \in C^\infty(\mathbb{R}^2)^2$ is solenoidal, $|\nabla^2 u(x)| \sim \frac{1}{r^3}$ for large r yielding $\nabla^2 u \in L^2(\mathbb{R}^2)^4$, $\operatorname{supp} \Delta u \subset \operatorname{supp} a$ and $\omega \wedge u = \frac{a(r)}{r}\binom{-\cos\theta}{-\sin\theta} = u_\theta$. Consequently $\omega \wedge u - u_\theta \equiv 0$ and the right-hand side $f = -\nu\Delta u \in L^2(\mathbb{R}^2)^2$, but $|\omega \wedge u| \sim \frac{1}{r} \notin L^2(\mathbb{R}^2)$. An analogous result holds in L^q-spaces, $q \neq 2$, when choosing $u(x) = a(r)r^{-\lambda}\, x^\perp$ for suitable $\lambda > 0$.

Proposition 2.4. *Let* $f \in L^q(\mathbb{R}^2)^2$ *satisfy the compatibility conditions*

$$f_m(r) := \frac{1}{2\pi}\int_0^{2\pi} O(\theta)^T f(r, \theta)\, d\theta = 0 \quad \textit{for a.a. } \ r > 0. \tag{2.11}$$

Then one can find a suitable representative u *of the unique solution in* $\hat{H}^{2,q}(\mathbb{R}^2)^2$ *of* (1.3) *given by Theorem* 1.1, *satisfying the estimate*

$$\|\nabla^2 u\|_q + \|\partial_\theta u\|_q + \|u\|_q \le c\|f\|_q.$$

An analogous result holds for $n = 3$ *where* (2.11) *is replaced by the assumption* $\frac{1}{2\pi}\int_0^{2\pi} O(\theta)^T f(r,\theta,x_3)\,d\theta = 0$ *for a.a.* $r = \sqrt{x_1^2 + x_2^2} > 0$, $x_3 \in \mathbb{R}$.

Proof. The main idea is to show that the integral mean

$$u_m(r) = \frac{1}{2\pi}\int_0^{2\pi} O(\theta)^T u(r,\theta)\,d\theta$$

vanishes for a.a. $r > 0$, for a suitable representative u; for $n = 3$ the integral mean $u_m(r, x_3)$ is defined analogously. Then the identity $O(\theta)\partial_\theta(O(\theta)^T u) = \partial_\theta u - \omega \wedge u$ and Wirtinger's inequality will imply that

$$\begin{aligned}\|u\|_q^q &= \int_0^\infty r \int_0^{2\pi} |O(\theta)^T u(r,\theta)|^q \,d\theta dr \\ &\le c\|\partial_\theta(O(\theta)^T u)\|_q^q \le c\|\partial_\theta u - \omega \wedge u\|_q^q,\end{aligned}$$

and Theorem 1.1(1) will complete the proof for $n = 2$ and also for $n = 3$.

In order to prove that $u_m(r) \equiv 0$ notice that, for $n = 2$, $\widetilde{u}(x) = O(\theta)u_m(r)$ satisfies (1.3) with f replaced by $f = 0$ since

$$L(\widetilde{u}) = L(O(\theta)u_m(r)) = O(\theta)(Lu)_m(r) = O(\theta)f_m(r) = 0.$$

Furthermore, since $\widetilde{u} \in \mathcal{S}'(\mathbb{R}^2)^2$, the proof of Theorem 1.1(2), see Section 3 below, implies that $\widetilde{u} \in \Pi_1^2$. Replacing u by $u - \widetilde{u}$, we may then assume that $u_m = 0$. This argument easily extends to the case $n = 3$. □

Remark 2.5. The difficulties in the proof of Theorem 1.1 when estimating Δu with u given by (2.5) arise from the corresponding integrals on $(0, \varepsilon)$, $\varepsilon > 0$. Actually, consider the operator S on $L^q(\mathbb{R}^n)$ given by

$$Sf(x) = \int_{2\pi}^\infty (-\Delta)O(t)^T E_t * f(O(t)\cdot)(x)dt,$$

i.e., in Fourier space

$$\widehat{Sf}(\xi) = \int_{2\pi}^\infty s^2 e^{-\nu s^2 t} O(t)^T \hat{f}(O(t)\xi)dt, \quad s = |\xi|.$$

Since $O(t)$ is 2π-periodic and $s^2\sum_{k=1}^\infty e^{-2k\pi\nu s^2} = s^2 e^{-2\pi\nu s^2}(1-e^{-2\pi\nu s^2})^{-1} =: m(\xi)$, we get that

$$\begin{aligned}\widehat{Sf}(\xi) &= m(\xi)\int_0^{2\pi} e^{-\nu s^2 t} O(t)^T \hat{f}(O(t)\xi)\,dt \\ &= m(\xi)\mathcal{F}\left(\int_0^{2\pi} O(t)^T E_t * f(O(t)\cdot)(x)dt\right).\end{aligned}$$

Obviously $m(\xi)$ satisfies the classical Michlin–Hörmander multiplier condition, cf. [**9**], and due to properties of the heat kernel

$$\left\|\int_0^{2\pi} O(t)^T E_t * f(O(t)\cdot)(x)dt\right\|_q \le \int_0^{2\pi} \|f(O(t)\cdot)\|_q\,dt = 2\pi\|f\|_q.$$

Then multiplier theory yields the estimate $\|Sf\|_q \leq c\|f\|_q$ for every $q \in (1,\infty)$ with a constant $c = c(m,q)$.

3. Proof of Theorem 1.1

Due to the well-known estimate $\|\partial_j \partial_k u\|_q \leq c\|\Delta u\|_q$, $1 < q < \infty$, $1 \leq j,k \leq n$, cf. [**9**], it suffices to consider only Δu. The main ideas are Littlewood–Paley theory and a decomposition of the integral operator

$$Tf(x) = \int_0^\infty (-\Delta) O(t)^T (E_t * f)(O(t)x)dt = \int_{\mathbb{R}^n} K(x,y)f(y)dy \tag{3.1}$$

in Fourier space where each integral kernel has compact support. Since

$$\mathcal{F}\big(-\Delta O(t)^T (E_t * f)(O(t)\cdot)\big)(\xi) = O(t)^T |\xi|^2 e^{-\nu|\xi|^2 t} \hat{f}(O(t)\xi)$$

define $\psi \in \mathcal{S}(\mathbb{R}^n)$ by

$$\hat{\psi}(\xi) = (2\pi)^{-n/2} |\xi|^2 e^{-\nu|\xi|^2} = \widehat{(-\Delta)E_1} \tag{3.2}$$

and

$$\psi_t(x) = t^{-n/2}\psi\left(\frac{x}{\sqrt{t}}\right), \quad \hat{\psi}_t(\xi) = \hat{\psi}(\sqrt{t}\xi) = (2\pi)^{-n/2} t|\xi|^2 e^{-\nu t|\xi|^2}. \tag{3.3}$$

Thus the kernel $K(x,y)$ may be written in the form

$$K(x,y) = \int_0^\infty O(t)^T\, \psi_t(O(t)x - y)\, \frac{dt}{t}. \tag{3.4}$$

To decompose $\hat{\psi}_t$ choose $\widetilde{\varphi}$, $\widetilde{\chi} \in C_0^\infty(\frac{1}{2}, 2)$ such that $0 \leq \widetilde{\varphi}, \widetilde{\chi} \leq 1$ and

$$\sum_{j=-\infty}^\infty \widetilde{\chi}(2^{-j}r) = 1, \quad \int_0^\infty \widetilde{\varphi}(sr)^2 \frac{ds}{s} = \frac{1}{2} \quad \text{for all } r > 0.$$

Then define for $\xi \in \mathbb{R}^n$ and for $j \in \mathbb{Z}$, $s > 0$

$$\hat{\chi}_j(\xi) = \widetilde{\chi}(2^{-j}|\xi|), \quad \hat{\varphi}_s(\xi) = \widetilde{\varphi}(\sqrt{s}|\xi|)$$

yielding

$$\begin{aligned} &\operatorname{supp} \hat{\chi}_j \subset A(2^{j-1}, 2^{j+1}) := \{\xi \in \mathbb{R}^n : 2^{j-1} < |\xi| < 2^{j+1}\}, \\ &\operatorname{supp} \hat{\varphi}_s \subset A\left(\frac{1}{2\sqrt{s}}, \frac{2}{\sqrt{s}}\right); \end{aligned} \tag{3.5}$$

moreover $\int_{\mathbb{R}^n} \varphi_s(x)dx = 0$ and

$$\sum_{j=-\infty}^\infty \hat{\chi}_j(\xi) = 1, \quad \int_0^\infty \hat{\varphi}_s(\xi)^2\, \frac{ds}{s} = 1 \quad (\xi \neq 0). \tag{3.6}$$

The family of functions $\{\varphi_s : s > 0\}$ will be used in Littlewood–Paley theory, see I§8.23 in [**10**], yielding the inequalities

$$c_1\|f\|_q \le \left\| \left(\int_0^\infty |\varphi_s * f(\cdot)|^2 \frac{ds}{s} \right)^{1/2} \right\|_q \le c_2\|f\|_q \tag{3.7}$$

with constants $c_1, c_2 > 0$ depending on $q \in (1,\infty)$, but independent of $f \in L^q(\mathbb{R}^n)^n$. Furthermore we decompose K by defining $\psi^j \in \mathcal{S}(\mathbb{R}^n)$ by

$$\psi^j = (2\pi)^{-n/2}\chi_j * \psi \quad \text{or equivalently} \quad \hat{\psi}^j = \hat{\chi}_j \cdot \hat{\psi}, \quad j \in \mathbb{Z}, \tag{3.8}$$

yielding $\psi = \sum_{j=-\infty}^{\infty} \psi_j$ and, cf. (3.4),

$$K_j(x,y) = \int_0^\infty O(t)^T\, \psi_t^j(O(t)x - y)\, \frac{dt}{t}, \quad j \in \mathbb{Z}. \tag{3.9}$$

Given K_j we define the operator

$$T_j f(x) = \int_{\mathbb{R}^n} K_j(x,y)\, f(y) dy = \int_0^\infty O(t)^T\, (\psi_t^j * f)(O(t)x) \frac{dt}{t} \tag{3.10}$$

such that formally and even w.r.t to the operator norm topology $T = \sum_{j=-\infty}^{\infty} T_j$, see the proof below.

Lemma 3.1. *The functions ψ_t^j have the following properties:*

(1) *For $j \in \mathbb{Z}$ and $t > 0$*

$$\operatorname{supp} \hat{\psi}_t^j \subset A\left(\frac{2^{j-1}}{\sqrt{t}}, \frac{2^{j+1}}{\sqrt{t}} \right).$$

(2) *For $m > \frac{n}{2}$ let $h(x) = (1+|x|^2)^{-m}$ and, cf. (3.3), $h_t(x) = t^{-n/2}\, h\left(\frac{x}{\sqrt{t}}\right)$. Then there exists a constant $c > 0$ independent of $j \in \mathbb{Z}$ such that*

$$|\psi^j(x)| \le c\, 2^{-2|j|} h_{2^{-2j}}(x) \quad \textit{for all } x \in \mathbb{R}^n.$$

In particular

$$\|\psi^j\|_1 \le c\, 2^{-2|j|}.$$

Proof. (1) is obvious due to (3.3), (3.5) and (3.8). To prove (2) we show first of all the pointwise estimate

$$|2^{j|\alpha|} \partial^\alpha \hat{\psi}^j(\xi)| \le c_\alpha\, 2^{-2|j|} \eta(2^{-j}|\xi|) \tag{3.11}$$

for all $\xi \in \mathbb{R}^n$, $j \in \mathbb{Z}$, for all multi-indices $\alpha \in \mathbb{N}_0^n$ and with a function $\eta \in C_0^\infty\left(\frac{1}{4}, 4\right)$, $0 \le \eta \le 1$. By the definition of $\hat{\chi}_j$, (3.5) and the pointwise estimates

$$|\partial^\beta \hat{\psi}(\xi)| \le c_{\beta,N} \begin{cases} |\xi|^{\max(0,2-|\beta|)} & , \ |\xi| < 1 \\ |\xi|^{-N} & , \ |\xi| \ge 1 \end{cases} , \quad \beta \in \mathbb{N}_0^n,$$

for every $N \in \mathbb{N}$, cf. (3.2), Leibniz's formula yields the estimate

$$|2^{j|\alpha|}\partial^\alpha \hat{\psi}^j(\xi)| \le c \sum_{0\le\beta\le\alpha} 2^{j|a|}|\partial^{\alpha-\beta}\widetilde{\chi}(2^{-j}|\xi|)|\,|\partial^\beta\hat{\psi}(\xi)|$$
$$\le c \sum_{0\le\beta\le\alpha} 2^{j|\beta|}\eta(2^{-j}|\xi|)\,|\partial^\beta\hat{\psi}(\xi)|.$$

For $j \ge 0$ where only $|\xi| \sim 2^j$ has to be considered, we get (3.11) immediately, even with $2^{-N|j|}$ replacing $2^{-2|j|}$. For $j < 0$ and $|\xi| \sim 2^j < 1$ the right-hand side of the last inequality is bounded by

$$c \sum_{0\le\beta\le\alpha} \eta(2^{-j}|\xi|)\, 2^{j\ \max(|\beta|,2)} \le c\, 2^{-2|j|}\eta(2^{-j|\xi|}).$$

Now (3.11) is proved.

To estimate $\psi^j(x)$ we use for $m > \frac{n}{2}$ the identity

$$(1+|2^jx|^2)^m\, \psi^j(x) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}(1-2^{2j}\Delta)^m\,\hat{\psi}_j(\xi)\, e^{ix\cdot\xi}\, d\xi.$$

By (3.11)

$$|(1-2^{2j}\Delta)^m\,\hat{\psi}^j(\xi)| \le C_{m,N}\, 2^{-2|j|}\eta(2^{-j}|\xi|)$$

for all $j \in \mathbb{Z}$ and $\xi \in \mathbb{R}^n$. Hence

$$\|(1-2^{2j}\Delta)^m\,\hat{\psi}^j\|_1 \le C_m\, 2^{nj-2|j|}$$

and consequently $|(1+|2^jx|^2)^m\,\psi^j(x)| \le c\, 2^{nj-2|j|}$ proving Part (2). □

Lemma 3.2. *For $j \in \mathbb{Z}$ let $\mathcal{M}^j$ denote the maximal operator*

$$\mathcal{M}^j g(x) = \sup_{r>0}\int_{A_r}(|\psi_t^j| * |g|)(O(t)^T x)\,\frac{dt}{t}$$

where $A_r = \left[\frac{r}{16}, 16r\right]$. Then for $q \in (2,\infty)$ the operator T_j satisfies the estimate

$$\|T_jf\|_q \le c\,\|\psi^j\|_1^{1/2}\,|||\mathcal{M}^j|||_{(q/2)'}^{1/2}\,\|f\|_q$$

with a constant $c > 0$ independent of $j \in \mathbb{Z}$. The term $|||\mathcal{M}^j|||_{(q/2)'}$ denotes the operator norm of the sublinear operator $\mathcal{M}^j$ on $L^{(q/2)'}(\mathbb{R}^n)$, where $\frac{1}{(q/2)'} + \frac{1}{q/2} = 1$.

Proof. To estimate $\|T_jf\|_q$ we use the Littlewood–Paley decomposition (3.7) of T_jf and find a function $0 \le g \in L^{(q/2)'}(\mathbb{R}^n)$ with $\|g\|_{(q/2)'} = 1$ (note that $q > 2$) such that

$$\|T_jf\|_q^2 \le \frac{1}{c_1^2}\Big\|\int_0^\infty |\varphi_s * T_jf(\cdot)|^2\,\frac{ds}{s}\Big\|_{q/2}$$
$$= \frac{1}{c_1^2}\int_0^\infty\int_{\mathbb{R}^n}|\varphi_s * T_jf|^2 g\,dx\,\frac{ds}{s}.$$

By (3.9), (3.10)

$$\varphi_s * T_j f(x) = \int_0^\infty O(t)^T (\varphi_s * \psi_t^j * f)(O(t)x) \, \frac{dt}{t},$$

where due to (3.5) $\varphi_s * \psi_t^j = 0$ unless $t \in A(s,j) := [2^{2j-4}s, \, 2^{2j+4}s]$. Since $\int_{t \in A(s,j)} \frac{dt}{t} = \log 2^8$ for every $j \in \mathbb{Z}$, $s > 0$, the inequality of Cauchy–Schwarz and the associativity of convolutions yield

$$\begin{aligned} |\varphi_s * T_j f(x)|^2 &\le c \int_{A(s,j)} |\big(\psi_t^j * (\varphi_s * f)\big)(O(t)x)|^2 \, \frac{dt}{t} \\ &\le c \, \|\psi^j\|_1 \int_{A(s,j)} (|\psi_t^j| * |\varphi_s * f|^2)(O(t)x) \, \frac{dt}{t}. \end{aligned}$$

Here we used the inequality

$$|\big(\psi_t^j * (\varphi_s * f)\big)(y)|^2 \le \|\psi_t^j\|_1 (|\psi_t^j| * |\varphi_s * f|^2)(y)$$

and that $\|\psi_t^j\|_1 = \|\psi^j\|$ for all $t > 0$. Thus

$$\|T_j f\|_q^2 \le c \|\psi^j\|_1 \int_0^\infty \int_{A(s,j)} \int_{\mathbb{R}^n} (|\psi_t^j| * |\varphi_s * f|^2)(x) \, g(O(-t)x) dx \, \frac{dt}{t} \, \frac{ds}{s}.$$

In the inner integral on $\mathbb{R}^n$ note that $\phi = |\psi_t^j|$ is radially symmetric; thus for arbitrary functions f and h we get $\int (\phi * f) h \, dx = \int f \, \phi * h \, dx$. Then the elementary identity $\phi * [g(O(-t)\cdot)] = (\phi * g)(O(-t)\cdot)$ implies that

$$\|T_j f\|_q^2 \le c \, \|\psi^j\|_1 \int_{\mathbb{R}^n} \int_0^\infty |\varphi_s * f|^2(x) \int_{A(s,j)} (|\psi_t^j| * g)(O(-t)x) \, \frac{dt}{t} \, \frac{ds}{s} \, dx.$$

Here the inner integral on $A(s,j)$ is bounded by $\mathcal{M}^j g(x)$ uniformly in $s > 0$. Now Hölder's inequality and (3.7) show that

$$\begin{aligned} \|T_j f\|_q^2 &\le c \, \|\psi^j\|_1 \left(\int_{\mathbb{R}^n} \left(\int_0 |\varphi_s * f|^2 \, \frac{ds}{s} \right)^{q/2} dx \right)^{2/q} \|\mathcal{M}^j g\|_{(q/2)'} \\ &\le c c_2 \|\psi^j\|_1 \, \|f\|_q^2 \, |||\mathcal{M}^j|||_{(q/2)'} \, \|g\|_{(q/2)'}. \end{aligned}$$

Since $\|g\|_{(q/2)'} = 1$, the proof is complete. □

Lemma 3.3. *Let $\mathcal{M}$ denote the classical Hardy–Littlewood maximal operator on $\mathbb{R}^n$, i.e.,*

$$\mathcal{M} g(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |g(y)| \, dy,$$

and let $\widetilde{\mathcal{M}}_\theta g$ denote the "angular" maximal operator

$$\widetilde{\mathcal{M}}_\theta g(x) = \sup_{r>0} \int_{A_r} |g(O(t)^T x)| \frac{dt}{t},$$

where $A_r = [\frac{r}{16}, 16r]$. *Then* $\mathcal{M}^j$ *in Lemma* 3.2 *satisfies the estimates*

$$\begin{aligned} \mathcal{M}^j g(x) &\le c\, 2^{-2|j|}\, \mathcal{M}(\widetilde{\mathcal{M}}_\theta g)(x) \qquad &&\text{for a.a. } x \in \mathbb{R}^n, \\ \|\mathcal{M}^j g\|_q &\le c\, 2^{-2|j|}\, \|g\|_q &&\text{for } 1 < q < \infty. \end{aligned}$$

Proof. By Lemma 3.1 (2) $|\psi_t^j(x)| \le c\, 2^{-2|j|} h_{t2^{-2j}}(x)$ and consequently

$$\mathcal{M}^j g(x) \le c\, 2^{-2|j|} \sup_{r>0} \int_{A_r} (h_{t2^{-2j}} * |g|)(O(t)^T x)\, \frac{dt}{t}.$$

There exists a constant $c > 0$ independent of r, j such that $h_{t2^{-2j}} \le c h_{r2^{-2j}}$ for all $t \in A_r$. Hence

$$\begin{aligned} \mathcal{M}^j g(x) &\le c\, 2^{-2|j|} \sup_{r>0} h_{r2^{-2j}} * \int_{A_r} |g|(O(t)^T x)\, \frac{dt}{t} \\ &\le c\, 2^{-2|j|} \sup_{t>0} h_t * \widetilde{\mathcal{M}}_\theta g(x). \end{aligned}$$

Note that h is a nonnegative, radially decreasing function and that $\int h_t\, dx \equiv c_0 > 0$ for all $t > 0$. Therefore we conclude by II§2.1 in [**10**] that

$$\sup_{t>0} h_t * \widetilde{\mathcal{M}}_\theta g(x) \le c_0 \mathcal{M}(\widetilde{\mathcal{M}}_\theta g)(x)$$

proving the first assertion.

For $q \in (1, \infty)$ the maximal operator $\mathcal{M}$ is bounded on $L^q(\mathbb{R}^n)$. Concerning $\widetilde{\mathcal{M}}_\theta$ we consider for given $g \in L^q(\mathbb{R}^n)$ its restriction

$$g_r(\theta) = g(r, \theta) \quad \text{or} \quad g_{r,x_3}(\theta) = g(r, \theta, x_3)$$

for $n = 2$ or $n = 3$, resp., when using polar or cylindrical coordinates. For $n = 2$ $g_r(\theta) \in L^q(0, 2\pi)$ for a.a. $r > 0$ by Fubini's theorem, and with the classical one-dimensional Hardy–Littlewood maximal operator $\mathcal{M}_1$ on $L^q(0, 2\pi)$

$$|\widetilde{\mathcal{M}}_\theta g(r, \theta)| \le c(\mathcal{M}_1 g_r)(\theta) \quad \text{for a.a. } r > 0. \tag{3.12}$$

Thus

$$\|\widetilde{\mathcal{M}}_\theta g\|_q^q \le c \int_0^\infty r \|\mathcal{M}_1 g_r\|_{L^q(0,2\pi)}^q dr \le c \int_0^\infty r \|g_r\|_{L^q(0,2\pi)}^q dr = c\, \|g\|_q^q$$

due to the L^q-boundedness of $\mathcal{M}_1$. For $n = 3$ the proof is analogous. □

End of the proof of Theorem 1.1 (1). Let $q \in (2, \infty)$. Then by Lemmata 3.1-3.3

$$\|T_j f\|_q \le c\, 2^{-|j|} \cdot 2^{-|j|}\, \|f\|_q.$$

Thus $\sum_{j\in\mathbb{Z}} T_j$ converges in the L^q-operator norm and $T = \sum_{j\in\mathbb{Z}} T_j$ is bounded on $L^q(\mathbb{R}^n)^n$ for $q > 2$.

Closely related to T is the operator $T^*f(x) = \int K^*(x,y)f(y)dy$ with kernel

$$K^*(x,y) = \int_0^\infty \psi_t(O(t)y - x)O(t)\,\frac{dt}{t}.$$

Analogous arguments as before show that T^* is bounded on $L^q(\mathbb{R}^n)^n$ for every $q > 2$. Now let $q \in (1,2)$. Then for $f \in L^q(\mathbb{R}^n)^n$, $g \in L^{q'}(\mathbb{R}^n)^n$

$$|\langle Tf, g\rangle| = |\langle f, T^*g\rangle| \le \|f\|_q\, c\|g\|_{q'}$$

implying the L^q-boundedness of T. The case $q = 2$ had been considered in Proposition 2.2. □

Proof of Theorem 1.1(*2*). It suffices to prove that every solution $u \in \mathcal{S}'(\mathbb{R}^3)^3$ of (1.3) when $f = 0$ and $\nabla^2 u \in L^q(\mathbb{R}^3)$ equals a polynomial of the form $\alpha\omega + \beta\omega \wedge x + (\gamma x_1, \gamma x_2, \delta x_3)^T$. Given u define $\hat{v}(s', \varphi, \xi_3) = O(\varphi)^T\,\hat{u}(s', \varphi, \xi_3) \in \mathcal{S}'(\mathbb{R}^3)^3$ using cylindrical coordinates for $\xi \in \mathbb{R}^3$ and $s' = \sqrt{(\xi_1^2 + \xi_2^2)}$. Then, cf. Section 2,

$$\nu|\xi|^2\hat{v} - \partial_\varphi \hat{v} = 0 \quad \text{in } \mathcal{S}'(\mathbb{R}^3)^3.$$

Let us show that $\langle \hat{v}, \psi\rangle = 0$ for all $\psi \in C_0^\infty(\mathbb{R}^3\backslash\{0\})^3$. Given ψ define

$$\psi_0(s', \varphi, \xi_3) = e^{-\nu|\xi|^2\varphi} \int_{-\infty}^{\varphi} e^{\nu|\xi|^2\varphi'}\psi(s', \varphi', \xi_3)\,d\varphi'.$$

Obviously $\psi_0 \in C_0^\infty(\mathbb{R}^3\backslash\{0\})^3$ and $(\nu|\xi|^2 + \partial_\varphi)\psi_0 = \psi$. Consequently

$$\langle \hat{v}, \psi\rangle = \langle \hat{v}, (\nu|\xi|^2 + \partial_\varphi)\psi_0\rangle = \langle (\nu|\xi|^2 - \partial_\varphi)\hat{v}, \psi_0\rangle = 0$$

proving that $\operatorname{supp}\hat{v} \subset \{0\}$ and also $\operatorname{supp}\hat{u} \subset \{0\}$. Hence u is a polynomial. Since $\nabla^2 u \in L^q(\mathbb{R}^3)$, u is even affine linear, $u(x) = a + Bx$ for $a \in \mathbb{R}^3$, $B \in \mathbb{R}^{3,3}$. Then (1.3) with $f = 0$, i.e., $(\omega \wedge x)\cdot\nabla u = \omega \wedge u$, shows that $\omega \wedge a = 0$ or equivalently $a = \alpha\omega$, $\alpha \in \mathbb{R}$. Furthermore Bx must be of the form $Bx = \beta\omega \wedge x + (\gamma x_1, \gamma x_2, \delta x_3)^T$ with constants $\beta, \gamma, \delta \in \mathbb{R}$. For $n = 2$ one easily obtains that $a = 0$ and $Bx = \beta\omega \wedge x + \gamma x$. □

Proof of Theorem 1.1(*3*). As explained in Section 1 problem (1.2) may be reduced to (1.3) by solving the equation

$$\Delta p = \operatorname{div} f + \nu\Delta g + \partial_\theta g = \operatorname{div} F \quad \text{in } \mathbb{R}^n \tag{3.13}$$

where $F = f + \nu\nabla g + (\omega \wedge x)g$ satisfies the estimate $\|F\|_q \le c(\|f\|_q + \|\nu\nabla g + (\omega \wedge x)g\|_q)$. Thus $\operatorname{div} F$ may be considered as a continuous linear functional on $\hat{H}^{1,q'}(\mathbb{R}^n)$. Since the operator Δ is easily seen to be an isomorphism from $\hat{H}^{1,q}(\mathbb{R}^n)$ to its dual $\hat{H}^{1,q'}(\mathbb{R}^n)^*$ there exists a unique $p \in \hat{H}^{1,q}(\mathbb{R}^n)$ solving $\Delta p = \operatorname{div} F$ and satisfying $\|\nabla p\|_q \le c\|F\|_q$. Then Part (1) yields a $u \in \hat{H}^{2,q}(\mathbb{R}^n)^n$ satisfying $-\nu\Delta u - \partial_\theta u + \omega \wedge u = f - \nabla p$ and the estimate $\|\nabla^2 u\|_q + \|\partial_\theta u - \omega \wedge u\|_q \le c(\|f\|_q + \|\nabla p\|_q)$. In particular $(-\nu\Delta - \partial_\theta)\operatorname{div} u = \operatorname{div} f - \Delta p$ and consequently $(-\nu\Delta - \partial_\theta)(\operatorname{div} u - g) = 0$. By the reasoning of Part (2) we may conclude that $\operatorname{div} u - g$ is a polynomial and due to the integrability assumptions even a constant. Replacing u by $u - \gamma(x_1, x_2, 0)^T$, if necessary, we get a solution (u, p) of (1.2) satisfying also $\operatorname{div} u = g$. The uniqueness assertion is proved as in Part (2). □

References

[1] W. Borchers, *Zur Stabilität und Faktorisierungsmethode für die Navier–Stokes-Gleichungen inkompressibler viskoser Flüssigkeiten*, Habilitation Thesis, Univ. of Paderborn, 1992.

[2] Z.M. Chen and T. Miyakawa, *Decay properties of weak solutions to a perturbed Navier–Stokes system in* $\mathbb{R}^n$, Adv. Math. Sci. Appl., **7** (1997), 741–770, MR 1476275 (98k:35147), Zbl 0893.35092.

[3] G.P. Galdi, *An Introduction to the Mathematical Theory of the Navier–Stokes Equations*, Vol. I, *Linearized Steady Problems*, Springer Tracts in Natural Philosophy, **38**, 2nd edition, 1998, MR 1284205 (95i:35216a), Zbl 0949.35004.

[4] ______, *On the motion of a rigid body in a viscous liquid: A mathematical analysis with applications*, in 'Handbook of Mathematical Fluid Mechanics' (S. Friedlander and D. Serre, eds.), Elsevier Science, 653–791, 2002, MR 1942470 (2003j:76024).

[5] ______, *Steady flow of a Navier–Stokes fluid around a rotating obstacle*, J. Elasticity, **71** (2003), 1–31.

[6] T. Hishida, *An existence theorem for the Navier–Stokes flow in the exterior of a rotating obstacle*, Arch. Rational Mech. Anal., **150** (1999), 307–348, MR 1741259 (2001b:76024), Zbl 0949.35106.

[7] ______, *The Stokes operator with rotation effect in exterior domains*, Analysis, **19** (1999), 51–67, MR 1690643 (2000c:35185), Zbl 0938.35114.

[8] Š. Nečasova, *Some remarks on the steady fall of body in Stokes and Oseen flow.* Acad. Sciences Czech Republic Math. Institute Preprint **143**, 2001.

[9] E.M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, NJ, 1970, MR 0290095 (44 #7280), Zbl 0207.13501.

[10] ______, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, Princeton, NJ, 1993, MR 1232192 (95c:42002), Zbl 0821.42001.

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