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In this paper we show that Poisson brackets linked to geometric flows of curves on flat Riemannian manifolds are Poisson reductions of the Kac–Moody bracket of $SO(n)$. The bracket is reduced to submanifolds defined by either the Riemannian or the natural curvatures of the curves. We show that these two cases are (formally) Poisson equivalent and we give explicit conditions on the coefficients of the geometric flow guaranteeing that the induced flow on the curvatures is Hamiltonian.

1. Introduction

The study of infinite dimensional Poisson geometry has traditionally been an important component in the study of completely integrable systems. In fact, the majority of completely integrable systems of PDEs are Hamiltonian with respect to two different but compatible infinite dimensional Hamiltonian structures, that is, they are biHamiltonian. This property allows the generation of a recursion operator that produces an infinite sequence of preserved quantities, effectively integrating the system. Infinite dimensional Poisson geometry is also linked to some analytic problems, in particular to the classification of normal forms of differential operators. These are found using versal deformations of the symplectic foliation (see [**LPa**] and [**M1**]).

The connection between finite dimensional differential geometry and completely integrable PDEs dates back to Liouville, Bianchi and Darboux ([**Li**], [**Bi**] and [**Da**]), but it was after Hasimoto's work in the vortex filament flow evolution that the close relation between integrable PDEs and the evolution of curvature and torsion (rather than the curve flow itself) was clear. In fact, Hasimoto ([**Ha**]) proved that the vortex filament flow induces a completely integrable evolution on the curvature and torsion of the flow. In particular, the evolution of curvature and torsion were biHamiltonian.

Langer and Perline pointed out in their papers on the subject (see [**LP1**] and [**LP2**]) that the Hamiltonian structures that were used to integrate some of these systems were defined directly from the Euclidean geometry of spatial curves. Indeed, the structures found in [**MSW**] for the evolution of Riemannian curves in three dimensions were all defined geometrically with the use of Frenet frames. So were the ones in [**DSa**] and others. The so-called natural frames were also used to integrate systems in [**LP2**]. A direct relationship between the evolution of differential invariants (curvatures) of the evolving curves and infinite dimensional Poisson structures exists not only in Riemannian geometry but also in projective geometry. In fact, KdV and its generalizations can be viewed as the evolution of projective differential invariants of a certain invariant flow of curves in $\mathbb{R}P^n$ ([**DS**]). The Hamiltonian structure used to integrate them can be defined directly from the projective geometry of the curves ([**M2**]).

In this paper we describe the evolution induced on the Riemannian curvatures of curves evolving invariantly on a Riemannian manifold with constant curvature. Our approach using Cartan connections allows us to establish directly the connection between these evolutions and Hamiltonian structures on the dual of the algebra of loops on $o(n)$, $Lo(n)^*$. In fact, we prove that there exists a Poisson structure on the quotient $Lo(n)^*/LSO(n-1)$ obtained through a standard Poisson reduction procedure as described in [**MR**]. The Poisson reduction procedure links directly the geometry of the curves and the quotient $Lo(n)^*/LSO(n-1)$. The reduced structure on $Lo(n)^*/LSO(n-1)$ can be found in the literature (see [**TU**] and references within), although defined from a different point of view. We prove that both Frenet frames and natural frames can be viewed as transverse sections of the foliation induced on $Lo(n)^*$ by the coadjoint action of $LSO(n-1)$. Therefore, there exist two natural Hamiltonian structures defined on the spaces of Frenet curvatures and natural curvatures. They are given through the identification of $Lo(n)^*/LSO(n-1)$ with its sections. The Poisson map (a gauge) that takes one structure to the other is a generalization of the Hasimoto transformation found in [**Ha**] for $n = 3$.

The emphasis of this paper is on the geometric description of these Poisson structures and their precise relationship with invariant evolutions of Riemannian curves, not on the classification and study of integrable systems. For more information about completely integrable systems appearing in this setting, please see [**LP2**] and [**TT**].

In Chapter 2 we have included some background definitions and information on Cartan connections, Riemannian geometry and moving frames, since they are used and some of the readers might not be familiar with it. We have also included basic information about Poisson geometry in Chapter 4. Chapter 3 obtains a general formula for the evolution of Frenet and natural curvatures, evolutions induced by an arc-length preserving invariant flow of curves on a Riemannian manifold with constant curvature. In Chapter 4 we define the relevant Poisson structures via the Poisson reduction method. Using the Poisson reduction method allows us to define explicitly these brackets in both the Frenet and natural cases. We show that both Frenet and natural cases are simply different choices of transverse sections in $Lo(n)^*/LSO(n-1)$. Chapter 5 stablishes the relation, in the flat case, between these reduced brackets and the evolution of curvatures that were found in Chapter 3. In the natural case the relation is far simpler than in the Frenet case. That might explain why the latter is a preferred choice in the treatment of the associated integrable systems.

2. A short introduction to Riemannian manifolds, Cartan connections and moving frames

In this section we will provide the background definitions and results in Differential Geometry that will be used along this paper. Much of it is stated as in [**Sh**] and [**K**].

Definition 1. Let G be a Lie group and let $H \subset G$ be a closed subgroup such that G/H is connected. The pair (G, H) is called a *Klein Geometry.*

Assume that a group G acts transitively on a manifold M, and let $H_p = \{g \in G, \text{such that } g.p = p\}$. The manifold $M = G/H_p$ is called a *homogeneous space.* Examples of homogeneous spaces are Euclidean and projective space and the Möbius sphere.

A *Cartan Geometry* (P, ω) on a manifold M modeled on the Klein geometry (G, H) consists of the following data:

(1) A smooth manifold M;
(2) a principal H-bundle P over M;
(3) a $\mathfrak{g}$-value 1-form ω on P satisfying the following conditions:
   (i) For each point $p \in P$, the linear map $\omega_p : T_pP \to \mathfrak{g}$ is an isomorphism;
   (ii) $(R_h)^*\omega = \mathrm{Ad}(h^{-1})\omega$ for all $h \in H$;
   (iii) $\omega(0, X) = X$ for all $X \in \mathfrak{h}$

where, as usual, R_h denotes the right multiplication map, $h \in H$, Ad represents the Adjoint action of the group, and $(0, X)$ is a trivialization of the element in P associated to $X \in \mathfrak{h}$. The form ω is usually called the *Cartan connection.*

If (M, P, ω) is a Cartan geometry, the $\mathfrak{g}$-valued 2-form on P given by

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega] \tag{2.1}$$

is called *the curvature*. Equation (2.1) is called the *structural equation.*

Several interesting facts are known about Cartan connections and Cartan curvature forms. The most relevant to us can be found in [**Sh**] pp. 187-188:

(1) The curvature form Ω can be regarded as a 2-form on the pullback of the tangent bundle of M to the principal bundle P.
(2) The restriction of the Cartan connection ω to each fiber of the principal bundle coincides with the Maurer–Cartan form on H (if one identifies each fiber with H). This is a direct consequence of Property (iii) above.

(3) Let $\pi : P \to M$ be the projection from P to M. For $x \in M$ and $p \in P$ with $\pi(p) = x$ there exists a canonical isomorphism ω_p

$$\omega_p : T_x M \to \mathfrak{g}/\mathfrak{h}$$

such that $\omega_{ph} = \mathrm{Ad}(h^{-1})\omega_p$ for any $h \in \mathfrak{h}$.

If $\rho : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ is the canonical projection, then $\rho(\Omega)$ is called the *torsion*.

Definition 2. The n-dimensional Euclidean space is a homogeneous space given by $\mathbb{R}^n = \mathrm{Euc}_n(\mathbb{R})/SO_n(\mathbb{R})$. By $\mathrm{Euc}_n(\mathbb{R})$ we denote the Euclidean group defined as

$$\mathrm{Euc}_n(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & 0 \\ v & \vartheta \end{pmatrix} \right\},$$

$\vartheta \in SO_n(\mathbb{R})$, where we are identifying $SO_n(\mathbb{R})$ with its copy inside $\mathrm{Euc}_n(\mathbb{R})$ (that is, $v = 0$). $\mathrm{Euc}_n(\mathbb{R})$ acts on $\mathbb{R}^n$ by multiplication of matrices if we identify $x \in \mathbb{R}^n$ with $\begin{pmatrix} 1 \\ x \end{pmatrix}$. The subgroup $SO_n(\mathbb{R})$ leaves the origin fixed.

Let M be a smooth manifold. A *Euclidean geometry* on (oriented) M is a Cartan geometry on M modeled on Euclidean space $(\mathrm{Euc}_n(\mathbb{R}), SO_n(\mathbb{R}))$. A *Riemannian geometry* on M is a Euclidean geometry with torsion equals zero. We will say M *is a Riemannian manifold*.

Definition 3. Let

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix} \right\} \subset \mathrm{euc}_n(\mathbb{R}).$$

Comment (3) above shows that there exists an isomorphism between $T_x M$ and $\mathfrak{p}$ given by ω_p, depending on a point $p \in P$ with $\pi(p) = x$. Therefore, if $v \in \mathfrak{p}$ then $\omega_p^{-1}(v) \in T_x M$, or rather it belongs to the pullback of $T_x M$ to the principal bundle P. We define the *curvature function* $K : P \to \mathrm{Hom}\,(\bigwedge^2 \mathfrak{p}, \mathfrak{h})$ on P as

$$K(p)(v_1, v_2) = \Omega_p(\omega_p^{-1}(v_1), \omega_p^{-1}(v_2)),$$

for any $v_1, v_2 \in \mathfrak{p}$.

We say that a Riemannian manifold has constant curvature $\varkappa$ if, whenever $\{e_i\}$ are basis of $\mathbb{R}^n$ and e_{ij} are generators of $so(n)$ (same as $so_n(\mathbb{R})$) such that $ad(e_{ij})e_k = \delta_j^k e_i - \delta_i^k e_j$, then

$$K(p) = \varkappa \sum_{i<j} e_i^* \wedge e_j^* \otimes e_{ij}. \tag{2.2}$$

In the case of a Riemannian manifold there are two kinds of invariant frames that will be relevant to this paper. These are the Frenet frame and the natural frame. While the Frenet frame is better known, the natural frame description can be found in [**B**] and can be summarized as follows: Assume we have a curve in $\mathbb{R}^n$. We say that a normal vector field along the curve is relatively parallel if its derivative is tangential to the curve. The following theorem can be found in [**B**] for the case $n = 3$ although the result yields identically for the general case, as the authors point out at the end of the paper.

Theorem 1. *Let γ be a regular C^2 curve in $\mathbb{R}^n$. Then, for any vector V_0 at $\gamma(t_0)$ there is a unique C^1 relatively parallel field V along γ such that $V(t_0) = V_0$. The scalar product of two relatively parallel fields is constant.*

Given a regular Euclidean curve γ, the tangent space along γ is divided into an oriented tangential component and the normal subspace. One can thus chose an orthonormal basis in the normal subspace formed by relatively parallel vector fields. This basis is determined up to a constant matrix in $O(n-1)$ and, together with the tangential vector field, formed a so-called *natural frame*. See [**B**] for more details. The evolution of this frame is given by the equation $F_x = FN$, where F contains the natural frame as columns, and where

$$N = \begin{pmatrix} 0 & -\mathbf{u}^T \\ \mathbf{u} & 0 \end{pmatrix}.$$

The vector $\mathbf{u}^T = (u_1, \dots, u_{n-1})$ is formed by what are known as *natural curvatures*, which are differential invariants for the curve. Notice that $\mathbf{u}$ is unique only up to the action of $O(n-1)$.

If F_N is a natural frame and F is the Frenet frame, then

$$g = F^{-1}F_N = \begin{pmatrix} 1 & 0 \\ 0 & \theta \end{pmatrix} \tag{2.3}$$

satisfies

$$-g'g^{-1} + gNg^{-1} = K$$

where K is as in (3.4) below. From here, if K_2 is such that

$$K = \begin{pmatrix} 0 & -\kappa_1 e_1^T \\ \kappa_1 e_1 & K_2 \end{pmatrix}, \tag{2.4}$$

then θ as in (2.3) satisfies

$$\mathbf{u} = \kappa_1 \theta^T e_1, \qquad \theta' = K_2 \theta \tag{2.5}$$

which determines $\mathbf{u}$ from κ up to a constant matrix in $O(n-1)$. Also, if $\mathbf{u}$ is known, (2.5) gives $\kappa_1 = \|\mathbf{u}\|$, and K_2 can be obtained from the first row of θ (that is $\frac{1}{\kappa_1}\mathbf{u}$) using a process analogous to the construction of a Frenet frame.

Example 1. In the case $n = 3$, let $\{T, N, B\}$ be the Frenet frame, κ and τ the curvature and torsion of the curve, respectively, and let $\{T, M_1, M_2\}$ be a natural frame. Then

$$M_i = \cos\alpha_i N + \sin\alpha_i B$$

where the relatively parallel condition determines $\alpha_i' = -\tau$. That is, up to a rotation, M_1 and M_2 are determined. If we further ask M_1 and M_2 to be orthogonal, we can choose $\alpha_2 = \frac{\pi}{2} - \alpha_1$ and the natural curvatures are then given by $u_1 = -\kappa \cos \alpha_1$, $u_2 = -\kappa \sin \alpha_1$.

3. Invariant evolutions of Riemannian curves and the evolution of their differential invariants

Let ϕ be a curve parametrized by the arc-length. Assume we have an evolution of the form

$$\phi_t = F\mathbf{h} = h_1 T + \sum_{k=1}^{n-1} h_k N_{k-1} \tag{3.1}$$

where $F = (T, N_1, \ldots, N_{n-1})$ is the matrix having in columns an invariant frame (for example the Frenet or natural frames) along the curve, and where $\mathbf{h} = (h_k)$ is a vector whose entries are functions of the Riemannian invariants associated to the invariant frame, k_i, $i = 1, \ldots, n-1$ and their derivatives with respect to arc-length (for example, the usual Riemannian curvatures in the case of a Frenet frame, or the natural curvatures in the natural case).

The following theorem describes the evolution induced on the invariants by evolution (3.1):

Theorem 2. *Let M be a Riemannian manifold, and let P be its associated principal bundle. Let ω and Ω be its associated Cartan connection and curvature tensor, respectively. Let*

$$\phi_t = F\mathbf{h} \tag{3.2}$$

be an evolution of curves on M as in (3.1)*. Assume the evolution is arc-length preserving. Then, the evolution induced on the Riemannian curvatures of ϕ by* (3.2) *can be found by evaluating the structural Equation* (2.1) *on the vector fields (ϕ_x, F_x) (ϕ_t, F_t) tangent to P along the family of curves $(\phi(t,x), F(t,x))$. Indeed, in the Frenet case,*

$$\begin{aligned} &\begin{pmatrix} 0 & 0 \\ 0 & \frac{d}{dt}K \end{pmatrix} \\ &= \frac{d}{dx}\begin{pmatrix} 0 & 0 \\ \mathbf{h} & F^T F_t \end{pmatrix} + \left[\begin{pmatrix} 0 & 0 \\ e_1 & K \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \mathbf{h} & F^T F_t \end{pmatrix}\right] + \Omega(\phi_x, \phi_t) \end{aligned} \tag{3.3}$$

where

$$K = \begin{pmatrix} 0 & -\kappa_1 & 0 & \ldots & 0 \\ \kappa_1 & 0 & -\kappa_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & \kappa_{n-2} & 0 & -\kappa_{n-1} \\ 0 & \ldots & 0 & \kappa_{n-1} & 0 \end{pmatrix}. \tag{3.4}$$

The matrix K is substituted by

$$N = \begin{pmatrix} 0 & -u_1 & -u_2 & \dots & -u_{n-1} \\ u_1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ u_{n-2} & 0 & \dots & 0 & 0 \\ u_{n-1} & 0 & \dots & 0 & 0 \end{pmatrix} \tag{3.5}$$

in the natural frame case.

Furthermore, in the Frenet case, $F^T F_t$ can be found directly from Equation (3.3) *itself using simple algebraic manipulations. In the natural case $F^T F_t$ is also determined by the equation but only up to a constant element in $o(n-1)$.*

Before proving this theorem we will prove a convenient lemma.

Lemma 1. *Let $B(x,t)$ be in $so(n)^*$ and assume that its first column is fixed, that is $Be_1 = \begin{pmatrix} 0 \\ \mathbf{f} \end{pmatrix}$, where $\mathbf{f}$ is given. Then, equation*

$$\frac{d}{dt}K = B' + [K, B] + \hat{\Omega}(\phi_x, \phi_t) \tag{3.6}$$

completely determines B, where $\Omega = \begin{pmatrix} 0 & 0 \\ 0 & \hat{\Omega} \end{pmatrix}$ and K is as in (3.4).

Proof. Assume Be_1 is fixed.

Let $so(n) = \sum_{i=1}^{n-1} [\mathfrak{g}_i \oplus \mathfrak{g}_{-i}]$ be the usual decomposition of $so(n)$ given by the standard gradation of $gl(n)$. That is, $\mathfrak{g}_i$ is given by matrices whose only nonzero entries are in place $(r, r+i)$, $r = 1, 2, \dots, n-i$. We will assume $\mathfrak{g}_i = 0$ whenever $i > n-1$ or $i < -n+1$. For x and t fixed we have that $K = K_1 + K_{-1} \in \mathfrak{g}_1 \oplus \mathfrak{g}_{-1}$. Decompose $B = \sum_{i=1}^{n-1} (B_i + B_{-i})$ in its components with respect to the gradation, so that $B_i^T = -B_{-i}$. Decompose also $\hat{\Omega}(\phi_x, \phi_t) = \sum_{i=1}^{n-1} \hat{\Omega}_i + \hat{\Omega}_{-i}$ in its components.

Since $\frac{d}{dt}K \in \mathfrak{g}_1 \oplus \mathfrak{g}_{-1}$, from equality (3.6) we get

$$B_i' + [K_1, B_{i-1}] + [K_{-1}, B_{i+1}] + \hat{\Omega}_i = 0 \tag{3.7}$$

for $i \neq 1, -1$, where by definition $B_i = 0$ for $i \geq n$ or $i \leq -n$. If the first row of B is determined, clearly B_{n-1} and B_{-n+1} are determined. Now, (3.7) for $i = n-1$ will determine B_{n-2}. Indeed, for $i = n-1$ we have

$$B_{n-1}' + [K_1, B_{n-2}] + \hat{\Omega}_{n-1} = 0 \tag{3.8}$$

and $[K_1, B_{n-2}] = (\kappa_{n-1} b_{1,n-1} - \kappa_1 b_{2,n}) E_{1,n}$, where $B = (b_{i,j})$ and $E_{i,j}$ is the matrix having a 1 in the (i,j) entry and zeroes elsewhere. Thus, if $b_{1,n-1}$ is known, $b_{2,n}$ can be found from (3.8).

A simple induction shows that, if B_r is known $r = n-1, \ldots, s$ and $b_{1,s}$ is known, then

$$B_s' + [K_1, B_{s-1}] + [K_{-1}, B_{s+1}] + \hat{\Omega}_s = 0 \tag{3.9}$$

determines B_{s-1}.

The last equation to be used from the group of Equations (3.7) is the case $i = 2$

$$B_2' + [K_1, B_1] + [K_{-1}, B_3] + \hat{\Omega}_2 = 0$$

which solves for B_1. Hence, we can solve for B using (3.6). $\square$

From now on we will denote by $B(f)$ the matrix determined by $\mathbf{f}$ via Equation (3.6) with $B(f)e_1 = \begin{pmatrix} 0 \\ \mathbf{f} \end{pmatrix}$.

Proof of Theorem 2. The first part of the proof is quite simple. Indeed, if evolution (3.2) is arc-length preserving, differentiation with respect to x and t commute and so $[(\phi_x, F_x), (\phi_t, F_t)] = 0$ as vector fields on the tangent to the bundle along the curve. We use now the formula

$$d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]) \tag{3.10}$$

for vector fields $X, Y \in TP$. Recall that, as vector fields, the application of the vectors fields (ϕ_x, F_x) and (ϕ_t, F_t) on a function amounts to x and t-differentiation, respectively, of the function evaluated along the curve. Therefore, the evaluation of the structural equation on the vector fields (ϕ_x, F_x), (ϕ_t, F_t) along the curve (ϕ, F) results in equation

$$\frac{d}{dt}\omega(\phi_x, F_x) = \frac{d}{dx}\omega(\phi_t, F_t) + [\omega(\phi_x, F_x), \omega(\phi_t, F_t)] + \Omega((\phi_x, F_x), (\phi_t, F_t)). \tag{3.11}$$

Thus, we need to show that along the curve

$$\omega(\phi_x, F_x) = \begin{pmatrix} 0 & 0 \\ e_1 & K \end{pmatrix}, \qquad \omega(\phi_t, F_t) = \begin{pmatrix} 0 & 0 \\ \mathbf{h} & F^T F_t \end{pmatrix}.$$

It is known (see [**Sh**]) that

$$\omega = \begin{pmatrix} 0 & 0 \\ \theta & w_H \end{pmatrix}$$

where $\theta^T = (\theta_1, \ldots, \theta_n)$ are coframe fields dual to the frame under consideration, along the curve, and where ω_H is the Maurer–Cartan of $SO_n(\mathbb{R})$. Therefore

$$\omega(\phi_x, F_x) = \begin{pmatrix} 0 & 0 \\ \theta(\phi_x) & F^T F_x \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ e_1 & K \end{pmatrix}$$

and

$$\omega(\phi_t, F_t) = \begin{pmatrix} 0 & 0 \\ \theta(\phi_t) & F^T F_t \end{pmatrix}.$$

We just need to apply that $\phi_t = F\mathbf{h}$ and so $\theta(\phi_t) = \mathbf{h}$.

The last part of the theorem is to show that one can find $F^T F_t$ directly from Equation (3.11) using algebraic computations. The Frenet case is a direct consequence of Lemma 1. Indeed, the first row of $F^T F_t$ is determined by the first column of (3.3) which reads $0 = \mathbf{h}' + K\mathbf{h} - F^T F_t e_1$, so that $F^T F_t e_1 = \mathbf{h}' + K\mathbf{h}$. Therefore, $F^T F_t = B(\hat{\pi}(\mathbf{h}' + K\mathbf{h}))$, where $\hat{\pi} : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is the projection on the last $n-1$ components. Notice that, since $B(\hat{\pi}(\mathbf{h}' + K\mathbf{h}))$ is in $so(n)$, the first entry of $\mathbf{h}' + K\mathbf{h}$ needs to be zero. Indeed, this implies $h_2 = \frac{h_1'}{\kappa_1}$ which is known to be the arc-length preserving condition on evolution (3.2).

The natural frame case is simpler. If N is given as in (3.5) and we denote $F^T F_t$ by S, Equation (3.11) in the natural case can be rewritten as

$$Se_1 = \mathbf{h}' + N\mathbf{h}, \;\; N_t = S' + [N, S].$$

Therefore, $Se_1 = \begin{pmatrix} 0 \\ \mathbf{r} \end{pmatrix} = \mathbf{h}' + N\mathbf{h}$ is determined. Furthermore, if

$$S = \begin{pmatrix} 0 & -\mathbf{r}^T \\ \mathbf{r} & \hat{S} \end{pmatrix} \tag{3.12}$$

then $N_t = S' + [N, S]$ becomes

$$N_t = \begin{pmatrix} 0 & -\mathbf{u}^T \hat{S} \\ -\hat{S}\mathbf{u} & \hat{S}' + \mathbf{r}\mathbf{u}^T - \mathbf{u}\mathbf{r}^T \end{pmatrix}.$$

Therefore, condition $\hat{S}' = \mathbf{u}\mathbf{r}^T - \mathbf{r}\mathbf{u}^T$ determines $\hat{S}$ in terms of $\mathbf{r}$ up to a constant matrix in $o(n-1)$. $\square$

The next step is to determine the value of $\Omega((\phi_x, F_x), (\phi_t, F_t))$. Of course, that will depend on the tensor Ω itself. The next lemma gives us an answer in the special case of manifolds with constant curvature tensor.

Lemma 2. *Let M be a Riemannian manifold with constant curvature and let Ω be its curvature 2-form. Let $\phi(x,t)$ be a family of curves evolving according to* (3.1). *Then, along the family of curves $(\phi(x,t), F(x,t))$ on the principal bundle P, we have*

$$\Omega((\phi_x, F_x), (\phi_t, F_t)) = \varkappa \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & h_2 & \dots & h_{n-1} \\ 0 & -h_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -h_{n-1} & 0 & \dots & 0 \end{pmatrix} \tag{3.13}$$

where $\varkappa$ is the curvature of the manifold, and where $\mathbf{h} = (h_i)$ *is as in* (3.1).

Proof. Again, the proof of this lemma can be found solely based on known descriptions of Riemannian manifolds with constant curvatures. Indeed, if $\theta = (\theta_1, \dots, \theta_n)$ are the dual coframe fields, it is known that a Riemannian manifold has constant curvature whenever

$$\Omega = \varkappa \begin{pmatrix} 0 & 0 \\ 0 & \theta \wedge \theta \end{pmatrix}.$$

Therefore, $\Omega((\phi_x, F_x), (\phi_t, F_t))$ along (ϕ, F) is determined by the application of $\theta \wedge \theta$ to $(\phi_x, \phi_t) = (T, F\mathbf{h})$. When applied we obtain directly the result of the lemma. $\square$

To finish this section, and to illustrate the simplicity of this method, we will apply the procedure described above to the special case of a 3-dimensional Riemannian manifold with constant curvature. Compare this procedure to the more traditional one used in [**MSW**].

Example 2. Let M be a 3-dimensional Riemannian manifold with constant curvature $\varkappa$. Let $\phi(x, t)$ be a family of curves on M, parametrized by arc-length, with associated curvatures and torsion given by κ and τ (κ_1 and κ_2 in the theorems above). Assume ϕ is solution of an evolution of the form

$$\phi_t = h_1 T + h_2 N + h_3 B$$

where T is the tangent to ϕ, N the normal, and B the binormal. Assume that the evolution is arc-length preserving.

From Theorem 2 we have the following equation to hold true:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -\kappa & 0 \\ 0 & \kappa & 0 & \tau \\ 0 & 0 & -\tau & 0 \end{pmatrix}_t = \begin{pmatrix} 0 & 0 & 0 & 0 \\ h_1 & 0 & \alpha & \beta \\ h_2 & -\alpha & 0 & \gamma \\ h_3 & -\beta & -\gamma & 0 \end{pmatrix}_x \tag{3.14}$$

$$+ \left[\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -\kappa & 0 \\ 0 & \kappa & 0 & \tau \\ 0 & 0 & -\tau & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ h_1 & 0 & \alpha & \beta \\ h_2 & -\alpha & 0 & \gamma \\ h_3 & -\beta & -\gamma & 0 \end{pmatrix} \right] + \varkappa \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & h_2 & h_3 \\ 0 & -h_2 & 0 & 0 \\ 0 & -h_3 & 0 & 0 \end{pmatrix}$$

where

$$F^T F_t = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix}$$

is to be found from the equation. We can rewrite (3.14) as

(3.15)
$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -\kappa & 0 \\ 0 & \kappa & 0 & -\tau \\ 0 & 0 & \tau & 0 \end{pmatrix}_t$$
$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ h_1' - \kappa h_2 & 0 & \alpha' + \tau\beta + \varkappa h_2 & \beta' + \tau\alpha - \kappa\gamma + \varkappa h_3 \\ h_2' + \kappa h_1 - \tau h_3 + \alpha & * & 0 & \gamma' + \kappa\beta \\ h_3' + \tau h_2 + \beta & * & * & 0 \end{pmatrix}$$

where $*$ represent entries that are determined by the matrix being an element of the Euclidean algebra, and where we denote $\frac{d}{dx}$ by $'$. Comparison of entries in the first column in both sides of the equality leads to the condition

$$h_2 = \frac{h_1'}{\kappa}$$

which is known to be the arc-length preserving condition for evolution (3.1). It also leads to the determination of α and β in terms of $\mathbf{h}$, κ and τ, namely

$$\alpha = -h_2' - \kappa h_1 + \tau h_3, \;\; \beta = -h_3' - \tau h_2.$$

Comparison of the $(2,4)$-entries in the equation determines γ also in terms of $\mathbf{h}$, κ and τ, namely

$$\gamma = \frac{1}{\kappa}(\beta' + \tau\alpha + \varkappa h_3).$$

Substituting the values of α, β, γ and h_2 into the entries $(3,2)$ and $(4,3)$ yields to the following evolution for κ and τ:

(3.16)
$$\kappa_t = \left(\frac{h_1'}{\kappa}\right)'' + (\kappa h_1)' + \tau^2\frac{h_1'}{\kappa} - \tau' h_3 + \varkappa\frac{h_1'}{\kappa}$$
$$\tau_t = \left(\frac{\tau}{\kappa}\left(\frac{h_1'}{\kappa}\right)'\right)' + \left(\frac{1}{\kappa}\left(\frac{\tau}{\kappa}h_1'\right)'\right)' + \tau h_1' + (\tau h_1)'$$
$$+ \left(\frac{1}{\kappa}h_3''\right)' - \left(\frac{\tau^2}{\kappa}h_3\right)' + \kappa h_3' + \varkappa\left(\frac{h_3}{\kappa}\right)'$$

evolution that can be found, for example, in [**MSW**].

In the natural frame case, assume the curve ϕ is evolving following an evolution of the form:

$$\phi_t = h_1 T + h_2 M_1 + h_3 M_2$$

where $\{T, M_1, M_2\}$ is a natural frame along the curve. Assume the evolution is arc-length preserving. Then the evolution of the natural curvatures is

$$N_t = S' + [N, S] + \varkappa \begin{pmatrix} 0 & h_2 & h_3 \\ -h_2 & 0 & 0 \\ -h_3 & 0 & 0 \end{pmatrix} \tag{3.17}$$

where

$$S = \begin{pmatrix} 0 & \mathbf{r}^T \\ -\mathbf{r} & \hat{S} \end{pmatrix} = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix}$$

and $\begin{pmatrix} 0 \\ \mathbf{r} \end{pmatrix} = \mathbf{h}' + N\mathbf{h}$. From here we have that $h_1' = u_1 h_2 + u_2 h_3$ (the arc-length preserving condition) and $\alpha = -h_2' - u_1 h_1$, $\beta = -h_3' - u_2 h_1$. Furthermore, (3.17) implies $\hat{S}' = \mathbf{u}\mathbf{r}^T - \mathbf{r}\mathbf{u}^T$ and so $\gamma' = \alpha u_2 - \beta u_1$. Putting all this information together yields the evolution

$$(u_1)_t = h_2'' + [u_1 D^{-1}(u_1 h_2 + u_2 h_3)]' - u_2 D^{-1}(u_1 h_3' - u_2 h_2') - \varkappa h_2$$

$$(u_2)_t = h_3'' + [u_2 D^{-1}(u_1 h_2 + u_2 h_3)]' + u_1 D^{-1}(u_1 h_3' - u_2 h_2') - \varkappa h_3$$

where D^{-1} represents formally the integral, determined up to a constant.

The advantages of using this formulation are not only calculational. Indeed this way of writing the evolutions of the curvatures give us the inside view of where the associated Poisson brackets come from, as we will readily see in the next section.

4. A family of Poisson structures

In this section we will first give a very brief description of Poisson manifolds in finite dimensions, since most readers will be more familiar with this case, and we will describe how the picture translates into infinite dimensions. We will then define the Poisson structure that seems to generate all the interesting known Poisson structures associated to geometric evolutions.

Definition 4. A *Hamiltonian structure or Poisson bracket* on a finite dimensional manifold is a bilinear and skew-symmetric map

$$\{,\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

such that the following two additional properties hold:

(1) $\{FG, H\} = F\{G, H\} + G\{F, H\}$ (Leibniz's property),
(2) $\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0$ (Jacobi's property),

for any $F, G, H \in C^\infty(M)$.

If $H \in C^\infty(M)$, the vector field $\{H, \cdot\} = \xi_H : C^\infty(M) \to C^\infty(M)$ is called the *Hamiltonian vector field* associated to *the Hamiltonian H*.

If $u : \mathbb{R} \to M$, the evolution

$$u_t = \xi_H(u(t)) \tag{4.1}$$

is called a *Hamiltonian evolution* associated to the Hamiltonian H.

The flow of Hamiltonian evolutions remains always on a certain submanifold for all times. These submanifolds foliate the original manifold and they are called *the symplectic leaves* of the Poisson structure.

In the special case where $M = \mathfrak{g}^*$ there is a very natural Poisson bracket defined by the Lie algebra structure of $\mathfrak{g}$. Denote by $[,]$ the Lie bracket on $\mathfrak{g}$ and let $\langle , \rangle$ be the pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$.

Definition 5. Let $F, G \in C^\infty(\mathfrak{g}^*)$ be two real functions defined on the dual of a Lie algebra $\mathfrak{g}$. The total derivatives of these functions at a point $L \in \mathfrak{g}^*$ can be naturally identified with elements in the Lie algebra, say f and g, respectively. Define

$$\{F, G\}(L) = \langle [f, g], L \rangle.$$

The bracket $\{,\}$ is clearly Poisson and it is called *the Lie–Poisson bracket on* $\mathfrak{g}^*$.

It is well-known, and it is crucial in the description that follows, that the *symplectic leaves of the Lie–Poisson bracket coincide with the coadjoint orbits of* $\mathfrak{g}^*$ *under the action of the Lie group.*

In the case of M being an infinite dimensional manifold a general definition is technically complicated to give, so I will limit myself to the definition of the bracket that interests us. Let G be a semisimple Lie group, and $\mathfrak{g}$ its Lie algebra. Let $LG = C^\infty(S^1, G)$ be the group of loops on G and let $L\mathfrak{g} = C^\infty(S^1, \mathfrak{g})$ be its Lie algebra. Let $L\mathfrak{g}^* = C^\infty(S^1, \mathfrak{g}^*)$ be its dual (it is not really its dual but what is called the regular part of the dual, dense in the dual of the algebra of loops). The space of loops could be replaced by functions from $\mathbb{R}$ to G vanishing at infinity, or any condition that ensures that no boundary terms will appear when we integrate by parts. Let Q be the Killing form associated to $\mathfrak{g}$. Define the following *cocycle* of the algebra $L\mathfrak{g}$:

$$w(L, M) = \int_{S^1} Q\left(L, \frac{dM}{dx}\right) dx$$

for any $L, M \in L\mathfrak{g}$. The form w is called a cocycle because it has the properties necessary to guarantee that $L\mathfrak{g} \oplus \mathbb{R}$ is a Lie algebra with Lie bracket given by

$$[(L, s), (M, r)] = ([L, M], w(L, M)).$$

This algebra is called the *central extension* of $L\mathfrak{g}$, also known as a *Kac–Moody algebra on the circle associated to* $\mathfrak{g}$ (we will denote it by $\mathrm{kac}(\mathfrak{g})$.)

The Poisson bracket we are interested in is the Lie–Poisson bracket on the dual of the Kac–Moody algebra. It is defined as follows: Let $G : L\mathfrak{g}^* \to \mathbb{R}$ be a functional and let $\frac{\delta G}{\delta L} \in L\mathfrak{g}$ be its variational derivative. The Lie–Poisson bracket on $L\mathfrak{g}^*$ is defined as

$$\{H, G\}(L, s) = \int_{S^1} \operatorname{tr}\left(\frac{\delta G}{\delta L}\left(-s\left(\frac{\delta H}{\delta L}\right)' + \left[L, \frac{\delta H}{\delta L}\right]\right)\right) dx \tag{4.2}$$

where tr denotes $-\frac{1}{2}$ times the trace. From this expression we readily see that the Hamiltonian vector field associated to the Hamiltonian H is given by

$$\xi_H = -s\left(\frac{\delta H}{\delta L}\right)' + \left[L, \frac{\delta H}{\delta L}\right]. \tag{4.3}$$

It is known ([**Ki**]) that the coadjoint action of the Kac–Moody group on $\mathrm{kac}^*(\mathfrak{g})$ reduces to the following action of the group of loops:

$$\mathrm{Ad}^*(g)(L, s) = (-sg^{-1}g' + g^{-1}Lg, s). \tag{4.4}$$

We can see from this formula that $\mathrm{kac}^*(\mathfrak{g})$ foliates into Poisson manifolds corresponding to a fix value of s. Also, it is customary to identify an element $(L, s) \in \mathrm{kac}^*(\mathfrak{g})$ with the differential operator

$$s\frac{d}{dx} + L \tag{4.5}$$

so that the coadjoint action corresponds to conjugation of such operator by $g \in LG$. This conjugation (or gauge) by g corresponds to the change of variable $X = Yg$ on the solutions of the system $sX' = XL$.

Compare Equation (4.3) to (3.3). The similarity between these equations suggests that geometric evolutions (3.3) for both Frenet and natural cases might be Hamiltonian evolutions with respect to the Lie–Poisson bracket on the dual of the Kac–Moody algebra associated to $so(n)$ (with $s = -1$), after being reduced to the submanifolds $\mathcal{K}$ of Frenet matrices of the form (3.4), or to the submanifold $\mathcal{N}$ of natural matrices of the form (3.5), respectively.

The following definition of Poisson reduction can be found in [**MR**]:

Definition 6. Let $(M, \{,\})$ be a Poisson manifold and P a submanifold of M, $i : P \to M$ the inclusion. Let E be a subbundle of TM along P.

Assume $E \cap TP$ is an integral subbundle of TP defining a foliation Φ on P. We say that $(P, \{,\}, E)$ is *Poisson reducible* if P/Φ has a Poisson structure $\{,\}_R$ defined the following way: For any (locally defined) smooth functions f, g on P/Φ and any smooth extensions F, G of $f \circ \pi$, $g \circ \pi$ with differentials vanishing on E, we have

$$\{f, g\}_R \circ \pi = \{F, G\} \circ i, \tag{4.6}$$

where $\pi : P \to P/\Phi$ is the projection.

Notice that I have not commented on the nature of P/Φ (whether or not it is a manifold and what kind of manifold) and its projection (its smooth character, etc.). These conditions are, of course, included in the original definition found in [**MR**]. They are difficult questions in the infinite dimensional case and I rather perform these reductions formally (the definition above can be taken as a formal definition of reduction). Later on, one can look at both the quotient P/Φ and these formal definitions of the brackets and wonder if they are well-defined. This way we also preserve the simplicity and beauty of the geometric picture behind the reductions.

Theorem 3. *Let* $\{,\}$ *be the Lie–Poisson bracket on* $\mathrm{kac}^*(o(n))$ *with central parameter* $s=-1$. *Let* E *be the subbundle of* $T(Lo(n)^*)$ *generated by the Hamiltonian vector fields of Hamiltonian functionals* H *such that*

$$\frac{\delta H}{\delta L} = \begin{pmatrix} 0 & 0 \\ 0 & \Gamma \end{pmatrix} \tag{4.7}$$

with $\Gamma \in Lo(n-1)$. *Let* Φ *be the foliation defined by the orbits of* $LSO(n-1)$, *the foliation known to be associated to* E.

Then, the submanifold $\mathcal{K}$ *given by the set of all Frenet matrices of the form* (3.4) *with* $k_i \in C^\infty(S^1, \mathbb{R})$, $k_i > 0$, $i = 1, \dots, n-1$, *is a section transverse to the orbits of* $LSO(n-1)$ *under the coadjoint action on an open set of* $Lo(n)^*$, *say* U. *That is, we can identify* U/Φ *with* $\mathcal{K}$. *Furthermore,* $(U, \{,\}, E)$ *is Poisson reducible, and, hence, there exists a Poisson bracket* $\{,\}_R$ *defined on* $\mathcal{K}$.

Proof. Consider $P = M = Lo(n)^*$ with the Lie–Poisson Kac–Moody bracket for $s=-1$. Let E be given as in the statement of the theorem. Given that $E \cap TLo(n)^* = E$, the foliation Φ associated to this intersection is simply the coadjoint orbits under the action (4.4) with

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \theta \end{pmatrix}$$

$\theta \in LSO(n-1)$.

We will first show that, for some open set $U \subset Lo(n)^*$, U/Φ can be identified with $\mathcal{K}$. The description of such an identification is very simple and pretty. Let's identify each element in $Lo(n)^*$ with its differential operator as in (4.5), $s=-1$. We need to show that the set U of coadjoint orbits intersecting $\mathcal{K}$ is an open set of $Lo(n)^*$ and that such an orbit intersects $\mathcal{K}$ at only one point.

Indeed, let $-\frac{d}{dx} + L$ be an element in an orbit $\Theta(L)$ intersecting $\mathcal{K}$. Let X be a fundamental matrix solution for equation $X' = XL$, with $X(0) = I$, the identity matrix. Let T be the first column of X. Clearly the Riemannian length of T is constant and equal to one. Besides, since L has periodic coefficient, there exists a matrix $M(L) \in O(n)$, the *monodromy of* L, such that

$$T(x+2\pi) = M(L)T(x).$$

Define ϕ to be the curve whose tangent is given by T. Clearly ϕ is determined up to a translation and it is uniquely determined if we ask ϕ to have the same monodromy property as T does. If $\Theta(L)$ intersects $\mathcal{K}$, there exists $g \in LSO(n-1)$ such that $Xg = Y$ satisfies $Y' = YK$, K as in (3.4). Indeed, $g \in LSO(n-1)$ since the tangent to the curve (the first column of X) is also the first vector in the Frenet frame. Clearly, Y must be the Frenet frame and K the Frenet matrix associated to ϕ. K is unique by the uniqueness of the Frenet matrix and must have periodic entries since ϕ has a monodromy. Summarizing, moving along the orbit corresponds to *changing the normal components of the frame and intersecting* $\mathcal{K}$ *corresponds to fixing the normal vectors to be the normal Frenet vectors.*

It is clear now that the set of all these orbits is an open set in $Lo(n)^*$. Each orbit can be identified up to translations with a curve ϕ with a certain monodromy matrix. A nearby orbit will yield a nearby tangent vector and a nearby tangent vector will produce a nearby curve. This curve can be chosen to have a monodromy matrix because of the periodicity of the equation that the tangent satisfies. But if a curve is nondegenerate in the sense of having a well-defined Frenet frame, any nearby curve will also be nondegenerate. Thus, its orbit will intersect $\mathcal{K}$.

We will finally show that $(U, \{,\}, E)$ is Poisson reducible. Let $f, g : U/\Phi \to \mathbb{R}$ be two Fréchet differentiable functionals and define

$$\{f,g\}_R \circ \pi = \{F,G\}|_U \tag{4.8}$$

for any two extensions of $f\circ\pi$ and $g\circ\pi$, F and G respectively, such that $\frac{\delta F}{\delta L}$ and $\frac{\delta G}{\delta L}$ vanish on E. The definition does not depend on F and G since any two different extensions will coincide on U. Also, (4.8) above is well-defined. Indeed, since F and G are constant on the leaves of Φ, $\{F,G\}$ will also be constant on the leaves, by Jacobi's identity. If the variational derivative of H is as in (4.7), then

$$\{\{F,G\},H\} = -\{\{G,H\},F\} - \{\{H,F\},G\} = 0.$$

Thus, $\{F,G\}$ defines a functional on U/Φ. The bracket $\{,\}_R$ is Poisson since it inherits its properties from $\{,\}$. We only need to point out that, if F and G are extensions of f and g, respectively, both of them constant on the leaves of Φ, then clearly $\{F,G\}$ is an extension of $\{f,g\}_R$, constant on the leaves of Φ. With this in mind the verification of the properties is straightforward. $\square$

Unfortunately, the natural case can be carried out only formally. Indeed, from (2.5) we see that neither θ nor $\mathbf{u}$ will be, in general, periodic.

Theorem 4. *The set $\mathcal{N}/O(n-1)$, where $\mathcal{N}$ is the manifold of matrices of the form* (3.5), *can also be formally identified with $Lo(n)^*/\Phi$, so there exist an additional reduced bracket denoted by $\{,\}_{NR}$, on $\mathcal{N}/O(n-1)$. Both reduced brackets $\{,\}_R$ and $\{,\}_{NR}$ are formally Poisson equivalent* (*formally since the gauge that takes one to the other will not be periodic in general*).

Proof. Notice that since moving along the orbit corresponds to changing the normal components of the frame along the curve, we will intersect $\mathcal{N}$ when a natural frame is reached (we are allowing ourselves the use of non-periodic elements $g(x) \in O(n-1)$). The only condition needed to do that is for the curve to be regular ($T \neq 0$). Thus, any orbit intersects $\mathcal{N}$. But a natural frame is determined only up to the action of $O(n-1)$ (see [**B**]). Therefore, $Lo(n)^*/\Phi$ can be identified with $\mathcal{N}/O(n-1)$. □

This simple picture gives us also a clear description of how the reduced bracket can be found explicitly. Indeed, the definition of $\{,\}_R$ is given by formula (4.6). Now, if f is defined on $\mathcal{K}$, there exists a unique local extension which is constant on the leaves of the foliation Φ, namely $F = f \circ \pi$. Since F is an extension for f, its variational derivative in the $\mathcal{K}$-direction, along $\mathcal{K}$, coincides with that of f. This is reflected in the following algebraic fact:

Proposition 1. *Let $A \in Lo(n)^*$ and assume all entries of A in the $\mathcal{K}$-direction $A_{i,i+1} = f_i$, $i = 1, \dots, n-1$ are fixed. Then, there exists a unique choice of A such that $A' + [K, A]$ is of the form* (3.5) (*that is, A vanishes on E*).

The proof of this proposition is very similar to that of Lemma 1 for $\varkappa = 0$ and we will not include it. We will denote by $A(f)$ the matrix determined by $\mathbf{f} = (f_i)$ via Proposition 1.

Now the explicit formula of the reduced bracket can be readily given. Let $f, g : \mathcal{K} \to \mathbb{R}$ be two Fréchet differentiable functionals. Let $f_i = \frac{\delta f}{\delta \kappa_i}$ and $g_i = \frac{\delta g}{\delta \kappa_i}$, $i = 1, \dots, n-1$. If we place f_i in the entry $(i, i+1)$ of a matrix $A(f)$ so that $A(f) = \frac{\delta (f \circ \pi)}{\delta L}(K)$, $L \in U, K \in \mathcal{K}$, the rest of the entries of $A(f)$ are uniquely determined by $A(f)' + [K, A(f)]$ being of the form (3.5). We would then have

$$\{f, g\}_R(K) = \int_{S^1} \operatorname{tr}\left(\left(A(f)' + [K, A(f)]\right) A(g)\right) dx. \tag{4.9}$$

The same description can be given for the natural case if, instead of $A(f)$ we use a matrix $S(f)$ with $S(f)e_1 = \begin{pmatrix} 0 \\ \mathbf{f} \end{pmatrix}$, with $\mathbf{f} = (f_i)$, and such that $S(f)' + [N, S(f)]$ is of the form (3.5). If $f, g : \mathcal{N} \to \mathbb{R}$ are two functionals which are constant on the E-orbits, then the reduced bracket on $\mathcal{N}$ is given by

$$\{f,g\}_{NR}(N) = \int_{S^1} \operatorname{tr}\left((S(f)' + [N, S(f)])S(g)\right) dx, \tag{4.10}$$

which, from the theorem, will be invariant under the action of $O(n-1)$.

Example 3. In the case $n = 3$, if $\frac{\delta f}{\delta \kappa_i} = f_i$, $\frac{\delta g}{\delta \kappa_i} = g_i$, and if $A(f) = \begin{pmatrix} 0 & f_1 & \alpha \\ -f_1 & 0 & f_2 \\ -\alpha & -f_2 & 0 \end{pmatrix}$, then $A(f)' + [K, A(f)]$ being tangent to $\mathcal{N}$ implies $\alpha = -\frac{f_2'}{\kappa}$. A short computation yields

$$A(f)' + [K, A(f)] = \begin{pmatrix} 0 & f_1' + \frac{\tau}{\kappa} f_2' & -\left(\frac{f_2'}{\kappa}\right)' - \kappa f_2 + \tau f_1 \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix}.$$

With this information, (4.9) above can be written as

$$\{f,g\}_R(K) = \int_{S^1} g_1 \left(f_1' + \frac{\tau}{\kappa} f_2'\right) - \frac{g_2'}{\kappa}\left(-\left(\frac{f_2'}{\kappa}\right)' - \kappa f_2 + \tau f_1\right) dx$$

which itself can be finally written as

$$\{f,g\}_R(K) = \int_{S^1} \left(\frac{\delta g}{\delta \kappa}\right)^T \begin{pmatrix} D & \frac{\tau}{\kappa} D \\ D\frac{\tau}{\kappa} & -D - D\frac{1}{\kappa} D \frac{1}{\kappa} D \end{pmatrix} \left(\frac{\delta f}{\delta \kappa}\right) dx. \tag{4.11}$$

This Poisson bracket has appeared in the literature ([**MSW**]) in connection to the study of integrable systems associated to invariant evolutions of Riemannian differential invariants.

In the natural case, the matrix $S(f)$ is given as in (3.12) with $\mathbf{r} = \mathbf{f}$. From here

$$S(f)' + [N, S(f)] = \begin{pmatrix} 0 & (-\mathbf{f}')^T - \mathbf{u}^T D^{-1}(\mathbf{u}\mathbf{f}^T - \mathbf{f}\mathbf{u}^T) \\ \mathbf{f}' + D^{-1}(\mathbf{u}\mathbf{f}^T - \mathbf{f}\mathbf{u}^T)\mathbf{u} & 0 \end{pmatrix}$$

and so bracket (4.10) is defined as

$$\{f,g\}_{NR}(N) = \int_{S^1} g_1 u_2 D^{-1}(u_2 f_1 - f_2 u_1) + g_2 u_1 D^{-1}(u_1 f_2 - f_1 u_2) + g_1 f_1' + g_2 f_2' dx$$

which can be rewritten as

$$\{f,g\}_{NR}(N) = \int_{S^1} \left(\frac{\delta g}{\delta u}\right)^T \begin{pmatrix} D + u_2 D^{-1} u_2 & -u_2 D^{-1} u_1 \\ -u_1 D^{-1} u_2 & D + u_1 D^{-1} u_1 \end{pmatrix} \left(\frac{\delta f}{\delta u}\right).$$

This bracket has appeared in association to the integrability of modified KdV equations.

The precise connection between these two brackets and the curve evolutions is described in the next section.

5. Relationship between the reduced brackets and the evolution of the Riemannian curvatures

In this section we will give, in the flat case, necessary conditions so that the evolutions induced on the Riemannian curvatures by arc-length preserving flows of the form (3.1) are Hamiltonian with respect to the reduced brackets.

First of all, notice that, if $A(f)$ is defined as in Proposition 1, it is immediate that all other entries in $A(f)$ are defined as differential polynomials in f_i, $i = 1, \dots, n-1$ whose coefficients are rational functions of κ_i and their derivatives. This is a simple consequence of the fact that the entries are obtained algebraically from $\mathbf{f}$ using that $A(f)' + [K, A(f)]$ belongs to $T\mathcal{N}$. Lets denote by $\mathcal{A}$ the differential operator holding

$$A(f)' + [K, A(f)] = \begin{pmatrix} 0 & (\mathcal{A}\mathbf{f})^T \\ -\mathcal{A}\mathbf{f} & 0 \end{pmatrix}.$$

Likewise, if $B(g)$ is the matrix given in Lemma 1 (with $\varkappa = 0$), the entries of $B(g)$ are also differential polynomials in g_i, $i = 1, \dots, n-1$ since they are obtained algebraically from $\mathbf{g} = (g_i)$ using the fact that $B(g)' + [K, B(g)]$ lies in $T\mathcal{K}$. Their coefficients will also be rational functions on κ_i and their derivatives. Let's denote by $\mathcal{B}$ and $\mathcal{C}$ the differential operators that associates to $\mathbf{g}$ the nonzero entries in the 1-graded component of $B(g)' + [K, B(g)] + \varkappa\hat{\Omega}$, that is, $(\mathcal{B}\mathbf{g})_i + \varkappa(\mathcal{C}\mathbf{g})_i$ is the $(i, i+1)$ entry in $B(g)' + [K, B(g)]$.

Also, directly from Equation (3.3) we see that the entries of the right-hand side of (3.3) can be written as differential polynomials on $\mathbf{h}$ with coefficients depending on κ_i and their derivatives with respect to x. That is, if $\varkappa = 0$, evolution (3.3) can be rewritten as

$$\kappa_t = \mathcal{P}\hat{\mathbf{h}} \tag{5.1}$$

for some matrix of differential operators $\mathcal{P}$, and where $\kappa = (\kappa_i)$ and $\hat{\mathbf{h}} = (h_1, h_3, \dots, h_n)$. Notice that h_2 can be eliminated from the equation using the arc-length preserving condition $h_2 = \frac{h_1'}{\kappa}$.

Lemma 3.

$$\mathcal{B}^* = -\mathcal{A}. \tag{5.2}$$

Proof. The proof of this lemma is a consequence of the following basic calculation:

$$\begin{aligned}\langle A(f)' + [K, A(f)], B(g)\rangle &= \int_{S^1} \mathcal{A}\mathbf{f}\cdot\mathbf{g}dx \\ &= -\langle A(f), B(g)' + [K, B(g)]\rangle \\ &= -\int_{S^1} \mathcal{B}\mathbf{g}\cdot\mathbf{f}dx = -\int_{S^1} \mathcal{B}^*\mathbf{f}\cdot\mathbf{g}dx\end{aligned} \tag{5.3}$$

for any $\mathbf{f}, \mathbf{g}$. □

Theorem 5. *Let*

$$\phi_t = F\mathbf{h} \tag{5.4}$$

be an arc-length preserving evolution of curves $\phi(x,t)$ parametrized by the arc-length, where $F(x,t)$ contains in columns the Frenet frame along $\phi(x,t)$. Assume $\mathbf{h}$ is a vector which depends on the Riemannian curvatures κ_i, $i = 1,\dots,n-1$ and their derivatives with respect to x. Furthermore, assume $\hat\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ to be the projection on the last $n-1$ components and assume that there exists a functional $g : \mathcal{K} \to \mathbb{R}$ such that

$$\hat\pi(\mathbf{h}' + K\mathbf{h}) = \mathcal{C}^*\left(\frac{\delta g}{\delta\kappa}\right), \tag{5.5}$$

where $\mathcal{C}$ is the matrix of differential operators associated to the constant curvature of the manifold (and whose value is independent of the value of the curvature $\varkappa$). Then, in the flat case $\varkappa = 0$, the evolution induced by (5.4) *on the Riemannian curvatures κ_i via Theorem* 2*, is Hamiltonian with respect to the reduced bracket $\{,\}_R$ and its associated Hamiltonian functional is g.*

Proof. Let's analyze the reduced evolutions a little further. Assume that $A(f)$ is given as in Proposition 1, and assume we can write it as

$$A(f) = \begin{pmatrix} 0 & (\mathcal{R}\mathbf{f})^T \\ -\mathcal{R}\mathbf{f} & * \end{pmatrix}$$

for some matrix of differential operators $\mathcal{R}$. In order to rewrite the reduced evolution in terms of matrices of differential operators, notice that if f and g are two functionals

(5.6)

$$\{f,g\}_R(K) = \langle A(f)' + [K, A(f)], A(g)\rangle = \int_{S^1} \mathcal{A}\mathbf{f}\cdot\mathcal{R}\mathbf{g}dx = \int_{S^1} \mathcal{R}^*\mathcal{A}\mathbf{f}\cdot\mathbf{g}dx.$$

From here it is clear that the Hamiltonian vector field associated to f is given by $\xi_f = \mathcal{R}^*\mathcal{A}\mathbf{f}$, where $\mathbf{f} = \frac{\delta f}{\delta\kappa}$.

Assume $\phi_t = F\mathbf{h}$ is arc-length preserving and that ϕ is parametrized by arc-length. We see from (3.3) that the evolution induced on the Riemannian curvatures is given by

$$K_t = (F^T F_t)' + [K, F^T F_t] \tag{5.7}$$

where, in the notation of Lemma 1, $F^T F_t = B(\hat{\pi}(\mathbf{h}' + K\mathbf{h}))$. Therefore, the evolution induced upon the Riemannian curvatures can be written as

$$\kappa_t = -\mathcal{B}(\hat{\pi}(\mathbf{h}' + K\mathbf{h})). \tag{5.8}$$

On the other hand, the reduced Hamiltonian vector field is given by $\xi_f = \mathcal{R}^*\mathcal{A}(f) = -\mathcal{A}^*\mathcal{R}(f)$ because of the skew-symmetry of the bracket. The result of Theorem 5 now follows from Lemma 3 and the following lemma:

Lemma 4.

$$\mathcal{C} = \mathcal{R}^*. \tag{5.9}$$

Proof. First of all, notice that $\mathcal{C}$ is defined the following way: Given an element of $T\mathcal{N}$, for example

$$\begin{pmatrix} 0 & \mathbf{f}^T \\ \mathbf{f} & 0 \end{pmatrix}$$

there exists a unique element of $T\mathcal{N}^0$ such that

$$\begin{aligned} &\begin{pmatrix} 0 & \mathbf{f}^T \\ -\mathbf{f} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix}' + \left[K, \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix}\right] \\ &= \begin{pmatrix} 0 & \mathbf{f}^T - \kappa_1 e_1^T C \\ -\mathbf{f} + \kappa_1 C e_1 & C' + [K_2, C] \end{pmatrix} \end{aligned} \tag{5.10}$$

is tangent to $\mathcal{K}$. That is, the $(i, i+1)$ diagonal of (5.10) defines $\mathcal{C}\hat{\pi}(\mathbf{h})$, whenever $\mathbf{f} = \hat{\pi}(\mathbf{h})$. The uniqueness of C is clear since (5.10) tangent to $\mathcal{K}$ implies that $\hat{\pi}(Ce_1) = \hat{\pi}(\frac{1}{\kappa_1}\mathbf{h})$ and that condition together with $C' + [K_2, C]$ having zeroes off the $(i, i+1)$ and $(i, i-1)$ diagonals completely determines C, as it was shown in Lemma 1 (K_2 is given as in (2.4)). On the other hand,

$$\int_{S^1} \mathcal{C}\mathbf{f} \cdot \mathbf{g} dx = \int_{S^1} \left\langle \begin{pmatrix} 0 & \mathbf{f}^T - \kappa_1 e_1^T C \\ -\mathbf{f} + \kappa_1 C e_1 & C' + [K_2, C] \end{pmatrix}, A(g) \right\rangle dx, \tag{5.11}$$

since $A(g)$ has $\mathbf{g}$ in its $+1$ component with respect to the standard gradation. But

$$\begin{aligned} \left\langle \begin{pmatrix} 0 & \mathbf{f}^T - \kappa_1 e_1^T C \\ -\mathbf{f} + \kappa_1 C e_1 & C' + [K_2, C] \end{pmatrix}, A(g) \right\rangle dx &= \left\langle \begin{pmatrix} 0 & \mathbf{f}^T \\ -\mathbf{f} & 0 \end{pmatrix}, A(g) \right\rangle dx \\ &= \int_{S^1} \mathbf{f} \cdot \mathcal{R}\mathbf{g} dx \end{aligned}$$

where we use that $A(g)' + [K, A(g)]$ is tangent to $\mathcal{N}$ and so

$$\left\langle \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix}' + \left[K, \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix}\right], A(g) \right\rangle$$
$$= -\left\langle \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix}, A(g)' + [K, A(g)] \right\rangle = 0.$$

□

If we use natural frames, the connection is even simpler.

Theorem 6. *Let*

$$\phi_t = N\mathbf{h} \tag{5.12}$$

be an arc-length preserving evolution of curves $\phi(x,t)$ parametrized by the arc-length, where $N(x,t)$ contains in columns the natural frame along $\phi(x,t)$. Assume $\mathbf{h}$ is a vector which depends on the natural curvatures u_i, $i = 1, \dots, n-1$ and their derivatives with respect to x. Furthermore, assume that there exists a functional $g : \mathcal{N} \to \mathrm{I\!R}$ such that

$$\hat{\pi}((\mathbf{h})' + N\mathbf{h}) = \frac{\delta g}{\delta \mathbf{u}}. \tag{5.13}$$

Then, in the flat case $\varkappa = 0$, the evolution induced by (5.12) *on the natural curvatures u_i via Theorem* 2, *is Hamiltonian with respect to the reduced bracket $\{,\}_{NR}$, and its associated Hamiltonian functional is g.*

Proof. The proof of this theorem is analogous to the proof of the previous theorem. One has only to notice that the evolution induced upon $\mathbf{u}$ by (5.12) equals

$$\mathbf{u}_t = \mathcal{S}(\hat{\pi}(\mathbf{h}' + N\mathbf{h})) \tag{5.14}$$

where $\mathcal{S}$ is the operator defining the natural reduced bracket. That is, if $S(f)$ is the matrix such that $S(f)e_1 = \begin{pmatrix} 0 \\ \mathbf{f} \end{pmatrix}$ and such that $S(f)' + [N, S(f)]$ lies on $T\mathcal{N}$, then

$$S(f)' + [N, S(f)] = \begin{pmatrix} 0 & \mathcal{S}\mathbf{f}^T \\ -\mathcal{S}\mathbf{f} & 0 \end{pmatrix}.$$

□

Condition (5.13) is far simpler than condition (5.5). This, I believe, explains why the natural frame has been favored in the study of integrable systems associated to Riemannian geometry. Geometrically, though, there is hardly any difference, except the fact that the Frenet case is uniquely and well-defined (from a Poisson reduction point of view), unlike the natural case which requires the choice of a section in $\mathcal{N}/O(n-1)$ and a formal approach. In fact, once the section is fixed, both brackets are formally equivalent and the gauge that takes the Frenet frame to the natural frame of our choice is a Poisson map between both Poisson manifolds. As we saw before, *any such a map would be a generalization of the well-known Hasimoto transformation.*

Finally, the Lie–Poisson bracket on $\mathrm{kac}^*(\mathfrak{g})$ is known to have compatible Poisson brackets given by the following formula:

$$\{F,G\}^1(L) = -\left\langle \left[\frac{\delta F}{\delta L}, \frac{\delta G}{\delta L}\right], H_0 \right\rangle \tag{5.15}$$

for a fixed element $H_0 \in \mathfrak{g}^*$ with some nondegeneracy conditions. One can easily check that these brackets are also reducible using the foliation Φ. Since $\{,\}$ and $\{,\}^1$ are compatible on $\mathrm{kac}^*(o(n))$, when reduced they will still be compatible. Different choices of H_0 will produce different companions that can be used to integrate PDE's.

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