Pacific Journal of Mathematics

RECURSIVE RELATIONS, JACOBI MATRICES, MOMENT PROBLEMS AND CONTINUED FRACTIONS

B. El Wahbi, M. Rachidi and E.H. Zerouali

Volume 216 No. 1 September 2004

PACIFIC JOURNAL OF MATHEMATICS
Vol. 216, No. 1, 2004

RECURSIVE RELATIONS, JACOBI MATRICES, MOMENT PROBLEMS AND CONTINUED FRACTIONS

B. El Wahbi, M. Rachidi and E.H. Zerouali

We investigate in this paper the link between the moment problem for recursive sequences, the associated Jacobi matrices and the associated analytic functions. We generalize some classical results by providing simple proofs that use functional calculus.

1. Introduction

Let $a_0, \dots, a_{r-1}$ ($r \geq 1$, $a_{r-1} \neq 0$) be real numbers and let $\gamma = \{\gamma_n\}_{n\geq 0}$ be the sequence defined by the following recursive relation of order r:

$$\gamma_{n+1} = a_0\gamma_n + a_1\gamma_{n-1} + \cdots + a_{r-1}\gamma_{n-r+1}, \quad \text{for } n \geq r-1, \tag{1}$$

where $\gamma_0, \gamma_1, \dots, \gamma_{r-1}$ are the given initial values (or conditions). The polynomial $P(X) = X^r - a_0X^{r-1} - a_1X^{r-2} - \cdots - a_{r-1}$, together with the initial values, is said to define the sequence γ. Note that if Q is any multiple of P, then Q also defines γ provided that $\gamma_0, \gamma_1, \dots, \gamma_{\deg Q-1}$ are taken as initial conditions. As observed in [**3**] among all polynomials defining γ, there exists a unique monic polynomial of minimal degree. This latter, denoted by P_γ, is called the characteristic polynomial of γ.

Let $\gamma = \{\gamma_n\}_{n\geq 0}$ be a sequence of complex numbers and K a closed subset of the complex plane. The purpose of the K-moment problem associated with γ is to find a positive measure μ such that

$$\gamma_n = \int_K t^n d\mu(t). \tag{2}$$

Since its introduction by Stieltjes in [**15**] for $K = \mathbb{R}^+$, the moment problem has been the subject of an extensive literature. In particular, Hamburger and Hausdorff have studied this problem for $K = \mathbb{R}$ and $K = [0,1]$ respectively. The main idea in computing the measure μ, solution of (2), for a given sequence $\gamma = \{\gamma_n\}_{n\geq 0}$ is to extend the linear form defined on polynomials by

$$S_\gamma(X^n) = \gamma_n \tag{3}$$

to a positive linear form on some Hilbert completion and to use the L^2-representation of Hilbert spaces. The construction of S_γ motivated different approaches in the treatment of the moment problem. The continued fractions, the positivity of Hankel matrices and the decomposition of positive polynomials played a crucial role in this treatment [**1**, **4**, **6**, **7**, **8**, **13**, **14**] and [**15**].

The linear moment problem associated with $\gamma = \{\gamma_n\}_{n\geq 0}$ entails finding a Hilbert space $\mathcal{H}$, a self-adjoint operator $A \in \mathcal{L}(\mathcal{H})$ and a nonzero vector $x \in \mathcal{H}$ satisfying

$$\gamma_n = \langle A^n x, x\rangle, \quad \text{for} \quad n \geq 0. \tag{4}$$

Using the spectral representation of self-adjoint operators, one can show easily that the moment problems (2) and (4) are equivalent (see [**5**] for example).

The study of the moment problem for recursive sequences is motivated by the so-called "truncated moment problems" treated by R. Curto and L. Fialkow in [**6**] and [**7**]. It is known that the moment problem for recursive sequences is equivalent to the truncated moment problem, and that a necessary condition for (2) (or for (4)) to have a solution is that P_γ has only simple roots. The moment problem (2) for recursive sequences corresponds to the case where K is a finite set (see [**3**], [**9**] and [**10**] for example). We will omit any reference to the set K in this paper.

We investigate in this paper the moment problem for recursive sequences. The main motivation of this work is the paper of Dieudonné [**8**]; we use the functional calculus to obtain some results of Dieudonné's paper and to give explicit formulas of coefficients from [**8**], p. 6.

Section 2 is devoted to Jacobi matrices associated with recursive moment sequences. We show that (4) has a solution in finite-dimensional spaces and that the associated Jacobi matrices are of finite order. The link with continued fractions is studied in Section 3; we prove that these fractions are terminating in this case. In Section 4, we show that the analytic function associated with a recursive moment sequence is rational and we use techniques from analytic functional calculus to provide some generalizations of results from [**8**]. We discuss in Section 5 some moment problems arising from continued fractions and we give a new formula for the linear form associated with a terminating fraction.

2. Jacobi matrices associated with moment problems for recursive sequences

2.1. Jacobi matrices associated with moment problems. Let $\gamma = \{\gamma_n\}_{n\geq 0}$ be a given sequence of real numbers. Define on $\mathbb{C}[X]$, the space of all polynomials, the bilinear form

$$\langle P, Q\rangle_\gamma = \sum_{n,m} \alpha_n \bar{\beta}_m \gamma_{n+m}$$

with $P = \sum_n \alpha_n X^n$ and $Q = \sum_m \beta_m X^m$. (We suppose the upper limits in the sums are equal by completing by some zero coefficients if necessary.)

Observe that $\langle P, Q\rangle_\gamma = \langle P, HQ\rangle$ where $\langle\,\cdot\,,\cdot\,\rangle$ is the usual Euclidean inner product and $H = [\gamma_{i+j}]_{i,j\geq 0}$ is the Hankel matrix associated with γ.

If $H \geq 0$ then $\langle P, P\rangle_\gamma = \langle P, HP\rangle \geq 0$ for all $P \in \mathbb{C}[X]$ and the bilinear form $\langle\,\cdot\,,\cdot\,\rangle$ is an inner product on $\mathbb{C}[X]$. This defines a norm $\|\cdot\|_\gamma$ on $\mathbb{C}[X]$ when H is positive definite. Denote by $\mathcal{H}^\gamma$ the Hilbert completion of $(\mathbb{C}[X], \|\cdot\|_\gamma)$ and by $\overline{A}$ the unique extension to $\mathcal{H}^\gamma$ of the densely defined operator A on $\mathbb{C}[X]$ by $AX^n = X^{n+1}$. If $\overline{A}$ is self-adjoint, A is called essentially self-adjoint and $\overline{A}$ answers positively to (4). Otherwise, $\overline{A}$ has self-adjoint extensions and (4) is again solved (see [**16**]). In any orthogonal basis obtained by the Gram–Schmidt process from $\{1, X, X^2, \dots\}$, the self-adjoint extension A^γ of A, solution of (4) has a semi-infinite Jacobi matrix of the form,

$$J_\gamma = \begin{pmatrix} b_0 & c_0 & 0 & 0 & . & . & . & . \\ c_0 & b_1 & c_1 & 0 & . & . & . & . \\ 0 & c_1 & b_2 & c_2 & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & c_{r-2} & . & . \\ . & . & . & . & c_{r-2} & b_{r-1} & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{pmatrix}.$$

From this point of view, the Hamburger moment problem and the theory of semi-infinite Jacobi matrices coincide.

2.2. Finite Jacobi matrices. Let $A \in \mathcal{L}(\mathcal{H})$ be a solution of the moment problem (4) associated with the sequence γ, where $\mathcal{H}$ is a given Hilbert space. For $x \in \mathcal{H}$ satisfying (4), let $\mathcal{H}_0 = \operatorname{Span}\{x, Ax, \dots, A^n x, \dots\}$ be the invariant subspace generated by x. By the recursive relation (1) we have $\langle P_\gamma(A)x, A^n x\rangle = 0$ for every $n \geq 0$; in particular $\|P_\gamma(A)x\| = 0$. Hence $A^n x \in \operatorname{Span}\{x, Ax, \dots, A^{r-1}x\}$ for every $n \geq r$ and $\mathcal{H}_0$ is finite-dimensional. The study of the moment problem for recursive sequences is then reduced to the case of finite-dimensional Hilbert spaces. Such a link has been observed and studied in [**9**]. More precisely, we have:

Proposition 2.1. *Let γ be a recursive sequence. Then* (4) *has a solution $A \in \mathcal{L}(\mathcal{H})$ for some Hilbert space if and only if it has a solution A on some r-dimensional Hilbert space.*

It is known that H is positive definite if and only if $\det H_n > 0$ for all $n \geq 0$ where $H_n = [\gamma_{i+j}]_{0\leq i,j\leq n-1}$ are the Hankel matrices associated with the sequence γ. In the case of recursive sequences, $\det H_n = 0$ whenever

$n \geq r+1$. The process used in [**16**] is hence obstructed. We provide in this section an alternative method to avoid this obstruction.

We begin by proving an auxiliary result:

Lemma 2.1. *If $\gamma = \{\gamma_n\}_{n\geq 0}$ is a recursive sequence with characteristic polynomial P_γ, then $\langle P, Q\rangle_\gamma = 0$ for every $Q \in \mathbb{C}[X]$ if and only if $P \in (P_\gamma)$, where (P_γ) is the ideal of $\mathbb{C}[X]$ generated by P_γ.*

Proof. The reverse implication is a direct consequence of the relation (1).

Suppose that $\langle P, Q\rangle_\gamma = 0$ for any $Q \in \mathbb{C}[X]$. Then by writing $P = QP_\gamma + R$ and $R = \sum_{i=0}^{p} \alpha_i X^i$ where $\alpha_p \neq 0$ and $p < r$, we obtain $\langle R, X^n\rangle_\gamma = \sum_{i=0}^{p} \alpha_i \gamma_{n+i} = 0$ for every $n \geq 0$. Hence $\gamma_{n+1} = \sum_{i=0}^{p-1} a_i \gamma_{n-p+i-1}$ with $a_i = (-\alpha_i/\alpha_p)$, which implies that R is a characteristic polynomial of γ with degree less than $r-1$. Contradiction.

An immediate consequence is the following corollary:

Corollary 2.1. *Suppose $P_1 = Q_1 P_\gamma + R_1 \in \mathbb{C}[X]$ and $P_2 = Q_2 P_\gamma + R_2 \in \mathbb{C}[X]$. Then*

$$\langle P_1, P_2\rangle_\gamma = \langle R_1, R_2\rangle_\gamma.$$

Set $\mathcal{H}^{(\gamma)} = \mathbb{C}[X]/(P_\gamma)$ and let π be the canonical surjection of $\mathbb{C}[X]$ onto $\mathbb{C}[X]/(P_\gamma)$. Seeking simplicity, we will write $P = \pi(P)$. If H_r is positive definite, then the bilinear form $\langle P, Q\rangle_\gamma := \langle \pi(P), \pi(Q)\rangle_\gamma$ for $P, Q \in \mathbb{C}[X]$, is an inner product on $\mathcal{H}^{(\gamma)}$.

Let $A \in \mathcal{L}(\mathcal{H}^{(\gamma)})$ be given by $AX^j = X^{j+1}$ for $j = 0, 2, \ldots, r-1$. We have

$$\langle P, AQ\rangle_\gamma = \langle P, S_r Q\rangle_\gamma,$$

where $S_r = [\gamma_{i+j+1}]_{0\leq i,j\leq r-1}$; in particular,

$$\langle A^n 1|1\rangle = \gamma_n \ \text{ for } \ n = 0, 1, \ldots, r-1.$$

On the other hand, $A^r 1 = X^r = \sum_{j=0}^{r-1} a_j X^{r-j-1}$. Consequently,

$$\langle A^r 1|1\rangle = \sum_{j=0}^{r-1} a_j \langle X^{r-j-1}|1\rangle = \sum_{j=0}^{r-1} a_j \gamma_{r-j-1} = \gamma_r.$$

By induction we establish that $\langle A^n 1|1\rangle = \gamma_n$, for $n \geq 0$.

Thus:

Proposition 2.2. *Let $\gamma = \{\gamma_n\}_{n\geq 0}$ be a recursive sequence with positive definite Hankel matrix H_r and P_γ as a characteristic polynomial. Then there exist a $(\deg P_\gamma)$-dimensional Hilbert space $\mathcal{H}^{(\gamma)}$ and a self-adjoint operator A on $\mathcal{H}^{(\gamma)}$ providing a solution of the Hamburger moment problem (4). Moreover, if S_r is positive definite, then $A \geq 0$, and this yields a solution of the Stieltjes moment problem.*

Let $\{P_0, P_1, \ldots, P_{r-1}\}$ be the orthogonal basis of $\mathcal{H}^{(\gamma)}$, obtained from the basis $\{1, X, X^2, \ldots, X^{r-1}\}$ by the Gram–Schmidt process of the form

$$P_i(X) = X^i + \textit{ lower order,} \quad \text{for } i = 0, 1, \ldots, r-1.$$

It is clear that $\langle XP_i, P_j\rangle = \langle P_i, XP_j\rangle = 0$, for $j > i+1$ and $j < i-1$. It follows that for suitable sequences $\{b_n\}_{0\le n\le r-1}$ and $\{c_n\}_{0\le n\le r-1}$ (with $P_{-1}(X) = 0$ and $P_r(X) = 0$), we have

$$XP_n(X) = c_nP_{n+1}(X) + b_nP_n(X) + c_{n-1}P_{n-1}(X), \quad \text{for } n = 0, 1, \ldots, r-1.$$

Thus, given a recursive sequence $\gamma = \{\gamma_n\}_{n\ge 0}$, with positive definite Hankel matrix H_r, we can find a finite-dimensional Hilbert space $\mathcal{H}^{(\gamma)}$ (with $\dim \mathcal{H}^{(\gamma)} = r$), an orthogonal basis $\{P_0, P_1, \ldots, P_{r-1}\}$, real numbers $b_0, b_1, \ldots, b_{r-1}$, and positive numbers $c_0, c_1, \ldots, c_{r-2}$ such that the moment problem (4) is associated to the self-adjoint operator A on $\mathcal{H}^{(\gamma)}$ with Jacobi matrix

$$J_\gamma = \begin{pmatrix} b_0 & c_0 & 0 & 0 & . & . \\ c_0 & b_1 & c_1 & 0 & . & . \\ 0 & c_1 & b_2 & c_2 & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & c_{r-2} \\ . & . & . & . & c_{r-2} & b_{r-1} \end{pmatrix}.$$

The matrix J_γ determines uniquely the moments, since the expansion $A^kP_0 = X^k = \sum_{j=0}^{k} \xi_{kj}P_j(X)$, for $k \ge 0$, implies

$$\gamma_k = \langle A^kP_0|P_0\rangle = \xi_{k0}.$$

3. Continued fractions associated with moment problems for recursive sequences

Let $x = \sum_{j=0}^{r-1} x_je_j \in \mathcal{H}^{(\gamma)}$ be an eigenvector of the matrix J_γ associated with the eigenvalue λ. We obtain the system of r linear equations

$$\text{(5)} \qquad \left\{ \begin{array}{cl} b_0x_0 + c_0x_1 & = \lambda x_0, \\ c_0x_0 + b_1x_1 + c_1x_2 & = \lambda x_1, \\ \vdots & \\ c_{r-3}x_{r-3} + b_{r-2}x_{r-2} + c_{r-1}x_{r-1} & = \lambda x_{r-2}, \\ c_{r-2}x_{r-2} + b_{r-1}x_{r-1} & = \lambda x_{r-1}. \end{array} \right.$$

By induction we derive

$$\text{(6)} \qquad x_j = P_j(\lambda)x_0, \quad \text{for } j = 0, 1, \ldots, r-1,$$

where $\{P_j\}_{0\le j\le r-1}$ is the family of polynomials defined by $P_0 = 1$, $P_1(X) = (X - b_0)/c_0$ and the recursive relation

$$c_jP_{j+1} = (u - b_j)P_j(u) - c_{j-1}P_{j-1}(u), \quad \text{for } j = 1, \ldots, r-2.$$

We associate to the system of equations (5) the terminating fraction given by

$$\frac{1|}{|u-b_0} - \frac{c_0^2|}{|u-b_1} - \frac{c_1^2|}{|u-b_2} - \cdots - \frac{c_{r-2}^2|}{|u-b_{r-1}} \tag{7}$$

and the j-th convergent

$$\frac{A_j(u)}{B_j(u)} := \frac{1|}{|u-b_0} - \frac{c_0^2|}{|u-b_1} - \frac{c_1^2|}{|u-b_2} - \cdots - \frac{c_{j-2}^2|}{|u-b_{j-1}}, \tag{8}$$

for $1 \le j \le r$. The family $\{B_j\}_{1 \le j \le r-1}$ of polynomials satisfies

$$B_j(u) = c_0 c_1 \cdots c_{j-1} P_j(u), \quad \text{for } j = 1, 2, \ldots, r-1.$$

By setting $B_0 := 1$ and using the recursive relation involving the P_j's, we obtain

$$B_{j+1}(u) = (u - b_j)B_j(u) - c_j^2 B_{j-1}(u),$$

for $1 \le j \le r-2$. The denominator of the terminating fraction (7) is

$$B_r(u) = (u - b_{r-1})B_{r-1}(u) - c_{r-2}^2 B_{r-2}(u).$$

The B_j's and A_j's are defined by (8) if we take $B_0 = 1$, $B_1(u) = u - b_0$, $A_0 = 0$, $A_1(u) = 1/c_0$ as initial conditions. They are called the polynomials of the first and second kind, respectively.

Replacing x_{r-1} by the expression (6) in the last line of the system (5), we obtain $B_r(\lambda) = 0$ for any λ in the spectrum of A. Hence B_r is the characteristic polynomial of the matrix J_γ (see also [**8**], for example). On the other hand, from (1) we get $P_\gamma(A) = 0$ via easy computations. Since $\deg P_\gamma = \deg B_r$ and the two polynomials are monic we obtain $P_\gamma = B_r$. Thus:

Proposition 3.1. *With the preceding notation, B_r is the characteristic polynomial of the operator A. In particular,*

- *B_r has only simple roots;*
- *$A \ge 0$ if and only if $Z(P_\gamma) \subset \mathbb{R}^+$, with $Z(P_\gamma)$ the set of zeros of P_γ.*

Proposition 3.1 can be regarded as the solution of the Stieltjes moment problem.

4. Analytic function associated with moment problems

4.1. Analytic functional calculus for recursive sequences. For a moment sequence $\gamma = \{\gamma_n\}_{n\ge 0}$, the formal series $f_\gamma(z) = \sum_{n\ge 0}(-1)^n \gamma_n z^n$ canonically associated with the moment sequence γ is called the Hamburger series in the case of the Hamburger moment problem. It is easy to check that

$$f_\gamma(z) = \int \frac{d\mu(t)}{1+tz},$$

where μ is the measure solution of (2) (see [**2**], p. 208, for details).

Proposition 4.1. *Let γ be a moment sequence. Then γ is a recursive sequence if and only if f_γ is a rational function. More precisely, $f_\gamma = P/Q$, where Q is a polynomial of degree r with only simple roots and* $\deg P < \deg Q$.

Proof. Suppose that γ is a recursive sequence. By [**3**] or [**9**], we have $\mu = \sum_{n=0}^{r-1} \rho_n \delta_{z_n}$. Hence,

$$f_\gamma(z) = \int \frac{d\mu(t)}{1+tz} = \sum_{n=0}^{r-1} \frac{\rho_n}{1+z_n z} = \frac{P(z)}{Q(z)},$$

with $Q(z) = \prod_{n=0}^{r-1}(1+z_n z)$, and $f_\gamma = P/Q$ is a rational function with $\deg P < \deg Q$. Conversely, write $f_\gamma = P/Q$ and set $Q(z) = 1 + a_0 z + \cdots + a_{r-1} z^r$. Then

$$P(z) = \sum_{n\geq 0} (-1)^n \gamma_n z^n (1 + a_0 z + \cdots + a_{r-1} z^r).$$

Thus, for $n \geq r$ we have

$$(-1)^n \gamma_n + (-1)^{n-1} a_0 \gamma_{n-1} + (-1)^{n-2} a_1 \gamma_{n-2} + \cdots + (-1)^{n-r} a_{r-1}\gamma_{n-r} = 0,$$

or equivalently

$$\gamma_n = a_0 \gamma_{n-1} - a_1 \gamma_{n-2} + \cdots + (-1)^r a_{r-1} \gamma_{n-r}. \tag{9}$$

The desired result is obtained.

If $f(z) = \sum_{n=0}^{\infty} \gamma_n z^n = P/Q$, with $\deg P \geq \deg Q$, we see by writing $fQ = P$ that (9) is valid for n large enough. The sequence γ is recursively defined by Q.

Corollary 4.1. *Let γ be a recursive sequence and f_γ its Hamburger associated function. Then*

$$\frac{1}{z} f_\gamma\left(\frac{1}{z}\right) = \frac{A_r(-z)}{B_r(-z)}.$$

Proof. Proposition 4.1 implies that $\frac{1}{z} f_\gamma(\frac{1}{z})$ is rational. By writing

$$\frac{A_j(z)}{B_j(z)} = \sum_{p=0}^{\infty} \frac{(-1)^p c_p^j}{z^{p+1}}$$

at infinity for $1 \leq j \leq r$, we have $c_p^j = \gamma_p$ for $p \leq j$, by [**8**]. In particular, $c_p^r = \gamma_p$ for $p \leq r$. Therefore, γ and $(c_p^r)_{p\geq 0}$ are recursive sequences, associated with the same initial conditions and characteristic polynomial. The required assertion is proved.

The following lemma will be used to prove the main result of this section:

Lemma 4.1. *Let A be as in (4) and let $z \in \mathbb{C}$ be such that $|z| > \|A\|$. Then*

$$\big\langle (A - zI)^{-1}x,\, x\big\rangle = \frac{-1}{z} f_\gamma\left(\frac{-1}{z}\right) = \frac{A_r(z)}{B_r(z)}. \tag{10}$$

Proof.

$$\begin{aligned}\big\langle (A - zI)^{-1}x,\, x\big\rangle &= \frac{1}{z}\left\langle \left(\frac{1}{z}A - I\right)^{-1} x, x\right\rangle = \frac{-1}{z}\sum_{n\geq 0}\langle A^n x, x\rangle\left(\frac{1}{z}\right)^n \\ &= \frac{-1}{z}\sum_{n\geq 0}\gamma_n\left(\frac{1}{z}\right)^n = \frac{-1}{z} f_\gamma\left(\frac{-1}{z}\right).\end{aligned}$$

The second equality is trivial from Corollary 4.1.

Using this lemma we obtain:

Proposition 4.2. *For any entire function f, denote by $f(A)$ the operator defined by the Riesz functional calculus. Then*

$$\langle f(A)x, x\rangle = \sum_{z_j\in\sigma(A)} f(z_j)\frac{A_r(z_j)}{{B_r}'(z_j)},$$

where $\sigma(A)$ is the spectrum of A.

Proof. For $R > \|A\|$, let $\Gamma_R = \{z \in \mathbb{C} \ :\ |z| = R\}$. We have

$$f(A) = \frac{1}{2i\pi}\int_{\Gamma_R} f(z)(A - zI)^{-1}\, dz.$$

Then

$$\begin{aligned}\langle f(A)x, x\rangle &= \frac{1}{2i\pi}\int_{\Gamma_R}\langle f(z)(A - zI)^{-1}x, x\rangle\, dz \\ &= \frac{1}{2i\pi}\int_{\Gamma_R} f(z)\langle (A - zI)^{-1}x, x\rangle\, dz \\ &= \frac{1}{2i\pi}\int_{\Gamma_R} f(z)\frac{A_r(z)}{B_r(z)}\, dz \qquad \text{(by (10))} \\ &= \sum_{z_j\in Z(B_r)=\sigma(A)} f(z_j)\frac{A_r(z_j)}{{B_r}'(z_j)} \qquad \text{(by the Residue Theorem).}\end{aligned}$$

Remark. As $\sigma(A) = Z(B_r)$ is a finite set of eigenvalues, the spectral measure associated with the self-adjoint operator A is given by the orthonormal projection onto the associated eigenspaces. This fact is also derived from Proposition 4.3.

For any entire function f and complex number u, we denote by $L_u(f)$ the holomorphic function defined as follows:

$$L_u(f)(z) = \begin{cases} \dfrac{f(z)-f(u)}{z-u} & \text{if } z \neq u, \\ f'(u) & \text{if } z = u. \end{cases} \tag{11}$$

Let S_γ be the linear form defined by (3), associated with the linear moment sequence γ.

The following proposition unifies some results of [**8**].

Proposition 4.3. *For any holomorphic function f, we have*

$$S_\gamma(L_u(fB_r)) = f(u)A_r(u).$$

Proof. As in the proof of Proposition 4.2, we have

$$\begin{aligned} S_\gamma(L_u(fB_r)) &= \langle (L_u(fB_r)(A)x, x\rangle \\ &= \frac{1}{2i\pi}\int\limits_{\Gamma_R} \frac{f(u)B_r(u)-f(z)B_r(z)}{u-z}\frac{A_r(z)}{B_r(z)}\,dz \\ &= \frac{1}{2i\pi}\int\limits_{\Gamma_R} \frac{f(u)B_r(u)}{u-z}\frac{A_r(z)}{B_r(z)}\,dz - \frac{1}{2i\pi}\int\limits_{\Gamma_R}\frac{f(z)}{u-z}A_r(z)\,dz \\ &= f(u)A_r(u) + \frac{f(u)B_r(u)}{2i\pi}\int\limits_{\Gamma_R}\frac{A_r(z)}{(u-z)B_r(z)}\,dz \\ &= f(u)A_r(u) - f(u)B_r(u)\left(\frac{A_r(u)}{B_r(u)} - \sum_{z_j\in Z(B_r)}\frac{A_r(z_j)}{B_r{}'(z_j)}\frac{1}{u-z_j}\right) \\ &= f(u)A_r(u), \end{aligned}$$

since the expression in big parentheses on the penultimate line is 0.

We derive two corollaries:

Corollary 4.2 ([**8**], Theorem 1 (17))**.** *With the same notation as in Proposition* 4.3*, we have*

$$S_\gamma(L_u(B_r)) = A_r(u).$$

Proof. $f \equiv 1$ in Proposition 4.3.

Corollary 4.3 ([**8**], p. 6)**.** *For any polynomial P, we have*

$$S_\gamma(P) = \sum_{z_j\in Z(B_r)} S_\gamma(L_{z_j}(B_r))\frac{P(z_j)}{B_r{}'(z_j)} = \sum_{z_j\in Z(B_r)} A_r(z_j)\frac{P(z_j)}{B_r{}'(z_j)}.$$

Proof. Combine Proposition 4.2 with the preceding corollary.

5. Moment problems associated with limited continued fractions

In this section, we use the preceding section to shed some light on the moment problem arising from the terminating fraction (7).

Consider the limited Jacobi fraction

$$\frac{1|}{|u-b_0} - \frac{c_0^2|}{|u-b_1} - \frac{c_1^2|}{|u-b_2} - \cdots - \frac{c_{r-2}^2|}{|u-b_{r-1}}, \tag{12}$$

where $b_0, b_1, \ldots, b_{r-1}$ are real and $c_0, c_1, \ldots, c_{r-2}$ are nonzero real numbers. Let

$$J = \begin{pmatrix} b_0 & c_0 & 0 & 0 & . & . \\ c_0 & b_1 & c_1 & 0 & . & . \\ 0 & c_1 & b_2 & c_2 & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & c_{r-2} \\ . & . & . & . & c_{r-2} & b_{r-1} \end{pmatrix}$$

be the Jacobi matrix associated with (12) and consider the operator A associated with the matrix J defined on an r-dimensional Hilbert space $\mathcal{H}$. For $x \in \mathcal{H}$ a nonzero vector, the sequence $\gamma_n(x) = \langle A^n x, x\rangle$ is clearly a recursive moment sequence defined by its r initial conditions and the characteristic polynomial of the matrix J.

Given $\gamma_0, \ldots, \gamma_{r-1}$ some real numbers, is there $x \in \mathcal{H}$ such that $\gamma_j = \gamma_j(x)$ for $0 \le j \le r-1$?

Suppose such an x exists and write

$$x = \sum_{j=0}^{r-1} \rho_j x_j,$$

where $\{x_j : j = 0, \ldots, r-1\}$ is the orthonormal basis of eigenvectors of A associated with the eigenvalues $\{z_j : j = 0, \ldots, r-1\}$. Therefore,

$$\langle A^n x, x\rangle = \sum_{j=0}^{r-1} \rho_j^2 z_j^n = \gamma_n.$$

Let $\{\gamma_n\}_{0\le n\le r-1}$ be the initial conditions of the recursive sequence $\{\gamma_n\}_{n\ge 0}$ associated with the characteristic polynomial $P(X) = \prod_{j=0}^{r-1}(X - z_j)$. By Theorem 3 of [**8**], the sequence $\{\gamma_n\}_{n\ge 0}$ is associated with a positive linear form if and only if the Hankel matrix $H_r = [\gamma_{i+j}]_{0\le i,j\le r-1}$ is positive definite.

Let $A_j(u)$ and $B_j(u)$ be defined as in (8). It is known that there exists a linear functional S, defined on polynomials, that orthogonalizes the B_j's. That is,

$$S(B_iB_j) = 0, \quad \text{for } 0 \le i < j \le r-1.$$

Under the additional assumption

$$S(B_n^2) = c_0c_1 \cdots c_n, \quad \text{for } 0 \le n \le r-1,$$

S is unique and satisfies the following property:

Theorem 5.1. *There exist positive numbers $\lambda_0, \ldots, \lambda_{r-1}$ such that*

$$S(P) = \sum_{j=0}^{r-1} \lambda_j P(z_j),$$

for every polynomial P.

In view of Corollary 4.3 we have $\lambda_j = A_r(z_j)/B_r{}'(z_j) > 0$. Thus:

Proposition 5.1. *With the same notations, we have*

$$S(X^n) = \langle A^n x, x\rangle = \int_{I\!R} t^n d\mu(t),$$

with

$$x = \sum_{j=0}^{r-1} \sqrt{\frac{A_r(z_j)}{B_r{}'(z_j)}} x_j \quad \text{and} \quad \mu = \sum_{j=0}^{r-1} \frac{A_r(z_j)}{B_r{}'(z_j)} \delta_{z_j}.$$

Moreover, B_r is the characteristic polynomial of $\{S(X^n)\}_{n\geq 0}$.

In the spirit of [**12**], it will be of interest to write the results of this paper for multivariable truncated moment sequences. The authors are thankful to the anonymous referee for pointing out reference [**12**] and for his suggestions and remarks.

Acknowledgment. The first and the third authors were supported by Abdus Salam International Centre for Theoretical Physics. They express their thanks to the ICTP Mathematics Section, especially to Prof. C. E. Chidume.

References

[1] N.I. Akhiezer, *The Classical Moment Problem and Some Related Questions in Analysis*, Hafner Publ. Co., New York, 1965, MR 0184042 (32 #1518), Zbl 0173.41001.

[2] G.A. Baker, Jr. and P. Graves-Morris, *Padé Approximants. Part* I. *Basic theory*, Encyclopedia of Mathematics and its Applications, **13**, Addison-Wesley Publishing Co., Reading, Mass., 1981, MR 0635619 (83a:41009a), Zbl 0603.30044.

[3] R. Ben Taher, M. Rachidi and E.H. Zerouali, *Recursive subnormal completion and the truncated moment problem*, Bull. London Math. Soc., **33**(**4**) (2001), 425–432, MR 1832554 (2003c:44009), Zbl 1027.44003.

[4] G. Cassier, *Problème des moments sur un compact de* $\mathbf{R}^n$ *et décomposition de polynômes à plusieurs variables*, J. Funct. Anal., **58**(**3**) (1984), 254–266, MR 0759099 (85m:44004), Zbl 0556.44006.

[5] J.B. Conway, *A Course in Functional Analysis*, Graduate Texts in Mathematics, **96**, second edition, Springer-Verlag, 1990, MR 1070713 (91e:46001), Zbl 0706.46003.

[6] R. Curto and L. Fialkow, *Recursiveness, positivity, and truncated moment problems*, Houston J. Math., **17**(**4**) (1991), 603–635, MR 1147276 (93a:47016), Zbl 0757.44006.

[7] R. Curto and L. Fialkow, *Flat extensions of positive moment matrices: Recursively generated relations*, Mem. Amer. Math. Soc., **136**(**648**) (1998), MR 1445490 (99d:47015), Zbl 0913.47016.

[8] J. Dieudonné, *Fraction continuées et polynômes orthogonaux dans l'œuvre de E.N. Laguerre*, Proceeding of the Laguerre symposium held at Bar-le-Duc (October 15-18, 1984), 1985, Lecture Notes in Math., **1171**, Springer, Berlin, 1985, 1–15, MR 0838968 (87g:01039c), Zbl 0665.42024.

[9] B. El Wahbi and M. Rachidi, *r-generalized Fibonacci sequences and the linear moment problem*, Fibonacci Quart., **38**(**5**) (2000), 386–394, MR 1796554 (2001h:11015), Zbl pre01584262.

[10] B. El Wahbi and M. Rachidi, *On r-generalized Fibonacci sequences and Hausdorff moment problem*, Fibonacci Quart., **39**(**1**) (2001), 5–11, MR 1812614 (2001k:44009), Zbl 0989.44003.

[11] Yu.I. Lyubich, *Linear Functional Analysis*, Funktional'nyj Analiz., Itogi Nauki i Tekhniki **19**, Publisher VINITI, Moscow 1988, MR 0981366 (90j:00004b), Zbl 0683.46001.

[12] M. Putinar, *A note on Tchakaloff's theorem*, Proc. Amer. Math. Soc., **125** (1997), 2409–2414, MR 1389533 (97j:65045), Zbl 0886.41025.

[13] M. Putinar and F.-H. Vasilescu, *Solving moment problems by dimensional extension*, Ann. of Math. (2), **149**(**3**) (1999), 1087–1107, MR 1709313 (2001c:47023), Zbl 0939.44003.

[14] J.A. Shohat and J.D. Tamarkin, *The Problem of Moments*, Amer. Math. Soc. Surveys, **2**, American Mathematical Society, New York, 1943, MR 0008438 (5,5c), Zbl 0063.06973.

[15] T. Stieltjes, *Recherches sur les fractions continues*, Anns. Fac. Sci. Univ. Toulouse, **8**, J1–J122; **9** (1894–1895), A5–A47, MR 1623484 (99d:01041), Zbl 0861.01036, MR 1607517 (99d:01040), Zbl 0861.01037.

[16] B. Simon, *The classical moment problem as a self-adjoint finite difference operator*, Adv. Math., **137** (1998), 82–203, MR 1627806 (2001e:47020), Zbl 0910.44004.

Received March 10, 2003 and revised October 29, 2003.

Département de mathématiques et informatique
Faculté des Sciences de Tétouan
Tétouan
Morocco
E-mail address: elwahbi@hotmail.com

Département de mathématiques et informatique
Faculté des Sciences de Rabat
Rabat
Morocco
E-mail address: rachidi@fsr.ac.ma

Département de mathématiques et informatique
Faculté des Sciences de Rabat
Rabat
Morocco
E-mail address: zerouali@fsr.ac.ma