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Córdoba–Fefferman collections are defined and used to characterize functions whose corresponding maximal functions are locally integrable. Córdoba–Fefferman collections are also used to show that, if M_x and M_y respectively denote the one-dimensional Hardy–Littlewood maximal operators in the horizontal and vertical directions in $\mathbf{R}^2$, M_{HL} denotes the standard Hardy–Littlewood maximal operator in $\mathbf{R}^2$, and f is a measurable function supported in the unit square $Q = [0,1]\times[0,1]$, then $\int_Q M_{\mathrm{HL}} f \sim \int_Q M_x f + \int_Q M_y f$.

We begin by introducing the following definition:

Definition 1. Let β be a countable collection of Lebesgue measurable subsets of the unit n-cube I^n in $\mathbf{R}^n$ of positive measure. A (possibly finite) subset $\{R_i\}$ of β is said to be a Córdoba–Fefferman collection with respect to β (denoted by $\{R_i\} \in \mathrm{CFC}(\beta)$) if and only if there exists an enumeration $\widetilde{R}_1, \widetilde{R}_2, \widetilde{R}_3, \dots$ of the elements of $\{R_i\}$ such that $\bigl|\widetilde{R}_i \cap \bigcup_{j<i} \widetilde{R}_j\bigr| \le \frac{1}{2}|\widetilde{R}_i|$ for each $i = 2, 3, 4, \dots$.

A. Córdoba and R. Fefferman used what we are now calling Córdoba–Fefferman collections in [1] to characterize geometric maximal operators that are of weak type (p,p) for $p > 1$. The purpose of this paper is to show that Córdoba–Fefferman collections may also be used to estimate the integrals of maximal functions. The primary result in this regard is the following:

Theorem 2. *Let β be a countable collection of Lebesgue measurable subsets of the unit n-cube I^n in $\mathbf{R}^n$ of positive measure. Let β be such that for any point x in I^n, $x \in R$ for some $R \in \beta$. Define the maximal operator M_β on $L^1(I^n)$ by*

$$M_\beta f(x) = \sup_{x \in R \in \beta} \frac{1}{|R|} \int_R |f(y)|\, dy. \tag{1}$$

Suppose M_β satisfies the (Tauberian) condition

$$\bigl|\{x \in I^n : M_\beta \chi_E(x) \ge \tfrac{1}{2}\}\bigr| \le C_\beta |E| \tag{2}$$

for all measurable sets $E \subset I^n$. Then if $f \in L^1(I^n)$,

$$\int_{I^n} M_\beta f \sim \sup_{\{R_i\}\in \mathrm{CFC}(\beta)} \int_{I^n} |f| \sum_i \chi_{R_i}. \tag{3}$$

In particular,

$$\frac{1}{2} \sup_{\{R_i\}\in \mathrm{CFC}(\beta)} \int_{I^n} |f| \sum_i \chi_{R_i} \leq \int_{I^n} M_\beta f$$
$$\leq 4\,C_\beta \sup_{\{R_i\}\in \mathrm{CFC}(\beta)} \int_{I^n} |f| \sum_i \chi_{R_i}.$$

Proof. We assume without loss of generality that $f \in L^\infty(I^n)$, $f \not\equiv 0$. We begin by showing

$$\int_{I^n} M_\beta f \leq 4\,C_\beta \sup_{\{R_i\}\in \mathrm{CFC}(\beta)} \int_{I^n} |f| \sum_i \chi_{R_i}.$$

Let $\epsilon > 0$. It suffices to show there exists $\{R_i\} \in \mathrm{CFC}(\beta)$ such that

$$4\,C_\beta \int_{I^n} |f| \sum_i \chi_{R_i} \geq \int_{I^n} M_\beta f \;-\; \epsilon.$$

Let m be a positive integer such that

$$0 \leq \int_{\left\{x\in I^n : M_\beta f(x) > 2^m\right\}} M_\beta f < \frac{\epsilon}{3}.$$

Let $R_1, R_2, R_3, \ldots$ be an enumeration of the elements of β. Let $R_{1,1}$ be the first element on the list of the R_i such that

$$\frac{1}{|R_{1,1}|} \int_{R_{1,1}} |f| > 2^m.$$

Assuming $R_{1,1}, R_{1,2}, \ldots, R_{1,k}$ have been chosen, let $R_{1,k+1}$ be the first element on the list of the R_i such that

$$\left| R_{1,k+1} \cap \bigcup_{i=1}^{k} R_{1,i} \right| \leq \tfrac{1}{2} \left| R_{1,k+1} \right|$$

and

$$\frac{1}{|R_{1,k+1}|} \int_{R_{1,k+1}} |f| > 2^m.$$

(If such an element of β does not exist, we stop the selection procedure at this point.) In this manner, a (possibly finite) sequence $R_{1,1}, R_{1,2}, \ldots$ is attained.

Let j_1 be an integer such that

$$\left| \bigcup_{i=1}^{j_1} R_{1,i} \right| > \tfrac{1}{2} \left| \bigcup_{i=1}^{\infty} R_{1,i} \right|.$$

We renumerate β (allowing for multiple counting of individual elements) as

$$(*_2) \qquad R_{1,1}, R_{1,2}, \ldots, R_{1,j_1}, R_1, R_2, R_3, \ldots .$$

Now let $R_{2,1} = R_{1,1}$. Let $R_{2,2}$ be the first element on the list $(*_2)$ such that $|R_{2,2} \cap R_{2,1}| \leq \frac{1}{2}|R_{2,2}|$ and

$$\frac{1}{|R_{2,2}|}\int_{R_{2,2}} |f| > 2^{m-1}.$$

Assuming $R_{2,1}, R_{2,2}, \ldots, R_{2,k}$ have been selected, let $R_{2,k+1}$ be the first element on the list $(*_2)$ such that

$$\left|R_{2,k+1} \cap \bigcup_{i=1}^{k} R_{2,i}\right| \leq \tfrac{1}{2}|R_{2,k+1}|$$

and

$$\frac{1}{|R_{2,k+1}|}\int_{R_{2,k+1}} |f| > 2^{m-1}.$$

In this manner the sequence $R_{2,1}, R_{2,2}, \ldots$ is generated.

Let $j_2 \geq j_1$ be an integer such that

$$\left|\bigcup_{i=1}^{j_2} R_{2,i}\right| > \tfrac{1}{2}\left|\bigcup_{i=1}^{\infty} R_{2,i}\right|.$$

Note that $R_{1,1} = R_{2,1}, R_{1,2} = R_{2,2}, \ldots, R_{1,j_1} = R_{2,j_1}$.

We continue inductively. Assume that $R_{n,1}, R_{n,2}, \ldots, R_{n,j_n}$ have been selected. We renumerate β as

$$(*_{n+1}) \qquad R_{n,1}, R_{n,2}, \ldots, R_{n,j_n}, R_1, R_2, R_3, \ldots .$$

Let $R_{n+1,1} = R_{n,1}$. Let $R_{n+1,2}$ be the first element on the list $(*_{n+1})$ such that $|R_{n+1,2} \cap R_{n+1,1}| \leq \frac{1}{2}|R_{n+1,2}|$ and

$$\frac{1}{|R_{n+1,2}|}\int_{R_{n+1,2}} |f| > 2^{m-n}.$$

Assuming $R_{n+1,1}, \ldots, R_{n+1,k}$ have been selected, let $R_{n+1,k+1}$ be the first element on the list $(*_{n+1})$ such that

$$\left|R_{n+1,k+1} \cap \bigcup_{i=1}^{k} R_{n+1,i}\right| \leq \tfrac{1}{2}|R_{n+1,k+1}|$$

and

$$\frac{1}{|R_{n+1,k+1}|}\int_{R_{n+1,k+1}} |f| > 2^{m-n}.$$

In this manner, a sequence $R_{n+1,1}, R_{n+1,2}, \ldots$ is selected. Let $j_{n+1} \geq j_n$ be an integer such that

$$\left|\bigcup_{i=1}^{j_{n+1}} R_{n+1,i}\right| > \tfrac{1}{2}\left|\bigcup_{i=1}^{\infty} R_{n+1,i}\right|.$$

Note that $R_{n,1} = R_{n+1,1},\ R_{n,2} = R_{n+1,2},\ \ \dots,\ R_{n,j_n} = R_{n+1,j_n}$. This is clear, as $R_{n,1}, R_{n,2}, \dots, R_{n,j_n}$ are the first j_n elements of β chosen in the procedure for selecting the $R_{n+1,i}$.

We now relate $\left|\bigcup_{i=1}^{j_{n+1}} R_{n+1,i}\right|$ to $|\{x \in I^n : M_\beta f(x) > 2^{m-n}\}|$. Suppose for some $p \in I^n$ that $M_\beta f(p) > 2^{m-n}$. Then $\frac{1}{|R|}\int_R |f| > 2^{m-n}$ for some $R \in \beta$. Then $\left|R \cap \bigcup_{i=1}^{\infty} R_{n+1,i}\right| \geq \frac{1}{2}|R|$. Hence

$$M_\beta\big(\chi_{\bigcup_{i=1}^{\infty} R_{n+1,i}}\big)(p) \geq \tfrac{1}{2}.$$

Since

$$\left|\left\{x \in I^n : M_\beta \chi_E(x) \geq \tfrac{1}{2}\right\}\right| \leq C_\beta |E|$$

for all measurable sets $E \subset I^n$, we see that

(4)
$$\left|\left\{x \in I^n : M_\beta f(x) > 2^{m-n}\right\}\right| \leq C_\beta \left|\bigcup_{i=1}^{\infty} R_{n+1,i}\right| \leq 2\, C_\beta \left|\bigcup_{i=1}^{j_{n+1}} R_{n+1,i}\right|.$$

We now let l be a positive integer such that $2^{m-l} < \epsilon/3$. Then

$$\left|\int_{\left\{x \in I^n : 2^{m-l} < M_\beta f(x) < 2^m\right\}} M_\beta f - \int_{I^n} M_\beta f\right| < \frac{2\epsilon}{3}.$$

We now compare $\int_{\left\{x \in I^n : 2^{m-l} < M_\beta f(x) < 2^m\right\}} M_\beta f$ to $\int_{I^n} |f| \sum_{i=1}^{j_{l+1}} \chi_{R_{l+1,i}}$. Set

$$\lambda(\alpha) = \left|\left\{x \in I^n : M_\beta f(x) > \alpha\right\}\right|,$$

$$\mu(\alpha) = \left|\left\{x \in I^n : |f(x)| \cdot \sum_{i=1}^{j_{l+1}} \chi_{R_{l+1,i}}(x) > \alpha\right\}\right|,$$

$$\omega(\alpha) = \left|\left\{x \in I^n : \sum_{i=1}^{j_{l+1}} \left(\frac{1}{|R_{l+1,i}|}\int_{R_{l+1,i}} |f|\right) \cdot \chi_{R_{l+1,i}}(x) > \alpha\right\}\right|.$$

Suppose $2^{m-l} \leq \alpha \leq 2^m$. Let r be the largest integer such that $2^r \leq \alpha$. Clearly $m - l \leq r \leq m$. Now

$$R_{l+1,1} = R_{m-r+1,1},\ R_{l+1,2} = R_{m-r+1,2},\ \dots,\ R_{l+1,j_{m-r+1}} = R_{m-r+1,\, j_{m-r+1}}.$$

Also, by (4) we have

$$\lambda(2^r) \leq 2\, C_\beta \left|\bigcup_{i=1}^{j_{m-r+1}} R_{m-r+1,i}\right|.$$

Since

$$\frac{1}{|R_{m-r+1,i}|}\int_{R_{m-r+1,i}} |f| > 2^r \quad \text{for } i = 1, 2, \dots, j_{m-r+1},$$

we get $\lambda(2^r) \le 2\,C_\beta\,\omega(2^r)$. Hence $\lambda(\alpha) \le 2\,C_\beta\,\omega\left(\frac{\alpha}{2}\right)$ for $2^{m-l} \le \alpha \le 2^m$. So

$$\begin{aligned}\int_{\left\{x\in I^n : 2^{m-l} < M_\beta f(x) < 2^m\right\}} M_\beta f &\le \int_{2^{m-l}}^{2^m} \lambda(\alpha)\,d\alpha + \frac{\epsilon}{3}\\ &\le 2\,C_\beta \int_{2^{m-l}}^{2^m} \omega\left(\frac{\alpha}{2}\right)\,d\alpha + \frac{\epsilon}{3}\\ &\le 4\,C_\beta \int_0^\infty \omega(\alpha)\,d\alpha + \frac{\epsilon}{3}\\ &= 4\,C_\beta \int_0^\infty \mu(\alpha)\,d\alpha + \frac{\epsilon}{3}\\ &= 4\,C_\beta \int_{I^n} |f| \cdot \sum_{i=1}^{j_{l+1}} \chi_{R_{l+1,i}} + \frac{\epsilon}{3}.\end{aligned}$$

Hence

$$\int_{I^n} M_\beta f \le 4\,C_\beta \int_{I^n} |f| \sum_{i=1}^{j_{l+1}} \chi_{R_{l+1,i}} + \epsilon.$$

As ϵ is an arbitrary positive real number and $\{R_{\ell+1,i}\} \in \mathrm{CFC}(\beta)$, we see that

$$\int_{I^n} M_\beta f \le 4\,C_\beta \sup_{\{R_i\}\in \mathrm{CFC}(\beta)} \int_{I^n} |f| \sum_i \chi_{R_i},$$

as desired.

We now show that

$$\int_{I^n} M_\beta f \ge \frac{1}{2} \sup_{\{R_i\}\in \mathrm{CFC}(\beta)} \int_{I^n} |f| \sum_i \chi_{R_i}.$$

Let $\{R_i\} \in \mathrm{CFC}(\beta)$. Without loss of generality, we assume

$$\left| R_i \cap \left(\bigcup_{j=1}^{i-1} R_j\right)\right| \le \tfrac{1}{2}\,|R_i| \quad \text{for } \; i = 2, 3, \ldots .$$

It suffices to show that

$$\int_{I^n} M_\beta f \ge \frac{1}{2} \int_{I^n} |f| \cdot \sum_i \chi_{R_i}.$$

Let $E_1 = R_1$ and $E_k = R_k \setminus \bigcup_{i=1}^{k-1} R_i$ for $k = 2, 3, \ldots$. Let

$$Tf(x) = \sum_k \left(\frac{1}{|R_k|} \int_{R_k} |f(y)|\,dy\right) \chi_{E_k}(x).$$

Clearly $Tf(x) \leq M_\beta f(x)$. Also,

$$\begin{aligned}\int_{I^n} Tf &= \int_{I^n} \sum_k \left(\frac{1}{|R_k|}\int_{R_k} |f(y)|\,dy\right)\chi_{E_k}(x)\,dx \\ &= \sum_k \int_{I^n} \left(\frac{1}{|R_k|}\int_{R_k} |f(y)|\,dy\right)\chi_{E_k}(x)\,dx \\ &\geq \frac{1}{2}\sum_k \int_{I^n} \left(\frac{1}{|R_k|}\int_{R_k} |f(y)|\,dy\right)\chi_{R_k}(x)\,dx \\ &\qquad\qquad\qquad\qquad\qquad (\text{since } |E_k| \geq \tfrac{1}{2}|R_k|) \\ &= \frac{1}{2}\sum_k \int_{R_k} |f(y)|\,dy \\ &= \frac{1}{2}\int_{I^n} |f| \sum_k \chi_{R_k}.\end{aligned}$$

So

$$\int_{I^n} |f| \sum_i \chi_{R_i} \leq 2\int_{I^n} M_\beta f,$$

as desired. □

To illustrate the role of the Tauberian condition in the above theorem, we consider the following example:

For $0 < \delta < \frac{1}{10}$, define β_δ by

$$\beta_\delta = \left\{A \subset [0,1] : A = [0,\delta] \cup [x, x+\delta^2] \text{ for some } x \in [0, 1-\delta^2]\right\}.$$

Note that $C_{\beta_\delta} \gtrsim \delta^{-1}$, since $M_{\beta_\delta}(\chi_{[0,\delta]})(x) > \frac{1}{2}$ for all $x \in [0,1]$. If $\{R_i\} \in \mathrm{CFC}\,(\beta_\delta)$, then $\{R_i\}$ has only one element, say $R_1 = [0,\delta] \cup [x, x+\delta^2]$ for some $x \in [0, 1-\delta^2]$. So

$$\int_0^1 \chi_{[0,1]} \cdot \chi_{R_1} \leq 2\delta.$$

However,

$$\int_0^1 M_{\beta_\delta}\chi_{[0,1]} = 1.$$

Note that although the ratio of

$$\int_0^1 M_{\beta_\delta}\chi_{[0,1]} \quad \text{to} \quad \sup_{\{R_i\}\in \mathrm{CFC}(\beta_\delta)} \int_Q \chi_{[0,1]} \cdot \sum_i \chi_{R_i}$$

may be arbitrarily large (depending on the value of δ), the ratio is still bounded by $4C_{\beta_\delta}$.

Before indicating applications of the preceding theorem, we list some basic definitions.

Definition 3 (Hardy–Littlewood maximal function)**.** Let f be a measurable function defined on $\mathbf{R}^n$. Let $B(p,r)$ denote the Euclidean ball of radius r in $\mathbf{R}^n$ centered at p, and let $|B(p,r)|$ denote the Lebesgue measure of $B(p,r)$. The Hardy–Littlewood maximal function of f is defined on $\mathbf{R}^n$ by

$$M_{\mathrm{HL}}f(p) = \sup_{r>0} \frac{1}{|B(p,r)|} \int_{B(p,r)} |f(z)|\, dz. \tag{5}$$

Definition 4 (Strong maximal function)**.** Let f be a measurable function defined on $\mathbf{R}^2$. The strong maximal function of f is defined on $\mathbf{R}^2$ by

$$M_S f(x,y) = \sup_{\substack{x_1<x<x_2\\ y_1<y<y_2}} \frac{1}{(x_2-x_1)(y_2-y_1)} \int_{x_1}^{x_2}\int_{y_1}^{y_2} |f(u,v)|\, dv\, du. \tag{6}$$

Definition 5 (Horizontal maximal function)**.** Let f be a measurable function defined on $\mathbf{R}^2$. The horizontal maximal function of f is defined on $\mathbf{R}^2$ by

$$M_x f(u,v) = \sup_{u_1<u<u_2} \frac{1}{u_2-u_1} \int_{u_1}^{u_2} |f(w,v)|\, dw. \tag{7}$$

Definition 6 (Vertical maximal function)**.** Let f be a measurable function defined on $\mathbf{R}^2$. The vertical maximal function of f is defined on $\mathbf{R}^2$ by

$$M_y f(u,v) = \sup_{v_1<v<v_2} \frac{1}{v_2-v_1} \int_{v_1}^{v_2} |f(u,w)|\, dw. \tag{8}$$

We now turn to one of the most useful applications of Córdoba–Fefferman collection theory. In this discussion we will denote the unit square I^2 in $\mathbf{R}^2$ by Q. Also, if a given maximal operator M_β is naturally associated to a collection β, as, say, M_S is associated to the set of rectangles with sides parallel to the axes, we will frequently denote the Córdoba–Fefferman collection $\mathrm{CFC}(\beta)$ by $\mathrm{CFC}(M_\beta)$.

Suppose we are given the maximal operators M_α, M_β, and M_γ, all of which satisfy the desired Tauberian condition. Suppose also we want to show that $\int_Q M_\alpha f \lesssim \int_Q M_\beta f + \int_Q M_\gamma f$ for some measurable function f supported on Q. One strategy for doing this would be to show that, given an arbitrary collection $\{A_i\} \in \mathrm{CFC}(M_\alpha)$, one can produce $\{B_i\} \in \mathrm{CFC}(M_\beta)$ and $\{C_i\} \in \mathrm{CFC}(M_\gamma)$ such that

$$\sum_i \chi_{A_i}(p) \lesssim \sum_i \chi_{B_i}(p) + \sum_i \chi_{C_i}(p)$$

for almost every p in Q. Theorem 2 would then yield the desired result. The primary difficulty in applying this strategy is that the production of the $\{B_i\} \in \mathrm{CFC}(M_\beta)$ and $\{C_i\} \in \mathrm{CFC}(M_\gamma)$ can be a complicated matter and in some situations may not be possible. However, in many cases one can modify this strategy by using the geometry of f to choose a particular $\{A_i\} \in \mathrm{CFC}(M_\alpha)$ in such a manner that not only $\int_Q M_\alpha f \sim \int_Q |f| \sum_i \chi_{A_i}$, but also the production of the $\{B_i\} \in \mathrm{CFC}(M_\beta)$ and $\{C_i\} \in \mathrm{CFC}(M_\gamma)$ can follow in a geometrically intuitive fashion. Actually proving that $\int_Q M_\alpha f \sim \int_Q |f| \sum_i \chi_{A_i}$ often requires a duplication of large parts of the proof of Theorem 2 in the special case determined by the geometry of f and the desired properties of the collection $\{A_i\}$. We illustrate these ideas in the proof of the following lemma:

Lemma 7. *Let f be a nonnegative measurable function supported on Q such that $f(x_1, y_1) \geq f(x_2, y_2)$ whenever $0 \leq x_1 \leq x_2$, $0 \leq y_1 \leq y_2$. Then*

$$\int_Q M_{\mathrm{HL}} f \leq C \left(\int_Q M_x f + \int_Q M_y f \right).$$

for some universal constant C.

Proof. Since M_{HL} and the operators M_x, M_y are bounded on $L^2(Q)$, we may assume without loss of generality that f is smooth. Hence without loss of generality we may assume $f \in L^\infty(Q)$. Let m be the largest integer such that $2^m \leq \|f\|_{L^\infty(Q)}$. Let

$$E(2^{m-j+1}) = \{x \in Q : M_{\mathrm{HL}} f(x) > 2^{m-j+1}\}.$$

For each positive integer j and to each $p \in E(2^{m-j+1})$ associate a square $Q_{p,m-j+1}$ containing p such that

$$\frac{1}{|Q_{p,m-j+1}|} \int_{Q_{p,m-j+1}} f \; > \; 2^{m-j+1}$$

and such that one of the edges of $Q_{p,m-j+1}$ is contained in one of the coordinate axes. Note that such a square exists, since f is nonincreasing in each variable separately. Now associate to $Q_{p,m-j+1}$ a dyadic subsquare ${Q'}_{p,m-j+1}$ contained in $Q_{p,m-j+1}$ which has an edge contained in one of the coordinate axes, such that $|{Q'}_{p,m-j+1}| \geq \frac{1}{16}|Q_{p,m-j+1}|$, and such that no dyadic subsquare of $Q_{p,m-j+1}$ of the same size contains a point closer to the origin than any point of ${Q'}_{p,m-j+1}$.

If ${Q'}_{p,m-j+1}$ intersects the origin, let ${Q''}_{p,m-j+1}$ be ${Q'}_{p,m-j+1}$. Otherwise, let ${Q''}_{p,m-j+1}$ be the dyadic square with the same area as ${Q'}_{p,m-j+1}$ such that ${Q''}_{p,m-j+1}$ and ${Q'}_{p,m-j+1}$ share an edge, ${Q''}_{p,m-j+1}$ has an edge contained in one of the coordinate axes, and such that ${Q''}_{p,m-j+1}$ contains a point q which is closer to the origin than any point of ${Q'}_{p,m-j+1}$. Note that

$$\frac{1}{|{Q''}_{p,m-j+1}|} \int_{{Q''}_{p,m-j+1}} f \; > \; 2^{m-j+1}. \tag{9}$$

Let $16{Q''}_{p,m-j+1}$ be the square concentric to ${Q''}_{p,m-j+1}$ whose sides are parallel to the axes and whose volume is 16^2 times that of ${Q''}_{p,m-j+1}$. Note that $Q_{p,m-j+1} \subset 16{Q''}_{p,m-j+1}$.

For each positive integer j, let $Q_{1,m-j+1}$, $Q_{2,m-j+1}$, ... enumerate the squares in $\{Q''_{p,m-j+1} : p \in E(2^{m-j+1})\}$ that are not properly contained in any of the other squares in $\{Q''_{p,m-j+1} : p \in E(2^{m-j+1})\}$. Since the $Q_{i,m-j+1}$ are dyadic, the interiors of the squares $Q_{1,m-j+1}$, $Q_{2,m-j+1}$, ... are disjoint. Also

$$\left|\bigcup_{i=1}^{\infty} Q_{i,m-j+1}\right| \geq 2^{-8}|E(2^{m-j+1})|, \tag{10}$$

since $E(2^{m-j+1}) \subset \bigcup_{i=1}^{\infty} 16Q_{i,m-j+1}$. Also, by (9), each $Q_{i,m-j+1}$ satisfies

$$\frac{1}{|Q_{i,m-j+1}|}\int_{Q_{i,m-j+1}} f \;>\; 2^{m-j+1}. \tag{11}$$

For each positive integer j, let ρ_{m-j+1} be a positive integer such that

$$\left|\bigcup_{i=1}^{\rho_{m-j+1}} Q_{i,m-j+1}\right| \geq \tfrac{1}{2}\left|\bigcup_{i=1}^{\infty} Q_{i,m-j+1}\right|. \tag{12}$$

We now form the following sequence of dyadic squares: let $Q_1 = Q_{1,m}$, $Q_2 = Q_{2,m}$, ... , $Q_{\rho_m} = Q_{\rho_m,m}$, $Q_{\rho_m+1} = Q_{1,m-1}$, $Q_{\rho_m+2} = Q_{2,m-1}$, ... , $Q_{\rho_m+\rho_{m-1}} = Q_{\rho_{m-1},m-1}$, $Q_{\rho_m+\rho_{m-1}+1} = Q_{1,m-2}$, ...

Let now $\widetilde{Q}_1 = Q_1$. Let $\widetilde{Q}_2$ be the first Q on the list $Q_1, Q_2, Q_3, \ldots$ such that $|Q \cap \widetilde{Q}_1| \leq \frac{1}{2}|Q|$. Assuming $\widetilde{Q}_1, \ldots, \widetilde{Q}_k$ have been chosen, let $\widetilde{Q}_{k+1}$ be the first Q on the list $Q_1, Q_2, Q_3, \ldots$ such that $\left|Q \cap \bigcup_{i=1}^{k} \widetilde{Q}_i\right| \leq \frac{1}{2}|Q|$. In this manner the sequence $\{\widetilde{Q}_i\}$ is generated.

Now, let $j_1 \leq j_2 \leq j_3 \leq \ldots$ be such that $Q \in \{\widetilde{Q}_1, \widetilde{Q}_2, \ldots, \widetilde{Q}_{j_k}\}$ implies $(1/|Q|)\int_Q |f| > 2^{m-k+1}$ and

$$\left|\bigcup_{i=1}^{j_k} \widetilde{Q}_i\right| \geq \tfrac{1}{4}\left|\bigcup_{i=1}^{\infty} Q_{i,m-k+1}\right|.$$

This is possible via (12) and the selection rule for the $\widetilde{Q}_i$.

Note that (10) implies

$$\left|\bigcup_{i=1}^{j_k} \widetilde{Q}_i\right| \geq \tfrac{1}{4}\cdot 2^{-8}\left|E(2^{m-k+1})\right|. \tag{13}$$

Pick $\epsilon > 0$. Let $\ell > 1$ be an integer such that $2^{m-\ell+1} < \epsilon$. Then

$$\left|\int_{\{x\in Q: 2^{m-\ell+1} < M_{\mathrm{HL}}f(x) < 2^{m+1}\}} M_{\mathrm{HL}}f - \int_Q M_{\mathrm{HL}}f\right| < \epsilon.$$

We compare $\int_{\{x\in Q: 2^{m-\ell+1}<M_{\mathrm{HL}}f(x)<2^{m+1}\}} M_{\mathrm{HL}}f$ to $\int_Q f \sum_{i=1}^{j_\ell} \chi_{\widetilde{Q}_i}$. Let

$$\lambda(\alpha) = \left|\{x \in Q : M_{\mathrm{HL}}f(x) > \alpha\}\right|,$$

$$\mu(\alpha) = \left|\left\{x \in Q : f(x) \sum_{i=1}^{j_\ell} \chi_{\widetilde{Q}_i}(x) > \alpha\right\}\right|,$$

$$\omega(\alpha) = \left|\left\{x \in Q : \sum_{i=1}^{j_\ell}\left(\frac{1}{|\widetilde{Q}_i|}\int_{\widetilde{Q}_i} f\right)\chi_{\widetilde{Q}_i}(x) > \alpha\right\}\right|.$$

Now, suppose $2^{m-\ell+1} \leq \alpha < 2^{m+1}$. Let r be the largest integer such that $2^r \leq \alpha$. Hence $m-\ell+1 \leq r \leq m$. By (13) and the remarks preceding (13) we see that $\lambda(2^r) \leq 4\cdot 2^8\omega(2^r)$. Hence $\lambda(\alpha) \leq 2^{10}\omega\left(\frac{\alpha}{2}\right)$ for $2^{m-\ell+1} \leq \alpha < 2^{m+1}$. So

$$\begin{aligned}
&\int_{\{x\in Q: 2^{m-\ell+1}<M_{\mathrm{HL}}f(x)<2^{m+1}\}} M_{\mathrm{HL}}f \\
&\quad \leq \int_{2^{m-\ell+1}}^{2^{m+1}} \lambda(\alpha)\,d\alpha + \epsilon \leq 2^{10}\int_{2^{m-\ell+1}}^{2^{m+1}} \omega\left(\frac{\alpha}{2}\right)d\alpha + \epsilon \leq 2^{11}\int_0^\infty \omega(\alpha)\,d\alpha + \epsilon \\
&\quad = 2^{11}\int_0^\infty \mu(\alpha)\,d\alpha + \epsilon = 2^{11}\int_Q f\left(\sum_{i=1}^{j_\ell}\chi_{\widetilde{Q}_i}\right) + \epsilon.
\end{aligned}$$

Hence

$$\int_Q M_{\mathrm{HL}}f \leq 2^{11}\int_Q f\left(\sum_{i=1}^{j_\ell}\chi_{\widetilde{Q}_i}\right) + 2\epsilon. \tag{14}$$

For $i = 1, 2$ we generate a finite sequence $\{\widetilde{Q}_{i,j}\}$ as follows: Let $\widetilde{Q}_{i,1}$ be the first square Q on the list $\widetilde{Q}_1, \widetilde{Q}_2, \ldots, \widetilde{Q}_{j_\ell}$ which contains an element whose i-th component is 0 (i.e. there is an element $p = (p_1, p_2) \in Q$ with $p_i = 0$.) For each positive integer j, let $\widetilde{Q}_{i,j}$ be the j-th square on the list with this property (if such a square exists). Since each of the $\widetilde{Q}_i$ intersect one of the coordinate axes, each cube $\widetilde{Q}_i$ will be an element in at least one of the two sequences. Suppose now that for each $i = 1, 2$ the sequence $\{\widetilde{Q}_{i,j}\}$ has q_i squares. Then by (14),

$$\int_Q M_{\mathrm{HL}}f \leq 2^{11}\int_Q f\left(\sum_{i=1}^{q_1}\chi_{\widetilde{Q}_{1,i}} + \sum_{i=1}^{q_2}\chi_{\widetilde{Q}_{2,i}}\right) + 2\epsilon. \tag{15}$$

For $i = 1, 2$ and each positive integer j, define $\mathcal{R}_{i,j}$ as the collection of $R \subseteq Q$ such that R is a rectangle with sides parallel to the axes, one of the edges of R with smallest length is parallel to the line $x_i = 0$, and one of the edges of R with smallest length has length 2^{-j}. For example, an element of

$\mathcal{R}_{1,3}$ would be the rectangle with corners at the points $\left(0,\frac{1}{8}\right)$, $\left(0,\frac{1}{4}\right)$, $\left(\frac{1}{2},\frac{1}{8}\right)$, and $\left(\frac{1}{2},\frac{1}{4}\right)$. Define the maximal operators $M_{i,j}$ by

$$M_{i,j}f(x) = \sup_{x\in R\in\mathcal{R}_{i,j}} \frac{1}{|R|}\int_R |f(y)|\,dy. \tag{16}$$

Since $f \in C^\infty(Q)$, it is clear that

$$\lim_{j\to\infty}\int_Q M_{1,j}f = \int_Q M_x f \quad \text{and} \quad \lim_{j\to\infty}\int_Q M_{2,j}f = \int_Q M_y f.$$

For convenience, we will frequently denote M_x by M_1 and M_y by M_2. Hence the equalities above become

$$\lim_{j\to\infty}\int_Q M_{i,j}f = \int_Q M_i f.$$

It follows that there exists an integer, designated by j_0, such that

$$\left|\int_Q M_{i,j_0}f - \int_Q M_i f\right| < \frac{\epsilon}{2^{13}} \quad \text{for } i = 1, 2. \tag{17}$$

We assume without loss of generality that

$$2^{-j_0} \le \inf\left\{|\widetilde{Q}_1|^{1/2}, |\widetilde{Q}_2|^{1/2}, \ldots, |\widetilde{Q}_{j_\ell}|^{1/2}\right\}.$$

We now show that for each $i = 1, 2$,

$$\int_Q f \sum_{j=1}^{q_i} \chi_{\widetilde{Q}_{i,j}} \le 2\int_Q M_{i,j_0} f.$$

Let $\widetilde{R}_{i,j,1}, \ldots, \widetilde{R}_{i,j,2^{j_0}\,|\widetilde{Q}_{i,j}|^{1/2}}$ be the $2^{j_0}\,|\widetilde{Q}_{i,j}|^{1/2}$ disjoint rectangles in $\mathcal{R}_{i,j_0}$ of equal area whose union is $\widetilde{Q}_{i,j}$. Let $\gamma_{ij} = 2^{j_0}\,|\widetilde{Q}_{i,j}|^{1/2}$. For $i = 1, 2$, let $\widetilde{R}_{i,1} = \widetilde{R}_{i,1,1}$, $\widetilde{R}_{i,2} = \widetilde{R}_{i,1,2}$, $\ldots$, $\widetilde{R}_{i,\gamma_{i1}} = \widetilde{R}_{i,1,\gamma_{i1}}$, $\widetilde{R}_{i,\gamma_{i1}+1} = \widetilde{R}_{i,2,1}$, $\ldots$, $\widetilde{R}_{i,\gamma_{i1}+\gamma_{i2}} = \widetilde{R}_{i,2,\gamma_{i2}}$, $\widetilde{R}_{i,\gamma_{i1}+\gamma_{i2}+1} = \widetilde{R}_{i,3,1}$, $\ldots$, $\widetilde{R}_{i,\gamma_{i1}+\cdots+\gamma_{iq_i}} = \widetilde{R}_{i,q_i,\gamma_{iq_i}}$.

Since for $i = 1, 2$ the squares $\widetilde{Q}_{i,j}$ for $j = 1, \ldots, q_i$ are all dyadic and intersect the line $\{x_i = 0\}$, the selection rule for the $\widetilde{Q}_k$ yields

$$\left|\widetilde{R}_{i,j} \cap \textstyle\bigcup_{k=1}^{j-1} \widetilde{R}_{i,k}\right| \le \frac{1}{2}|\widetilde{R}_{i,j}| \tag{18}$$

for $j = 2, 3, \ldots, \gamma_{i1} + \cdots + \gamma_{iq_i}$.

Let $\gamma_i = \gamma_{i1} + \cdots + \gamma_{iq_i}$. Now, (18) implies that $\{R_{i,1}, R_{i,2}, \ldots, R_{i,\gamma_i}\} \in \mathrm{CFC}\,(\mathcal{R}_{i,j_0})$. Hence Theorem 2 tells us that

$$\int_Q |f| \sum_{j=1}^{\gamma_i} \chi_{\widetilde{R}_{i,j}} \le 2\int_Q M_{i,j_0} f. \tag{19}$$

Now by the construction of the $\widetilde{R}_{i,j}$ it is clear that

$$\sum_{j=1}^{\gamma_i} \chi_{\widetilde{R}_{i,j}} = \sum_{j=1}^{q_i} \chi_{\widetilde{Q}_{i,j}}. \tag{20}$$

(19) and (20) then yield

$$\int_Q |f| \sum_{j=1}^{q_i} \chi_{\widetilde{Q}_{i,j}} \leq 2 \int_Q M_{i,j_0} f. \tag{21}$$

(15), (17), and (21) yield

$$\int_Q M_{\mathrm{HL}} f \leq 2^{12} \left(\int_Q M_1 f + \int_Q M_2 f \right) + 3\epsilon. \tag{22}$$

As ϵ is arbitrarily small, we see that

$$\int_Q M_{\mathrm{HL}} f \leq C \left(\int_Q M_1 f + \int_Q M_2 f \right) \tag{23}$$

for some universal constant C, completing the proof. □

The preceding lemma and the following rearrangement result will enable us to prove that if f is a measurable function supported on Q, then $\int_Q M_{\mathrm{HL}} f \sim \int_Q M_x f + \int_Q M_y f$.

Lemma 8. *Let f be a nonnegative measurable function supported on Q. Let $\widetilde{f}$ be the function supported on Q which is nonincreasing in x (i.e., $\widetilde{f}(x_1, y) \geq \widetilde{f}(x_2, y)$ whenever $0 \leq x_1 \leq x_2 \leq 1$, $0 \leq y \leq 1$) and such that, for each $y \in [0,1]$, $\widetilde{f}(\cdot, y)$ and $f(\cdot, y)$ are equidistributed. Then*

$$\int_Q M_y \widetilde{f} \leq c \int_Q M_y f,$$

where c is a universal constant.

Proof. Let $\alpha > 0$. Let $\lambda(\alpha) = \left|\{(u,v) \in Q : M_y f(u,v) > \alpha\}\right|$. Define $\widetilde{\lambda}(\alpha)$ similarly. It suffices to show that $\widetilde{\lambda}(\alpha) \leq 400\, \lambda(\alpha/64)$.

Without loss of generality, assume f is smooth on Q. Take the Calderón–Zygmund decomposition of f with respect to α on each vertical segment in $\{s \times [0,1], s \in [0,1]\}$ of Q, yielding for each $x \in [0,1]$ disjoint sets $Q_{x,j,\alpha} \subseteq [0,1]$ such that

$$\alpha < \frac{1}{|Q_{x,j,\alpha}|} \int_{Q_{x,j,\alpha}} f(x,z)\, dz \leq 2\alpha.$$

(In the case that $\int_0^1 f(x,z)\, dz > 2\alpha$, set $Q_{x,1,\alpha} = [0,1]$.) Note that $f(p) \leq \alpha$ for almost every p in the complement of

$$\bigcup_{\substack{x \in [0,1] \\ j \in \mathbf{Z}_+}} \left(x \times Q_{x,j,\alpha}\right).$$

For $\widetilde{f}$ one may produce the associated sets $\widetilde{Q}_{x,j,\alpha}$ in a similar fashion.

Let $E_\alpha = \{(x,y) \in Q : y \in \bigcup_{x\in[0,1],\, j\in\mathbf{Z}_+} Q_{x,j,\alpha}\}$. Define $\widetilde{E}_\alpha$ similarly. It suffices to show that $|\widetilde{E}_{4\alpha}| \le 2|E_\alpha|$. Now, it is easily seen that if g is a measurable function supported on $[0,1]$, $g \ge 0$, $\int_0^1 g \le 2\alpha$, $\lambda_{\mathrm{HL}}(\alpha) = \big|\{x \in [0,1] : M_{\mathrm{HL}} g(x) > \alpha\}\big|$, and $E_{\mathrm{HL},\alpha} = \bigcup Q_{j,\alpha}$, where the $Q_{j,\alpha}$ are the intervals obtained by taking the Calderón–Zygmund decomposition of g with respect to α, then $|E_{\mathrm{HL},\alpha}| \le \lambda_{\mathrm{HL}}(\alpha/2) \le 200|E_{\mathrm{HL},\alpha/8}|$. From this we readily conclude that $|\widetilde{E}_{4\alpha}| \le 2|E_\alpha|$ implies $\widetilde{\lambda}(\alpha) \le 200|\widetilde{E}_{\alpha/8}| \le 400\,|E_{\alpha/32}| \le 400\,\lambda(\alpha/64)$. Hence $\widetilde{\lambda}(\alpha) \le 400\,\lambda(\alpha/64)$, as desired.

To show that $|\widetilde{E}_{4\alpha}| \le 2|E_\alpha|$ we proceed as follows. First consider the special case in which $\int_0^1 f(x,y)\,dy \le \alpha$ for any $x \in [0,1]$. Having taken the Calderón–Zygmund decomposition of f with respect to α described above, we have the disjoint sets $Q_{x,j,\alpha} \subset [0,1]$ for each $x \in [0,1]$ and the associated set E_α. Now, $f(p) \le \alpha$ for almost every p in the complement of E_α. So if S is a measurable subset of Q and $|S| > 2|E_\alpha|$, then $\int_S f \le 2\alpha|S|$. Now let $\phi : Q \to Q$ be a measure-preserving bijection such that $\widetilde{f}(\phi(p)) = f(p)$ for any $p \in Q$. Using ϕ we see that $|\widetilde{E}_{4\alpha}| \le 2|E_\alpha|$. Otherwise, if $|\widetilde{E}_{4\alpha}| > 2|E_\alpha|$, we would have

$$\frac{1}{|\widetilde{E}_{4\alpha}|}\int_{\widetilde{E}_{4\alpha}} \widetilde{f} = \frac{1}{|\widetilde{E}_{4\alpha}|}\int_{\phi^{-1}(\widetilde{E}_{4\alpha})} f \le 2\alpha$$

by the above; but the left-hand side is greater than 4α by the construction of $\widetilde{E}_{4\alpha}$. So $|\widetilde{E}_{4\alpha}| \le 2|E_\alpha|$ if $\int_0^1 f(x,y)\,dy \le \alpha$ for all $x \in [0,1]$.

Now we let f be an arbitrary nonnegative smooth function on Q. Without loss of generality assume there exists $c \in (0,1)$ such that $\int_0^1 f(x,y)\,dy > \alpha$ if $x < c$, and $\int_0^1 f(x,y)\,dy \le \alpha$ if $x \ge c$. Form the Calderón–Zygmund decomposition of f with respect to α as before, obtaining the $Q_{x,j,\alpha}$ and E_α. Note that $Q_{x,1,\alpha} = [0,1]$ if $x < c$.

For each $y \in [0,1]$ we define the functions $f_y(x)$ on $[0,1]$ by $f_y(x) = f(x,y)$. We construct a function $f'_y(x)$ on $[0,1]$ equidistributed to $f_y(x)$ such that $f'_y(x_2) \le f'_y(x_1)$ if $x_2 \ge c \ge x_1$ and $f'_y(x) \le f_y(x)$ if $x \ge c$ as follows:

Let $B_y = \{x \in [0,c) : f_y(x) < \widetilde{f}_y(c)\}$.

Let $A_y \subset \{x \in [c,1] : f_y(x) \ge \widetilde{f}_y(c)\}$ be such that the measure of its interior is equal to $|B_y|$. Let A_y° and B_y° denote the interiors of A_y and B_y, respectively. Let $\phi_y : A_y^\circ \to B_y^\circ$ be a measure-preserving bijection such that $\big|\{b \in B_y^\circ : b < \phi_y(x)\}\big| = \big|\{a \in A_y^\circ : a < x\}\big|$ if $x \in A_y^\circ$. Define $f'_y(x)$ by

$$f'_y(x) = \begin{cases} f_y(x) & \text{if } x \notin A_y^\circ \cup B_y^\circ, \\ f_y\left(\phi_y^{-1}(x)\right) & \text{if } x \in B_y^\circ, \\ f_y\left(\phi_y(x)\right) & \text{if } x \in A_y^\circ. \end{cases}$$

Note that $f'_y(x) \leq f_y(x)$ if $x > c$. Define a function f' on Q by $f'(x,y) = f'_y(x)$. Form the Calderón–Zygmund decomposition of f' with respect to α as above, obtaining the associated sets $Q'_{x,j,\alpha}$, E'_α. Note that $E_\alpha \supseteq E'_\alpha$, so without loss of generality we may assume $f = f'$. Hence, without loss of generality, $f(x_1, y) \geq f(x_2, y)$ if $0 \leq x_1 < c \leq x_2 \leq 1$, and $\int_0^1 f(x,y)\,dy > \alpha$ if and only if $x < c$.

Let $f_1 = f\chi_{[0 \leq x < c]}$, $f_2 = f\chi_{[c \leq x \leq 1]}$. So $f = f_1 + f_2$. Let $\widetilde{f_1}$ be a rearrangement of f_1 such that, for each $y \in [0,1]$, $f_1(\cdot, y)$ and $\widetilde{f}_1(\cdot, y)$ are equidistributed and $\widetilde{f}_1(x,y)$ is nonincreasing in x. Define $\widetilde{f}_2$ to be the rearrangement of f_2 within $\{Q \cap \{(x,y) : c \leq x \leq 1\}\}$ such that, for each $y \in [0,1]$, the functions $f_2(\cdot, y)$ and $\widetilde{f}_2(\cdot, y)$ are equidistributed and such that $\widetilde{f}_2(x,y)$ is nonincreasing in x in $\{x : c \leq x \leq 1\}$. Now $\widetilde{f} = \widetilde{f}_1 + \widetilde{f}_2$. Let

$$E_{1,\alpha} = \bigcup_{\substack{x \in [0,c) \\ j \in \mathbf{Z}_+}} (x \times Q_{x,j,\alpha}), \qquad E_{2,\alpha} = \bigcup_{\substack{x \in [c,1] \\ j \in \mathbf{Z}_+}} (x \times Q_{x,j,\alpha}).$$

Define $\widetilde{E}_{1,\alpha}$ and $\widetilde{E}_{2,\alpha}$ similarly. Note that $|\widetilde{E}_{1,4\alpha}| \leq 2|E_{1,\alpha}|$ trivially (since $Q_{x,1,\alpha} = [0,1]$ if $x < c$) and $|\widetilde{E}_{2,4\alpha}| \leq 2|E_{2,\alpha}|$ by the special case argument, since $\int_0^1 f(x,y)\,dy \leq \alpha$ for $x \geq c$. Since $|E_\alpha| = |E_{1,\alpha}| + |E_{2,\alpha}|$ and $|\widetilde{E}_\alpha| = |\widetilde{E}_{1,\alpha}| + |\widetilde{E}_{2,\alpha}|$, we see that $|\widetilde{E}_{4\alpha}| \leq 2|E_\alpha|$, as desired. □

Theorem 9. *Suppose f is a measurable function supported on Q. Then*

$$c \int_Q M_{\mathrm{HL}} f \leq \int_Q M_x f + \int_Q M_y f \leq C \int_Q M_{\mathrm{HL}} f \tag{24}$$

for universal constants $0 < c, C < \infty$.

Proof. From [4] we see that $\int_Q M_x f \lesssim \int_Q M_{\mathrm{HL}} f$ and $\int_Q M_y f \lesssim \int_Q M_{\mathrm{HL}} f$. Hence it suffices to show that $\int_Q M_{\mathrm{HL}} f \lesssim \int_Q M_x f + \int_Q M_y f$. We may assume without loss of generality that f is nonnegative. Let $\widetilde{f}(x,y)$ be the function supported on Q which is nonincreasing in x and such that $\widetilde{f}(\cdot, y)$ and $f(\cdot, y)$ are equidistributed for each $y \in [0,1]$. Let $f^*(x,y)$ be the function supported on Q which is nonincreasing in y and such that $f^*(x, \cdot)$ and $\widetilde{f}(x, \cdot)$ are equidistributed for each $x \in [0,1]$. Note that $f^*(x_1, y_1) \geq f^*(x_2, y_2)$ whenever $0 \leq x_1 \leq x_2 \leq 1$, $0 \leq y_1 \leq y_2 \leq 1$. As $\int_Q M_{\mathrm{HL}} f^* \sim \int_Q M_{\mathrm{HL}} f$, $\int_Q M_y f^* \sim \int_Q M_y \widetilde{f}$, and $\int_Q M_x \widetilde{f} \sim \int_Q M_x f$ by Stein's $L \log L$

result [**5**], we see that

$$\begin{aligned}\int_Q M_{\mathrm{HL}} f &\sim \int_Q M_{\mathrm{HL}} f^* \\ &\lesssim \int_Q M_x f^* + \int_Q M_y f^* &&\text{(Lemma 7)} \\ &\lesssim \int_Q M_x \widetilde{f} + \int_Q M_y \widetilde{f} &&\text{(Lemma 8)} \\ &\lesssim \int_Q M_x f + \int_Q M_y f &&\text{(Lemma 8)},\end{aligned}$$

as desired. □

Building upon these ideas, more sophisticated applications of Córdoba–Fefferman collections are used collectively in [**2**] and [**3**] to prove that if f is a function supported on Q such that $\int_Q M_y M_x f < \infty$ but $\int_Q M_x M_y f = \infty$, there exists a set A of finite measure in $\mathbf{R}^2$ such that $\int_A M_S f = \infty$.

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