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We study quantum moment maps of G-invariant star products, a quantum analogue of the moment map for classical Hamiltonian systems. Introducing an integral representation, we show that any quantum moment map for a G-invariant star product is differentiable. This property gives us a new method for the classification of G-invariant star products on regular coadjoint orbits of compact semisimple Lie groups.

1. Introduction

Deformation quantization was introduced in the 1970s by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer [2]. It is one of the important methods for quantizing classical systems. This quantization scheme provides an autonomous theory based on deformations of the ring of classical observables on a phase space (Poisson algebra), and does not involve a radical change in the nature of the observables.

Star products invariant under the action of a Lie group G have been studied with increasing generality from the beginning of the deformation quantization. They appear naturally in the quantization of classical systems with group symmetries, or in the star representation theory of Lie groups.

Quantum moment maps have been introduced in [21], and are the natural quantum analogue of moment maps on Hamiltonian G-spaces [16]; see Definition 3.1. A quantum moment map plays an important role for the study of G-invariant star products, similar to the one played by a (classical) moment map for classical systems. One of the interesting applications of quantum moment maps is to provide an example of *quantum dual pair* [21, 20]. Another remarkable result is the *quantum reduction theorem*, which says that a quantization commutes with reduction [9]. We also give an application of quantum moment map by providing an invariant, called c_* in [12], for a G-invariant star product $*$ on a G-transitive symplectic manifold [12]. This c_* is computed with the help of a quantum moment map and depends only on the class of G-equivalent star products. In [12, 13], we give a few examples of c_* for a SO(3)-invariant star product on the coadjoint orbit S^2.

But there are serious problems with quantum moment maps. First, there is no obvious way to compute an explicit expression for a quantum moment map for a given G-invariant star product. We provide a partial answer to this problem in [13]. Another important problem is the differentiability of quantum moment maps. Originally, a quantum moment map is defined only on the universal enveloping algebra $\mathfrak{U}(\mathfrak{g}_\lambda)$, that is, the set of polynomials on $\mathfrak{g}^*$. But this definition of quantum moment maps does not directly imply its differentiability. A priori, a quantum moment map has only an algebraic meaning, and cannot be studied in the category of differentiable deformations, which can be inconvenient.

In this article, we give another expression for quantum moment maps which is differentiable. This expression is an analogue of Weyl correspondence that can be formally written as

$$\Phi_*(u) = \int \mathfrak{F}u(\xi) \exp_*(i\xi\Phi_*(X))\, d\xi,$$

where $\mathfrak{F}u$ and Φ_* denote the Fourier transform of u and the quantum moment map of $*$ on $\mathfrak{g}$ respectively. To make sense of this formula, it is necessary to address two questions: defining the function $\exp_*(i\xi\Phi_*(X))$ and giving a meaning to the integral.

For the first, we simply define $\exp_*(i\xi\Phi_*(X))$ by power series with respect to the star product. We show that this naive definition of $\exp_*(i\xi\Phi_*(X))$ is well-defined and it is a product of $e^{i\xi\Phi_0(X)}$ and a polynomial in ξ. This is an ingredient to make the quantum moment map differentiable. For the second question, since the domain of a quantum moment map contains any polynomial, u in the formula above should be considered as a tempered distribution. In fact, for any slowly increasing infinitely differentiable function u, one can provide the integration as

$$\mathfrak{F}_x^{-1}\left[\mathfrak{F}_\xi[u](\xi) \exp_*(i\xi\Phi_*(X))e^{-i\xi\Phi_0(X)}\right]|_{x=\Phi_0(X)} \quad [18].$$

We prefer to use oscillatory integrals rather than tempered distributions in order to make computations easier. We give a brief review on oscillatory integrals in Appendix A; see also [15].

As an application of the differentiability of a quantum moment map we give a structure theorem for G-invariant star products on a coadjoint orbit of compact semisimple Lie groups. The class of G-invariant star products is parametrized by G-invariant Weyl curvature, that is, the second G-invariant de Rham cohomology [3]. However, this classification does not give enough information on the structure of these star products.

Regarding the structure of star products, there is an interesting study in [10]. It provides a family of algebraic star products on a coadjoint orbit of semisimple Lie group by a quotient algebra of the Gutt star product. This work has the advantage of giving an explicit representation for this kind of star products.

We provide here a similar structure theory of G-invariant star products on such orbits in the differentiable category as an application of the differentiability of quantum moment maps. So we have another classification of such star products by using quantum moment maps. Moreover, as a corollary of the structure theorem, we answer the problem we introduced in [**13**]: Does c_* parametrize the class of G-invariant star products? The answer is yes for regular coadjoint orbits of compact semisimple Lie groups.

The paper is organized as follows: in Section 2, we recall basic concepts and results in deformation quantization, λ-formal analytic functions and the Gutt star product on $\mathfrak{g}$. The main results are in Section 3: the exponential $\exp_*(i\xi\Phi_*(X))$, an integral expression for Φ_*, and a proof of the differentiability of Φ_*. In Section 4, we show the structure theorem of G-invariant star products on a coadjoint orbit.

2. Preliminaries

2.1. Star products. Let (M,ω) be a symplectic manifold and $C^\infty(M)$ the set of smooth functions on M. The Poisson bracket on $C^\infty(M)$ associated to ω is denoted by $\{\ ,\ \}$. Let $C^\infty(M)[\![\lambda]\!]$ be the space of power series in a formal parameter λ with coefficients in $C^\infty(M)$.

A (differentiable) star product is an associative multiplication $*$ on the space $C^\infty(M)[\![\lambda]\!]$, having the form

$$u * v = uv + \sum_{n=1}^{\infty}\left(\frac{\lambda}{2}\right)^n C_n(u,v) \quad \text{for any } u,v \in C^\infty(M),$$

where each C_k is a bidifferential operator annihilating constants and

$$C_1(u,v) - C_1(v,u) = 2\{u,v\}.$$

In the situation where a Lie group G acts on M, a star product $*$ is said to be G-invariant if $g(u*v) = gu * gv$ for any $u,v \in C^\infty(M)[\![\lambda]\!]$ and $g \in G$, where $gu(x) = u(g^{-1}x)$ for $x \in M$. There exists a star product on any symplectic manifold [**5**, **17**, **7**], and the existence of G-invariant star products is equivalent to the existence of a G-invariant connection on M [**21**, **8**]. When G is compact, G-invariant connections always exist and consequently there always exist G-invariant star products on M.

Two star products $*_1$ and $*_2$ on $C^\infty(M)[\![\lambda]\!]$ are said to be formally equivalent if there is a formal series

$$T = \mathrm{Id} + \sum_{n=1}^{\infty}\lambda^n T_n$$

of differential operators on $C^\infty(M)$ annihilating constants such that $u *_2 v = T(T^{-1}u *_1 T^{-1}v)$. In this case, T is called an equivalence between $*_1$ and $*_2$, and $*_2$ is denoted by $*_1^T$. If $*_1$ and $*_2$ are equivalent G-invariant star products and if the equivalence T is G-invariant, then $*_1$ and $*_2$ are said to be formally G-equivalent and T is called G-equivalence; see also [4, 3].

2.2. Formal analytic functions. We now make some simple but useful remarks on the convergence of the power series valued in $\mathbb{C}[\![\lambda]\!]$, that will be needed for calculus of functions in $C^\omega(\mathbb{R}^n)[\![\lambda]\!]$.

Definition 2.1. A function $u = u_0 + \lambda u_1 + \cdots \in C^\infty(\mathbb{R}^n)[\![\lambda]\!]$ is called formal analytic if each u_i is analytic on $\mathbb{R}^n$. We denote the set of formal analytic functions by $C^\omega(\mathbb{R}^n)[\![\lambda]\!]$.

Let u and v be formal analytic functions. We define the composition $u(v)$. If u is a polynomial, there is no difficulty: just substitute v in u. For the general case, we define the composition by using power series. We begin with the following definition:

Definition 2.2. Let $a_J = \sum_{k=0}^\infty a_{J,k}\lambda^k \in \mathbb{C}[\![\lambda]\!]$ be a multi-indexed sequence with respect to $J = (j_1, \ldots, j_n)$. The series $\sum_J a_J$ is said to converge formally absolutely if, for any k, the series $\sum_J a_{J,k}$ converges absolutely.

If a power series $\sum_J a_J y^J$ converges formally absolutely for some radius $\rho > 0$, it defines a formal analytic function on $|y| < \rho$.

Let $p^j(x) = \sum_{k=1}^\infty p_k^j(x)\lambda^k : \mathbb{R}^m \to \lambda\mathbb{R}^n[\![\lambda]\!]$, $j = 1, \ldots n$, be a formal analytic map. A formal differential operator $p\partial$ is defined by $((p\partial)u)(y) = \sum p^j(x)(\partial_j u)(y)$ for any smooth function $u : \mathbb{R}^n \to \mathbb{R}$. We define a formal operator $e^{p\partial}$ for u by

$$(e^{p\partial}u)(y) = u(y) + \sum_{0<|J|} \frac{1}{|J|!} p^J(x)(\partial_J u)(y). \tag{1}$$

Here $|J| = j_1 + \cdots + j_n$, $p^J = (p^1)^{j_1} \ldots (p^n)^{j_n}$ and $\partial_J = (\partial/\partial y_1)^{j_1} \ldots (\partial/\partial y_n)^{j_n}$. Note that the right-hand side converges with respect to the filtration of λ since $\deg p > 0$. It is easy to show that $e^{p\partial}u$ is an automorphism, that is, $e^{p\partial}(u_1u_2) = (e^{p\partial}u_1)(e^{p\partial}u_2)$.

If u is a polynomial on $\mathbb{R}^n$ then $u(y+p(x))$ is a function of (x, y) that can be defined by substituting $y+p(x)$ in u, and we have $u(y+p(x)) = (e^{p\partial}u)(y)$. If u is given by a power series, $u(y) = \sum_J a_J y^J$, one can see that the series $\sum_J a_J(y + p(x))^J$ is equal to $(e^{p\partial}u)(y)$ as formal power series in y. Since $(e^{p\partial}u)(y)$ converges formally absolutely on the same domain of $y \in \mathbb{R}^n$ where $\sum_J a_J y^J$ converges, we can define $u(y+p(x)) = \sum_J a_J(y+p(x))^J$ as a formal analytic function. Therefore we can define $u(v)$ for any formal analytic map $v : \mathbb{R}^m \to \mathbb{R}^n[\![\lambda]\!]$, and $u(v) = (e^{(v-v_0)\partial}u)(v_0)$ holds, where $v = v_0 + v_1\lambda + \cdots$.

Definition 2.3. Let $u : \mathbb{R}^n \to \mathbb{R}$ be an analytic map and $v : \mathbb{R}^m \to \mathbb{R}^n[[\lambda]]$ a formal analytic map. Then we define a formal analytic map $u(v) : \mathbb{R}^m \to \mathbb{R}[\![\lambda]\!]$ by the power series

$$u(v(x)) = \sum_J a_J(v(x))^J,$$

where $u = \sum_J a_J y^J$.

Remark. The equation

$$u(v(x)) = (e^{(v-v_0)(x)\partial} u)(v_0(x)) \tag{2}$$

holds for any formal analytic map, and it gives the Taylor theorem for formal analytic functions.

2.3. The Gutt star product. Let $\mathfrak{g}$ be a real Lie algebra and $\mathfrak{g}^*$ its dual. The universal enveloping algebra of $\mathfrak{g}$ is denoted by $\mathfrak{U}(\mathfrak{g})$, and the universal symmetric algebra of $\mathfrak{g}$ by $\mathfrak{S}(\mathfrak{g})$. We also denote the space of polynomials on $\mathfrak{g}^*$ by $\mathrm{Pol}(\mathfrak{g}^*)$. Let $\mathfrak{g}[\![\lambda]\!]$ be the set of formal power series in λ with coefficients in $\mathfrak{g}$. We define a Lie algebra structure $[\ ,\ ]_\lambda$ on $\mathfrak{g}[\![\lambda]\!]$ by $[\xi,\eta]_\lambda = \lambda[\xi,\eta]$ for any $\xi,\eta \in \mathfrak{g}$ and extend it by λ-linearity, where $[\ ,\ ]$ is the Lie bracket of $\mathfrak{g}$. We denote this Lie algebra by $\mathfrak{g}_\lambda$. One can introduce a grading on $\mathfrak{g}_\lambda$ by assigning degree 2 to $\xi \in \mathfrak{g}$ and to λ, and $[\ ,\ ]_\lambda$ has degree 0. This grading induces a grading on the universal enveloping algebra $\mathfrak{U}(\mathfrak{g}_\lambda)$ of $\mathfrak{g}_\lambda$.

It is well-known that the space of smooth functions on $\mathfrak{g}^*$ admits a natural Poisson structure defined by the Kirillov–Poisson bracket Π. For any smooth functions u and v on $\mathfrak{g}^*$, Π is given by $\Pi(u,v)(\mu) = \big\langle [du(\mu), dv(\mu)], \mu \big\rangle$, where $du(\mu)$ is an element of $\mathfrak{g}$ considered as 1-form on $\mathfrak{g}^*$.

S. Gutt has defined a star product on $\mathfrak{g}^*$ [**11**]. We shall call this product the Gutt star product, denoted by $*^G$. The Gutt star product can be directly obtained by transposing the algebraic structure of $\mathfrak{U}(\mathfrak{g}_\lambda)$ to $C^\infty(\mathfrak{g}^*)[\![\lambda]\!]$. This is achieved through the natural isomorphism between $\mathrm{Pol}(\mathfrak{g}^*)[\![\lambda]\!]$ and $\mathfrak{S}(\mathfrak{g}_\lambda)$ and with the help of the symmetrization map $s : \mathfrak{S}(\mathfrak{g}_\lambda) \to \mathfrak{U}(\mathfrak{g}_\lambda)$. For polynomials u and v, the Gutt star product is given by

$$u *^G v = s^{-1}(s(u) \cdot s(v)), \tag{3}$$

where $\cdot$ is the product of $\mathfrak{U}(\mathfrak{g}_\lambda)$. Formula (3) defines an associative differentiable deformation of the usual product on $\mathrm{Pol}(\mathfrak{g}^*)$ which admits a unique extension to $C^\infty(\mathfrak{g}^*)[\![\lambda]\!]$.

As a direct consequence of Equation (3) $*^G$ is a Weyl star product, that is, for any linear function ξ on $\mathfrak{g}^*$, we have $\xi^{*^G k} = \xi^k$, where $\xi^{*^G k} = \xi *^G \cdots *^G \xi$ (k factors). Moreover, $*^G$ is $\mathfrak{g}$-covariant,

$$\xi *^G \eta - \eta *^G \xi = 2\lambda\Pi(\xi,\eta) \quad \text{for } \xi,\eta \in \mathrm{Lin}(\mathfrak{g}^*),$$

and $\mathrm{Ad}^*(G)$-invariant,

$$g(u *^G v) = (gu) *^G (gv) \quad \text{for } u,v \in C^\infty(\mathfrak{g}^*)[\![\lambda]\!],\ g \in G.$$

There is a characterization of the Gutt star product:

Proposition 2.1 ([**6**]). *The Gutt star product is the unique $\mathfrak{g}$-covariant Weyl star product on $(\mathfrak{g}^*, \Pi)$. Any $\mathfrak{g}$-covariant star product on $(\mathfrak{g}^*, \Pi)$ is equivalent to the Gutt star product.*

Let $\xi = \sum_{k=0}^{\infty} \xi_k \lambda^k \in \mathrm{Lin}(\mathfrak{g}^*)[\![\lambda]\!] \cong \mathfrak{g}_\lambda$. Then a power series

$$e^{\xi} = \sum_{k=0}^{\infty} \frac{1}{k!} \xi^k \tag{4}$$

can be defined in the sense of formal absolute convergence, and satisfies Equation (2). A simple computation implies that there are polynomials $p_k(\xi_1, \xi_2, \ldots, \xi_k)$ such that

$$e^{\xi} = e^{\xi_0} \sum_{k=0}^{\infty} p_k(\xi_1, \xi_2, \ldots, \xi_k) \lambda^k. \tag{5}$$

Since $*^G$ is a Weyl star product, we also have

$$e^{\xi} = \exp_{*^G}(\xi) \equiv \sum_{k=0}^{\infty} \frac{1}{k!} \xi^{*^G k}.$$

For any $\xi, \eta \in \mathfrak{g}_\lambda$, we denote by $\mathrm{CH}_\lambda(\xi, \eta)$ the Campbell–Hausdorff series of a Lie algebra $\mathfrak{g}_\lambda$. We note that $\mathrm{CH}_\lambda(\xi, \eta)$ is an element of $\mathfrak{g}_\lambda$ since $[\ ,\ ]_\lambda$ has degree 0 and CH_λ converges with respect to the filtration of λ.

Since $*^G$ is $\mathfrak{g}$-covariant, we have $\exp_{*^G}(\xi) *^G \exp_{*^G}(\eta) = \exp_{*^G}(\mathrm{CH}_\lambda(\xi, \eta))$, that is,

$$e^{\xi} *^G e^{\eta} = e^{\mathrm{CH}_\lambda(\xi, \eta)} \quad \text{for } \xi, \eta \in \mathfrak{g}_\lambda. \tag{6}$$

Therefore, the set $G_\lambda \equiv \{e^{\xi} : \xi \in \mathfrak{g}_\lambda\} \subset C^{\omega}(\mathfrak{g}^*)[\![\lambda]\!]$ is closed under multiplication by $*^G$. It is also easy to show that G_λ is a group.

2.4. Oscillatory integral formula for star products. For later use, we provide *an oscillatory integral expression* of the Gutt star product. We shall use the notations and the definitions given in Appendix A for oscillatory integrals; see also [**15**].

Definition 2.4. A function $u \in C^{\infty}(\mathbb{R}^n)$ has polynomial growth of degree $\tau > 0$ if for any multi-index $I = (i_1, i_2, \ldots, i_n)$, there is a constant C_I such that

$$|\partial_{\zeta}^{I} u(\zeta)| \leq C_I \langle \zeta \rangle^{\tau},$$

where $\langle \zeta \rangle = \sqrt{1 + |\zeta|^2}$. We denote the set of such functions by A_τ^0.

Let $\mathcal{A}^0 = \bigcup_{\tau \geq 0} \mathcal{A}_\tau$. If we identify $\mathfrak{g}^*$ with $\mathbb{R}^n$, $\mathcal{A}^0[\![\lambda]\!]$ is a subalgebra of $(C^{\infty}(\mathfrak{g}^*)[\![\lambda]\!], *^G)$, which contains all polynomials.

Definition 2.5. Let $u \in \mathcal{A}^0$. The oscillatory integral expression of u is given by

$$u(\zeta) = \mathrm{Os}\text{-}\!\int e^{i\alpha(\zeta - \beta)} u(\alpha)\, d\alpha\, d\beta,$$

where the right-hand side means oscillatory integral.

Since $*^G$ is differentiable, the $*^G$ operation commutes with integration. Therefore we have the oscillatory integral expression of the Gutt star product as follows: for any $u, v \in A^0$,

$$\begin{aligned} u *^G v(x) &= \text{Os-}\int e^{-i(\alpha\beta+\alpha'\beta')} f(\alpha) g(\alpha') e^{i\beta x} *^G e^{i\beta' x}\, d\alpha\, d\alpha'\, d\beta\, d\beta' \\ &= \text{Os-}\int e^{-i(\alpha\beta+\alpha'\beta')} f(\alpha) g(\alpha') e^{\mathrm{CH}_\lambda(i\beta x, i\beta' x)}\, d\alpha\, d\alpha'\, d\beta\, d\beta'. \end{aligned}$$

We remark that $e^{\mathrm{CH}_\lambda(i\beta x, i\beta' x)} \in \mathcal{A}^0[\![\lambda]\!] \times \mathcal{A}^0[\![\lambda]\!]$ because of Equation (5). Hence the computation above makes sense and $\mathcal{A}^0[\![\lambda]\!]$ is a subalgebra of $(C^\infty(\mathfrak{g}^*)[\![\lambda]\!], *^G)$.

3. Differentiability of quantum moment maps

This section is devoted to the study of quantum moment maps, a main subject of this paper. The definition of quantum moment maps adopted here is given in [**21**]; see Definition 3.1. This definition is a natural analogue of the definition of classical moment maps in Hamiltonian systems.

However, quantum moment maps differ from their classical counterparts in a significant feature, locality. In the classical case, giving a ring morphism of $C^\infty(\mathfrak{g}^*)$ into $C^\infty(M)$ is equivalent to giving a differential map of M into $\mathfrak{g}^*$; this is a consequence from the locality of the ring of functions and its ring morphisms. So this implies that any ring morphism of $\mathrm{Pol}(\mathfrak{g}^*)$ into $C^\infty(M)$ has a natural extension to $C^\infty(\mathfrak{g}^*)$. But the problem is not clear for the quantum case. There is no guarantee that a homomorphism of star algebras is local or differentiable.

We show here that any quantum moment map is differentiable.

3.1. Definition of quantum moment maps.

Let (M, ω) be a symplectic G-space and $*$ a G-invariant star product. We denote the star commutator by $[a, b]_* = a * b - b * a$.

Definition 3.1 ([**21**])**.** A quantum moment map is a homomorphism of associative algebras

$$\Phi_* : \mathfrak{U}(\mathfrak{g}_\lambda) \to C^\infty(M)[\![\lambda]\!] \tag{7}$$

that satisfies

$$[\Phi_*(\xi), u]_* = \lambda \xi u, \tag{8}$$

where the right-hand side of (8) is the infinitesimal action of $\xi \in \mathfrak{g}$ on $C^\infty(M)[\![\lambda]\!]$.

It is easy to see that (7) is equivalent to

$$\Phi_*([\xi, \eta]_\lambda) = [\Phi_*(\xi), \Phi_*(\eta)]_* \quad \text{for any } \xi, \eta \in \mathfrak{g}. \tag{9}$$

On the existence and the uniqueness of quantum moment maps, some simple criteria are known.

Theorem 3.1 ([21])**.** *Let* $\mathrm{H}^*_{\mathrm{dR}}(M)$ *be the de Rham cohomology group and* $\mathrm{H}^*(\mathfrak{g}, \mathbb{R})$ *the Lie algebra cohomology group with coefficients in* $\mathbb{R}$*. There exists a quantum moment map if* $\mathrm{H}^1_{\mathrm{dR}}(M) = 0$ *and* $\mathrm{H}^2(\mathfrak{g}, \mathbb{R}) = 0$*.*

Theorem 3.2 ([21])**.** *The set of quantum moment maps of a G-invariant star product is parametrized by* $\mathrm{H}^1(\mathfrak{g}, \mathbb{R})$*.*

The following proposition says that quantum moment maps are natural analogues of the classical ones.

Proposition 3.1 ([21])**.** *Let* $\Phi_* : \mathrm{Pol}(\mathfrak{g}^*[\![\lambda]\!]) \to C^\infty(M)[\![\lambda]\!]$ *be a quantum moment map. Then M is a Hamiltonian G-space. Moreover Φ_* satisfies*

$$\Phi_*(u) = \Phi_0(u) + O(\lambda) \text{ for any } u \in \mathrm{Pol}(\mathfrak{g}^*),$$

where $\Phi_0 : \mathrm{Pol}(\mathfrak{g}^*) \to C^\infty(M)$ *is the corresponding classical moment map.*

An important property of Φ_* is its covariance under G-equivalence.

Proposition 3.2. *Let $*$ be a G-invariant star product and Φ_* a quantum moment map of $*$. If $*'$ is a G-invariant star product which is G-equivalent to $*$, then $T\Phi_*$ is a quantum moment map of $*'$, where T is a G-equivalence between $*$ and $*'$.*

Proof. It is enough to show that $\left[T\Phi_*(X), f\right]_{*'} = \lambda X f$, since $T\Phi_*$ is an algebra homomorphism from $(\mathrm{Pol}(\mathfrak{g}^*)[\![\lambda]\!], *^G)$ to $(C^\infty(M)[\![\lambda]\!], *')$.

$$[T\Phi_*(\xi), f]_{*'} = T\left[\Phi_*(\xi), T^{-1} f\right]_* = T(\lambda \xi T^{-1} f) = \lambda \xi f. \qquad \square$$

Since quantum moment maps are parametrized by $H^1(\mathfrak{g}, \mathbb{R})$ we have:

Corollary 3.1. *Assume $H^1(\mathfrak{g}, \mathbb{R}) = \{0\}$. Let $*, *'$ be G-invariant star product and $\Phi_*, \Phi_{*'}$ quantum moment maps of $*, *'$ respectively. If $*'$ is G-equivalent to $*$ then $T\Phi_* = \Phi_{*'}$.*

3.2. Exponential function of a quantum moment map. We define a function $\exp_*(\Phi_*(X))$ in $C^\infty(M)[\![\lambda]\!]$ for $X \in \mathfrak{g}_\lambda$. This function "generates" $\Phi_*(\mathrm{Pol}(\mathfrak{g}^*))$, and we will use it to obtain another expression for Φ_*. An important property of $\exp_*(\Phi_*(X))$ is that it is a product of e^X and a polynomial of X. This is essential for the differentiability of Φ_*.

Assume that $*$ is a G-invariant star product of Fedosov type. Recall that Q, σ and $\circ$ denote the Fedosov quantization procedure corresponding to $*$, the projection of W_D onto $C^\infty(M)[[\lambda]]$ and the Weyl product on ΓW respectively. See also Appendix B, where we give a brief summary of these terms.

Lemma 3.1. *Let $\xi = \xi_0 + \xi_1\lambda + \cdots \in \mathfrak{g}_\lambda$. The series*

$$\sum_{k=0}^{\infty} \frac{1}{k!} Q(\Phi_*(\xi))^{\circ k} \tag{10}$$

converges λ-formally absolutely and uniformly on any compact subset of M, and defines an element of W_D. Moreover, (10) has the expression

$$\sum_{k=0}^{\infty} \frac{1}{k!} Q(\Phi_*(\xi))^{\circ k} = e^{\Phi_0(\xi_0)} \sum_{I,j} p_{I,j}(\Phi_*(\xi), \partial\Phi_*(\xi), \dots) y^I \lambda^j, \tag{11}$$

where the $p_{I,j}$ are polynomials in $\{\Phi_*(\xi), \partial\Phi_*(\xi), \dots\}$.

Proof. Decompose $Q(\Phi_*(\xi)) = \Phi_0(\xi_0) + R(\xi)$, where Φ_0 is the classical moment map, and $\deg R(\xi) \geq 1$. Since $\Phi_0(\xi_0)$ is a central element of ΓW, we have $[\Phi_0(\xi_0), R(\xi)]_\circ = 0$, where $[\cdot,\cdot]_\circ$ denotes the Weyl product commutator. Therefore we have the (formal) equation

$$\sum_{k=0}^{\infty} \frac{1}{k!} Q(\Phi_*(\xi))^{\circ k} = \sum_{k=0}^{\infty} \frac{1}{k!} \Phi_0(\xi_0)^k \sum_{k=0}^{\infty} \frac{1}{k!} R(\xi)^{\circ k}. \tag{12}$$

The last sum on the right converges with respect to the filtration of ΓW since $\deg R(\xi) \geq 1$. So, it is easy to see that the right-hand side of (12) converges absolutely and uniformly on any compact subset of M. Applying the Weyl derivation D on (10) term by term, we see that (10) is a flat section.

To show the last statement, we express $R(\xi)$ as

$$R(\xi) = \sum_{|I|+j\geq 1} r_{I,j}(\Phi_*(\xi), \partial\Phi_*(\xi), \dots) y^I \lambda^j. \tag{13}$$

Each $r_{I,j}$ is a polynomial in $\{\Phi_*(\xi), \partial\Phi_*(\xi), \dots\}$, since it is obtained by the Fedosov quantization procedure. Thus each coefficient of $y^I\lambda^j$ in the series

$$\sum_{k=0}^{\infty} \frac{1}{k!} R(\xi)^{\circ k} \tag{14}$$

is also a polynomial. $\square$

Definition 3.2. For any $\xi \in \mathfrak{g}_\lambda$, the function $\exp_*(\Phi_*(\xi))$ in $C^\infty(M)[\![\lambda]\!]$ is defined by

$$\exp_*(\Phi_*(\xi)) = \sigma\left(\sum_{k=0}^{\infty} \frac{1}{k!} Q(\Phi_*(\xi))^{\circ k} \right).$$

In the proof of Lemma 3.1, setting $\xi = \alpha^l X_l$, where $\{\alpha^l\} \in \mathbb{C}^n[\![\lambda]\!]$ and $\{X_l\}$ is a basis of $\mathfrak{g}$, we have:

Corollary 3.2. *$\exp_*(\Phi_*(\alpha^l X_l))$ is a product of $e^{\alpha_0^l \Phi_0(X_l)}$ and a polynomial in α^l taking values in $C^\infty(M)[\![\lambda]\!]$.*

Proof. Since a quantum moment map is linear with respect to $\xi = \alpha^l X_l \in \mathfrak{g}[\![\lambda]\!]$, each $r_{I,j}$ in (13) is also linear with respect to ξ and (14) is a polynomial in α^l. $\square$

Lemma 3.2. *Assume $\xi, \eta \in \mathfrak{g}_\lambda$. Then*

$$\exp_*(\Phi_*(\xi)) * \exp_*(\Phi_*(\eta)) = \exp_*\big(\Phi_*(\mathrm{CH}_\lambda(\xi,\eta))\big). \tag{15}$$

Proof. By the definition of $\exp_*(\Phi_*(\xi))$,

$$\begin{aligned} Q\big(\exp_*(\Phi_*(\mathrm{CH}_\lambda(\xi,\eta)))\big) &= \sum \frac{1}{k!} Q\big(\Phi_*(\mathrm{CH}_\lambda(\xi,\eta))\big)^{\circ k} \\ &= \sum \frac{1}{k!}\big(\mathrm{CH}_\circ(Q(\Phi_*(\xi)), Q(\Phi_*(\eta)))\big)^{\circ k}, \end{aligned}$$

where $\mathrm{CH}_\circ$ denotes the Campbell–Hausdorff series with respect to the Weyl product $\circ$ of the Weyl bundle ΓW. Since

$$\begin{aligned} &\sum \frac{1}{k!}\big(\mathrm{CH}_\circ(Q(\Phi_*(\xi)), Q(\Phi_*(\eta)))\big)^{\circ k} \\ &\qquad = \sum \frac{1}{k!}\big(Q(\Phi_*(\xi))\big)^{\circ k} \circ \sum \frac{1}{k!}\big(Q(\Phi_*(\eta))\big)^{\circ k} \end{aligned}$$

in ΓW, we have the lemma. $\square$

For each multi-index $J = (j_1, j_2, \ldots, j_n)$, define a differential operator

$$D_\alpha^J = \Big(-i\frac{\partial}{\partial\alpha^1}\Big)^{j_1} \cdots \Big(-i\frac{\partial}{\partial\alpha^n}\Big)^{j_n}.$$

Lemma 3.3. *Assume $\{\alpha^l\} \in \mathbb{R}^n$. Then*

$$\big(D^J \exp_*(\Phi_*(i\alpha^l X_l))\big)|_{\alpha=0} = \Phi_*(X^J). \tag{16}$$

Proof. Let $\widetilde{X}_l = Q(\Phi_*(X_l))$.

$$\begin{aligned} D^J\big(Q(\exp_*(\Phi_*(i\alpha^l X_l)))\big)|_{\alpha=0} &= D^J\Big(\sum_{k=0}^{\infty}\frac{1}{k!}(i\alpha^l\widetilde{X}_l)^{\circ k}\Big)\Big|_{\alpha=0} \\ &= D^J\left(\frac{1}{|J|!}(i\alpha^l\widetilde{X}_l)^{\circ|J|}\right)\Big|_{\alpha=0} \\ &= Q(\Phi_*(X^J)). \end{aligned}$$ $\square$

3.3. Oscillatory integral expression for Φ_*. We shall provide another expression for a quantum moment map Φ_* by using $\exp_*(\Phi_*(X))$ and an oscillatory integral. This expression gives us a clear understanding of quantum moment maps and enables us to show the differentiability of Φ_*.

Definition 3.3. Let $\{X_l\}$ be a basis of $\mathfrak{g}$ and $\{X^l\}$ its dual basis. We define the map $\overline{\Phi}_*$ from $\mathcal{A}^0$ into $C^\infty(M)[\![\lambda]\!]$ as follows:

$$\overline{\Phi}_*(u) = \text{Os-}\!\int u(\mu X)e^{-i\nu\mu}\exp_*(\Phi_*(i\nu X))\,d\mu\,d\nu, \quad u \in \mathcal{A}^0, \tag{17}$$

where $\mu X = \mu_l X^l$ and $\nu X = \nu^l X_l$.

This definition makes sense since $u(\mu X)\exp_*(\Phi_*(i\nu X)) \in \mathcal{A}$. It is easy to see that the definition does not depend on a choice of a basis $\{X_l\}$.

Lemma 3.4. *$\overline{\Phi}_*$ coincides with Φ_* on polynomials.*

Proof. Let X^J be a monomial on $\mathfrak{g}^*$. Then

$$\begin{aligned}\overline{\Phi}_*(X^J) &= \text{Os-}\int \beta^J e^{-i\alpha\beta}\exp_*(\Phi_*(i\alpha X))\,d\alpha\,d\beta\\ &= \text{Os-}\int e^{-i\alpha\beta} D^J_\alpha \exp_*(\Phi_*(i\alpha X))\,d\alpha\,d\beta\\ &= D^J_\alpha \exp_*(\Phi_*(i\alpha X))|_{\alpha=0} = \Phi_*(X^J),\end{aligned}$$

where we have applied Equation (16) in the last line. □

So we shall also use the notation Φ_* for $\overline{\Phi}_*$.

The following proposition says that $\exp_*(\Phi_*(X))$ can be considered as the image of e^X under a quantum moment map.

Proposition 3.3. *Assume that $p^k = p^k_j\lambda^j \in \mathbb{C}[\![\lambda]\!]$ satisfies $p^k_0 \in i\mathbb{R}$. Then $e^{pX} = e^{p^k X_k} \in \mathcal{A}$ and $\Phi_*(e^{pX}) = \exp_*(\Phi_*(pX))$.*

Proof. Let $pX = iaX + rX$, where $a^k \in \mathbb{R}$ and $r \in \lambda\mathbb{C}[\![\lambda]\!]$. Then $e^{pX} \in \mathcal{A}$ by Equation (5). By the definition of Φ_*,

$$\begin{aligned}\Phi_*(e^{pX}) &= \text{Os-}\int e^{p\mu}e^{-i\mu\nu}\exp_*(\Phi_*(i\nu X))\,d\mu\,d\nu\\ &= \text{Os-}\int e^{ia\mu}e^{r\mu}e^{-i\mu\nu}\exp_*(\Phi_*(i\nu X))\,d\mu\,d\nu\\ &= \text{Os-}\int e^{i(a-\nu)\mu}(e^{rD_\nu}\exp_*(\Phi_*(i\nu X)))\,d\mu\,d\nu\\ &= \text{Os-}\int e^{i(a-\nu)\mu}\exp_*(\Phi_*(i(\nu - ir)X))\,d\mu\,d\nu\\ &= \exp_*(\Phi_*(i(a-ir)X)) = \exp_*(\Phi_*(pX)),\end{aligned}$$

where we have applied Equation (2). □

As a corollary of Proposition 3.3 and Lemma 3.2, we have

$$\Phi_*(e^{i\xi}) * \Phi_*(e^{i\eta}) = \Phi_*(e^{i\xi} *^G e^{i\eta}).$$

Theorem 3.3. *The quantum moment map Φ_* is differentiable. Moreover, if $*$ is of Fedosov type, there are functions $S_{I,j} \in C^\infty(M)$, $I = (i_1, \dots, i_n)$, $j = 0, 1, \dots$ such that*

$$\Phi_*(u) = \sum_{j=0}^{\infty}\lambda^j \sum_{0\le|I|\le 2j} S_{I,j}\Phi_0(D^I_\mu u) \quad \text{for any } u(\mu) \in C^\infty(\mathfrak{g}^*). \tag{18}$$

Proof. First we assume that $*$ is of Fedosov type. By Corollary 3.2, we have an expression for $\exp_*(\Phi_*(i\alpha^l X_l))$:

$$e^{i\alpha^l\Phi_0(X_l)}\sum_{j=0}^{\infty}\lambda^j\sum_{0<|I|\le 2j}S_{I,j}\alpha^I,$$

where $S_{I,j}\in C^\infty(M)$ depends on $\Phi_*(X_i)$ and on the Weyl connection D of $*$. By the definition of Φ_*, we have

$$\begin{aligned}
\Phi_*(u) &= \text{Os-}\int u(\mu)e^{-i\mu_l\nu^l}\exp_*(\Phi_*(i\nu^l X_l))\,d\mu\,d\nu \\
&= \sum_{j=0}^{\infty}\lambda^j\sum_{0<|I|\le 2j}\text{Os-}\int u(\mu)e^{-i\mu_l\nu^l}e^{i\nu^l\Phi_0(X_l)}S_{I,j}\nu^I\,d\mu\,d\nu \\
&= \sum_{j=0}^{\infty}\lambda^j\sum_{0<|I|\le 2j}\text{Os-}\int u(\mu)e^{-i\nu^l(\mu_l-\Phi_0(X_l))}S_{I,j}\nu^I\,d\mu\,d\nu \\
&= \sum_{j=0}^{\infty}\lambda^j\sum_{0<|I|\le 2j}\text{Os-}\int S_{I,j}(D_\mu^I u)(\mu)e^{-i\nu^l(\mu_l-\Phi_0(X_l))}\,d\mu\,d\nu \\
&= \sum_{j=0}^{\infty}\lambda^j\sum_{0<|I|\le 2j}S_{I,j}\Phi_0(D_\mu^I u).
\end{aligned}$$

For a general G-invariant star product $*'$, we have a G-invariant star product $*$ of Fedosov type which is G-equivalent to $*'$ by Theorem B.3. If T denotes a G-equivalence between $*$ and $*'$, then any quantum moment map $\Phi_{*'}$ of $*'$ has the form $T\Phi_*$. Therefore $\Phi_{*'}$ is differentiable. □

By Theorem 3.3, Φ_* admits a unique extension to $C^\infty(\mathfrak{g}^*)[\![\lambda]\!]$. Since Φ_* is an algebra homomorphism on polynomials on $\mathfrak{g}^*$, the differentiability of Φ_* implies that Φ_* is an algebra homomorphism on $C^\infty(\mathfrak{g}^*)[\![\lambda]\!]$.

It is not difficult to compute $S_{I,j}$ for lower degrees in I, j. For instance, $S_{0,0}=1$, $S_{l,1}=\Phi_1(X_l)$, $S_{lm,1}=\{\Phi_0(X_l),\Phi_0(X_m)\}$, and so on. Therefore $\Phi_*(u)=\Phi_0(u)+o(\lambda)$ for any $u\in C^\infty(\mathfrak{g}^*)$.

3.4. Properties of Φ_*.

Proposition 3.4. *A quantum moment map is a $\mathfrak{g}$-equivariant map from $C^\infty(\mathfrak{g}^*)[\![\lambda]\!]$ to $C^\infty(M)[\![\lambda]\!]$. Therefore, Φ_* is G-equivariant if G is connected.*

Proof. This is a direct consequence of the definition of quantum moment maps and the G-invariance of $*^G$ and $*$: for any $u\in C^\infty(\mathfrak{g}^*)$ and $\xi\in\mathfrak{g}$,

$$\Phi_*(\lambda\xi u)=\Phi_*([\xi,u]_{*^G})=[\Phi_*(\xi),\Phi_*(u)]_*=\lambda\xi\Phi_*(u). \qquad \square$$

Proposition 3.5. *If $f\in C^\infty(M)[\![\lambda]\!]$ commutes with any $\Phi_*(u)$, $u\in C^\infty(\mathfrak{g}^*)$, then f is a G-invariant function.*

Proof. It is easy. □

Proposition 3.6. *A quantum moment map Φ_* is surjective if and only if its classical part Φ_0 is surjective.*

Proof. Assume that Φ_0 is surjective. For $u = \sum u_i \lambda^i \in C^\infty(\mathfrak{g}^*)[\![\lambda]\!]$ and $\varphi \in C^\infty(M)$ the equation $\Phi_*(u) = \varphi$ is equivalent to

$$\Phi_0(u_0) = \varphi, \tag{19}$$

$$\Phi_0(u_k) = -\sum_{j=1}^{k} \Phi_j(u_{k-j}) \quad \text{for any } k > 0. \tag{20}$$

One can solve this system of equations by induction, since Φ_0 is surjective. The converse is trivial. □

Lemma 3.5. *Let φ be a smooth function on M. Assume there exists a solution $u \in C^\infty(\mathfrak{g}^*)[\![\lambda]\!]$ of the equation $\Phi_*(u) = \varphi$. Then u depends locally on φ. More precisely, the dependence of u at $J(q)$ on φ is described by differentials of φ at $q \in M$, where $J : M \to \mathfrak{g}^*$ is the dual form of Φ_*, that is, $(\Phi_0(u))(q) = u(J(q))$.*

Proof. Equation (19) says that $u_0(J(q))$ depends on $\varphi(q)$. Since Φ_* is differentiable, the right-hand side of Equation (20) also depends on differentials of φ if $u_0, \ldots, u_{k-1}$ depend on differentials of φ. □

3.5. Invariants for G-invariant star products on transitive spaces. In this subsection we review the results of [**12**, **13**], where we define invariants for G-invariant star products on G-transitive symplectic manifolds.

Let M be a G-transitive symplectic manifold and $*$ a G-invariant star product on M. We assume that there is a unique quantum moment map Φ_* for $*$. Let $\mathfrak{Z}$ be the center of $C^\infty(\mathfrak{g}^*)$, that is, the set of functions that commute with any smooth function on $\mathfrak{g}^*$ with respect to the Gutt star product. One can show that $\mathfrak{Z}$ is equal to the set of G-invariant functions on $\mathfrak{g}^*$. For any $l \in \mathfrak{Z}$ we have $\left[\Phi_*(l), \Phi_*(C^\infty(\mathfrak{g}^*))\right]_* = \Phi_*\big([l, C^\infty(\mathfrak{g}^*)]_{*G}\big) = 0$, so that Proposition 3.5 implies $\Phi_*(l)$ is a G-invariant function on M. Since M is transitive, $\Phi_*(l)$ is constant. Consequently, we make the following definition:

Definition 3.4. Let M be a G-transitive symplectic manifold and $*$ a G-invariant star product admitting a unique quantum moment map Φ_*. Define a map $c_* : \mathfrak{Z} \to \mathbb{C}[\![\lambda]\!]$ by $c_*(l) \equiv \Phi_*(l)$ for any $l \in \mathfrak{Z}$.

The following simple proposition is important:

Proposition 3.7. *If* $\ker \Phi_* = \ker \Phi_{*'}$ *then $c_* = c_{*'}$.*

Proof. Any function on $C^\infty(\mathfrak{g}^*)[\![\lambda]\!]$ of the form $l - c_*(l)$ for $l \in \mathfrak{Z}$ is an element of $\ker \Phi_*$. Therefore if $\ker \Phi_* = \ker \Phi_{*'}$, we have $\Phi_{*'}(l - c_*(l)) = 0$, that is, $c_{*'}(l) = c_*(l)$. □

Corollary 3.3. *If $*'$ is G-equivalent to $*$ then $c_* = c_{*'}$.*

This means that c_* depends only on the class of G-equivalent star products. We have computed c_* for a few examples in [**13**].

There is a natural question to ask: Does c_* parametrize the class of G-invariant star products on a G-transitive space? In the next section, we give a complete answer of this question when M is a coadjoint orbit of a compact semisimple Lie group.

4. Star products on regular coadjoint orbits of compact semisimple Lie groups

Let G be a real compact semisimple Lie group and $\mathcal{O} \subset \mathfrak{g}^*$ a regular coadjoint orbit of G. $\mathcal{O}$ has a natural symplectic structure that is induced from the Kirillov–Poisson structure Π.

Recall that there is a G-invariant star product on $\mathcal{O}$ since G is compact. Since G is semisimple, Theorems 3.1 and 3.2 imply that for each G-invariant star product $*$ on $\mathcal{O}$ there is a unique quantum moment map of $*$.

We study here G-invariant star products on $\mathcal{O}$, and our goal is to present a structure theory for them.

4.1. Structure theory. Let $*$ be a G-invariant star product on $\mathcal{O}$ and $\Phi_* = \Phi_0 + \Phi_1\lambda^1 + \cdots$ a quantum moment map of $*$. The classical moment map Φ_0 is given simply by the pullback of the embedding map of $\mathcal{O}$ in $\mathfrak{g}^*$; that is, Φ_0 is surjective. Therefore, Proposition 3.6 implies that the quantum moment map Φ_* is also surjective. As a direct consequence:

Proposition 4.1. *We have a G-equivariant isomorphism*

$$C^\infty(\mathfrak{g}^*)[\![\lambda]\!]/\ker\Phi_* \cong C^\infty(\mathcal{O})[\![\lambda]\!]. \tag{21}$$

Let $*$ and $*'$ be G-invariant star products. If we assume $\ker\Phi_* = \ker\Phi_{*'}$, Proposition 4.1 defines a morphism $S : (C^\infty(\mathcal{O})[\![\lambda]\!], *) \to (C^\infty(\mathcal{O})[\![\lambda]\!], *')$. Let φ be a smooth function on $\mathcal{O}$. There exists $u \in C^\infty(\mathfrak{g}^*)[\![\lambda]\!]$ such that $\Phi_*(u) = \varphi$, and we define $S(\varphi) \equiv \Phi_{*'}(u)$. Lemma 3.5 and the differentiability of $\Phi_{*'}$ imply that the morphism S is differentiable. Therefore:

Lemma 4.1. $\ker\Phi_* = \ker\Phi_{*'}$ *if and only if $*$ is G-equivalent to $*'$.*

As we have seen, $\ker\Phi_* = \ker\Phi_{*'}$ implies $c_* = c_{*'}$. The next proposition shows that if there are "good coordinates" on $\mathfrak{g}^*$, the converse also holds:

Proposition 4.2. *Assume there are functionally independent G-invariant functions $p_i : \mathfrak{g}^* \to \mathbb{R}$, $1 \le i \le r$, such that $\mathcal{O}$ is given by the level set $\{\xi \in \mathfrak{g}^* : p_i(\xi) = c_i\}$ for some regular value $\{c_i\}$ of $\{p_i\}$. Then* $\ker\Phi_*$ *is equal to the ideal of $(C^\infty(\mathfrak{g}^*)[\![\lambda]\!], *^G)$ generated by $\{p_i - c_*(p_i) : 1 \le i \le r\}$.*

Proof. Let $f = f_0 + f_1\lambda + \cdots \in \ker \Phi_*$. It is easy to see that $f_0 \in \ker \Phi_0$, that is, f_0 is null on $\mathcal{O}$. So there are functions $g_i \in C^\infty(\mathfrak{g}^*)$ such that

$$f_0 = \sum_{i=1}^{r} g_i(p_i - c_i). \tag{22}$$

Setting

$$f^{(0)} = \sum_{i=1}^{r} g_i *^G (p_i - c_*(p_i)), \tag{23}$$

we have $f^{(0)} \in \ker \Phi_*$ and $f_0^{(0)} = f_0$. Applying the same argument to $(f - f^{(0)})/\lambda$ inductively we find a sequence of functions $f^{(k)}$ satisfying

$$f = \sum_{k=0}^{\infty} f^{(k)}\lambda^k. \tag{24}$$

Since each $f^{(k)}$ has the form (23), this completes the proof. □

Let $I \subset \mathrm{Pol}(\mathfrak{g}^*)$ be the set of polynomials on $\mathfrak{g}^*$ invariant under G. By Chevalley's theorem [**19**] one has $I = \mathbb{C}[p_1, \ldots, p_r]$, where $p_1, \ldots, p_r$ are algebraically independent homogeneous polynomials and r is the rank of $\mathfrak{g}$. One can also see that any regular coadjoint orbit $\mathcal{O}$ is given by the level set $\{\xi \in \mathfrak{g}^* : p_1(\xi) = c_1, \ldots, p_r(\xi) = c_r\}$ for some regular value $\{c_j\}$ [**14**]. Therefore, $\{p_j\}$ satisfies the condition of Proposition 4.2. So we have the inverse of Proposition 3.7.

Proposition 4.3. *Let $*, *'$ be G-invariant star products and let $\Phi_*, \Phi_{*'}$ be quantum moment maps of $*, *'$ respectively. Then $c_* = c_{*'}$ implies* $\ker \Phi_* = \ker \Phi_{*'}$. *Moreover, if we take $\{p_j\}$ as the algebraically independent homogeneous polynomials obtained from Chevalley's theorem, we have*

$$\ker \Phi_* = \langle p_j - \Phi_*(p_j) \rangle,$$

where $\langle p_j - \Phi_*(p_j) \rangle$ denotes the ideal of $C^\infty(\mathfrak{g}^*)[\![\lambda]\!]$ generated by $p_j - \Phi_*(p_j)$.

And we have also the following structure theorem:

Theorem 4.1. *For any G-invariant star product $*$ on $\mathcal{O}$, there are constants $c_{*,j} \in \mathbb{C}[\![\lambda]\!], j = 1, 2, \ldots, r$ such that*

$$(C^\infty(\mathcal{O})[\![\lambda]\!], *) \cong C^\infty(\mathfrak{g}^*)[\![\lambda]\!]/\langle p_i - c_{*,j} \rangle.$$

Moreover, this isomorphism is G-equivariant.

Proof. Let Φ_* be a quantum moment map of $*$. If we set $c_{*,j} = \Phi_*(p_j)$, the theorem is a direct consequence of Propositions 4.1 and 4.3. □

Corollary 4.1. *There is a one-to-one correspondence between the classes of G-invariant star products on $\mathcal{O}$ and c_*.*

4.2. Example. Coadjoint orbits of SO(3). Let $\mathcal{O}$ be a regular coadjoint orbit of $\mathrm{SO}(3)$. It is well-known that $\mathcal{O}$ is a two-dimensional sphere in $\mathfrak{so}(3)^*$ given by the Casimir polynomial $p(x,y,z) = x^2 + y^2 + z^2$ on $so(3)^* = \mathbb{R}^3$; that is, there is a real number $r > 0$ such that

$$\mathcal{O} = \big\{(x,y,z) \in \mathfrak{g}^* : p(x,y,z) = r^2\big\}.$$

The class of $\mathrm{SO}(3)$-invariant star products is parametrized by the second equivariant de Rham cohomology $H^2_{\mathrm{dR}}(\mathcal{O},\mathbb{R})^{\mathrm{SO}(3)}$. Let $*$ be a $\mathrm{SO}(3)$-invariant star product on $\mathcal{O}$. Then p satisfies the conditions of Proposition 4.2, so that $\ker \Phi_*$ is described by $c_*(p)$ and we obtain

$$\big(C^\infty(\mathcal{O})[\![\lambda]\!], *\big) \cong \big(C^\infty(\mathfrak{g}^*)[\![\lambda]\!], *^G\big)/\big\langle p - c_*(p)\big\rangle. \tag{25}$$

Hence, G-invariant star products on $\mathcal{O}$ have the form of the right-hand side of (25) and are parametrized by $c_*(p)$. This gives another classification of G-invariant star products on $\mathcal{O}$.

Appendices

A. Oscillatory integrals

We provide here a brief review on *oscillatory integrals* in order to fix some definitions and notations. The following is based on [**15**], with little modifications adapted to our problem:

Definition A.1. A function $a(\xi,x) \in C^\infty(\mathbb{R}^n_\xi \times \mathbb{R}^n_x)$ is said to be of $\mathcal{A}^m_\tau$-class, where $-\infty < m < \infty$ and $0 \le \tau$, if for any multi-indices I and J, there exists a constant $C_{I,J}$ such that

$$\big|\partial^I_\xi \partial^J_x a(\xi,x)\big| \le C_{I,J}\langle\xi\rangle^m\langle x\rangle^\tau,$$

where $\langle\xi\rangle = \sqrt{1+|\xi|^2}$. Set

$$\mathcal{A} = \bigcup_{-\infty<m<\infty}\ \bigcup_{0\le\tau} \mathcal{A}^m_\tau.$$

For $a(\xi,x) \in \mathcal{A}^m_\tau$, we define a family of seminorms $|a|_l$, $l = 0,1,\ldots,$ by

$$|a|_l = \max_{|I+J|\le l}\ \sup_{(\xi,x)} \big\{|\partial^I_\xi\partial^J_x a(\xi,x)|\langle\xi\rangle^{-m}\langle x\rangle^{-\tau}\big\}.$$

Then $\mathcal{A}^m_\tau$ becomes a Fréchet space. A subset B of $\mathcal{A}$ is called bounded if there is a $\mathcal{A}^m_\tau$ such that $B \subset \mathcal{A}^m_\tau$ and $\sup_{a\in B}\{|a|_l\} < \infty$ for any $l = 0,1,\ldots.$

Definition A.2. For any $a(\xi, x) \in \mathcal{A}$, we define the oscillatory integral $\mathrm{Os}[e^{-i\xi x}a]$ by

$$\begin{aligned}\mathrm{Os}[e^{-i\xi x}a] &\equiv \mathrm{Os}\text{-}\int e^{-i\xi x}a(\xi, x)\, d\xi\, dx \\ &= \frac{1}{(2\pi)^n}\lim_{\varepsilon\to 0}\int \chi(\varepsilon\xi, \varepsilon x)e^{-i\xi x}a(\xi, x)\, d\xi\, dx,\end{aligned}$$

where $\chi(\xi, x)$ is any function of $\mathcal{S}(\mathbb{R}^{2n}_{\xi,x})$ satisfying $\chi(0, 0) = 1$.

Lemma A.1. *If $\chi(x) \in \mathcal{S}(\mathbb{R}^n)$ satisfies $\chi(0) = 1$ then*

$$\chi(\varepsilon x) \underset{\varepsilon\to 0}{\to} 1 \quad (\textit{uniformly on compact sets}), \tag{26}$$

$$\partial_x^I\chi(\varepsilon x) \underset{\varepsilon\to 0}{\to} 0 \quad (\textit{uniformly on } \mathbb{R}^n,\ |I| > 0), \tag{27}$$

and for any multi-index I there is a constant C_I independent of $0 < \varepsilon < 1$ such that for any σ such that $0 \le \sigma \le |I|$,

$$\left|\partial_x^I\chi(\varepsilon x)\right| \le C_I\varepsilon^\sigma\langle x\rangle^{-(|I|-\sigma)}. \tag{28}$$

Proof. (27) is clear since $\partial_x^I\chi(\varepsilon x) = \varepsilon^{|I|}\partial_y^I\chi(y)|_{y=\varepsilon x}$. If $|x| \le 1$ then (28) is obtained from the equality $\left|\partial_x^I\chi(\varepsilon x)\right| = \varepsilon^\sigma\big(\varepsilon^{(|I|-\sigma)}|\partial_y^I\chi(y)|_{|y=\varepsilon x}\big)$. If $|x| > 1$ and $0 \le \sigma \le |I|$, we have

$$\begin{aligned}\varepsilon^{(|I|-\sigma)}\left|\partial_y^I\chi(y)_{|y=\varepsilon x}\right| &= \big(|y|^{(|I|-\sigma)}|\partial_y^I\chi(y)|\big)_{|y=\varepsilon x}|x|^{-(|I|-\sigma)} \\ &\le C_I\langle x\rangle^{-(|I|-\sigma)}\end{aligned}$$

□

Theorem A.1. *For any $a \in \mathcal{A}$, $\mathrm{Os}[e^{-i\xi x}a]$ is independent of the choice of $\chi \in \mathcal{S}$ satisfying $\chi(0, 0) = 1$. For $a \in \mathcal{A}^m_\tau$, if we take integers l, l' satisfying*

$$-2l + m < -n, \quad -2l' + \tau < -n, \tag{29}$$

then

$$\left|\langle x\rangle^{-2l'}\langle D_\xi\rangle^{2l'}\{\langle\xi\rangle^{-2l}\langle D_x\rangle^{2l}a(\xi, x)\}\right| \in L_1(\mathbb{R}^{2n})$$

and

$$\mathrm{Os}[e^{-i\xi x}a] = \mathrm{Os}\text{-}\int e^{-i\xi x}\langle x\rangle^{-2l'}\langle D_\xi\rangle^{2l'}\big(\langle\xi\rangle^{-2l}\langle D_x\rangle^{2l}a(\xi, x)\big)\, d\xi\, dx.$$

Moreover, for $a \in \mathcal{A}^m_\tau$ there is a constant C such that

$$\left|\mathrm{Os}[e^{-i\xi x}a]\right| \le C|a|_{2(l+l')}. \tag{30}$$

Proof. Fix $0 < \varepsilon < 1$. Integrating by parts, we have

$$\begin{aligned} I_\varepsilon &\equiv \int e^{-i\xi x}\chi(\varepsilon\xi, \varepsilon x)a(\xi, x)\, d\xi\, dx \\ &= \int e^{-i\xi x}\langle\xi\rangle^{-2l}\langle D_x\rangle^{2l}(\chi(\varepsilon\xi, \varepsilon x)a(\xi, x))\, d\xi\, dx \\ &= \int e^{-i\xi x}\langle x\rangle^{-2l'}\langle D_\xi\rangle^{2l'}\big(\langle\xi\rangle^{-2l}\langle D_x\rangle^{2l}(\chi(\varepsilon\xi, \varepsilon x)a(\xi, x))\big)\, d\xi\, dx. \end{aligned}$$

Lemma A.1 implies the set $\{\chi(\varepsilon\xi, \varepsilon x)\}_{0<\varepsilon<1}$ is a bounded subset of $\mathcal{A}_0^0$, so that for any I and J there is a constant $C_{I,J}$ independent of $\varepsilon, a \in \mathcal{A}_\tau^m$ such that

$$\big|\partial_\xi^I \partial_x^J(\chi(\varepsilon\xi, \varepsilon x)a(\xi, x))\big| \leq C_{I,J}|a|_{(|I|+|J|)}\langle\xi\rangle^m\langle x\rangle^\tau.$$

On the other hand, for any s, there is a constant $C_{s,I}$ such that

$$\big|\partial_\xi^I\langle\xi\rangle^s\big| \leq C_{s,I}\langle\xi\rangle^{s-|I|},$$

obtained by induction from $\partial_{\xi_j}\langle\xi\rangle^s = s\xi_j\langle\xi\rangle^{s-2}$. From these facts we deduce that there is for any I a constant $C_{l,I}$ independent of ε and $a \in \mathcal{A}_\tau^m$ and such that

$$\big|\partial_\xi^I\{\langle\xi\rangle^{-2l}\langle D_x\rangle^{2l}(\chi(\varepsilon\xi, \varepsilon x)a(\xi, x))\}\big| \leq C_{l,I}|a|_{(2l+|I|)}\langle\xi\rangle^{m-2l}\langle x\rangle^\tau.$$

Hence there is a constant $C_{l,l'}$ independent of $\varepsilon, a \in \mathcal{A}_\tau^m$ such that

$$\langle x\rangle^{-2l'}\langle D_\xi\rangle^{2l'}\big\{\langle\xi\rangle^{-2l}\langle D_x\rangle^{2l}(\chi(\varepsilon\xi, \varepsilon x)a(\xi, x))\big\} \leq C_{l,l'}|a|_{(l+l')}\langle\xi\rangle^{m-2l}\langle x\rangle^{\tau-2l'}. \tag{31}$$

The right-hand side of (31) is in $L_1(\mathbb{R}^{2n}_{\xi,x})$ because of (29). Hence Lebesgue's convergence theorem gives

$$\begin{aligned} \mathrm{Os}[e^{i\xi x}a] &= \lim_{\varepsilon\to 0}\frac{I_\varepsilon}{(2\pi)^2} \\ &= \frac{1}{(2\pi)^2}\int e^{-i\xi x}\langle x\rangle^{-2l'}\langle D_\xi\rangle^{2l'}\{\langle\xi\rangle^{-2l}\langle D_x\rangle^{2l}a(\xi, x))\}\, d\xi\, dx, \end{aligned}$$

and proves (30). □

Theorem A.2. *Assume* $\{a_j\}_{j=1}^\infty$ *is a bounded set of* $\mathcal{A}$ *and there is* $a \in \mathcal{A}$ *such that*

$$a_j(\xi, x) \to a(\xi, x) \quad \text{uniformly on compact sets of } \mathbb{R}^{2n}.$$

Then

$$\lim_{j\to\infty}\mathrm{Os}[e^{-i\xi x}a_j] = \mathrm{Os}[e^{-i\xi x}a].$$

Theorem A.3. *The oscillatory integral satisfies the following formulas:*

$$\begin{aligned}\mathrm{Os}[e^{-i\xi x}a(\xi,x)] &= \mathrm{Os}\big[e^{-i(\xi-\xi_0)(x-x_0)}a(\xi-\xi_0,x-x_0)\big], \quad (\xi_0,x_0)\in\mathbb{R}^{2n},\\ \mathrm{Os}[e^{-i\xi x}x^I a] &= \mathrm{Os}[(-D_\xi)^I e^{-i\xi x}a] = \mathrm{Os}[e^{-i\xi x}D_\xi^I a],\\ \mathrm{Os}[e^{-i\xi x}\xi^I a] &= \mathrm{Os}[(-D_x)^I e^{-i\xi x}a] = \mathrm{Os}[e^{-i\xi x}D_x^I a].\end{aligned}$$

Theorem A.4. *Let $a = a(x) \in \mathcal{A}$ be a function depending only on x. Then*

$$\mathrm{Os}\text{-}\!\int e^{-i\xi(x-y)}a(x)\,d\xi\,dx = a(y).$$

B. The Fedosov construction of star products

We provide here a brief summary of the Fedosov construction, which is one of the most useful method of constructing a star product on a symplectic manifold (M,ω). For details see [7, 8].

A formal Weyl algebra W_x associated with T_xM for $x \in M$ is an associative algebra with unit over $\mathbb{C}$ defined as follows: each element of W_x is a formal power series in λ with coefficients being formal polynomials in T_xM, that is, each element has the form

$$a(y,\lambda) = \sum_{k,J}\lambda^k a_{k,J}y^J,$$

where $y = (y^1,\dots,y^{2n})$ are linear coordinates on T_xM, $J = (j_1,\dots,j_{2n})$ is a multi-index and $y^J = (y^1)^{j_1}\cdots(y^{2n})^{j_{2n}}$. The product $\circ$ is defined by the Moyal–Weyl rule,

$$a\circ b = \sum_{k=0}^{\infty}\Big(\frac{\lambda}{2}\Big)^k\frac{1}{k!}\omega^{i_1j_1}\cdots\omega^{i_kj_k}\frac{\partial^k a}{\partial y^{i_1}\cdots\partial y^{i_k}}\frac{\partial^k b}{\partial y^{j_1}\cdots\partial y^{j_k}},$$

where the ω^{lm} are the coefficients of ω with respect to y^j. If we assign $\deg y^j = 1$ and $\deg\lambda = 2$, the algebra W_x becomes a filtered algebra.

Let $W = \bigcup_{x\in M}W_x$. Then W is a bundle of algebras over M, called the Weyl bundle over M. Each section of W has the form

$$a(x,y,\lambda) = \sum_{k,\alpha}\lambda^k a_{k,\alpha}(x)y^\alpha, \tag{32}$$

where $x \in M$. We call $a(x,y,\lambda)$ smooth if each coefficient $a_{k,\alpha}(x)$ is smooth in x. We denote the set of smooth sections by ΓW. It constitutes an associative algebra with unit under the fibrewise multiplication.

Let ∇ be a torsion-free symplectic connection on M, which always exists and $\partial : \Gamma W \to \Gamma W\otimes\Lambda^1$ be its induced covariant derivative. Consider a connection on W of the form

$$Da = -\delta a + \partial a - \frac{1}{\lambda}[\gamma,a] \quad \text{for } a\in\Gamma W \tag{33}$$

with $\gamma \in \Gamma W \otimes \Lambda^1$, where

$$\delta a = dx^k \wedge \frac{\partial a}{\partial y^k}.$$

Clearly, D is a derivation for the Moyal–Weyl product. A simple computation shows that

$$D^2 a = \frac{1}{\lambda}[\Omega, a] \quad \text{for any } a \in \Gamma W,$$

where

$$\Omega = \omega - R + \delta\gamma - \partial\gamma + \frac{1}{\lambda}\gamma^2.$$

Here $R = \frac{1}{4} i R_{ijkl} y^i y^j \, dx^k \wedge dx^l$ and $R_{ijkl} = \omega_{im} R^m_{jkl}$ is the curvature tensor of the symplectic connection.

A connection of the form (33) is called Abelian if Ω is a scalar 2-form, that is, $\Omega \in \Lambda^2[\![\lambda]\!]$. We call D a Fedosov connection if it is Abelian and $\deg\gamma \geq 3$. For an Abelian connection, the Bianchi identity implies that $d\Omega = D\Omega = 0$, that is, Ω is closed. In this case, we call Ω a Weyl curvature.

Theorem B.1 ([7]). *Let ∇ be any torsion-free symplectic connection, and $\Omega = \omega + \lambda\omega_1 + \cdots \in Z^2(M)[\![\lambda]\!]$ a perturbation of the symplectic form ω. There exists a unique $\gamma \in \Gamma W \otimes \Lambda^1$ such that D given by Equation (33) is a Fedosov connection which has Weyl curvature Ω and satisfies $\delta^{-1}\gamma = 0$.*

We denote W_D the set of smooth and flat sections, that is, sections a in ΓW satisfying $Da = 0$. The space W_D becomes a subalgebra of ΓW. Let σ denote the projection of W_D onto $C^\infty(M)[\![\lambda]\!]$ defined by $\sigma(a) = a|_{y=0}$.

Theorem B.2 ([7]). *Let D be an Abelian connection. For any $a_0(x, \lambda) \in C^\infty(M)[\![\lambda]\!]$ there exists a unique section $a \in W_D$such that $\sigma(a) = a_0$. Thus σ establishes an isomorphism between W_D and $C^\infty(M)[\![\lambda]\!]$ as $\mathbb{C}[\![\lambda]\!]$-vector spaces.*

We denote the inverse map of σ by Q and call it a quantization procedure. The Weyl product $\circ$ on W_D is translated to $C^\infty(M)[\![\lambda]\!]$, yielding a star product $*$. Namely, we set for $a, b \in C^\infty(M)[\![\lambda]\!]$

$$a * b = \sigma(Q(a) \circ Q(b)).$$

For G-invariant star products, there is a simple criterion.

Proposition B.1 ([8]). *Let ∇ be a G-invariant connection, Ω a G-invariant Weyl curvature and D the Fedosov connection corresponding to (∇, Ω). The star product corresponding to D is G-invariant.*

We study mainly star products of Fedosov type because of the following theorem:

Theorem B.3 ([3]). *Every G-invariant star product is G-equivalent to a Fedosov star product.*

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