DAVENPORT PAIRS OVER FINITE FIELDS

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We call a pair of polynomials \( f, g \in F_q[T] \) a Davenport pair (DP) if their value sets are equal, \( V_f(F_q^t) = V_g(F_q^t) \), for infinitely many extensions of \( F_q \). If they are equal for all extensions of \( F_q \) (for all \( t \geq 1 \)), then we say \((f, g)\) is a strong Davenport pair (SDP). Exceptional polynomials and SDP’s are special cases of DP’s. Monodromy/Galois-theoretic methods have successfully given much information on exceptional polynomials and SDP’s. We use these methods to study DP’s in general, and analogous situations for inclusions of value sets.

For example, if \((f, g)\) is an SDP then \( f(T) - g(S) \in F_q[T, S] \) is known to be reducible. This has interesting consequences. We extend this to DP’s (that are not pairs of exceptional polynomials) and use reducibility to study the relationship between DP’s and SDP’s when \( f \) is indecomposable. Additionally, we show that DP’s satisfy \((\deg f, q^t - 1) = (\deg g, q^t - 1)\) for all sufficiently large \( t \) with \( V_f(F_q^t) = V_g(F_q^t) \). This extends Lenstra’s theorem (Carlitz–Wan conjecture) concerning exceptional polynomials.

1. Introduction

Let \( F_q \) be a finite field with \( q \) elements, and let \( p \) denote its characteristic. For any \( f \in F_q[T] \) and finite extension \( F_q^t \) of \( F_q \), define the value set \( V_f(F_q^t) \) to be \( \{ f(a) \mid a \in F_q^t \} \). Call \((f, g)\) a Davenport pair over \( F_q \) if \( V_f(F_q^t) = V_g(F_q^t) \) for infinitely many values of \( t \). For brevity, we use the acronym DP. We will see that \((f, g)\) is automatically a Davenport pair (DP) if \( V_f(F_q^t) = V_g(F_q^t) \) for one sufficiently large value of \( t \). Call \((f, g)\) a strong Davenport pair (SDP) over \( F_q \) if \( V_f(F_q^t) = V_g(F_q^t) \) for all \( t \geq 1 \).

The name Davenport pair honors a problem formulated by H. Davenport in the 1960’s on a characteristic zero analogue of what we call SDP’s. He asked which pairs \((f, g) \in \mathbb{Q}[T]\) have equal value sets mod \( l \), for almost all primes \( l \). (See Section 3.2 below for more details.)

1.1. Examples, summary of results, and problems. Call \( f \in F_q[T] \) an exceptional polynomial if \( V_f(F_q^t) = F_q^t \) for infinitely many values of \( t \). So \( f \) is exceptional if and only if \((f, T)\) is a DP. Thus both SDP’s and exceptional polynomials are special types of Davenport pairs. One way to
create a DP which is not an SDP is to compose an SDP with exceptional polynomials.

**Definition 1.1.** Suppose \((f,g)\) is an SDP and \((h_1,h_2)\) is a pair of exceptional polynomials. Then \((f \circ h_1, g \circ h_2)\) is a DP, which we call an SDP-Ex composition.

SDP-Ex compositions have equal value sets over the base field \(\mathbb{F}_q\), a property not possessed by all DP’s.

**Problem 1.2.** Suppose \((f,g)\) is a DP over \(\mathbb{F}_q\), where \(q\) is sufficiently large and \(V_f(\mathbb{F}_q) = V_g(\mathbb{F}_q)\). When is \((f,g)\) an SDP-Ex composition?

Here sufficiently large means larger than a bound depending on the degrees of \(f\) and \(g\). Condition (1.1) can be replaced with a condition not requiring large \(q\). By Corollary 4.4 there is a natural union of arithmetic progressions, defined Galois theoretically, containing all but finitely many of the values \(t\) for which \(V_f(\mathbb{F}_{q^t}) = V_g(\mathbb{F}_{q^t})\). We can replace (1.1) with the hypothesis that 1 is in this union of arithmetic progressions: \(1 \in D_{f,g}\) (see Definition 4.3).

Examples of Müller (see Remark 3.16) illustrate the phenomenon of masking, which suggests an approach for finding DP’s satisfying (1.1) which are not SDP-Ex compositions.

**Definition 1.3.** Let \(f,g,h \in \mathbb{F}_q[x]\). We say that \(h\) masks differences between value sets of \(f\) and \(g\) if \(V_f(\mathbb{F}_q) \neq V_g(\mathbb{F}_q)\) but \(V_{h \circ f}(\mathbb{F}_{q^t}) = V_{h \circ g}(\mathbb{F}_{q^t})\) for an infinite number of \(t\).

We now describe the key results of this paper from the point of view that several properties held by SDP-Ex compositions extend to DP’s in general.

For example, if \((f,g)\) is an SDP with \(\deg f > 1\), then \(f(T) - g(S)\) is known to be reducible in \(\mathbb{F}_q[S,T]\). It follows that, \(f \circ h_1(T) - g \circ h_2(S)\) is also reducible in \(\mathbb{F}_q[S,T]\) for any pair \((h_1,h_2)\). This gives a property of SDP-Ex composition which extends: If \((f,g)\) is a DP satisfying (1.1), and \(f\) is not an exceptional polynomial, then \(f(T) - g(S) \in \mathbb{F}_q[S,T]\) is reducible over \(\mathbb{F}_q\) (Corollary 4.12).

As another example, consider this theorem of Lenstra [CF95], conjectured by Carlitz and Wan: If \(h \in \mathbb{F}_q[T]\) is exceptional, then \(\deg h\) is relatively prime to \(q - 1\). It is also known that if \((f,g)\) is an SDP, and if the degrees of \(f\) and \(g\) are prime to the characteristic \(p\), then \(\deg f = \deg g\). Thus if \(f = f' \circ h_1\) and \(g = g' \circ h_2\) where \((f',g')\) is an SDP, \((h_1,h_2)\) is a pair of exceptional polynomials, and \(\deg f\) and \(\deg g\) are prime to \(p\), then \(\gcd(\deg f, q - 1) = \gcd(\deg g, q - 1)\). This property of SDP-Ex composition holds for all DP’s satisfying (1.1). It is a consequence of Theorem 5.4 (which is stronger since it makes no assumption on the degrees of \(f\) and \(g\)).
Finally, consider our Theorem 8.1, a result consistent with SDP-Ex composition. Suppose that \((f, g)\) is a DP and that \(f\) is indecomposable. Suppose also that \(f\) has degree prime to the characteristic \(p\), and is neither an exceptional polynomial nor linearly related to a cyclic polynomial. Then \(g = g' \circ h\) for some SDP \((f, g')\).

We end this introduction with other problems related to DP’s.

**Problem 1.4.** If \((h_1, h_2)\) is a pair of polynomials such that \((f \circ h_1, g \circ h_2)\) is a DP for all SDP’s \((f, g)\), must \(h_1\) and \(h_2\) be exceptional polynomials?

Other problems involve multiplicities of values. Call \((f, g)\) a DP *with multiplicity* if there are an infinite number of \(t\) so that \(f\) and \(g\) not only have the same value sets over \(\mathbb{F}_{q^t}\), but the values occur with the same multiplicities. That is, \(f(T) - b\) and \(g(T) - b\) have the same number of zeros in \(\mathbb{F}_{q^t}\) for each \(b \in \mathbb{F}_{q^t}\). Similarly, call \((f, g)\) an SDP *with multiplicity* if the multiplicity condition occurs for all values of \(t\).

**Problem 1.5.** Are there SDP’s which are not SDP’s with multiplicity? Are there DP’s which are not DP’s with multiplicity?

[Müll98, Conjecture 5.2] considered the characteristic zero analogue of the first part of this question. Müller conjectures that Kronecker conjugate polynomials (the analogue of SDP’s) are arithmetically equivalent (i.e., have the same multiplicities).

### 1.2. A bigger context for DP’s

A polynomial \(f \in \mathbb{F}_q[T]\) gives an algebraic map \(f : \mathbb{A}^1 \to \mathbb{A}^1\), or, by adding points at infinity, an algebraic map \(f : \mathbb{P}^1 \to \mathbb{P}^1\). Our approach, via the arithmetic and geometric monodromy groups associated with the map \(f\), or pairs of maps \((f, g)\), extends considerably to include maps between algebraic curves defined over \(\mathbb{F}_q\), and even to finite maps between higher-dimensional varieties. We concentrate on polynomial maps, as these offer a sufficient challenge while showing us their considerable structure without forcing excessive notation. Also, more can be proven for such maps since they have a totally ramified point, infinity, and the maps are between curves of genus zero. Still, we now briefly discuss a natural program that will benefit from the investigations of this paper, but requires considering the challenges of extending beyond polynomial maps.

We describe, in particular, the link between Davenport pairs and such topics as Weil vectors, Galois stratification, and Chow motives. Here, a *Weil vector* is the sequence of coefficients of a Poincaré series associated to a number-theoretic counting problem. For example, if \(V\) is a projective variety over \(\mathbb{F}_q\), we get the familiar Weil vector \(N = (N_1, N_2, \ldots)\) where \(N_t\) is the number of \(\mathbb{F}_{q^t}\)-rational points of \(V\). The associated Poincaré series is \(P_V(X) = \sum_{t=1}^{\infty} N_t X^t\), and the associated zeta function is \(Z_V(X) = \exp \left(\sum_{t=1}^{\infty} N_t X^t/t\right)\).
Weil vectors also arise from other counting problems. For example, let $V$ be a scheme (reduced, separated) of finite type over $\mathbb{Z}$. Consider the Weil vector $\mathcal{N} = (N_1, N_2, \ldots)$, where $N_t$ is the number of $\mathbb{Z}/p^t\mathbb{Z}$-rational points which lift to $\mathbb{Z}_p$-rational points. The rationality of the associated Poincaré series was established by Denef [Den84].

Galois stratification is a tool for studying Weil vectors in a wide variety of counting problems (see [FS76] and [FJ86]). Denef and Loeser [DL] link Galois stratification and Chow motives. Given two Weil vectors $\mathcal{N} = (N_1, N_2, \ldots)$ and $\mathcal{N}' = (N'_1, N'_2, \ldots)$, the characteristic set $\chi(\mathcal{N}, \mathcal{N}')$ is $\{t \in \mathbb{N}^+ | N_t = N'_t\}$. Such characteristic sets, when the Weil vectors arise from Galois stratification, form Frobenius progressions (Definition 4.5).

To consider the link between DP’s and these topics, consider your favorite equation $\Phi(T, U) = 0$, where $\Phi \in \mathbb{F}_q[T, U]$ and $U = (U_1, \ldots, U_s)$. Consider also the Weil vector $\mathcal{N}(\Phi) = (N_1(\Phi), N_2(\Phi), \ldots)$, where $N_t(\Phi)$ is the number of solutions over $\mathbb{F}_q$. You often substitute a polynomial or rational function $f(T)$ for $T$ to get the related equation $\Phi(f(T), U) = 0$. Write $\Phi_f$ for $\Phi(f(T), U)$. Let $(f, g)$ be a pair of polynomials, and let $\chi(f, g)$ be the set of $t$ with the property that $\mathcal{N}_f(\mathbb{F}_q^t)$ and $\mathcal{N}_g(\mathbb{F}_q^t)$ are equal, and every value occurs with the same multiplicity. We assume $\chi(f, g)$ is infinite. In other words, $(f, g)$ is a DP with multiplicity. Observe that $\chi(f, g) \subseteq \chi(\mathcal{N}(\Phi_f), \mathcal{N}(\Phi_g))$.

This gives us a procedure for generating nontrivial (nonfinite) characteristic sets relating many different pairs of Weil vectors. The resulting characteristic sets must contain a common Frobenius progression $\chi(f, g)$ regardless of your choice of favorite equation. This suggests the importance of the study of Frobenius progressions of the form $\chi(f, g)$ from the more general Weil vector viewpoint.

For any pair of Weil vectors, attached to any elementary problem (as in [FS76]), there is a characteristic set at which the two Weil vectors are equal. The argument of [Fri94, Riem. Hyp. Lem. 2.2] extends to show that such a characteristic set is always, modulo finite sets, a union of Frobenius progressions. We consider such a characteristic set a relation among Weil vectors. It is a fundamental problem to consider how such relations arise and to what extent they arise from sets $\chi(f, g)$ as in our problem above.

2. Notations and conventions

Note that $(f(T), f(T^p))$ is an SDP (as above, $f \in \mathbb{F}_q[T]$ and $p$ is the characteristic of $\mathbb{F}_q$). So, for value set problems, it is harmless to replace any polynomial of the form $f(T^p)$ by $f(T)$. By repeating this process starting with a given polynomial, we obtain a polynomial whose derivative is not the zero polynomial, and whose value set, in all finite extensions, is the same as the original polynomial. This justifies the following convention. Assume all polynomials appearing in this paper have nonzero derivatives.
Let $F$ be a field. We are most interested in $F = \mathbb{F}_q$, especially when we are considering value sets, but many of our results hold for more general $F$. Fix an algebraic closure $\overline{F}(z)$ of $F(z)$, where $z$ is a fixed transcendental element over $F$, and regard $\overline{F}$ as a subfield of $\overline{F}(z)$. We use the letter $T$ (as above) for a general transcendental element not in $\overline{F}(z)$. We use $S$ and $T$ when we need two independent transcendental elements (neither in $\overline{F}(z)$).

For any $f \in F[T]$, let $\Omega_f \subseteq \overline{F}(z)$ be the splitting field of $f(T) - z$. Since $f(T) - z$ has $z$-degree 1, it is irreducible in $F(z)[T]$. It is also separable (the derivative $f'$ is not the zero polynomial). Call

$$\hat{G}_f = \text{Gal}(\Omega_f/F(z))$$

the arithmetic monodromy group of $f$. Let $\hat{F}_f = \Omega_f \cap F$. Call

$$G_f = \text{Gal}(\Omega_f/\hat{F}_f(z)) \subseteq \hat{G}_f$$

the geometric monodromy group. Let $n = \deg f$, and let $\{x_1, x_2, \ldots, x_n\}$ be the zeros of $f(T) - z$ in $\Omega_f$. If $H$ is $\hat{G}_f$ or a subgroup, denote the elements of $H$ which fix $x_i$ by $H(x_i)$. For example, $\hat{G}_f(x_i) = \text{Gal}(\Omega_f/F(x_i))$.

Think of $f \in F[T]$ as an algebraic map $f : \mathbb{A}^1 \to \mathbb{A}^1$. By adding a point at infinity, also regard a polynomial (or rational function) as an algebraic covering map $f : \mathbb{P}^1 \to \mathbb{P}^1$.

Now consider the case $F = \mathbb{F}_q$. Here we abuse notation and write $\hat{F}_f$ for $\hat{F}_f$. The quotient $\hat{G}_f/G_f$ is isomorphic to the cyclic group $\text{Gal}(\mathbb{F}_q^n/\mathbb{F}_q)$, where $d = [\hat{F}_f : \mathbb{F}_q]$. Not only is $\text{Gal}(\mathbb{F}_q^n/\mathbb{F}_q)$ cyclic, but it is canonically isomorphic to $\mathbb{Z}/d$ by the map sending the Frobenius automorphism $a \mapsto a^q$ to 1. Let $\hat{G}_{f,t}$ be the $G_f$-coset of elements $\sigma \in \hat{G}_f$ for which $\sigma|_{\hat{F}_f}$ is the map $a \mapsto a^q$. So $\hat{G}_{f,t}$ consists of elements of $\hat{G}_f$ whose image in $\mathbb{Z}/d$ is congruent to $t$. Thus $\hat{G}_{f,t}$ depends only on $t$ modulo $d$.

Now consider analogous definitions for pairs of polynomials $(f, g)$, first for a general field $F$. Let $\Omega_{f,g} = \Omega_f \cdot \Omega_g \subseteq \overline{F}(z)$ be the splitting field of the product $(f(T) - z)(g(T) - z)$. Let $\hat{F}_{f,g} = \Omega_{f,g} \cap F$. Define the arithmetic monodromy group of the pair as $\hat{G}_{f,g} = \text{Gal}(\Omega_{f,g}/F(z))$ and the geometric monodromy group as $G_{f,g} = \text{Gal}(\Omega_{f,g}/\hat{F}_{f,g}(z))$.

Let $\{x_1, x_2, \ldots, x_n\}$ be the zeros of $f(T) - z$, and $\{y_1, y_2, \ldots, y_m\}$ those of $g(T) - z$. Then $\hat{G}_{f,g}$ acts on $\{x_i\}$, on $\{y_j\}$, and on the Cartesian product $\{x_i\} \times \{y_j\}$. For $H$ equal to $\hat{G}_{f,g}$ or a subgroup, $H(x_i)$, $H(y_j)$, and $H(x_i, y_j)$ have the usual meanings for stabilizer subgroups.

Note that $\hat{G}_{f,g}$ is the fiber product of $\hat{G}_f$ and $\hat{G}_g$ over the common quotient group $\text{Gal}(\Omega_f \cap \Omega_g/F(z))$. 
Now consider the case $F = \mathbb{F}_q$. We abuse notation and write $\hat{F}_{f,g}$ for $\hat{F}_{f,g}$.

As before, we have the exact sequence

$$1 \to G_{f,g} \to \hat{G}_{f,g} \to \mathbb{Z}/d \to 1,$$

where $d = [\hat{F}_{f,g} : \mathbb{F}_q]$. Denote the elements of $\hat{G}_{f,g}$ mapping to $t \mod d$ by $\hat{G}_{f,g,t}$. So $\hat{G}_{f,g,t}$ is the $G_{f,g}$-coset of all $\sigma$ that restrict on $\mathbb{F}_q^d$ to the automorphism $x \mapsto x^t$.

Again consider a general field $F$. Call $f \in F[T]$ decomposable over $F$ if $f = f_1 \circ f_2$ with $f_1, f_2 \in F[T]$, deg $f_i > 1$, $i = 1, 2$. Otherwise, $f$ is indecomposable over $F$.

If $f, l_1, l_2 \in F[T]$ are polynomials with deg $l_1 = \deg l_2 = 1$, then we say $f$ and $l_1 \circ f \circ l_2$ are linearly related over $F$. Linearly related polynomials have isomorphic monodromy groups and equivalent actions of their monodromy groups on their respective zero sets.

When comparing value sets, we are interested in a special type of linearly related polynomial pairs. If $f, l \in F[T]$ are polynomials such that deg $l = 1$, then we say that $f$ and $f \circ l$ are linearly related on the inside over $F$. For example, a pair of polynomials $f, g \in \mathbb{F}_q[T]$ linearly related on the inside over $\mathbb{F}_q$ clearly forms an SDP. We call such SDP’s trivial. As explained in the next section, there are examples of nontrivial SDP’s.

If $n$ is a positive integer, we consider the statement $n$ is prime to the characteristic of $F$ to be vacuously true if $F$ has characteristic zero.

3. Review of earlier results

We summarize some of what is known concerning value sets, exceptional polynomials, SDP’s, and DP’s.

3.1. Value sets from the monodromy point of view. Consider a polynomial map as a covering map $f : \mathbb{P}^1 \to \mathbb{P}^1$. Suppose $b \in \mathbb{F}_q^t = \mathbb{A}^1(\mathbb{F}_q^t)$ is not a branch point for this map. Then $b \in \mathcal{V}_f(\mathbb{F}_q^t)$ if and only if the associated Frobenius element $\text{Frob}_b(b) \in \hat{G}_f$ fixes at least one zero of $f(T) - z$. Further, the number of $a \in \mathbb{F}_q^t$ satisfying $f(a) = b$ is equal to the number of fixed points of $\text{Frob}_b(b)$ acting on the zeros $\{x_i\}$. We call this the Frobenius Principle. It follows from an early result of Artin [Art23, §2]. Here

$$\text{Frob}_b(b) = i\left(\frac{\Omega_f \cdot \mathbb{F}_q^t / \mathbb{F}_q^t(z)}{P_b}\right),$$

where $P_b$ is the place of $\mathbb{F}_q^t(z)$ associated to $b \in \mathbb{A}^1(\mathbb{F}_q^t)$, $(L/K)$ is the Artin symbol, and $i : \text{Gal}(\Omega_f \cdot \mathbb{F}_q^t / \mathbb{F}_q^t(z)) \to \hat{G}_f$ is the natural inclusion induced by restriction. The Artin symbol is defined up to conjugacy, so the number of fixed points of $\text{Frob}_b(b)$ is well-defined.

Observe that $\text{Frob}_b(b) \in \hat{G}_{f,t}$. Conversely, the nonregular analog of the Chebotarev Density Theorem implies the proportion of $b \in \mathbb{F}_q^t$ with $\text{Frob}_b(b)$
in a given conjugacy class $C \subseteq \hat{G}_{f,t}$ is approximately $|C|/|\hat{G}_{f,t}|$. More precisely, if $p(C)$ is the proportion of $b \in \mathbb{F}_{q^t}$ such that $\text{Frob}_t(b) \in C$ and $b$ is not a branch point, then

$$p(C) - \frac{|C|}{|\hat{G}_{f,t}|} < B|q^{-t/2}|.$$ 

The best $B$ depends on $f$, but we can find a $B$ depending only on $n = \deg f$. For example, the bound of Proposition 5.16 of [FJ86], specialized to the current situation, gives $B = 4(g + 2)$, where $g$ is the genus of $\Omega_f$. There is a bound in $n$ for this genus $g$, and hence for $B$. (From Riemann–Hurwitz, bounding higher ramification group orders bounds the $\Omega_f/\mathbb{F}_f(z)$ different divisor degree. To bound the nontrivial higher ramification groups in $G_f$, combine an obvious bound on the $\mathbb{F}_f(x_i)/\mathbb{F}_f(z)$ different degree with the corollary to Proposition 4, Chapter IV, §1, of [Ser79].)

Let $N(\sigma)$ be the cardinality of those $\{x_1, \ldots, x_n\}$ fixed by $\sigma \in \hat{G}_{f,t}$. Then

$$\sum_{\sigma \in \hat{G}_{f,t}} N(\sigma) = |\hat{G}_{f,t}|.$$ 

This is a corollary of the Chebotarev Density Theorem, taking $t' \equiv t \pmod{d}$, where $d = [\mathbb{F}_f : \mathbb{F}_q]$ and $t'$ is large. It is also a consequence of the following group-theoretical lemma [GW97, Lemma 3.1], taking $H = G_f, H^* = \hat{G}_{f,t}, G \subseteq \hat{G}_f$ the group generated by $G_f$ and $\hat{G}_{f,t}$, and $r = 1$.

**Lemma 3.1.** Let $G$ be a finite group acting on a finite set $S$. Let $H$ be a normal subgroup of $G$ such that $G/H$ is cyclic. Finally, let $H^*$ be a coset whose image generates $G/H$. Then

$$\frac{1}{|H^*|} \sum_{\sigma \in H^*} N(\sigma) = r,$$

where $r$ is the number of $H$-orbits in $S$ which are also $G$-orbits, and where $N(\sigma)$ is the number of points in $S$ that $\sigma \in G$ fixes.

(The case when $H = G$ is well-known; see Lemma 7.1.)

From (3.1), the following are equivalent:

(3.2) Every element of $\hat{G}_{f,t}$ fixes at least one element of $\{x_i\}$.

(3.3) Every element of $\hat{G}_{f,t}$ fixes at most one element of $\{x_i\}$.

(3.4) Every element of $\hat{G}_{f,t}$ fixes exactly one element of $\{x_i\}$.

**Remark 3.2.** Suppose any of (3.2), (3.3) or (3.4) hold. Then, for any $b \in \mathbb{F}_{q^t}$ not a branch point, $\text{Frob}_t(b)$ fixes exactly one zero. So, by the Frobenius Principle, $f : \mathbb{F}_{q^t} \to \mathbb{F}_{q^t}$ is bijective on the set of points mapping to nonbranch points.
When \( b \in \mathbb{F}_{q^t} \) is a branch point, one has a Frobenius coset instead of a Frobenius element. To determine the number of \( a \in \mathbb{F}_{q^t} \) satisfying \( f(a) = b \), consider the action of the associated decomposition group \( D \) and inertia group \( I \) on the zeros \( \{x_i\} \). It is well-known that one counts \( I \)-orbits which are also \( D \)-orbits (for example, \[\text{vdW35}\]). Lemma 3.1, with \( G = D, H = I \), and \( H^* \) the Frobenius coset, shows that the number of \( a \in \mathbb{F}_{q^t} \) mapping to \( b \) is the average number of \( \{x_i\} \) fixed by \( \sigma \) as \( \sigma \) varies over the Frobenius coset. We call this the Strong Frobenius Principle. So, (3.4) implies bijectivity even when we allow points above branch points. (Note: in the Frobenius Principle or the Strong Frobenius Principle, we can replace \( \hat{G}_f \) with the Galois group of any normal extension of \( \mathbb{F}_q(z) \) containing \( \Omega_f \).)

**Definition 3.3.** Let \( 0 \leq \epsilon \leq 1 \). Call a polynomial map \( f : \mathbb{F}_{q^t} \to \mathbb{F}_{q^t} \) \( \epsilon \)-almost injective if the proportion of points \( b \in \mathbb{F}_{q^t} \) which either have at most one \( a \in \mathbb{F}_{q^t} \) satisfying \( f(a) = b \) or are branch points is at least \( 1 - \epsilon \). Similarly, call \( f : \mathbb{F}_{q^t} \to \mathbb{F}_{q^t} \) \( \epsilon \)-almost surjective if the proportion of points \( b \in \mathbb{F}_{q^t} \) which are either in the value set \( V_f(\mathbb{F}_{q^t}) \) or are branch points is at least \( 1 - \epsilon \).

The above considerations lead easily to the following theorem.

**Theorem 3.4.** Let \( 0 \leq \epsilon < 1/|\hat{G}_{f,t}| \), and let \( \delta = 1/|\hat{G}_{f,t}| - \epsilon \). If \( q^t \geq (B/\delta)^2 \), where \( B \) is the constant in the Chebotarev Density Theorem, then the following are equivalent:

\[
\begin{align*}
(3.5) & \quad f : \mathbb{F}_{q^t} \to \mathbb{F}_{q^t} \text{ is } \epsilon \text{-almost surjective.} \\
(3.6) & \quad f : \mathbb{F}_{q^t} \to \mathbb{F}_{q^t} \text{ is } \epsilon \text{-almost injective.} \\
(3.7) & \quad f : \mathbb{F}_{q^t} \to \mathbb{F}_{q^t} \text{ is bijective.} \\
(3.8) & \quad \text{Every element of } \hat{G}_{f,t} \text{ fixes exactly one zero of } f(T) - z.
\end{align*}
\]

For general \( q^t \), large or small, (3.8) implies (3.7).

**Remark 3.5.** See [Fri74, Lemma 2 and Theorem 1] for a generalization to multivariable polynomial maps \( \mathbb{A}^n \to \mathbb{A}^n \). This theorem has also been generalized [FGS93, p. 186] to covering maps \( X \to Y \) between absolutely irreducible curves over \( \mathbb{F}_q \). (The statement in [FGS93] is essentially the case where \( \epsilon = 0 \), but the methods clearly work for small \( \epsilon > 0 \).)

The upper bound for \( \epsilon \) in the implication (3.5) \( \Rightarrow \) (3.8) can be replaced by \( 1/\deg f \). With \textit{a priori} restrictions on the monodromy groups involved, one can often do better (see [GW97]).

**Corollary 3.6.** A polynomial \( f \in \mathbb{F}_q[T] \) is exceptional if and only if any of the equivalent conditions (3.2) to (3.7) hold for a suitable value of \( t \) and \( \epsilon \).

If (3.7) holds for \( t \), then it holds for any divisor of \( t \). This yields:

**Corollary 3.7.** If \( f \in \mathbb{F}_q[T] \) is an exceptional polynomial and any of the (equivalent) conditions (3.2) to (3.4) are true of \( t = t_0 \), then these conditions are true of any \( t \) satisfying \( \gcd(t,d) \mid \gcd(t_0,d) \) where \( d = |\text{Frob}| : \mathbb{F}_q \).
A similar analysis gives a monodromy interpretation for $V_f(\mathbb{F}_{q^t}) = V_g(\mathbb{F}_{q^t})$.

**Theorem 3.8.** Let $f, g \in \mathbb{F}_q[T]$. Suppose that, for some $t$,

$(3.9)$ every $\sigma \in \widehat{G}_{f,g,t}$ fixes an element of $\{x_i\}$ if and only if it fixes an element of $\{y_j\}$ (as usual, $\{x_i\}$ are the zeros of $f(T) - z$ and $\{y_j\}$ are the zeros of $g(T) - z$).

Then $V_f(\mathbb{F}_{q^t}) = V_g(\mathbb{F}_{q^t})$.

Conversely, if $V_f(\mathbb{F}_{q^t}) = V_g(\mathbb{F}_{q^t})$ for $t$ sufficiently large, then (3.9) holds.

**Remark 3.9.** The Chebotarev Density Theorem together with the Frobenius Principle gives the converse above, even generalizing it by replacing the hypothesis $V_f(\mathbb{F}_{q^t}) = V_g(\mathbb{F}_{q^t})$ with an $\epsilon$-almost equality (analogous to Theorem 3.4).

To prove that (3.9) implies $V_f(\mathbb{F}_{q^t}) = V_g(\mathbb{F}_{q^t})$ one can use the Strong Frobenius Principle (as in Remark 3.2) to cover both branch points and nonbranch points. Alternatively, one can use the following argument, a straightforward adaptation to the current situation of the second part of the proof of [FJ86, Lem. 19.27]. Let $b \in V_f(\mathbb{F}_{q^t})$, and let $a \in \mathbb{F}_{q^t}$ be a zero of $f(T) - b$. Consider the homomorphism $\mathbb{F}_q[x_1] \rightarrow \mathbb{F}_{q^t}$ with $x_1 \mapsto a$ (and so $z \mapsto b$). Extend this to a homomorphism $\varphi : R \rightarrow \mathbb{F}_{q^t}$, where $R$ is the integral closure of $\mathbb{F}_q[z]$ in $\Omega_{f,g}$. Let $D(\varphi) \subseteq \widehat{G}_{f,g}(x_1)$ be the decomposition group associated to $\varphi$ (the subgroup fixing ker $\varphi$). Since $D(\varphi)$ is a decomposition group, the homomorphism $D(\varphi) \rightarrow \text{Gal}(\mathbb{F}_{q^t}/\mathbb{F}_q(a))$ associated to the residue maps is surjective, where $\mathbb{F}_{q^t}$ is the image of $\varphi$. Thus, some $\tau \in D(\varphi)$ has image in $\text{Gal}(\mathbb{F}_{q^t}/\mathbb{F}_q(a))$ the $q^t$-power Frobenius map $u \mapsto u^{q^t}$. Note that $\tau$ fixes $x_1$ and that $\tau \in \widehat{G}_{f,g,t}$. From (3.9), $\tau$ fixes some $y_j$. Let $c = \varphi(y_j)$. The image of $\tau$ acting on $\mathbb{F}_{q^t}$ fixes $c$. Thus, $c \in \mathbb{F}_{q^t}$. Since $g(c) = b$, conclude $b \in V_g(\mathbb{F}_{q^t})$.

For inclusions of value sets we have:

**Theorem 3.10.** Let $f, g \in \mathbb{F}_q[T]$. Suppose that, for some $t$,

$(3.10)$ every $\sigma \in \widehat{G}_{f,g,t}$ that fixes some $x_i$ also fixes some $y_j$.

Then $V_f(\mathbb{F}_{q^t}) \subseteq V_g(\mathbb{F}_{q^t})$.

Conversely, if $V_f(\mathbb{F}_{q^t}) \subseteq V_g(\mathbb{F}_{q^t})$ for $t$ sufficiently large, then (3.10) holds.

**Remark 3.11.** We can replace (3.10) with:

$(3.11)$ Every $\sigma \in \widehat{G}_{f,g,t}(x_1)$ fixes an element of $\{y_j\}$.

**3.2. Strong Davenport pairs.** [Fri99] discusses the theory of SDP’s starting with the following characterization (a corollary of Theorem 3.8).

**Corollary 3.12.** The pair $(f, g)$ in $\mathbb{F}_q[T]$ is an SDP if and only if

$(3.12)$ for $\sigma \in \widehat{G}_{f,g}$, fixing an element of $\{x_i\}$ is equivalent to fixing an element of $\{y_j\}$. 
An analogous result holds for polynomials over number fields ([FJ86, Lemma 19.27] or [Mül98, Theorem 2.3]). Then (3.12) is equivalent to \( f \) and \( g \) being Kronecker conjugate over a number field \( K \): their value sets are equal modulo all but a finite number of nonzero prime ideals of \( K \).

We generalize the following well-known result ([Fri73, Proposition 3], [FJ86, Lemma 19.31], and [Fri99]) to DP’s (see Corollary 4.12).

**Theorem 3.13.** Let \( f, g \in \mathbb{F}_q[T] \). If \( (f, g) \) is an SDP where \( \deg f > 1 \), then \( f(T) - g(S) \in \mathbb{F}_q[S, T] \) is reducible.

This gives several immediate corollaries. For example, if \( f \) and \( g \) have relatively prime degrees, then \( (f, g) \) is not an SDP. As another example, if \( (f, g) \) is an SDP with each degree at most 3, then \( (f, g) \) is a trivial SDP: reducibility implies the existence of a linear factor, which implies that \( f \) and \( g \) are linearly related on the inside.

When \( f : \mathbb{P}^1 \to \mathbb{P}^1 \) and \( g : \mathbb{P}^1 \to \mathbb{P}^1 \) are tamely ramified, the results in [Fri73] in characteristic 0 are relevant, implying major restrictions on the pair \( (f, g) \). We now review these results.

Let \( K \) be a number field and let \( f, g \in K[T] \). If \( f \) and \( g \) are Kronecker conjugate and \( \deg f > 1 \), the analogue of Theorem 3.13 holds: \( f(T) - g(S) \) is reducible. When \( f \) is indecomposable, the reducibility of \( f(T) - g(S) \) forces the geometric monodromy group of \( f \) to be one of a small list, and \( \deg f \) to be one of 7, 11, 13, 15, 21, and 31. That \( f \) and \( g \) are Kronecker conjugate also forces \( \deg f = \deg g \). This together with the Grothendieck Lifting Theorem gives the following theorem in positive characteristic.

**Theorem 3.14.** Consider an SDP \( (f, g) \) over \( \mathbb{F}_q \) with these properties:

1. (3.13) \( f : \mathbb{P}^1 \to \mathbb{P}^1 \) is tamely ramified.
2. (3.14) \( f \) is indecomposable.

Then \( \deg f = \deg g \), and both \( \deg f \) and \( G_f \) satisfy the above restrictions.

The following result from [Fri99, Thm 5.7] shows degrees are not bounded when we allow \( f : \mathbb{P}^1 \to \mathbb{P}^1 \) to have wild ramification.

**Theorem 3.15.** Over any field \( \mathbb{F}_q \) there are infinitely many \( n \) prime to the characteristic for which nontrivial SDP’s \( (f, g) \) exist with \( n = \deg f = \deg g \) and \( f \) indecomposable.

The monodromy groups appearing in these examples are subgroups of the projective linear groups over finite fields of characteristic \( p \).

Finally we mention what is known concerning Davenport’s original question. If \( K = \mathbb{Q} \) there are no nontrivial Kronecker conjugate polynomials with \( f \) indecomposable [Fri73], or with \( f \) and \( g \) each compositions of two indecomposable polynomials of degree at least 2 [Mül98]. Still, \( f(T) = T^8 \) and \( g(T) = 16T^8 \) are Kronecker conjugate polynomials, each the composition of three indecomposable polynomials. Over \( \mathbb{Q} \), Müller suggests this is a singular anomaly.
Remark 3.16 (Müller’s work on masking). If \((f, g)\) is an SDP, then the pair \((h \circ f, h \circ g)\) is also an SDP for all \(h \in \mathbb{F}_q[T]\). More surprisingly, there are pairs \((f, g)\) which are not SDP’s (not even DP’s) and \(h \in \mathbb{F}_q[T]\) of positive degree such that \((h \circ f, h \circ g)\) is an SDP. That is, \(h\) masks (see Definition 1.3) the difference between \(f\) and \(g\). Müller [Müll98, §4] gave examples of this over number fields and they apply over suitable \(\mathbb{F}_q\). So this, in addition to the case where the pairs are indecomposable in Theorem 3.15 shows there are many nontrivial SDP’s. Müller’s examples give polynomials with equivalent permutation characters, so they yield SDP’s with multiplicity.

3.3. Other related value set work. Earlier related work (not exclusive to SDP’s or exceptional polynomials) did not concern DP’s per se, but rather polynomials with equal value sets over the ground field \(\mathbb{F}_q\). Note: for \(q\) large, such pairs are DP’s (Corollary 4.2).

For example, [Coh81] studies pairs of rational functions \(f, g \in \mathbb{F}_q(T)\) satisfying \(V_f(\mathbb{F}_q) \subseteq V_g(\mathbb{F}_q)\). The main result is a classification of such \(f\) and \(g\) with \(\deg g \leq 4\), where the characteristic is greater than 3 and \(q\) is large (lower bounds depending on \(\deg f\)). Other much earlier work: McCann and Williams (value set equalities for polynomials of degree 3), Mordell (also for degree 3), and Carlitz (value set inclusions with \(g(T) = T^m\)).

Finally, [Ait98] studies the overlap between \(V_f(\mathbb{F}_q)\) and \(V_g(\mathbb{F}_q)\) when the two sets are not equal, which, for large \(q\), yields a criterion for whether or not two polynomials form a DP.

4. Basic results concerning Davenport pairs

Let \(f, g \in \mathbb{F}_q[T]\), and let \(d = \lceil \mathbb{F}_q : \mathbb{F}_q \rceil\). Below are corollaries of Theorem 3.8.

Corollary 4.1. The pair \((f, g)\) is a DP if and only if, for some \(t\), (3.9) holds.

Corollary 4.2. The pair \((f, g)\) is a DP if and only if \(V_f(\mathbb{F}_{q^t}) = V_g(\mathbb{F}_{q^t})\) for a sufficiently large \(t\).

Here, sufficiently large means that \(q^t\) exceeds some bound depending only on the maximum of the degrees of \(f\) and \(g\).

Condition (3.9) depends only on \(t \mod d\). Thus, if (3.9) holds for one \(t\), it holds for infinitely many \(t\); the set of such \(t\) forms a union of arithmetic progressions. For any integer \(t\), denote its image in \(\mathbb{Z}/d\) by \(\hat{t}\).

Definition 4.3. Let \(\mathcal{D}_{f, g} = \{ \hat{t} \in \mathbb{Z}/d \mid (3.9) \text{ holds for } t \}\). So \((f, g)\) is a DP if and only if \(\mathcal{D}_{f, g}\) is not empty, and \((f, g)\) is an SDP if and only if \(\mathcal{D}_{f, g} = \mathbb{Z}/d\).

Corollary 4.4. For \(t\) sufficiently large, \(V_f(\mathbb{F}_{q^t}) = V_g(\mathbb{F}_{q^t})\) if and only if \(\hat{t} \in \mathcal{D}_{f, g}\). For all \(t\), large or small, \(\hat{t} \in \mathcal{D}_{f, g}\) implies \(V_f(\mathbb{F}_{q^t}) = V_g(\mathbb{F}_{q^t})\).
The set $D_{f,g}$ is an example of a Frobenius set:

**Definition 4.5.** A Frobenius set (mod $d$) is a subset $S$ of $\mathbb{Z}/d$ with the following property. If $a \in S$, then so is $ua$, where $u$ is a unit in $\mathbb{Z}/d$. Equivalently, if $a, b$ have the same order in $\mathbb{Z}/d$, then $a \in S$ if and only if $b \in S$. So $S$ is completely determined by the data $(d, D)$, where $D$ is the set of divisors of $d$ representing the orders in $\mathbb{Z}/d$ of the elements in $S$.

Call a subset $A$ of $\mathbb{N}^+$ (or $\mathbb{N}$ or $\mathbb{Z}$) a pure Frobenius progression if there exists a Frobenius set $S \subseteq \mathbb{Z}/d$ so that $a \in A$ if and only if $\overline{a} \in S$. Finally, call a subset $A$ of $\mathbb{N}^+$ a Frobenius progression if it differs from a pure Frobenius progression by only a finite number of elements.

**Remark 4.6.** If $(f, g)$ is a pair of polynomials, then $D_{f,g}$ is a Frobenius set.

The set of $t$ satisfying (3.9) forms a pure Frobenius progression. Finally, the set of $t$ where $V_f(\mathbb{F}_{q^t}) = V_g(\mathbb{F}_{q^t})$ is a Frobenius progression (containing the associated pure Frobenius progression).

For exceptional polynomials, the associated Frobenius set has additional structure: if $d_i \in D$, where $D$ is the set of divisors characterizing the Frobenius set, and $k$ is a positive integer such that $kd_i | d$, then $kd_i \in D$. This follows from Corollary 3.7. One consequence is that $D_{f,T}$ contains $(\mathbb{Z}/d)^*$. In particular, $1 \in D_{f,T}$.

When we require $V_f(\mathbb{F}_{q^t}) = V_g(\mathbb{F}_{q^t})$ with multiplicity, we also get Frobenius progressions. Later we discuss Frobenius progressions in the context of the reducibility of $f(T) - g(S)$.

The following lemma, a basic application of the Riemann Hypothesis, is needed to prove reducibility.

**Lemma 4.7.** Suppose $\Phi(S,T) \in \mathbb{F}_q[S,T]$ has $A_t$ irreducible factors over $\mathbb{F}_{q^t}[S,T]$, of which $N_i$ are absolutely irreducible. Then $M_t$, the number of $\mathbb{F}_{q^t}$-points in the algebraic set $\Phi(S,T) = 0$, is approximately $N_t \cdot q^t$. More precisely, $|M_t/q^t - N_t| < cq^{-t/2}$ for some constant $c$ which depends only on the total degree of $\Phi$.

**Proof.** Factor $\Phi$ over $\mathbb{F}_{q^t}[S,T]$ as $\Phi_1 \cdots \Phi_{A_t}$. Index the factors so $\Phi_1, \ldots, \Phi_{N_t}$ are absolutely irreducible. Let $M_i$ be the number of $\mathbb{F}_{q^t}$-points of the variety $\Phi_i = 0$. Bezout’s Theorem bounds $|M_i - \Sigma M_i|$. For $i > N_t$, $|M_i|$ is bounded (use Bezout’s Theorem here as well). For $i \leq N_t$ let $\tilde{X}_i$ be the nonsingular projective curve corresponding to the affine curve $\Phi_i = 0$. Let $M_i$ be the number of $\mathbb{F}_{q^t}$-points on $\tilde{X}_i$. Then $|M_i - \tilde{M}_i|$ is bounded. All these bounds depend on the total degree of $\Phi$, not on $q^t$. Finally, the Riemann Hypothesis bounds $|\tilde{M}_i - q^t|$, giving the desired bound for $|M_t - N_t \cdot q^t|$.

**Theorem 4.8.** Suppose $V_f(\mathbb{F}_{q^t}) = V_g(\mathbb{F}_{q^t})$ for sufficiently large $t$. Let $N_t$ be the number of absolutely irreducible factors of $f(T) - g(S) \in \mathbb{F}_q[S,T]$ defined
over \( \mathbb{F}_{q^t} \). Then \( N_t \geq 1 \). Furthermore, \( N_t = 1 \) if and only if
\[
V_f(\mathbb{F}_{q^t}) = \mathbb{F}_{q^t} = V_g(\mathbb{F}_{q^t}).
\]
(So \( N_t = 1 \) implies that \( f \) and \( g \) are both exceptional polynomials.)

Here, \( t \) sufficiently large means that \( q^t \) is larger than an effectively computable bound which depends only on \( \deg f \) and \( \deg g \).

**Proof.** Let \( M_t \) be the number of \( \mathbb{F}_{q^t} \)-solutions of \( f(T) - g(S) = 0 \). Then
\[
V_f(\mathbb{F}_{q^t}) = V_g(\mathbb{F}_{q^t}) \implies M_t \geq q^t.
\]
Lemma 4.7 shows \( N_t \geq 1 \). Furthermore, if \( V_f(\mathbb{F}_{q^t}) = \mathbb{F}_{q^t} \), then \( M_t = q^t \), so \( N_t = 1 \).

Now suppose \( V_f(\mathbb{F}_{q^t}) \neq \mathbb{F}_{q^t} \). Theorem 3.4 gives an \( A > 0 \) (independent of \( t \)) with at least \( A \cdot q^t \) elements of \( V_f(\mathbb{F}_{q^t}) \) having at least two elements of \( \mathbb{F}_{q^t} \) mapping to it under \( f \). This implies \( M_t \geq q^t \cdot (A + 1) \). Thus \( N_t > 1 \). \( \square \)

**Remark 4.9.** Let \( f, g \in F[T] \). Gauss’ Lemma implies that the factorization of \( f(T) - g(S) \) into irreducibles in \( F[S,T] \) gives a factorization of \( f(T) - g(y_j) \) into irreducibles in \( F(y_j)[T] \) (with all factors having positive \( T \)-degree). By basic Galois theory, these irreducible factors of \( f(T) - g(y_j) \) over \( \mathbb{F}_q(y_j) \) correspond to the orbits of \( \{x_i\} \) under the action of \( \hat{G}_{f,g}(y_j) \).

Conclude that the \( F \)-irreducible factors of \( f(T) - g(S) \) correspond to the orbits of \( \{x_i\} \) under the action of \( \hat{G}_{f,g}(y_j) \). Further, if \( \Phi \) is a factor associated with an orbit \( O \) then \( |O| = \deg_T \Phi \). Similar statements apply for the \( \hat{G}_{f,g}(x_i) \)-action on \( \{y_j\} \).

**Remark 4.10.** When \( f, g \in \mathbb{F}_q[T] \), let \( G_t \) be the subgroup of \( \hat{G}_{f,g} \) generated by elements of \( \hat{G}_{f,g,t} \cup G_{f,g} \). In other words, \( G_t \) is the subgroup generated by \( G_{f,g} \) and an element lifting the \( q^t \)-power Frobenius automorphism. Since \( G_t \) is canonically isomorphic to the Galois group of \( \Omega_{f,g} \mathbb{F}_{q^t} \) over \( \mathbb{F}_{q^t}(z) \), Remark 4.9 gives a natural correspondence between divisors \( \Phi \in \mathbb{F}_{q^t}[S,T] \) of \( f(T) - g(S) \) (up to multiplication by constants in \( \mathbb{F}_{q^t}^* \)) and subsets \( B \subseteq \{y_j\} \) on which \( G_t(x_i) \) acts. Also, the divisor \( \Phi \) is absolutely irreducible if and only if the corresponding subset \( B \) is an orbit under the action of \( G_{f,g}(x_i) \). A similar statement applies, reversing the roles of \( \{y_j\} \) vs. \( \{x_i\} \) and \( S \) vs. \( T \).

**Remark 4.11.** Suppose \( \Phi \in \mathbb{F}_{q^t}[S,T] \) is a divisor of \( f(T) - g(S) \). Since \( G_t = G_{d'} \) with \( d' = \gcd(d,t) \), the above shows that, up to multiplication by a nonzero constant, \( \Phi \in \mathbb{F}_{q^{d'}}[S,T] \).

The above theorem and remarks give the following:

**Corollary 4.12.** For \( (f, g) \) a DP and \( f \) not exceptional, \( f(T) - g(S) \) is reducible over \( \mathbb{F}_q \). In fact, if \( t \in D_{f,g} \), then \( f(T) - g(S) \) is reducible over \( \mathbb{F}_{q^t} \).

**Proof.** The first statement is clear. The second statement is clear for \( t \) sufficiently large, though the above remarks show that reducibility is not actually a property of \( t \), large or small, but a property of \( t \mod d \). \( \square \)
Remark 4.13. The $t$ such that $f(T) - g(S)$ is reducible over $\mathbb{F}_{q^t}$ form a pure Frobenius progression, with associated Frobenius set a \textit{subgroup} of $\mathbb{Z}/d$. Let $(d, D)$ be the data defining this Frobenius set, where $D$ is a set of divisors of $d$. Then, in contrast with the Frobenius set of an exceptional polynomial, if $d_1 | d_2$ are divisors of $d$ with $d_2 \in D$, then $d_1 \in D$.

Now we consider the analogous situation for inclusions $\mathcal{V}_f(\mathbb{F}_{q^t}) \subseteq \mathcal{V}_g(\mathbb{F}_{q^t})$.

**Proposition 4.14.** Let $\mathcal{V}_f(\mathbb{F}_{q^t}) \subseteq \mathcal{V}_g(\mathbb{F}_{q^t})$ for $t$ sufficiently large, and let $N_t$ be the number of absolutely irreducible factors of $f(T) - g(S)$ defined over $\mathbb{F}_{q^t}$. Then $N_t \geq 1$. Furthermore, $N_t = 1$ if and only if $g$ is bijective over $\mathcal{V}_f(\mathbb{F}_{q^t})$ in the sense that every nonbranch point $b \in \mathcal{V}_f(\mathbb{F}_{q^t})$ has exactly one $a \in \mathbb{F}_{q^t}$ mapping to it under $g$.

**Proof.** Let $G_t$ be as in Remark 4.10. Also from this remark, the number $N_t$ of absolutely irreducible factors of $f(T) - G(S)$ defined over $\mathbb{F}_{q^t}$ equals the number of $G_t(x_1)$-orbits which are also $f,g(1)$-orbits.

Use Lemma 3.1 to count such orbits. Conclude that $N_t = r$, where $r$ is the average number of elements of $\{y_j\}$ fixed by $\sigma$, as $\sigma$ varies in $\tilde{G}_{f,g}(x_1)$. By Theorem 3.10, $r \geq 1$, and $r = 1$ if and only if every $\sigma \in \tilde{G}_{f,g}(x_1)$ fixing $x_1$ fixes exactly one element of $\{y_j\}$. So, by the Frobenius Principle and the Chebotarev Density Theorem, $r = 1$ is equivalent to every nonbranch point $b \in \mathcal{V}_f(\mathbb{F}_{q^t})$ being the image of exactly one $a \in \mathbb{F}_{q^t}$ under the map induced by $g$.

**Remark 4.15.** This generalizes Theorem 4.8 since, if $\mathcal{V}_f(\mathbb{F}_{q^t}) = \mathcal{V}_g(\mathbb{F}_{q^t})$, bijectivity of $f$ over $\mathcal{V}_f(\mathbb{F}_{q^t})$ is equivalent to $\mathcal{V}_f(\mathbb{F}_{q^t}) = \mathcal{V}_g(\mathbb{F}_{q^t})$ (use Theorem 3.4). In fact, we may view the above proof as an alternate proof of Theorem 4.8.

We end with a generalization of Theorem 3.13.

**Proposition 4.16.** If $\mathcal{V}_f(\mathbb{F}_{q^t}) \subseteq \mathcal{V}_g(\mathbb{F}_{q^t})$ for all $t$, and $\deg g > 1$, then $f(T) - g(S)$ is reducible over $\mathbb{F}_q$.

**Proof.** By Remark 4.9, the number of factors of $f(T) - g(S)$ is the number of $\tilde{G}_{f,g}(x_1)$-orbits of $\{y_j\}$. By (3.10), each element of $\tilde{G}_{f,g}(x_1)$ fixes at least one element of $\{y_j\}$. If $\{y_j\}$ has only one $\tilde{G}_{f,g}(x_1)$-orbit, then $\tilde{G}_{f,g}(x_1, y_j)$, as $y_j$ varies, are conjugate subgroups of $\tilde{G}_{f,g}(x_1)$. The conjugates, however, of a proper subgroup of a finite group cannot cover the group.

**Remark 4.17.** Section 7 continues the topic of reducibility.

5. Behavior at infinity

Many results above generalize to nonpolynomial maps. The main distinction is that polynomial maps $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ totally ramify above the place at infinity. This section considers the consequences of total ramification.
We begin with a lemma concerning the special case of tame ramification. It is similar to results in the literature (for example [Mül98, Section 2.2]). However, for generality and the convenience of the reader, we give a proof. The setup is as follows. Let $K$ be a field with discrete valuation $v$ and associated residue field $k$, and let $L$ be a degree $n$ separable extension of $K$ with valuation $w$ extending $v$ to $L$. Let $M$ be the normal closure of $L$ over $K$, and let $\omega$ be a valuation of $M$ extending $w$ with residue field $k(\omega)$. For a subgroup $H$ of a group $G$, we denote the group of permutations of the $n$ cosets $G/H$ by $\text{Perm}(G/H)$.

**Lemma 5.1.** Let $G = \text{Gal}(M/K)$ and $H = \text{Gal}(M/L)$. Suppose $(L, w)$ is tamely and totally ramified over $(K, v)$. Then:

1. $(M, \omega)$ is tamely ramified over $(K, v)$ and unramified over $(L, w)$.
2. The inertia group $I_\omega \subseteq G$ is cyclic and acts transitively and effectively on $G/H$; any generator of $I_\omega$ corresponds to an $n$-cycle in $\text{Perm}(G/H)$.
3. $k(\omega) = k(\zeta_n)$, where $\zeta_n$ is a primitive $n$-th root of 1.
4. The decomposition group $G_\omega \subseteq G$ is isomorphic to the semidirect product $\mu_n \rtimes \tilde{G}$ where $\mu_n \subseteq k(\omega)^\times$ is the group of $n$-th roots of 1, $\tilde{G}$ is $\text{Gal}(k(\omega)/k)$, and $\varphi : \tilde{G} \to \text{Aut}(\mu_n)$ is the natural Galois action on the $n$-th roots of unity.
5. The isomorphism $G_\omega \to \mu_n \rtimes \tilde{G}$ can be chosen so that the inertia group $I_\omega$ corresponds to $\mu_n$, and $H \cap G_\omega$ corresponds to $\tilde{G}$.

**Proof.** Let $\pi = \pi_L$ be a uniformizer for $(L, w)$ and let $h_1 \in K[T]$ be its minimal, monic polynomial. This polynomial is Eisenstein of degree $n$ (its nonleading coefficients have positive valuation, and its constant term is a uniformizer for $(K, v)$).

Since $(L, w)$ is totally ramified over $(K, v)$, $I_\omega$ acts transitively on $G/H$. The action is effective ($G$ acts effectively on $G/H$). The property of tame ramification behaves well under composita, and $(L, w)$ is tamely ramified over $(K, v)$. Thus $(M, \omega)$ is also tamely ramified over $(K, v)$. So $I_\omega$ is cyclic. It acts transitively and effectively on $G/H$, so its generator acts as an $n$-cycle. In particular $|I_\omega| = |G/H| = n$, forcing $(M, \omega)$ to be unramified over $(L, w)$.

Let $K_v$, $L_w$, and $M_\omega$ be the completions associated with $(K, v)$, $(L, w)$, and $(M, \omega)$. So $G_\omega$ is canonically isomorphic to $\text{Gal}(M_\omega/K_v)$. Since $h_1$ remains irreducible over $K_v$, $L_w = K_v(\pi)$ and $M_\omega$ is the splitting field of $h_1$ over $K_v$. Let $h_2(T) = T^n - \pi_K$, where $\pi_K = -h_1(0)$. Note that $\pi_K$ is a uniformizer for $K_v$. Let $M'$ be the splitting field of $h_2$ over $K_v$. We show that $M_\omega = M'$.

Let $\beta \in M'$ be a zero of $h_2$, and $\zeta \in M'$ a primitive $n$-th root of unity. Note that $h_3(T) \overset{\text{def}}{=} h_1(\beta T)/\pi_K$ is a monic polynomial in $M'$ with coefficients...
of nonnegative valuation. By Hensel’s Lemma, all the zeros \( r_1, \ldots, r_n \) of \( h_3 \) are in \( M' \). Thus \( \{ r_i \beta \} \), the zeros of \( h_1 \), are in \( M' \). Conclude \( M_\omega \subseteq M' \).

The zeros of \( h_1 \) correspond to the zeros of \( h_2 \) as follows. If \( \alpha \) is a zero of \( h_1 \), expand \( \alpha \) in \( M' \) in terms of the uniformizer \( \beta \) as \( \alpha = \zeta^i \beta \) plus higher-order terms. The correspondence sends \( \alpha \) to \( \zeta^i \beta \). This correspondence is compatible with the \( \text{Gal}(M'/K_v) \) action. Conclude that \( M_\omega = M' \).

Clearly \( I_\omega = \text{Gal}(M_\omega/K_v(\zeta)) \), and so \( \text{Gal}(K_v(\zeta)/K_v) \) is canonically isomorphic to \( \tilde{G} = \text{Gal}(k(\omega)/k) \). Conclude that \( k(\omega) = k(\zeta_\omega) \).

Replace \( \beta \) by \( \beta \zeta^i \), if necessary, so that \( \pi \) corresponds to \( \beta \). So \( H_\omega = H \cap G_\omega \) is the subgroup of \( G_\omega \) fixing \( \beta \), and \( L_\omega = K_v(\beta) \). Clearly \( H_\omega \cap I_\omega = 1 \) and \( |H_\omega| = |\tilde{G}| \). So, restricting the natural homomorphism \( G_\omega \rightarrow \tilde{G} \) gives an isomorphism \( H_\omega \rightarrow \tilde{G} \). The inverse isomorphism splits the exact sequence

\[
1 \rightarrow I_\omega \rightarrow G_\omega \rightarrow \tilde{G} \rightarrow 1.
\]

Thus \( G_\omega \) is isomorphic to a semi-direct product \( I_\omega \rtimes \tilde{G} \) with an isomorphism which sends \( H_\omega \) to \( \tilde{G} \).

The rule \( \gamma \mapsto \tilde{\gamma}(\beta)/\beta \) defines a natural isomorphism \( I_\omega \rightarrow \mu_n \) \cite[Section 8]{}, where \( a \mapsto \tilde{a} \) is the residue map. If \( \gamma \mapsto \zeta_\omega^i \), then clearly \( \sigma \gamma \sigma^{-1} \mapsto \tilde{\sigma}(\zeta_\omega^i) \), where \( \tilde{\sigma} \) is the image of \( \sigma \) in \( \tilde{G} \). The result follows.

\[ \square \]

**Example 5.2.** Let \( f \in F[T] \) be a polynomial of degree prime to the characteristic of \( F \). The following imply the hypotheses of Lemma 5.1:

- \( K = F(z) \) with \( v = \infty_z \), the place at infinity. (So \( k = F \).
- \( L = F(x_1) \) with \( w = \infty_{x_1} \). (Here, \( x_1 \) is a fixed zero of \( f(T) - z \).
- \( M = \Omega_f \) with \( \omega \) any place above \( \infty_{x_1} \).
- \( G = \tilde{G}_f \) and \( H \) is the subgroup fixing \( x_1 \).

Note: We can identify the zeros \( \{ x_1, \ldots, x_n \} \) with \( G/H \), where a given zero \( x_j \) corresponds to the coset of elements sending \( x_1 \) to \( x_j \).

Let \( \zeta_n \in F \) be a primitive \( n \)-th root of 1, and \( \mu_n \in F^\times \) the group of \( n \)-th roots of 1. Applying Lemma 5.1 to the above example yields:

**Corollary 5.3.** Suppose \( n = \deg f \) is prime to the characteristic of \( F \).

- The geometric monodromy group \( G_f \) contains an element which acts on the set \( \{ x_i \} \) as an \( n \)-cycle.
- The field \( \tilde{F}_f \) is a subfield of \( F(\zeta_n) \). In particular, if \( F = \mathbb{F}_q \) and \( q \equiv 1 \mod n \), then \( \tilde{F}_f = \mathbb{F}_q \) and \( \tilde{G}_f = G_f \).
- The arithmetic monodromy group \( \hat{G}_f \) contains a subgroup isomorphic to \( \mu_n \rtimes \text{Gal}(F(\mu_n)/F) \), and the geometric monodromy group \( G_f \) contains a subgroup isomorphic to \( \mu_n \rtimes \text{Gal}(F(\mu_n)/F) \).

We now give the main theorem of this section. Here \( \mathcal{D}_{f,g} \subseteq \mathbb{Z}/d \) is as in Definition 4.3 and \( d = [\tilde{F}_{f,g} : \mathbb{F}_q] \),
Theorem 5.4. Let \( f, g \in \mathbb{F}_q[T] \) with \( n = \deg f \) and \( m = \deg g \). If \( (f, g) \) is a DP, then \( \gcd(n, q^t - 1) = \gcd(m, q^t - 1) \) for all positive \( t \) with \( t \in \mathcal{D}_{f, g} \).

Proof. Let \( t \) be a positive integer with \( t \in \mathcal{D}_{f, g} \). Let \( n = n_0p^u \) and \( m = m_0p^v \) with \( n_0 \) and \( m_0 \) prime to \( p = \text{char}(\mathbb{F}_q) \). We must show

\[
\gcd(n_0, q^t - 1) = \gcd(m_0, q^t - 1).
\]

Let \( \infty \) be the infinite place of \( \mathbb{F}_q(z) \) and \( K \) the completion. Fix a place \( \omega \) of \( \Omega_{f, g} \) above \( \infty \). Let \( \mathcal{G}_\omega \subseteq \mathcal{G}_{f, g} \) be the decomposition group associated to \( \omega \), and \( I \subseteq \mathcal{G}_\omega \) the inertia group. Thus \( \mathcal{G}_\omega \) is canonically isomorphic to \( \text{Gal}(\Omega_\omega/K) \), where \( \Omega_\omega \) is the completion of \( \Omega_{f, g} \) at \( \omega \). Choose \( \phi_t \in \mathcal{G}_\omega \) that induces the automorphism \( x \mapsto x^{q^t} \) of the residue field. Since \( \mathbb{F}_q(x_1) \) is totally ramified over \( \mathbb{F}_q(z) \) at \( \infty \), the group \( I \) acts transitively on \( \{x_i\} \). So, after replacing \( \phi_t \) by \( \sigma \phi_t \) for a suitable \( \sigma \in I \), we can assume \( \phi_t \) fixes \( x_1 \).

Note: \( \phi_t \in \mathcal{G}_{f, g}\{x_1\} \), so \( \phi_t \) must also fix an element of \( \{y_j\} \).

Let \( I_1 \subseteq I \) be the first higher ramification group. Thus \( I_1 \) is a normal \( p \)-Sylow subgroup of \( I \) with cyclic quotient. Let \( \gamma \in I \) be an element whose image in \( I/I_1 \) is a generator.

Let \( R_x \) be \( \mathcal{G}_\omega/I_1\mathcal{G}_\omega(x_1) \) and consider the map \( \{x_i\} \to R_x \) sending \( x_i \) to the coset \( \sigma I_1\mathcal{G}_\omega(x_1) \), where \( \sigma \in \mathcal{G}_\omega \) is chosen so that \( \sigma(x_1) = x_i \). The fibers of this map are exactly the \( I_1 \)-orbits of \( \{x_i\} \). Since \( I_1 \) is normal in \( I \) and \( I \) acts transitively on \( \{x_i\} \), the \( I_1 \)-orbits all have the same size; that size is a power of \( p \), and the number of \( I_1 \)-orbits is prime to \( p \). Since \( n = |\{x_i\}| \) is the product of \( |R_x| \) and the fiber size, it follows that \( R_x \) has \( n_0 \) elements, and the fibers have size \( p^u \). Likewise, let \( R_y \) be \( \mathcal{G}_\omega/I_1\mathcal{G}_\omega(y_1) \) and consider the corresponding map \( \{y_j\} \to R_y \). Conclude that \( |R_y| = m_0 \) and the fibers have size \( p^v \).

Let \( L_x \subseteq \Omega_\omega \) be the fixed field of \( I_1\mathcal{G}_\omega(x_1) \) and \( L_y \) that of \( I_1\mathcal{G}_\omega(y_1) \). Let \( M_x \subseteq \Omega_\omega \) be the normal closure of \( L_x \) over \( K \) and \( M_y \) that of \( L_y \) over \( K \). We can identify \( R_x \) with \( \text{Gal}(M_x/K)/\text{Gal}(M_x/L_x) \).

So \( [L_x : K] = n_0 \). Since \( I \) acts transitively on \( R_x \), the extension \( L_x/K \) is totally and tamely ramified. A similar conclusion holds for \( L_y/K \).

Apply Lemma 5.1 to \( L_x/K \) and \( L_y/K \). For example, identify \( R_x \) with \( \mathbb{Z}/n_0 \) so \( \phi_t \) fixes \( 0 \in \mathbb{Z}/n_0 \) and \( \gamma \) acts as the map \( c \mapsto c + 1 \). Consequently, \( \gamma^b\phi_t \) acts on \( \mathbb{Z}/n_0 \) as the map \( c \mapsto q^t c + b \). Identify \( R_y \) with \( \mathbb{Z}/m_0 \) in a similar manner.

Now suppose \( a = \gcd(q^t - 1, n_0) \) is not a multiple of \( \gcd(q^t - 1, m_0) \). Then \( \gamma^a\phi_t \), viewed as \( c \mapsto q^t c + a \) modulo \( n_0 \), clearly fixes an element of \( R_x \). Yet \( \gamma^a\phi_t \), viewed as \( c \mapsto q^t c + a \) modulo \( m_0 \), fixes no element of \( R_y \). Suppose \( \gamma^a\phi_t \) fixes \( \rho \in R_x \). Let \( x_{i_0} \) be an element of the fiber of \( \{x_i\} \to R_x \). Since fibers of this map are \( I_1 \)-orbits, there is a \( \tau \in I_1 \) such that \( \tau\gamma^a\phi_t \) fixes \( x_{i_0} \). As
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\[ \tau \gamma^a \phi_t \] and \[ \gamma^a \phi_t \] act on \( R_y \) in the same way, neither has a fixed point in \( R_y \). Thus, \( \tau \gamma^a \phi_t \) fixes no element of \( \{ y_j \} \), contradicting \( t \in D_{f,g} \). Conclude that \( a \) is a multiple of \( \gcd(q^t - 1, m) \).

Similarly, conclude \( \gcd(q^t - 1, m_0) \) is a multiple of \( \gcd(q^t - 1, n_0) \). Therefore, \( \gcd(q^t - 1, m_0) = \gcd(q^t - 1, n_0) \).

\[ \square \]

**Remark 5.5.** Although we have adopted the convention that polynomials in this paper have nonzero derivatives, the above theorem (and its corollaries) remain valid for polynomials with zero derivatives.

A corollary is Lenstra’s theorem [CF95]:

**Corollary 5.6.** Let \( f \in \mathbb{F}_q[T] \) with \( n = \deg f \). If \( f \) is an exceptional polynomial, then \( \gcd(n, q - 1) = 1 \).

**Proof.** Apply the theorem to \((f, g)\), where \( g(T) = T \). Take \( t = 1 \) and recall that \( t \in D_{f,g} \) since \( f \) is exceptional. \( \square \)

**Corollary 5.7.** Let \( f, g \in \mathbb{F}_q[T] \), where \( \deg f = n_0 p^u \) and \( \deg g = m_0 p^v \) with \( n_0 \) and \( m_0 \) prime to the characteristic \( p \). If \((f, g)\) is an SDP, then \( n_0 = m_0 \).

**Proof.** Let \( t \) be the order of \( q \) modulo \( n_0 m_0 \). Thus \( n_0 m_0 | (q^t - 1) \). By Theorem 5.4,

\[
\begin{align*}
n_0 &= \gcd(q^t - 1, \deg f) = \gcd(q^t - 1, \deg g) = m_0.
\end{align*}
\]

The above theorem and corollary easily generalize to value set inclusions.

**Proposition 5.8.** Let \( f, g \in \mathbb{F}_q[T] \), where \( \deg f = n \) and \( \deg g = m \). For all \( t \) such that (3.10) holds, \( \gcd(q^t - 1, m) \) divides \( \gcd(q^t - 1, n) \).

**Proposition 5.9.** Let \( f, g \in \mathbb{F}_q[T] \), where \( \deg f = n_0 p^u \) and \( \deg g = m_0 p^v \) with \( n_0 \) and \( m_0 \) prime to the characteristic \( p \). If \( V_f(\mathbb{F}_{q^t}) \subseteq V_g(\mathbb{F}_{q^t}) \) for all \( t \), then \( m_0 \) divides \( n_0 \).

6. Factoring separated variable polynomials and rational functions

This section reviews properties of *induced decompositions* and how they affect factorizations of variables separated rational functions. For simplicity we stay with a notation where \( f \) and \( g \) are polynomials, until Remarks 6.4 and 6.9. Most results are in [Fri73] or are extensions of results there.

**6.1. Statements on reducibility.** This subsection concentrates on factorization, while the next concentrates on conclusions about SDPs and DPs.

**Lemma 6.1.** Let \( f, g \in F[T] \) be a pair of polynomials. There is a decomposition \( f = f_1 \circ f_2 \) with \( f_1, f_2 \in F[T] \) having the following properties:

(6.1) \( F(x_i) \cap \Omega_g = F(f_2(x_i)) \) for all \( x_i \) in \( \{ x_1, \ldots, x_n \} \).
(6.2) \( \deg f_2 = 1 \) if and only if \( \Omega_f \subseteq \Omega_g \).

(6.3) For all \( x_i \), \( f_2(T) - f_2(x_i) \) is an irreducible polynomial over \( \Omega_g \).

These properties characterize \( f_1 \) and \( f_2 \) up to composition with linear polynomials (actually (6.1) suffices). More specifically, if \( f = f_1 \circ f_2 = f'_1 \circ f'_2 \) are two such decompositions, then \( f'_1 = f_1 \circ l^{-1} \) and \( f'_2 = l \circ f_2 \), where \( l \in F[T] \) is a linear polynomial.

Call this decomposition and the analogous decomposition of \( g \) the induced decompositions associated to the pair \((f,g)\).

**Proof.** Fix a particular zero \( x_i \). By Lüroth’s Theorem, \( F(x_i) \cap \Omega_g = F(w_i) \) for some \( w_i \in F(x_i) \). Adjust \( w_i \) by a suitable linear fractional transformation so that \( w_i = f_2(x_i) \) and \( z = f_1(w_i) \) for some \( f_1, f_2 \in F[T] \). Thus \( f = f_1 \circ f_2 \).

Any other choice \( w'_i \) has the form \( aw_i + b \), where \( a, b \in F \) and \( a \neq 0 \). So \( f_1 \) and \( f_2 \) are unique up to composition with a linear polynomial. Now let \( x_j \) be any element of \( \{x_1, \ldots, x_n\} \) and let \( \sigma \in \tilde{G}_{f,g} \) send \( x_i \) to \( x_j \). Then \( F(x_j) \cap \Omega_g = F(\sigma(x_i)) \cap \Omega_g = F(f_2(\sigma(x_i))) = F(f_2(x_j)) \). So (6.1) holds.

To see (6.2), note that \( \deg f_2 = 1 \) implies \( F(x_i) \cap \Omega_g = F(x_i) \). Thus \( F(x_i) \subseteq \Omega_g \), so \( \Omega_f \subseteq \Omega_g \). The converse is clear.

To see (6.3), consider \( f_2(T) - f_2(x_i) \). By (6.1), this polynomial is defined over \( \Omega_g \). We will show it is irreducible by showing \( \text{Gal}(\overline{\Omega_g}/\Omega_g) \) acts transitively on its zeros. Any zero equals some \( x_j \) satisfying \( f_2(x_i) = f_2(x_j) \). Let \( \sigma \in \text{Gal}(\overline{F(z)}/F(z)) \) satisfy \( \sigma(x_i) = x_j \). Let \( \bar{\sigma} \) be the restriction of \( \sigma \) to \( \Omega_g \). Clearly, \( \bar{\sigma}(f_2(x_i)) = f_2(x_i) \), so \( \bar{\sigma} \in \text{Gal}(\Omega_g/\Omega_g \cap F(x_i)) \). The restriction map

\[
\text{Gal}(F(x_i) \cdot \Omega_g/F(x_i)) \to \text{Gal}(\Omega_g/\Omega_g \cap F(x_i))
\]

is an isomorphism. Use this to lift \( \bar{\sigma} \) to \( F(x_i) \cdot \Omega_g \), and then to \( \overline{\Omega_g} \) so the lifting \( \tau \) fixes \( x_i \). Then \( \sigma \circ \tau^{-1} \in \text{Gal}(\overline{\Omega_g}/\Omega_g) \) and \( \sigma \circ \tau^{-1}(x_i) = \sigma(x_i) = x_j \). \( \square \)

An important feature of these induced decompositions is that they respect the factorization of \( f(T) - g(S) \).

**Lemma 6.2.** Suppose \( f(T) - g(S) \) is reducible over \( F \), and \( f = f_1 \circ f_2 \) is the induced decomposition. Then \( f_1(T) - g(S) \) is reducible over \( F \). Moreover, substituting \( f_2(T) \) for \( T \) into the factorization of \( f_1(T) - g(S) \) gives the factorization of \( f(T) - g(S) \). In particular, \( \deg f_1 > 1 \).

**Proof.** Fix \( x_i \). As in Remark 4.9, factoring \( g(S) - f(T) \) over \( F[S,T] \) corresponds to finding the orbits of \( \{y_j\} \) under the action of

\[
G_{x_i} \overset{\text{def}}{=} \text{Gal}(\overline{F(z)}/F(x_i)).
\]

Similarly, factoring \( g(S) - f_1(T) \) over \( F[S,T] \) corresponds to finding the orbits of \( \{y_j\} \) under the action of

\[
G_{f_2(x_i)} \overset{\text{def}}{=} \text{Gal}(\overline{F(z)}/F(f_2(x_i))).
\]
Decompose \( \{y_j\} \) into orbits with both groups. Clearly the \( G_{f_2(x_i)} \)-orbits contain the \( G_{x_i} \)-orbits. We show, in fact, they are equal. Let \( \sigma \in G_{f_2(x_i)} \) send \( y_j \) to \( y_k \). If \( \sigma \) sends \( x_i \) to \( x_i \), then \( x_i \) and \( x_i \) are both zeros of the polynomial \( f_2(T) - f_2(x_i) \). By (6.3), there is a \( \tau \in \text{Gal}(F(z)/\Omega_g) \) sending \( x_i \) to \( x_i \). Thus \( \tau^{-1} \circ \sigma \in G_{x_i} \) sends \( y_j \) to \( y_k \).

Let \( O \subseteq \{y_j\} \) be such an orbit, \( \Phi(S,T) \) the corresponding irreducible factor of \( g(S) - f_1(T) \), and \( \Phi'(S,T) \) the corresponding irreducible factor of \( g(S) - f(T) \). The correspondence of Remark 4.9 yields the equation

\[
\prod_{y_j \in O} (S - y_j) = c \Phi(S, x_i) = c' \Phi'(S, f_2(x_i))
\]

for some \( c, c' \in F \). Thus \( c \Phi(S, T) = c' \Phi'(S, f_2(T)) \).

\[\square\]

**Corollary 6.3** ([Fri73], Lemma 7). Suppose \( f(T) - g(S) \) is reducible over \( F \). Then there are decompositions \( f = f' \circ f'' \) and \( g = g' \circ g'' \) with \( f', f'', g', g'' \) in \( F[T] \) such that:

(i) \( f'(T) - g'(S) \) is reducible.

(ii) \( \Omega_f' = \Omega_g' \).

(iii) Substituting \( f''(T) \) for \( T \) and \( g''(S) \) for \( S \) into the factorization of \( f(T) - g(S) \) gives the the factorization of \( f(T) - g(S) \).

Furthermore, if either \( \deg f' \) or \( \deg g' \) is prime to \( p \), then \( \deg f' = \deg g' \).

**Proof.** To prove this, repeatedly use the previous lemma applied to induced decompositions of \( f \) and \( g \). (Replace \( f \) and \( g \) with the outer composites as you go along.) Eventually you will obtain \( f_2 \) and \( g_2 \) of degree 1, which implies that \( \Omega_f' = \Omega_g' \).

Now if \( \deg f' \) or \( \deg g' \) is prime to \( p \), then the place at infinity is tamely ramified in \( \Omega_f' = \Omega_g' \). Conclude that both \( \deg f' \) and \( \deg g' \) give the order of the inertia group at infinity, so they are equal. (See Lemma 5.1 above.) \[\square\]

**Remark 6.4.** Suppose \( f = u_1/u_2 \) and \( g = v_1/v_2 \), with \( u_1, u_2, v_1, v_2 \in F[T] \) and \((u_1, u_2) = 1 = (v_1, v_2) \). We think of the factors of \( f(T) - g(S) \) as being the factors of the polynomial \( u_2v_2(f(T) - g(S)) \). Geometrically, these are the components of the fiber product of the two maps \( f : \mathbb{P}_T^1 \to \mathbb{P}_z^1 \) and \( g : \mathbb{P}_s^1 \to \mathbb{P}_z^1 \) over the sphere \( \mathbb{P}_z^1 \) uniformized by \( z \). Recall that the degree of \( f \) is the maximum of the degrees of \( u_1 \) and \( u_2 \).

Lemma 6.1 and Lemma 6.2 hold exactly as stated for rational functions \( f \) and \( g \) (though in the proof one uses linear fractional changes of variables instead of just affine changes). In Corollary 6.3, the only result that doesn’t hold for rational functions is the conclusion about \( \deg f = \deg g \) when one of the degrees are prime to \( p \). That requires using total tame ramification over \( \infty \).
6.2. Statements on value sets. Now we show that induced decompositions behave well in certain types of value set situations. Since we are dealing with value sets, we restrict to $F = \mathbb{F}_q$ for the remainder of this section.

**Proposition 6.5.** Suppose $V_f(\mathbb{F}_{q^t}) \subseteq V_g(\mathbb{F}_{q^t})$ for all $t$. Let $f = f_1 \circ f_2$ be the induced decomposition associated to the pair $(f, g)$ and let $g = g_1 \circ g_2$ be any decomposition (for example, the induced decomposition). Then

$\mathcal{V}_{f_1}(\mathbb{F}_{q^t}) \subseteq \mathcal{V}_{g_1}(\mathbb{F}_{q^t})$

for all $t$.

**Proof.** All zeros of $f_1(T) - z$ have the form $f_2(x_i)$. By Theorem 3.10, we can show $V_{f_1}(\mathbb{F}_{q^t}) \subseteq V_g(\mathbb{F}_{q^t})$ for all $t$ by showing that any $\sigma \in \text{Gal}(\mathbb{F}_q(z)/\mathbb{F}_q(z))$ fixing $f_2(x_i)$ must also fix some $y_j$. If $\sigma \in \text{Gal}(\mathbb{F}_q(z)/\mathbb{F}_q(z))$ fixes $f_2(x_i)$, then $x_i = \sigma(x_i)$ is a zero of $f_2(T) - f_2(x_i)$. By (6.3), there is a $\tau \in \text{Gal}(\mathbb{F}_q(z)/\mathbb{F}_q)$ sending $x_i$ to $x_i$. So $\tau \circ \sigma$ fixes $x_i$, and by hypothesis and Theorem 3.10, it must fix some $y_j$. Since $\tau$ fixes $y_j$, conclude that $\sigma$ also fixes $y_j$.

Clearly, $V_g(\mathbb{F}_{q^t}) \subseteq V_{g_1}(\mathbb{F}_{q^t})$. \qed

**Corollary 6.6.** Suppose $(f, g)$ is an SDP with $\deg g > 1$, so (as in Proposition 4.16) $f(T) - g(S)$ is reducible. Then the decompositions of Corollary 6.3 can be chosen so that $(f', g')$ is an SDP.

Suppose, instead, $V_f(\mathbb{F}_{q^t}) \subseteq V_g(\mathbb{F}_{q^t})$ for all $t$. Then the decompositions of Corollary 6.3 can be chosen so that $V_{f'}(\mathbb{F}_{q^t}) \subseteq V_{g'}(\mathbb{F}_{q^t})$ for all $t$.

**Proposition 6.7.** Suppose $(f, g)$ is an SDP with $n = \deg f$ and $m = \deg g$ prime to $p$. Then $\Omega_f = \Omega_g$ and $\deg f = \deg g$.

**Proof.** By Corollary 5.7, $n = m$. Let $f = f_1 \circ f_2$ be the induced decomposition associated to the pair $(f, g)$. By Proposition 6.5, $(f_1, g)$ is also an SDP. By Proposition 5.9 again, $\deg f_1 = m$. Hence, $\deg f_2 = 1$. Thus, by (6.2), $\Omega_f \subseteq \Omega_g$. A similar argument gives the other inclusion. \qed

Finally, we show that in some circumstances the induced decompositions behave well for DP’s.

**Proposition 6.8.** Suppose $(f, g)$ is a DP with $\mathcal{E}_{f,g} = \mathcal{E}_g$. Let $f = f_1 \circ f_2$ be the induced decomposition associated to the pair $(f, g)$. Then $(f_1, g)$ is a DP. Furthermore, we have $\mathcal{D}_f \subseteq \mathcal{D}_{f_1,g}$, both being subsets of $\mathbb{Z}/d$ where $d = [\mathcal{E}_g : \mathcal{E}_f]$. An analogous result holds for inclusions of value sets replacing the DP hypothesis.

**Proof.** We need to verify (3.9) with $(f_1, g)$ for all $t$ such that $t \in \mathcal{D}_f$ (Definition 4.3). So, let $\sigma \in \mathcal{G}_{f_1,g}$ with $t \in \mathcal{D}_f$, and let $\sigma \in \mathcal{G}_{f_1,g}$ be an element restricting to $\sigma$. Note that the zeros of $f_1(T) - z$ have the form $f_2(x_i)$.

First, suppose $\sigma$ fixes $y_j$. So $\sigma$ fixes $y_j$, and, by property (3.9), $\sigma$ fixes some $x_i$. Thus $\sigma$, and hence $\sigma$, fix $f_2(x_i)$.
Now suppose that \( \sigma \) fixes \( f_2(x_i) \). Let \( x_j = \overline{\sigma}(x_i) \) (so that \( x_j \) is a zero of \( f_2(T) - f_2(x_i) \)). By (6.3) there is a \( \tau \in \text{Gal}(\Omega_g/\Omega_g) \) sending \( x_i \) to \( x_j \). Since \( \tau \) fixes \( \Omega_g \), it also fixes \( \hat{\mathbb{F}}_g = \hat{\mathbb{F}}_{f,g} \). Hence \( \tau^{-1} \circ \overline{\sigma} \in \hat{G}_{f,g,t}(x_i) \). Since \( l \in D_{f,g} \), property (3.9) applies, and \( \tau^{-1} \circ \overline{\sigma} \) must fix some \( y_j \). Since \( \tau \) fixes \( y_j \in \Omega_g \), conclude that \( \sigma \) also fixes \( y_j \).

\[ \square \]

**Remark 6.9.** As with Remark 6.4, we may allow \( f \) and \( g \) to be rational functions, rather than polynomials. For the discussion of value sets this means we formally add \( \infty \) to the domain and range. The only exception is in Proposition 6.7, where even the conclusion \( \Omega_f = \Omega_g \) uses the total tame ramification over \( \infty \) (as in the direct argument of [Fri73, Prop. 3]).

### 7. Reducibility and representations

This section links the reducibility of \( f(T) - g(S) \) to the behavior of the associated Galois representations. It builds on the characteristic zero results of [Fri73] and the positive characteristic results of [Fri99].

#### 7.1. Representation lemmas.

Let \( G \), a finite group, act on a set \( \mathcal{S} = \{ s_i \} \) with \( N \) elements. This permutation action of \( G \) has an associated linear action of \( G \) on a complex vector space \( V_\mathcal{S} \) as follows. Let \( V_\mathcal{S} \) be an \( N \)-dimensional complex vector space with a chosen basis \( \{ s_i \} \). Have \( \sigma \in G \) act on \( V_\mathcal{S} \) by the unique linear transformation that sends \( s_i \) to \( s_{i\sigma} \) if and only if \( \sigma \) (acting on \( \mathcal{S} \)) sends \( s_i \) to \( s_{i\sigma} \).

Let \( \chi_\mathcal{S} \) be the character of the action of \( G \) on \( V_\mathcal{S} \). The following lemma, a special case of Lemma 3.1, is easy and well-known.

**Lemma 7.1.** For all \( \sigma \in G \), the value of the character \( \chi_\mathcal{S}(\sigma) \) is the number of elements of \( \mathcal{S} \) fixed by \( \sigma \). Furthermore, \[
\langle \chi_\mathcal{S}, 1 \rangle = \frac{1}{|G|} \sum_{\sigma \in G} \chi_\mathcal{S}(\sigma) = \text{number of orbits in } \mathcal{S}.
\]

Here we use the standard Hermitian inner product on the vector space \( \mathbb{C}^{|G|} \) of functions from \( G \) to \( \mathbb{C} \):

\[
\langle f_1, f_2 \rangle \overset{\text{def}}{=} \frac{1}{|G|} \sum_{\sigma \in G} f_1(\sigma) \overline{f_2(\sigma)}.
\]

The irreducible characters form an orthogonal basis.

The \( \mathbb{C}[G] \)-module \( V_\mathcal{S} \) decomposes as \( 1_\mathcal{S} \oplus V'_\mathcal{S} \), where \( 1_\mathcal{S} \) is the submodule generated by \( \sum_i s_i \) and where \( V'_\mathcal{S} \) is the kernel of the augmentation map \( \eta : V_\mathcal{S} \to \mathbb{C} \) defined by \( \sum_i \lambda_i s_i \mapsto \sum \lambda_i \). Let \( \chi'_\mathcal{S} \) be the character associated to \( V'_\mathcal{S} \). In particular, \( \chi_\mathcal{S} = 1 + \chi'_\mathcal{S} \).

**Lemma 7.2.** If \( G \) acts transitively on \( \mathcal{S} \), then \( 1_\mathcal{S} \) consists of all elements of \( V_\mathcal{S} \) fixed by \( G \). Furthermore \( (1, \chi'_\mathcal{S}) = 0 \), and so \( \langle \chi'_\mathcal{S}, \chi'_\mathcal{S} \rangle = \langle \chi_\mathcal{S}, \chi_\mathcal{S} \rangle = 1 \).
Proof. The identity character appears exactly once in the permutation representation of each orbit of $G$ acting on $S$. So, transitivity means that $1$ doesn’t appear in $\chi'_S$. Apply the inner product of $1 + \chi'_S$ to itself to get the given relation. □

Remark 7.3. In general, $\langle \chi'_S, \chi'_S \rangle = \langle \chi_S, \chi_S \rangle - (2r - 1)$, where $r$ is the number of orbits in $S$.

Now let the finite group $G$ act transitively on two finite sets $A$ and $B$. Consider also the associated $G$-action on $A \times B$. The following easy lemma is the starting point for our analysis of reducibility.

Lemma 7.4. There are $\langle \chi_A, \chi_B \rangle$ orbits for $G$ acting on the product $A \times B$.

Proof. The character $\chi_{A \times B}$ associated to the action of $G$ on $A \times B$ is $\chi_A \cdot \chi_B$. By Lemma 7.1, the number of orbits in $A \times B$ is

$$\frac{1}{|G|} \sum_{\sigma \in G} \chi_{A \times B}(\sigma) = \frac{1}{|G|} \sum_{\sigma \in G} \chi_A(\sigma) \chi_B(\sigma) = \langle \chi_A, \chi_B \rangle.$$  
(Note: this proof does not require the transitivity assumption.) □

The following well-known characterization of double transitivity is an immediate consequence of the above results.

Corollary 7.5. Suppose the action of $G$ on $S$ is transitive where $|S| \geq 2$. Then the following are equivalent:

(7.1) The action of $G$ on $S$ is doubly transitive.

(7.2) There are exactly two orbits in $S \times S$ under the action of $G$.

(7.3) $\langle \chi_S, \chi_S \rangle = 2$.

(7.4) $\langle \chi'_S, \chi'_S \rangle = 1$.

(7.5) $V'_S$ is an irreducible $\mathbb{C}[G]$-module.

Remark 7.6. In Corollary 7.5, we can replace the hypothesis that $G$ acts transitively on $S$ with the alternate hypothesis $|S| > 2$.

The following is also an easy consequence of Lemma 7.4.

Lemma 7.7. If $G$ acts doubly transitively on $A$ and $|A| \geq 2$, then the multiplicity of $V'_A$ in the decomposition of $V'_B$ is one less than the number of $G$-orbits of $A \times B$.

Corollary 7.8. Suppose $|A| \geq 2$ and $G$ acts doubly transitively on $A$. Suppose also that $|A| = |B|$. Then the following are equivalent:

(7.6) $\chi_A = \chi_B$.

(7.7) $A \times B$ has more than one orbit.

(7.8) $A \times B$ has exactly two orbits.

(7.9) $V_A$ and $V_B$ are isomorphic as $\mathbb{C}[G]$-modules.
Remark 7.9. If (7.6) (or its equivalents) hold, then $G$ must act doubly transitively on $B$ as well (by Corollary 7.5).

We refine (7.9) above by explicitly constructing a natural isomorphism from $V_A$ to $V_B$ when the following hold:

(i) $G$ acts doubly transitively on $A$.
(ii) $|A| = |B|$.
(iii) $A \times B$ has more than one orbit.

We define some maps $V_A \to V_B$ without using hypotheses (i), (ii) or (iii). Then we show one gets an isomorphism when the hypotheses hold.

First choose a $G$-invariant subset $\Gamma$ of $A \times B$, for example a $G$-orbit. For convenience, label elements: $A = \{a_i\}$, $B = \{b_j\}$. Consider the matrix $E = [\epsilon_{i,j}]$, where $\epsilon_{i,j}$ is 1 if $(a_i, b_j) \in \Gamma$, and 0 otherwise. Consider the linear map $\psi: V_A \to V_B$ defined by the matrix $E$:

$$\psi(a_i) \overset{\text{def}}{=} \sum_j \epsilon_{i,j} b_j,$$

so

$$\psi \left( \sum_i \lambda_i a_i \right) = \sum_j \left( \sum_i \lambda_i \epsilon_{i,j} \right) b_j.$$

Here and below, $(a_i)$ is the basis of $V_A$ associated to $A = \{a_i\}$ and $(b_j)$ is the basis of $V_B$ associated to $B = \{b_j\}$.

The following lemma follows directly from the definition (and does not depend on transitivity).

Lemma 7.10. The map $\psi: V_A \to V_B$ is a $\mathbb{C}[G]$-module morphism.

Now we investigate some of the consequences of transitivity.

Lemma 7.11. If $\Gamma$ is nonempty, restricting $\psi$ to $1_A$ gives an isomorphism $1_A \to 1_B$ of $\mathbb{C}[G]$-modules.

Proof. Check that $\psi$ sends $\sum_i a_i$ to $C \sum_j b_j$, where, for each $b_j$, $C = C_j$ is the number of $a_i \in A$ with the property that $(a_i, b_j) \in \Gamma$. ($C_j$ is independent of $j$ by transitivity). \hfill $\square$

Lemma 7.12. Restricting $\psi$ to $V'_A$ gives a $\mathbb{C}[G]$-module morphism

$$\psi': V'_A \to V'_B.$$

Proof. Check that $\eta_B \circ \psi = D \cdot \eta_A$, where $\eta_A$ and $\eta_B$ are the augmentation maps and, for each $a_i$, $D = D_i$ is the number of $b_j \in B$ with the property that $(a_i, b_j) \in \Gamma$. ($D_i$ is independent of $i$ by transitivity). \hfill $\square$

Lemma 7.13. If $\Gamma \subseteq A \times B$ is neither empty nor all of $A \times B$, then $\psi': V'_A \to V'_B$ (defined above) has nontrivial image.

Proof. Fix a basis vector $a_i$ of $V_A$. Since $\Gamma$ is nonempty, $\epsilon_{i,j_1} = 1$ for some $j_1$. Since $\Gamma$ is a proper subset of $A \times B$, $\epsilon_{i,j_2} = 0$ for some $j_2$. Let $\sigma \in G$ be an element such that $\sigma(b_{j_1}) = b_{j_2}$. Then $\psi'(\sigma(a_i) - a_i) \neq 0$. \hfill $\square$
Lemma 7.14. Suppose that $G$ acts doubly transitively on $A$ and that $\Gamma$ is neither empty nor all of $A \times B$. Then $\psi : V_A \to V_B$ is injective.

Proof. By Corollary 7.5, $V'_A$ is irreducible, and by the previous lemma, the map $\psi' : V'_A \to V'_B$ is not trivial. Thus $\psi'$ is injective. By Lemma 7.11, the map $1_A \to 1_B$ induced by $\psi$ is an isomorphism. Thus $\psi : 1_A \oplus V'_A \to 1_B \oplus V'_B$ is injective. $\square$

Proposition 7.15. Suppose that:

(i) $G$ acts doubly transitively on $A$.

(ii) $|A| = |B|$.

(iii) $\Gamma$ is a nonempty proper subset of $A \times B$ invariant under $G$.

Then $\psi : V_A \to V_B$ is an isomorphism.

Proof. By the previous lemma, $\psi$ is injective. Since $V_A$ and $V_B$ have the same dimension, $\psi$ is an isomorphism. $\square$

Also of interest is the following [Fri73, Lemma 2]:

Lemma 7.16. Suppose $G$ acts doubly transitively on $A$, and $|A| = |B| \geq 2$. Suppose also that, for all $\sigma \in G$, $\chi_A(\sigma) > 0$ if and only if $\chi_B(\sigma) > 0$. Then $\chi_A = \chi_B$.

Proof. Recall $\chi'_A = \chi_A - 1$ and $\chi'_B = \chi_B - 1$. By hypothesis, for all $\sigma \in G$, $\chi'_A(\sigma) < 0$ if and only if $\chi'_B(\sigma) < 0$. If $\sigma = 1$ then $\chi'_A(\sigma) > 0$ and $\chi'_B(\sigma) > 0$. Thus $\langle \chi_A, \chi_B \rangle = \langle \chi'_A, \chi'_B \rangle + 1 \geq 2$. The result follows from Lemma 7.4 and Corollary 7.8. $\square$

Remark 7.17. This shows that if $(f, g)$ is an SDP and if $\hat{G}_f$ acts doubly transitively on $\{x_i\}$, then $(f, g)$ is actually an SDP with multiplicity. (This can also be seen as a corollary of Proposition 7.26 and Theorem 3.13.)

Remark 7.18. Lemma 7.16 uses this hypothesis: For all $\sigma \in G$, $\chi_A(\sigma) > 0$ if and only if $\chi_B(\sigma) > 0$. We can replace it with this hypothesis: For all $\sigma \in G$, $\chi_A(\sigma) \leq 1$ if and only if $\chi_B(\sigma) \leq 1$.

We end with one more consequence of double transitivity which we need later.

Lemma 7.19. Let $\Gamma$ be an orbit of $A \times B$ where $G$ is transitive on $A$ and $B$. Suppose that $G$ acts doubly transitively on $A$, where $|A| \geq 2$. Then

$$\frac{|A||B|(|A|-1)}{|\Gamma|(|\Gamma|-|B|)}.$$ 

Proof. For $b \in B$, let $\Gamma_b \overset{\text{def}}{=} \{a \mid (a, b) \in \Gamma\}$. Note that $k \overset{\text{def}}{=} |\Gamma_b|$ is independent of $b \in B$ since $G$ acts transitively on $B$.

Now consider the set

$$\Gamma' = \{(a, a', b) \mid (a, b), (a', b) \in \Gamma \text{ and } a \neq a' \}.$$
For distinct elements $a, a'$ of $A$, let

$$\Gamma'_{a,a'} \stackrel{\text{def}}{=} \{ b \mid (a, a', b) \in \Gamma' \}.$$  

Note that $l \stackrel{\text{def}}{=} |\Gamma'_{a,a'}|$ is independent of $a$ and $a'$ since $G$ acts doubly transitively on $A$.

We count the number of element of $\Gamma'$ in two ways:

$$|\Gamma'| = |A|(|A| - 1)l = |B|k(k - 1).$$

Now multiply both sides by $|B|$ and use the equation $k|B| = |\Gamma|$.  

\[\square\]

### 7.2. Reducibility

In this section, unless otherwise stated, $F$ is a general field and $f, g \in F[T]$. Remark 4.9 describes the factorization of $f(T) - g(S)$ in $F[S,T]$ in terms of $\hat{G}_{f,g}(y_j)$-orbits of $\{x_i\}$. There is, however, another description of the factorization of $f(T) - g(S)$ in $F[S,T]$ that follows from Remark 4.9.

**Proposition 7.20.** Consider the action of $\hat{G}_{f,g}$ on $\{x_1\} \times \{y_j\}$ induced by the natural actions of $\hat{G}_{f,g}$ on $\{x_i\}$ and $\{y_j\}$. Irreducible factors of $f(T) - g(S)$ in $F[S,T]$ naturally correspond to the orbits of $\{x_1\} \times \{y_j\}$. This correspondence sends an irreducible factor $\Phi$ of $f(T) - g(S)$ to the orbit consisting of all pairs $(x_i, y_j)$ satisfying $\Phi(x_i, y_j) = 0$ in $\Omega_{f,g}$. For $O \subseteq \{x_1\} \times \{y_j\}$ such an orbit, with $\Phi \in F[S,T]$ the corresponding factor of $f(T) - g(S)$:

$$|O| = \deg f \cdot \deg_S \Phi = \deg g \cdot \deg_T \Phi. \tag{7.10}$$

**Corollary 7.21.** Let $w = \gcd(\deg f, \deg g)$. The $T$-degree (resp. $S$-degree) of any irreducible factor of $f(T) - g(S)$ is a multiple of $\deg f / w$ (resp. of $\deg g / w$). So, $w$ bounds the number of irreducible factors of $f(T) - g(S)$ in $F[S,T]$. This result holds even if $f$ and $g$ are rational functions (see Remark 6.4).

**Proof.** By assumption, $\gcd(\deg g / w, \deg f / w) = 1$. So, (7.10) shows $\deg(f)/w$ divides the $T$-degree of any irreducible factor of $f(T) - g(S)$. Let $r$ be the number of irreducible factors of $f(T) - g(S)$. Then, the sum of their respective $T$ degrees (each a multiple of $\deg(f) / w$) adds up to $\deg f$. Therefore $r \leq w$.  

\[\square\]

**Remark 7.22.** The above corollary generalizes the well-known result of Ehrenfeucht that $\gcd(\deg f, \deg g) = 1$ implies $f(T) - g(S)$ is irreducible.

**Corollary 7.23.** Let $\Phi$ be an irreducible divisor of $f(T) - g(S)$ in the ring $F[S,T]$. If $deg f = \deg g$, then

$$\deg \Phi = \deg_T \Phi = \deg_S \Phi,$$

where the first of these is is the total degree of $\Phi$. 

Now consider the special case $F = \mathbb{F}_q$. Factoring $f(T) - g(S)$ over $\mathbb{F}_q$ amounts to describing the orbits of $\{x_i\} \times \{y_j\}$ under the action of the arithmetic monodromy group $\hat{G}_{f,g}$. Now use the canonical isomorphism between $\text{Gal}(\Omega_{f,g}/\mathbb{F}_q[T])$ and $G_{f,g}$, and then apply Proposition 7.20 with $F = \mathbb{F}_q$. Conclude that factoring $f(T) - g(S)$ over $\mathbb{F}_q$ amounts to describing the orbits of $\{x_i\} \times \{y_j\}$ under the action of the geometric monodromy group $G_{f,g}$.

In what follows, let $d = [\mathbb{F}_{f,g} : \mathbb{F}_q]$.

Proposition 7.24. Let $\Phi$ be an irreducible factor of $f(T) - g(S)$ over $\mathbb{F}_q[S,T]$, and let $(x_{i_0},y_{j_0})$ be in the corresponding orbit under $G_{f,g}$. Then a nonzero constant multiple of $\Phi$ is defined over $\mathbb{F}_q'$ if and only if $\hat{t}$ is in the subgroup of $\mathbb{Z}/d$ generated by the image of $\hat{G}_{f,g}(x_{i_0},y_{j_0})$ under $G_{f,g} \to \mathbb{Z}/d$.

Proof. Let $G_t$ consist of the elements in $\hat{G}_{f,g}$ whose image in $\mathbb{Z}/d$ is in the subgroup generated by $\hat{t}$. Note: $G_t$ is isomorphic to $\text{Gal}(\mathbb{F}_q'\Omega_{f,g}/\mathbb{F}_q'(T))$ and the action on $\{x_i\}$ and $\{y_j\}$ are preserved by this isomorphism. Thus, by Proposition 7.20, irreducible factors of $f(T) - g(S)$ in $\mathbb{F}_q'[S,T]$ correspond to $G_t$-orbits of $\{x_i\} \times \{y_j\}$.

Let $\Phi' \in \mathbb{F}_q'[S,T]$ be the irreducible factor corresponding to the $G_t$-orbit containing $(x_{i_0},y_{j_0})$. The nature of the correspondence in Proposition 7.20 implies that $\Phi$ divides $\Phi'$ in $\mathbb{F}_q'[S,T]$. The degrees of $\Phi$ and $\Phi'$ are determined by the sizes of the associated orbits, so $\Phi'$ is a nonzero constant multiple of $\Phi$ if and only if these orbits are the same size. This in turn is equivalent to

$$\frac{|G_t|}{|G_t(x_{i_0}, y_{j_0})|} = \frac{|G_{f,g}|}{|G_{f,g}(x_{i_0}, y_{j_0})|},$$

or yet to

$$\frac{d}{a} = \frac{|G_t|}{|G_{f,g}|} = \frac{|G_t(x_{i_0}, y_{j_0})|}{|G_{f,g}(x_{i_0}, y_{j_0})|},$$

where $a = \text{gcd}(d,t)$. The ratio $|G_t(x_{i_0}, y_{j_0})|/|G_{f,g}(x_{i_0}, y_{j_0})|$ determines the image of $G_t(x_{i_0}, y_{j_0})$ in $\mathbb{Z}/d$, and the above equation holds if and only if this image is the subgroup generated by $\hat{t}$. Finally, this occurs if and only if the image of $\hat{G}_{f,g}(x_{i_0}, y_{j_0})$ in $\mathbb{Z}/d$ contains $\hat{t}$. \qed

We return to the case that $F$ is a general field. Let $V_f$ and $V_g$, respectively, be the $\mathbb{C}[\hat{G}_{f,g}]$-modules associated to the action of $\hat{G}_{f,g}$ on $\{x_i\}$ and $\{y_j\}$. Let $\chi_f$ and $\chi_g$ be the associated characters. Lemma 7.4 and Proposition 7.20 give the following:

Proposition 7.25. The number of irreducible factors of $f(T) - g(S)$ in $F[S,T]$ is equal to $\langle \chi_f, \chi_g \rangle$.

Corollary 7.8 and Proposition 7.20 give the following:
Proposition 7.26. Suppose that the degrees of \( f \) and \( g \) are equal and greater than one, and that the action of \( \hat{G}_{f,g} \) on \( \{x_i\} \) is doubly transitive. Then the following are equivalent:

1. \( \chi_f = \chi_g \).
2. \( f(T) - g(S) \) is reducible in \( F[S,T] \).
3. \( f(T) - g(S) \) factors into exactly two irreducible factors in \( F[S,T] \).
4. \( V_f \) and \( V_g \) are isomorphic as \( \mathbb{C}[\hat{G}_{f,g}] \)-modules.

We note that if \( F = \mathbb{F}_q \) and \( \chi_f = \chi_g \), then Corollary 3.12 (together with the observation in Lemma 7.1) implies that \((f,g)\) is an SDP. Thus we get:

Corollary 7.27. Let \( F = \mathbb{F}_q \). Suppose:

1. The degrees of \( f \) and \( g \) are equal.
2. The action of \( \hat{G}_{f,g} \) on \( \{x_i\} \) is doubly transitive.
3. \( f(T) - g(S) \) is reducible in \( \mathbb{F}_q[S,T] \).

Then \((f,g)\) is an SDP with multiplicity.

We can also use Proposition 7.26 to prove the following:

Lemma 7.28. Suppose \( f, g \in F[T] \) are polynomials of degree at least three which are linearly related on the inside over the separable closure \( F^{\text{sep}} \). Suppose also that the action of \( \hat{G}_{f,g} \) on \( \{x_i\} \) is doubly transitive. Then \( f \) and \( g \) are linearly related on the inside over \( F \).

Proof. Let \( E \) be a finite Galois extension of \( F \) over which \( f \) and \( g \) are linearly related on the inside. This implies that \( f(T) - g(S) \) has a linear factor defined over \( E \). Proposition 7.26 implies \( f(T) - g(S) \) has exactly two factors defined over \( E \), one of which is linear, so the other must be of total degree greater than 1. Hence the factors are invariant under the natural Gal\((E/F)\) action. Since \( f(T) - g(S) \) has a linear factor defined over \( F \), the polynomials \( f \) and \( g \) are linearly related on the inside over \( F \). \( \square \)

Lemma 7.19 gives the following:

Proposition 7.29. Let \( \Phi \) be a factor of \( f(T) - g(S) \) of total degree \( k > 1 \) which is irreducible in \( F[S,T] \). Suppose that the degrees of \( f \) and \( g \) are both equal to \( n > 1 \). Suppose also that the action of \( \hat{G}_{f,g} \) on \( \{x_i\} \) is doubly transitive. Then

\[
\begin{align*}
1 + n - 1 & \mid k(k - 1).
\end{align*}
\]

Proof. Let \( O \) be the orbit corresponding to \( \Phi \) via Proposition 7.20. Note that \( \deg_S \Phi = \deg \Phi = k \) by Corollary 7.23. Apply Lemma 7.19 with \( A = \{x_i\} \), \( B = \{y_j\} \) and \( \Gamma = O \). By Proposition 7.20, \( |O| = nk \). \( \square \)

Corollary 7.30. Suppose the degrees of \( f \) and \( g \) both equal \( n > 2 \), the action of \( \hat{G}_{f,g} \) on \( \{x_i\} \) is doubly transitive, and \( f(T) - g(S) \) is reducible over \( F \). Then the two irreducible factors of \( f(T) - g(S) \) have nonequal degrees.
Proof. There are exactly two factors by Proposition 7.26. Suppose they both have degree \( k \), i.e., \( n = 2k \). Then \( n - 1 = 2k - 1 \) is prime to \( k \) and \( k - 1 \). Thus \( n - 1 \) cannot divide \( k(k - 1) \), contradicting the previous proposition.

7.3. Polynomials with doubly transitive monodromy groups. Many of the above results (Proposition 7.26 to Corollary 7.30) depend on the double transitivity of monodromy groups. The classification of polynomials with doubly transitive geometric monodromy groups is well-known, at least when the degree is prime to the characteristic. We describe this classification. Throughout this section, let \( f \in F[T] \) have degree \( n \) at least 2, and let \( V_f \) be the associated \( \mathbb{C}[G_f] \) module with character \( \chi_f \).

Lemma 7.31. Suppose the arithmetic monodromy group \( \hat{G}_f \) acts doubly transitively on \( \{x_i\} \). Then \( f \) is indecomposable over \( F \).

Proof. Assume \( f = f_1 \circ f_2 \) with \( f_1, f_2 \in F[T] \) of degrees at least two. Then

\[
f(T) - f(S) = (T - S) \Phi_1(f_2(S), f_2(T)) \Phi_2(S, T),
\]

where

\[
\Phi_i(S, T) \overset{\text{def}}{=} \frac{f_i(T) - f_i(S)}{T - S}.
\]

Thus \( f(T) - f(S) \) has at least three irreducible factors, contradicting Proposition 7.26.

The following argument of [Fri70] gives a partial converse.

Lemma 7.32. Suppose \( f \in F[T] \) is indecomposable over \( F \) with \( n = \deg f \) composite and prime to the characteristic of \( F \). Then the arithmetic and geometric monodromy groups of \( f \) act doubly transitively on \( \{x_i\} \).

Proof. A theorem of Fried and MacRae implies that, since \( n \) is prime to the characteristic of \( F \), \( f \) is indecomposable over \( \overline{F} \). Thus \( G_f \) acts primitively on \( \{x_i\} \). By Corollary 5.3, there is an element of \( G_f \) which acts as an \( n \)-cycle on \( \{x_i\} \). Schur proved that a finite group \( G \) acting on a set with \( N \) elements acts doubly transitively if:

(i) The action is primitive.

(ii) \( G \) contains an element acting as an \( N \)-cycle.

(iii) \( N \) is composite.

The above lemmas allow us to concentrate on the case \( n \) a prime. In the case \( n = 2 \) the action is trivially doubly transitive, thus we can restrict \( \deg f = n \) to odd primes (different from the characteristic of \( F \)). Before finishing the classification, we describe important families of polynomials whose geometric monodromy groups do not act doubly transitively on \( \{x_i\} \).

Consider the cyclic polynomials \( f(T) = T^n \). Here \( G_f \) is a cyclic group of order \( n \) with generator acting on the zeros \( \{x_i\} \) as an \( n \)-cycle. Furthermore,
\langle \chi_f, \chi_f \rangle = n \text{ and } f(T) - g(S) \text{ factors into } n \text{ linear factors. So, when } n > 2, \text{ the action of } G_f \text{ on } \{x_i\} \text{ is not doubly transitive.}

The other main family of examples is the Chebyshev polynomials:

**Definition 7.33.** The Chebyshev polynomial \( \tau_n \) of degree \( n \) is defined to be the polynomial in \( F[T] \) satisfying

\[
\tau_n \left( T + \frac{1}{T} \right) = T^n + \frac{1}{T^n}.
\]

The following well-known result is easily verified (the recursion can be used to prove existence).

**Lemma 7.34.** For every \( n \geq 1 \) the \( n \)-th Chebyshev polynomial \( \tau_n \) exists, is unique (for any given characteristic), and is monic. Moreover \( \tau_1(T) = T \), \( \tau_2(T) = T^2 - 2 \), and

\[
\tau_{n+2}(T) = T \cdot \tau_{n+1}(T) - \tau_n(T) \quad \text{for all } n \geq 1.
\]

**Remark 7.35.** When \( F = \mathbb{Q} \) we get \( \tau_n \in \mathbb{Z}[T] \). Such Chebyshev polynomials arise from the trigonometric identity \( 2 \cos(nT) = \tau_n(2 \cos(T)) \).

The following is well-known, and the second part is easily verified.

**Lemma 7.36.** Let \( n \) be an odd prime which is prime to the characteristic of \( F \). Then the \( n \)-th Chebyshev polynomial \( \tau_n \in F[T] \) has a dihedral geometric monodromy group of order \( 2n \), and this group acts on \( \{x_i\} \) via the standard dihedral action.

In particular, \( \tau_n(T) - \tau_n(S) \) has \( \langle \chi_{\tau_n}, \chi_{\tau_n} \rangle = (n+1)/2 \) irreducible factors. All are quadratic, except for the linear factor \( T - S \). So the action of the geometric monodromy group on \( \{x_i\} \) is doubly transitive only for \( n = 3 \).

The following result of Burnside is an important piece in the classification.

**Lemma 7.37.** Suppose \( G \) acts effectively and transitively, but not doubly transitively, on a set \( S \) of prime order \( l \). Then \( G \) is isomorphic to a subgroup of the affine group \( \mathbb{F}_l \rtimes \mathbb{F}_l^\times \).

The last piece is provided by the following:

**Lemma 7.38.** Let \( f \in F[T] \) be a polynomial whose degree \( l \) is a prime distinct from the characteristic of \( F \). If \( G_f \subseteq \mathbb{F}_l \rtimes \mathbb{F}_l^\times \), then \( f \) is linearly related, over \( F \), to either a cyclic polynomial or a Chebyshev polynomial.

**Remark 7.39.** See [Fri70] for the tame case. [FGS93] strengthens the result to general polynomials. See also [Mül97] (under the hypothesis that \( G_f \) is solvable).

Putting all this together gives the following classification.
Proposition 7.40. Suppose \( f \in F[T] \) has degree prime to the characteristic of \( F \). Then the geometric monodromy group acts doubly transitively on the zeros \( \{x_i\} \) if and only if one of the following hold:

\[
\begin{align*}
(7.15) & \text{ } f \text{ is indecomposable of composite degree.} \\
(7.16) & \text{ } f \text{ has degree } 2. \\
(7.17) & \text{ } f \text{ has degree } 3 \text{ and is not linearly related to the cyclic polynomial } T^3. \\
(7.18) & \text{ } f \text{ has prime degree } n > 3 \text{ and is not linearly related over } \overline{F} \text{ to either the cyclic polynomial or the Chebyshev polynomial of degree } n.
\end{align*}
\]

Remark 7.41. Suppose \( f \in F[T] \) has degree \( n \) prime to the characteristic of \( F \). It is easy to show that if \( f \) is linearly related over \( F \) to a cyclic polynomial, then \( f \) is linearly related over \( F \) to a cyclic polynomial. However, \( f \in F[T] \) can be linearly related over \( \overline{F} \) to the Chebyshev polynomial \( \tau_n \) but not be linearly related over \( F \).

This motivates the introduction of Dickson polynomials. For any \( a \in F^\times \) and any positive integer \( n \) define the Dickson polynomial

\[
D_{n,a}(T) = a^{n/2} \tau_n(a^{-1/2}T).
\]

Then, for \( n \) odd, \( f \in F[T] \) is linearly related over \( F \) to the Chebyshev polynomial \( \tau_n \) if and only if it is linearly related over \( F \) to a Dickson polynomial.

Note that if \( F \) is a finite field of odd characteristic then there are two nonlinearly related Dickson polynomials of each degree \( (n > 2) \); if \( F \) is a finite field of characteristic 2 there is only one.

Remark 7.42. The result quoted in the previous remark has been known for some time (see [Fri70] and [FGS93]; it can also be deduced from [Turn95, Lemma 1.9]). For the convenience of the reader we sketch an argument. Assume \( n > 1 \) since \( n = 1 \) is trivial.

First assume the characteristic of \( F \) is not 2, and check that the branch points of the covering map \( \tau_n : \overline{F} \rightarrow \overline{F} \) (the elements \( b \in \overline{F} \) where \( \tau_n(T) - b \) has multiple roots) are \( b_1 = 2 \) and \( b_2 = -2 \). Next, check that the unique point unramified above \( b_1 = 2 \) is \( a_1 = 2 \) and the unique point unramified above \( b_2 = -2 \) is \( a_2 = -2 \). So, if \( f \) is linearly related over \( \overline{F} \) to \( \tau_n \), there are \( a_1, a_2, b_1, b_2 \in \overline{F} \) such that \( b_1, b_2 \) are the branch points for the cover \( f : \overline{F} \rightarrow \overline{F} \) and such that \( a_i \) is unramified over \( b_i \).

If \( a_1 \) and \( a_2 \) are in the base field \( F \), observe that the linear polynomials of \( \overline{F}[T] \) sending \( \{a_1, a_2\} \) to \( \{-2, 2\} \) are in \( F[T] \). If \( a_i \in F \) then \( b_i = f(a_i) \) is in \( F \), so the linear polynomials of \( \overline{F}[T] \) sending \( \{b_1, b_2\} \) to \( \{-2, 2\} \) are in \( F[T] \). Conclude that \( f \) is linearly related over \( F \) to \( \tau_n \). So assume that some \( a_i \notin F \).

The derivative \( \tau_n' \) has distinct roots of the form \( \zeta + 1/\zeta \), where \( \zeta \neq -1, 1 \) is a \( 2n \)-th root of unity. Thus the derivative \( f' \) has distinct zeros. This implies that all ramification points are separable over \( F \), and so \( a_1, a_2, b_1, b_2 \) are also in the separable closure of \( F \). Since \( f \) has coefficients in \( F \), \( a_1 \) and \( a_2 \) must be conjugate and contained in a quadratic extension \( F' \) of \( F \). After
composing on the right by a linear polynomial in $F[T]$, we reduce to the case where $a_1 = \alpha$ and $a_2 = -\alpha$ where $\alpha^2 \in F$. Note that the images $b_1, b_2$ cannot be in $F$, and are in fact conjugate elements of $F'$. After composing on the left with an element of $F[T]$ we can assume $f$ is monic and $b_2 = -b_1$.

Finally, check that such $f$ must be a Dickson polynomial.

Now, if the characteristic of $F$ is 2, then $\tau_n : F \to F$ has a single branch point $b = 0$, and $a = 0$ is the unique point unramified above $b$. So any polynomial map $f : \overline{F} \to F$ which is linearly related to $\tau_n$ must have a single branch point $b$, and a single point $a$ unramified over $b$. Since $f$ is defined over $F$, both $a$ and $b$ must be in the base field $F$. After linear compositions, we can assume $a = b = 0$ and $f$ is monic. Thus $f = l_1 \circ \tau_n \circ l_2$ where $l_1(T) = ctT$ and $l_2 = c_2T$. A simple consequence of the recursion for $\tau_n$ is that $\tau_n(T) = T^n + T^{n-2} + \cdots + 1$ plus lower order terms. This implies that $(c_2)^2 \in F$. Conclude that $f$ is a Dickson polynomial.

**7.4. A special class of Davenport pairs.** Recall that one way to construct a DP $(f, g)$ is as an SDP-Ex composition (Definition 1.1). Such DP’s have the property that $1 \in D_{f,g}$.

How does one construct DP’s $(f, g)$ with $1 \not\in D_{f,g}$? One strategy is to consider $f, g \in F_q[T]$ with $g = f \circ l$ for some linear polynomial $l \in \mathbb{F}_q[T]$ not in $\mathbb{F}_q[T]$. We see the only examples of this type, when $f$ is indecomposable of degree prime to the characteristic of $\mathbb{F}_q$, are essentially of the form $(T^n, aT^n)$ where $a \in \mathbb{F}_q$ is not an $n$-th power in $\mathbb{F}_q$ (Corollary 7.45). In this case $f(T) = T^n$ and $l(T) = a^{1/n}T$.

**Lemma 7.43.** Let $f \in F[T]$ be linearly related over $\overline{F}$ to a Chebyshev polynomial of odd degree prime to the characteristic of $F$. Suppose that $f(\alpha T + \beta) \in F[T]$ for some $\alpha, \beta \in \mathbb{F}_q \setminus \{0\}$. Then $\alpha, \beta \in F$.

**Proof.** See [Turn95, Lemma 1.9]: Our $\tau_n(T)$ is equal to Turnwald’s $D_n(1, T)$.

**Proposition 7.44.** Let $f, g \in F[T]$ be indecomposable polynomials of degree $n$ prime to the characteristic of $F$. Suppose $F$ is a perfect field. If $f$ and $g$ are linearly related on the inside over $\overline{F}$ then either:

(i) $f$ and $g$ are linearly related on the inside over $F$, or
(ii) $f$ and $g$ are both linearly related over $F$ to the cyclic polynomial of degree $n$.

In either case, $f$ and $g$ are linearly related over $F$.

**Proof.** Observe that $n < 3$ is trivial. If $G_{f,g}$ acts doubly transitively on $\{x_i\}$, use Lemma 7.28. Otherwise, use Proposition 7.40 to reduce to the Chebyshev or cyclic case. In the case where $f$ and $g$ are linearly related over $\overline{F}$ to the Chebyshev polynomial and $n$ is an odd prime, use the previous lemma. Finally, in the cyclic case, Remark 7.41 says that $f$ and $g$ are linearly related to $T^n$ over the base field $F$. 

Corollary 7.45. Suppose that \( f \in F[T] \) is indecomposable of degree \( n \) prime to the characteristic of \( F \) where \( F \) is a perfect field. If \( g \in F[T] \) is linearly related to \( f \) on the inside over \( F \), but not over \( F \), then there are linear \( l_1, l_2, l_3 \in F[T] \) such that \( l_1 \circ f \circ l_2 = T^n \) and \( l_1 \circ g \circ l_3 = aT^n \) with \( a \in F \) not an \( n \)-th power in \( F \).

8. Main results concerning indecomposability

The following results hold when one of the polynomials, \( f \), say, of the pair \( (f, g) \) is indecomposable with degree prime to the characteristic. There are essentially two cases, depending on whether or not \( f \) is linearly related to a cyclic polynomial. Recall \( f \) is linearly related to a cyclic polynomial if and only if it is linearly related to a cyclic polynomial over \( F \).

Theorem 8.1. Let \( f \in \mathbb{F}_q[T] \) be indecomposable over \( \mathbb{F}_q \), nonexceptional, and of degree prime to the characteristic of \( \mathbb{F}_q \). Let \( g \in \mathbb{F}_q[T] \) be any polynomial where \( (f, g) \) forms a Davenport Pair, and let \( g = g_1 \circ g_2 \) be the induced decomposition over \( \mathbb{F}_q \) associated to \( (f, g) \).

If \( f \) is not linearly related to a cyclic polynomial, \( (f, g_1) \) is an SDP with multiplicity: the associated characters \( \chi_f, \chi_{g_1} \) are equal.

If \( f \) is linearly related to a cyclic polynomial, \( g = f \circ h \) for some \( h \in E[T] \) with \( E \) a finite extension of \( \mathbb{F}_q \). Also, \( g = l \circ f \circ h' \) for some \( l, h' \in \mathbb{F}_q[T] \) with \( l \) linear, and \( f \) and \( l \circ f \) are linearly related on the inside over \( \mathbb{F}_q \).

Proof. Let \( g = h_1 \circ h_2 \) be the induced decomposition over \( \mathbb{F}_q \) associated with the pair \( (f, g) \). (It turns out, at least in the noncyclic cases, that the two induced decompositions, \( g = g_1 \circ g_2 \) and \( g = h_1 \circ h_2 \), are equivalent.)

Since \( (f, g) \) is a DP and \( f \) is nonexceptional, \( f(T) - g(S) \) is reducible in \( \mathbb{F}_q[S, T] \) (Corollary 4.12). So \( f(T) - h_1(S) \) is also reducible in \( \mathbb{F}_q[S, T] \) (Lemma 6.2). Since \( f \) is indecomposable over \( \mathbb{F}_q \), \( f \) is indecomposable over \( \mathbb{F}_q \) (Theorem 3.5 of [FM69]). Thus the induced decompositions of both \( f \) and \( h_1 \), associated to the pair \( (f, h_1) \) over \( \mathbb{F}_q \), are trivial. Lemma 6.1, especially property (6.2), implies \( \overline{\mathbb{F}}_q\Omega_f = \overline{\mathbb{F}}_q\Omega_{h_1} \) and \( G_f = G_{h_1} = G_{f, h_1} \). Finally, \( \deg f = \deg h_1 \) (Corollary 6.3).

Now we show, assuming that \( f(T) - g(S) \) is reducible over \( \mathbb{F}_q \), that we can take \( h_i \) to be \( g_i \) for \( i = 1, 2 \). Note: The argument that \( \deg f = \deg h_1 \) modifies to show \( \deg f = \deg g_1 \) under this reducibility assumption. Now, by Lemma 6.1,

\[
\mathbb{F}_q(y_1) \cap \Omega_f = \mathbb{F}_q(g_2(y_1)) \quad \text{and} \quad \mathbb{F}_q(y_1) \cap (\mathbb{F}_q\Omega_f) = \mathbb{F}_q(h_2(y_1)).
\]

Since \( \mathbb{F}_q(\mathbb{F}_q(y_1) \cap \Omega_f) \subseteq \mathbb{F}_q(y_1) \cap (\mathbb{F}_q\Omega_f) \),

\[
\mathbb{F}_q(g_2(y_1)) \subseteq \mathbb{F}_q(h_2(y_1)).
\]

In particular, \( g_2 = h' \circ h_2 \) for some polynomial \( h' \in \mathbb{F}_q \). Since \( \deg h_1 = \deg g_1 \), \( \deg h' = 1 \). So, after adjusting \( h_1 \) and \( h_2 \) by a linear map, \( h_i = g_i \), for \( i = 1, 2 \).
We divide the remaining proof into three cases, using Proposition 7.40.

Case 1: $G_f$ acts doubly transitively on the zeros $\{x_i\}$ and $\deg f > 2$. By Proposition 7.26 and Corollary 7.30, $f(T) - h_1(S)$ has exactly two irreducible factors over $\overline{\mathbb{F}}_q$, and these factors have nonequal degrees. Substitute $h_2(S)$ for $S$ in the factorization of $f(T) - h_1(S)$ to get the factorization of $f(T) - g(S)$ (Lemma 6.2). Thus the two irreducible factors of $f(T) - g(S)$ have nonequal $T$-degrees, so the action of $\text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ fixes them. Conclude that the factorization of $f(T) - g(S)$ is defined over $\mathbb{F}_q$. As above, this means we can take $h_1 = g_1$. The result follows from Proposition 7.26 and Corollary 7.27.

Case 2: $f$ is linearly related over $\overline{\mathbb{F}}_q$ to a Chebyshev polynomial and $n = \deg f$ is an odd prime. Let $G = G_f = G_{h_1}$. By Lemma 7.36, $G$ is isomorphic to a dihedral group of order $2n$. Note: $G$ acts transitively on both $\{x_i\}$ and on the zeros $\{u_j\}$ of $h_1(T) - z$. Clearly, any two transitive actions of such a dihedral group on sets of order $n$ are equivalent as permutation representations. Thus $G(x_1)$ is $G(u_j)$ for some $j$. Use the description of factorization of Remark 4.9 applied to $G(u_j) = G(x_1)$ acting on $\{x_i\}$ to conclude that the factorization of $f(T) - h_1(S)$ has exactly one linear factor $\Phi$ and $(n-1)/2$ irreducible quadratic factors in $\mathbb{F}_q[S,T]$.

Recover the factorization of $f(T) - g(S)$ by substituting $h_2(S)$ for $S$ in the factorization of $f(T) - h_1(S)$ (Lemma 6.2). Since $\Phi(T, h_2(S))$ is the unique irreducible factor of $f(T) - g(S)$ of $T$-degree one, the action of $\text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ fixes it. So $f(T) - g(S)$ is reducible in $\mathbb{F}_q[S,T]$. As discussed above, we can conclude that $h_1 = g_1$. Also, $\Phi$, the only linear factor of $f(T) - g_1(S)$, must be defined over $\mathbb{F}_q$. The existence of $\Phi$ implies $f$ and $g_1$ are linearly related on the inside over $\mathbb{F}_q$. Thus $(f, g_1)$ forms a trivial SDP.

Case 3: $f$ is linearly related over $\overline{\mathbb{F}}_q$ to a cyclic polynomial. (This automatically includes the case $\deg f = 2$.) Let $G = G_f = G_{h_1}$. So $G$ is isomorphic to a cyclic group of order $n$ acting transitively on both $\{x_i\}$ and on zeros $\{u_j\}$ of $h_1(T) - z$. Clearly, any two transitive actions of $G$ on sets of order $n$ are equivalent as permutation representations. Thus $G(x_1)$ is $G(u_j)$ for some $j$. Use Remark 4.9 to conclude that $f(T) - h_1(S)$ factors over $\overline{\mathbb{F}}_q$ as the product of $n$ linear factors. This implies that $h_1 = f \circ l_0$ for some $l_0 \in \overline{\mathbb{F}}_q[T]$ of degree 1. So $g = f \circ h$, where $h = l_0 \circ h_2$.

Using [FGS93, Lemma 4.1] and that $\deg f$ is prime to $p$, we get a linear polynomial $l' \in \overline{\mathbb{F}}_q[T]$ such that $f' = f \circ l'$ and $h' = (l')^{-1} \circ h$ are in $\mathbb{F}_q[T]$, giving a decomposition $g = f' \circ h'$ over $\mathbb{F}_q$. By Proposition 7.44, $f' = l \circ f \circ l''$ for some linear $l, l'' \in \overline{\mathbb{F}}_q[T]$. By replacing $h'$ with $l'' \circ h'$, we obtain the decomposition $g = l \circ f \circ h'$. \qed
Remark 8.2. If we replace the hypotheses $(f, g)$ is a DP and $f$ is not exceptional with the alternate hypothesis $f(T) - g(S)$ reducible over $\overline{\mathbb{F}_q}$ (keeping all the other hypotheses as they are) we get a variant of Theorem 8.1.

Remark 8.3. This paper has adopted the convention that polynomials have nonzero derivatives. Theorems 8.1 and 8.4 hold for $g$ with zero derivative (with a suitable definition of induced decomposition).

Theorem 8.4. Let $f, g \in \mathbb{F}_q[T]$ be two polynomials with $\mathcal{V}_g(\mathbb{F}_{q^t}) \subseteq \mathcal{V}_f(\mathbb{F}_{q^t})$ for all $t$. Suppose $f$ is indecomposable over $\mathbb{F}_q$ and has degree prime to the characteristic of $\mathbb{F}_q$. Then there are $g_1, g_2 \in \mathbb{F}_q[T]$ with $g = g_1 \circ g_2$ and $(f, g_1)$ is an SDP with multiplicity.

Proof. Consider Remark 8.2 together with Proposition 4.16. The case where $f$ is not linearly related to a cyclic polynomial follows immediately.

In the cyclic case, consider the decomposition $g = l \circ f \circ h'$ of Theorem 8.1, where $f$ and $f' = l \circ f$ are linearly related on the inside over $\mathbb{F}_q$. We claim $f$ and $f'$ are linearly related on the inside over $\mathbb{F}_q$, and so we can take $g_1 = f$. Suppose otherwise and use Corollary 7.45 to reduce to the case $f = T^n$ and $f' = aT^n$, where $a \in \mathbb{F}_q$ is not an $n$-th power. Choose $t$ so that $q^t > \deg h'$ and $a$ is not an $n$-th power in $\mathbb{F}_{q^t}$. Then $\mathcal{V}_f(\mathbb{F}_{q^t})$ contains only $n$-th powers, but if $c \in \mathbb{F}_{q^t}$ is not a zero of $h'$ then $g(c)$ is not an $n$-th power, a contradiction. $\square$

Remark 8.5. In the above theorems, we can often conclude that $(f, g_1)$ is actually a trivial SDP. In other words, we can choose the decomposition $g = g_1 \circ g_2$ in such a way that $g_1 = f$ itself.

For example, in case 2 of the above proof we concluded that $(f, g_1)$ is a trivial SDP if $n = \deg f$ is an odd prime, and $f$ is linearly related to a Chebyshev polynomial. In this case $G_f$ is dihedral. In fact, from Proposition 7.44, having $(f, g_1)$ a nontrivial SDP requires $G_f$ to have two nonequivalent permutation representations on $n$ elements whose associated characters are equal. This excludes most $G_f$.

Part of the classification of finite simple groups includes the classification of doubly transitive representations [CKS76]. This applies to classify groups $G$ with two nonequivalent faithful permutation representations acting on a set with $n$ elements such that:

(i) The characters of the two actions are equal.

(ii) The actions are doubly transitive.

(iii) Some element of $G$ acts as an $n$-cycle under the two actions.

The conclusion is that

$$G = \text{PSL}_2(\mathbb{F}_{11}) \quad \text{and} \quad n = 11,$$

$$\text{PSL}_k(\mathbb{F}_s) \subseteq G \subseteq \text{PGL}_k(\mathbb{F}_s) \quad \text{and} \quad n = (s^k - 1)/(s - 1) \text{ for some } k \geq 3.$$
[Fri73] conjectured this; [Fri99, Thm. 2.7 and §9] has complete details, including historical information. The field $F_s$ appearing in the above list is called the characteristic field of the Chevalley group $G$.

This result allows us to strengthen the above theorems: if $G = G_f$ and $n$ are not of the above form, then the conclusion $(f, g_1)$ is an SDP, can be replaced by the stronger conclusion $g = f \circ h$ for some $h \in F_q[T]$.

Not all of the above groups are expected to occur as geometric monodromy groups of polynomials for a given $F_q$. Guralnick has conjectured the following: the finite simple groups appearing as composition factors of geometric monodromy groups $G_f$ as $f$ varies over all polynomials, or even all rational functions, are, with finitely many exceptions (depending on the characteristic), the cyclic groups, the alternating groups, and the Chevalley groups with characteristic field containing $F_p$. Thus, we can expect among $f \in F_q[T]$ with $F_q$ of fixed characteristic $p$, that the fields $F_s$ appearing as $G_f$ as in the above classification should, with a finite number of exceptions depending on $p$, also be of characteristic $p$.

By way of contrast, in the case where $F_q$ and $F_s$ have the same characteristic, examples abound. [Fri99, Thm. 5.2] (dependent on [Abh97]) states that, for any finite field $F_q$, any $s$ a power of the characteristic of $F_q$, and any $k \geq 3$, there is a nontrivial SDP $(f, g)$ with $\chi_f = \chi_g$ whose geometric monodromy group is $G_f = G_g = \text{PSL}_k(F_s)$.

References


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We investigate in this paper the link between the moment problem for recursive sequences, the associated Jacobi matrices and the associated analytic functions. We generalize some classical results by providing simple proofs that use functional calculus.

1. Introduction

Let \( a_0, \ldots, a_{r-1} \ (r \geq 1, \ a_{r-1} \neq 0) \) be real numbers and let \( \gamma = \{ \gamma_n \}_{n \geq 0} \) be the sequence defined by the following recursive relation of order \( r \):

\[
\gamma_{n+1} = a_0 \gamma_n + a_1 \gamma_{n-1} + \cdots + a_{r-1} \gamma_{n-r+1}, \quad \text{for} \quad n \geq r - 1,
\]

where \( \gamma_0, \gamma_1, \ldots, \gamma_{r-1} \) are the given initial values (or conditions). The polynomial \( P(X) = X^r - a_0 X^{r-1} - a_1 X^{r-2} - \cdots - a_{r-1} \), together with the initial values, is said to define the sequence \( \gamma \). Note that if \( Q \) is any multiple of \( P \), then \( Q \) also defines \( \gamma \) provided that \( \gamma_0, \gamma_1, \ldots, \gamma_{\deg Q - 1} \) are taken as initial conditions. As observed in [3] among all polynomials defining \( \gamma \), there exists a unique monic polynomial of minimal degree. This latter, denoted by \( P_\gamma \), is called the characteristic polynomial of \( \gamma \).

Let \( \gamma = \{ \gamma_n \}_{n \geq 0} \) be a sequence of complex numbers and \( K \) a closed subset of the complex plane. The purpose of the \( K \)-moment problem associated with \( \gamma \) is to find a positive measure \( \mu \) such that

\[
\gamma_n = \int_K t^n d\mu(t).
\]

Since its introduction by Stieltjes in [15] for \( K = \mathbb{R}^+ \), the moment problem has been the subject of an extensive literature. In particular, Hamburger and Hausdorff have studied this problem for \( K = \mathbb{R} \) and \( K = [0, 1] \) respectively. The main idea in computing the measure \( \mu \), solution of (2), for a given sequence \( \gamma = \{ \gamma_n \}_{n \geq 0} \) is to extend the linear form defined on polynomials by

\[
S_\gamma(X^n) = \gamma_n
\]

to a positive linear form on some Hilbert completion and to use the \( L^2 \)-representation of Hilbert spaces. The construction of \( S_\gamma \) motivated different
approaches in the treatment of the moment problem. The continued fractions, the positivity of Hankel matrices and the decomposition of positive polynomials played a crucial role in this treatment [1, 4, 6, 7, 8, 13, 14] and [15].

The linear moment problem associated with \( \gamma = \{ \gamma_n \}_{n \geq 0} \) entails finding a Hilbert space \( \mathcal{H} \), a self-adjoint operator \( A \in \mathcal{L}(\mathcal{H}) \) and a nonzero vector \( x \in \mathcal{H} \) satisfying

\[
\gamma_n = \langle A^n x, x \rangle, \quad \text{for} \quad n \geq 0.
\]

Using the spectral representation of self-adjoint operators, one can show easily that the moment problems (2) and (4) are equivalent (see [5] for example).

The study of the moment problem for recursive sequences is motivated by the so-called “truncated moment problems” treated by R. Curto and L. Fialkow in [6] and [7]. It is known that the moment problem for recursive sequences is equivalent to the truncated moment problem, and that a necessary condition for (2) (or for (4)) to have a solution is that \( P_\gamma \) has only simple roots. The moment problem (2) for recursive sequences corresponds to the case where \( K \) is a finite set (see [3], [9] and [10] for example). We will omit any reference to the set \( K \) in this paper.

We investigate in this paper the moment problem for recursive sequences. The main motivation of this work is the paper of Dieudonné [8]; we use the functional calculus to obtain some results of Dieudonné’s paper and to give explicit formulas of coefficients from [8], p. 6.

Section 2 is devoted to Jacobi matrices associated with recursive moment sequences. We show that (4) has a solution in finite-dimensional spaces and that the associated Jacobi matrices are of finite order. The link with continued fractions is studied in Section 3; we prove that these fractions are terminating in this case. In Section 4, we show that the analytic function associated with a recursive moment sequence is rational and we use techniques from analytic functional calculus to provide some generalizations of results from [8]. We discuss in Section 5 some moment problems arising from continued fractions and we give a new formula for the linear form associated with a terminating fraction.

2. Jacobi matrices associated with moment problems for recursive sequences

2.1. Jacobi matrices associated with moment problems. Let \( \gamma = \{ \gamma_n \}_{n \geq 0} \) be a given sequence of real numbers. Define on \( \mathbb{C}[X] \), the space of all polynomials, the bilinear form

\[
\langle P, Q \rangle_\gamma = \sum_{n,m} \alpha_n \beta_m \gamma_{n+m}
\]
with \( P = \sum_n \alpha_n X^n \) and \( Q = \sum_m \beta_m X^m \). (We suppose the upper limits in the sums are equal by completing by some zero coefficients if necessary.)

Observe that \( \langle P, Q \rangle_\gamma = \langle P, HQ \rangle \) where \( \langle \cdot, \cdot \rangle \) is the usual Euclidean inner product and \( H = [\gamma_{i+j}]_{i,j \geq 0} \) is the Hankel matrix associated with \( \gamma \).

If \( H \geq 0 \) then \( \langle P, P \rangle_\gamma = \langle P, HP \rangle \geq 0 \) for all \( P \in \mathbb{C}[X] \) and the bilinear form \( \langle \cdot, \cdot \rangle \) is an inner product on \( \mathbb{C}[X] \). This defines a norm \( \| \cdot \|_\gamma \) on \( \mathbb{C}[X] \) when \( H \) is positive definite. Denote by \( H_\gamma \) the Hilbert completion of \( (\mathbb{C}[X], \| \cdot \|_\gamma) \) and by \( \overline{A} \) the unique extension to \( H_\gamma \) of the densely defined operator \( A \) on \( \mathbb{C}[X] \) by \( AX^n = X^{n+1} \). If \( \overline{A} \) is self-adjoint, \( A \) is called essentially self-adjoint and \( A \) answers positively to (4). Otherwise, \( \overline{A} \) has self-adjoint extensions and (4) is again solved (see [16]). In any orthogonal basis obtained by the Gram–Schmidt process from \( \{1, X, X^2, \ldots\} \), the self-adjoint extension \( A_\gamma \) of \( A \), solution of (4) has a semi-infinite Jacobi matrix of the form,

\[
J_\gamma = \begin{pmatrix}
  b_0 & c_0 & 0 & 0 & \cdots \\
  c_0 & b_1 & c_1 & 0 & \cdots \\
  0 & c_1 & b_2 & c_2 & \cdots \\
  & \ddots & \ddots & \ddots & \ddots \\
  & & & c_{r-2} & \cdots \\
  & & & c_{r-2} & b_{r-1} \\
  & & & \cdots & \cdots & \cdots 
\end{pmatrix}
\]

From this point of view, the Hamburger moment problem and the theory of semi-infinite Jacobi matrices coincide.

### 2.2. Finite Jacobi matrices.

Let \( A \in \mathcal{L}(\mathcal{H}) \) be a solution of the moment problem (4) associated with the sequence \( \gamma \), where \( \mathcal{H} \) is a given Hilbert space. For \( x \in \mathcal{H} \) satisfying (4), let \( \mathcal{H}_0 = \text{Span}\{x, Ax, \ldots, A^n x, \ldots\} \) be the invariant subspace generated by \( x \). By the recursive relation (1) we have \( \langle P_\gamma(A)x, A^n x \rangle = 0 \) for every \( n \geq 0 \); in particular \( \| P_\gamma(A)x \| = 0 \). Hence \( A^n x \in \text{Span}\{x, Ax, \ldots, A^{r-1} x\} \) for every \( n \geq r \) and \( \mathcal{H}_0 \) is finite-dimensional. The study of the moment problem for recursive sequences is then reduced to the case of finite-dimensional Hilbert spaces. Such a link has been observed and studied in [9]. More precisely, we have:

**Proposition 2.1.** Let \( \gamma \) be a recursive sequence. Then (4) has a solution \( A \in \mathcal{L}(\mathcal{H}) \) for some Hilbert space if and only if it has a solution \( A \) on some \( r \)-dimensional Hilbert space.

It is known that \( H \) is positive definite if and only if \( \det H_n > 0 \) for all \( n \geq 0 \) where \( H_n = [\gamma_{i+j}]_{0 \leq i, j \leq n-1} \) are the Hankel matrices associated with the sequence \( \gamma \). In the case of recursive sequences, \( \det H_n = 0 \) whenever
$n \geq r + 1$. The process used in [16] is hence obstructed. We provide in this section an alternative method to avoid this obstruction.

We begin by proving an auxiliary result:

**Lemma 2.1.** If $\gamma = \{\gamma_n\}_{n \geq 0}$ is a recursive sequence with characteristic polynomial $P_\gamma$, then $\langle P, Q \rangle_\gamma = 0$ for every $Q \in \mathbb{C}[X]$ if and only if $P \in (P_\gamma)$, where $(P_\gamma)$ is the ideal of $\mathbb{C}[X]$ generated by $P_\gamma$.

**Proof.** The reverse implication is a direct consequence of the relation (1).

Suppose that $\langle P, Q \rangle_\gamma = 0$ for any $Q \in \mathbb{C}[X]$. Then by writing $P = QP_\gamma + R$ and $R = \sum_{i=0}^{p} \alpha_i X^i$ where $\alpha_p \neq 0$ and $p < r$, we obtain $\langle R, X^n \rangle_\gamma = \sum_{i=0}^{p} \alpha_i \gamma_{n+i} = 0$ for every $n \geq 0$. Hence $\gamma_{n+1} = \sum_{i=0}^{p-1} a_i \gamma_{n-i-1}$ with $a_i = (-\alpha_i/\alpha_p)$, which implies that $R$ is a characteristic polynomial of $\gamma$ with degree less than $r - 1$. Contradiction.

An immediate consequence is the following corollary:

**Corollary 2.1.** Suppose $P_1 = Q_1 P_\gamma + R_1 \in \mathbb{C}[X]$ and $P_2 = Q_2 P_\gamma + R_2 \in \mathbb{C}[X]$. Then

$$\langle P_1, P_2 \rangle_\gamma = \langle R_1, R_2 \rangle_\gamma.$$ 

Set $\mathcal{H}(\gamma) = \mathbb{C}[X]/(P_\gamma)$ and let $\pi$ be the canonical surjection of $\mathbb{C}[X]$ onto $\mathbb{C}[X]/(P_\gamma)$. Seeking simplicity, we will write $P = \pi(P)$. If $H_r$ is positive definite, then the bilinear form $\langle P, Q \rangle_\gamma := \langle \pi(P), \pi(Q) \rangle_\gamma$ for $P, Q \in \mathbb{C}[X]$, is an inner product on $\mathcal{H}(\gamma)$.

Let $A \in \mathcal{L}(\mathcal{H}(\gamma))$ be given by $AX^j = X^{j+1}$ for $j = 0, 2, \ldots, r - 1$. We have

$$\langle P, A Q \rangle_\gamma = \langle P, S_r Q \rangle_\gamma,$$

where $S_r = [\gamma_{i+j+1}]_{0 \leq i, j \leq r-1}$; in particular,

$$\langle A^n 1 \rangle = \gamma_n \text{ for } n = 0, 1, \ldots, r-1.$$

On the other hand, $A^n 1 = X^n = \sum_{j=0}^{r-1} a_j X^{r-j-1}$. Consequently,

$$\langle A^n 1 \rangle = \sum_{j=0}^{r-1} a_j \langle X^{r-j-1} \rangle 1 = \sum_{j=0}^{r-1} a_j \gamma_{r-j-1} = \gamma_r.$$

By induction we establish that $\langle A^n 1 \rangle = \gamma_n$, for $n \geq 0$.

Thus:

**Proposition 2.2.** Let $\gamma = \{\gamma_n\}_{n \geq 0}$ be a recursive sequence with positive definite Hankel matrix $H_r$ and $P_\gamma$ as a characteristic polynomial. Then there exist a $(\deg P_\gamma)$-dimensional Hilbert space $\mathcal{H}(\gamma)$ and a self-adjoint operator $A$ on $\mathcal{H}(\gamma)$ providing a solution of the Hamburger moment problem (4). Moreover, if $S_r$ is positive definite, then $A \geq 0$, and this yields a solution of the Stieltjes moment problem.
Let \( \{P_0, P_1, \ldots, P_{r-1}\} \) be the orthogonal basis of \( \mathcal{H}^{(\gamma)} \), obtained from the basis \( \{1, X, X^2, \ldots, X^{r-1}\} \) by the Gram–Schmidt process of the form

\[
P_i(X) = X^i + \text{lower order}, \quad \text{for } i = 0, 1, \ldots, r - 1.
\]

It is clear that \( \langle XP_i, P_j \rangle = \langle P_i, XP_j \rangle = 0 \), for \( j > i + 1 \) and \( j < i - 1 \). It follows that for suitable sequences \( \{b_n\}_{0 \leq n \leq r-1} \) and \( \{c_n\}_{0 \leq n \leq r-1} \) (with \( P_{-1}(X) = 0 \) and \( P_r(X) = 0 \)), we have

\[
XP_n(X) = c_n P_{n+1}(X) + b_n P_n(X) + c_{n-1} P_{n-1}(X), \quad \text{for } n = 0, 1, \ldots, r-1.
\]

Thus, given a recursive sequence \( \gamma = \{\gamma_n\}_{n \geq 0} \), with positive definite Hankel matrix \( H_\gamma \), we can find a finite-dimensional Hilbert space \( \mathcal{H}^{(\gamma)} \) (with \( \dim \mathcal{H}^{(\gamma)} = r \)), an orthogonal basis \( \{P_0, P_1, \ldots, P_{r-1}\} \), real numbers \( b_0, b_1, \ldots, b_{r-1} \), and positive numbers \( c_0, c_1, \ldots, c_{r-2} \) such that the moment problem (4) is associated to the self-adjoint operator \( A \) on \( \mathcal{H}^{(\gamma)} \) with Jacobi matrix

\[
J_\gamma = \begin{pmatrix}
 b_0 & c_0 & 0 & 0 & \cdots \\
 c_0 & b_1 & c_1 & 0 & \cdots \\
 0 & c_1 & b_2 & c_2 & \cdots \\
 & & \ddots & \ddots & \ddots \\
 & & & c_{r-2} & b_{r-1}
\end{pmatrix}.
\]

The matrix \( J_\gamma \) determines uniquely the moments, since the expansion \( A^k P_0 = X^k = \sum_{j=0}^{k} \xi_k P_j(X) \), for \( k \geq 0 \), implies

\[
\gamma_k = \langle A^k P_0 | P_0 \rangle = \xi_{k0}.
\]

3. Continued fractions associated with moment problems for recursive sequences

Let \( x = \sum_{j=0}^{r-1} x_j e_j \in \mathcal{H}^{(\gamma)} \) be an eigenvector of the matrix \( J_\gamma \) associated with the eigenvalue \( \lambda \). We obtain the system of \( r \) linear equations

\[
\begin{cases}
 b_0 x_0 + c_0 x_1 = \lambda x_0, \\
 c_0 x_0 + b_1 x_1 + c_1 x_2 = \lambda x_1, \\
 \vdots \\
 c_{r-3} x_{r-3} + b_{r-2} x_{r-2} + c_{r-1} x_{r-1} = \lambda x_{r-2}, \\
 c_{r-2} x_{r-2} + b_{r-1} x_{r-1} = \lambda x_{r-1}.
\end{cases}
\]

By induction we derive

\[
x_j = P_j(\lambda) x_0, \quad \text{for } j = 0, 1, \ldots, r-1,
\]

where \( \{P_j\}_{0 \leq j \leq r-1} \) is the family of polynomials defined by \( P_0 = 1, P_1(X) = (X - b_0)/c_0 \) and the recursive relation

\[
c_j P_{j+1} = (u - b_j) P_j(u) - c_{j-1} P_{j-1}(u), \quad \text{for } j = 1, \ldots, r-2.
\]
We associate to the system of equations (5) the terminating fraction given by
\[
\frac{1}{|u-b_0| - \frac{c_0^2}{|u-b_1|} - \frac{c_1^2}{|u-b_2|} - \cdots - \frac{c_{r-2}^2}{|u-b_{r-1}|}}
\]
and the \( j \)-th convergent
\[
\frac{A_j(u)}{B_j(u)} := \frac{1}{|u-b_0| - \frac{c_0^2}{|u-b_1|} - \frac{c_1^2}{|u-b_2|} - \cdots - \frac{c_{j-2}^2}{|u-b_{j-1}|}},
\]
for \( 1 \leq j \leq r \). The family \( \{B_j\}_{1 \leq j \leq r-1} \) of polynomials satisfies
\[
B_j(u) = c_0 c_1 \cdots c_{j-1} P_j(u), \quad \text{for} \quad j = 1, 2, \ldots, r - 1.
\]

By setting \( B_0 := 1 \) and using the recursive relation involving the \( P_j \)'s, we obtain
\[
B_{j+1}(u) = (u-b_j)B_j(u) - c_j^2 B_{j-1}(u),
\]
for \( 1 \leq j \leq r - 2 \). The denominator of the terminating fraction (7) is
\[
B_r(u) = (u-b_{r-1})B_{r-1}(u) - c_{r-2}^2 B_{r-2}(u).
\]

The \( B_j \)'s and \( A_j \)'s are defined by (8) if we take \( B_0 = 1, \ B_1(u) = u-b_0, \ A_0 = 0, \ A_1(u) = 1/c_0 \) as initial conditions. They are called the polynomials of the first and second kind, respectively.

Replacing \( x_{r-1} \) by the expression (6) in the last line of the system (5), we obtain \( B_r(\lambda) = 0 \) for any \( \lambda \) in the spectrum of \( A \). Hence \( B_r \) is the characteristic polynomial of the matrix \( J_\gamma \) (see also [8], for example). On the other hand, from (1) we get \( P_\gamma(A) = 0 \) via easy computations. Since \( \deg P_\gamma = \deg B_r \) and the two polynomials are monic we obtain \( P_\gamma = B_r \). Thus:

**Proposition 3.1.** With the preceding notation, \( B_r \) is the characteristic polynomial of the operator \( A \). In particular,

- \( B_r \) has only simple roots;
- \( A \geq 0 \) if and only if \( Z(P_\gamma) \subset \mathbb{R}^+ \), with \( Z(P_\gamma) \) the set of zeros of \( P_\gamma \).

Proposition 3.1 can be regarded as the solution of the Stieltjes moment problem.

4. **Analytic function associated with moment problems**

4.1. **Analytic functional calculus for recursive sequences.** For a moment sequence \( \gamma = \{\gamma_n\}_{n \geq 0} \), the formal series \( f_\gamma(z) = \sum_{n \geq 0}(-1)^n \gamma_n z^n \) canonically associated with the moment sequence \( \gamma \) is called the Hamburger series in the case of the Hamburger moment problem. It is easy to check that
\[
f_\gamma(z) = \int \frac{d\mu(t)}{1+tz},
\]
where $\mu$ is the measure solution of (2) (see [2], p. 208, for details).

**Proposition 4.1.** Let $\gamma$ be a moment sequence. Then $\gamma$ is a recursive sequence if and only if $f_\gamma$ is a rational function. More precisely, $f_\gamma = P/Q$, where $Q$ is a polynomial of degree $r$ with only simple roots and $\deg P < \deg Q$.

**Proof.** Suppose that $\gamma$ is a recursive sequence. By [3] or [9], we have $\sum_{n=0}^{r-1} \rho_n \delta_{z^n}$. Hence,

$$f_\gamma(z) = \int \frac{d\mu(t)}{1 + tz} = \sum_{n=0}^{r-1} \frac{\rho_n}{1 + z^n} = \frac{P(z)}{Q(z)},$$

with $Q(z) = \prod_{n=0}^{r-1} (1 + z^n)$, and $f_\gamma = P/Q$ is a rational function with $\deg P < \deg Q$. Conversely, write $f_\gamma = P/Q$ and set $Q(z) = 1 + a_0 z + \cdots + a_{r-1} z^r$. Then

$$P(z) = \sum_{n=0}^{\infty} (-1)^n \gamma_n z^n (1 + a_0 z + \cdots + a_{r-1} z^r).$$

Thus, for $n \geq r$ we have

$$(-1)^n \gamma_n + (-1)^{n-1} a_0 \gamma_{n-1} + (-1)^{n-2} a_1 \gamma_{n-2} + \cdots + (-1)^{n-r} a_{r-1} \gamma_{n-r} = 0,$$

or equivalently

$$\gamma_n = a_0 \gamma_{n-1} - a_1 \gamma_{n-2} + \cdots + (-1)^r a_{r-1} \gamma_{n-r}.$$  \hspace{1cm} (9)

The desired result is obtained.

If $f(z) = \sum_{n=0}^{\infty} \gamma_n z^n = P/Q$, with $\deg P \geq \deg Q$, we see by writing $fQ = P$ that (9) is valid for $n$ large enough. The sequence $\gamma$ is recursively defined by $Q$.

**Corollary 4.1.** Let $\gamma$ be a recursive sequence and $f_\gamma$ its Hamburger associated function. Then

$$\frac{1}{z} f_\gamma \left( \frac{1}{z} \right) = \frac{A_r(-z)}{B_r(-z)}.$$

**Proof.** Proposition 4.1 implies that $\frac{1}{z} f_\gamma \left( \frac{1}{z} \right)$ is rational. By writing

$$\frac{A_j(z)}{B_j(z)} = \sum_{p=0}^{\infty} \frac{(-1)^p c_p^j}{z^{p+1}}$$

at infinity for $1 \leq j \leq r$, we have $c_p^j = \gamma_p$ for $p \leq j$, by [8]. In particular, $c_p^r = \gamma_p$ for $p \leq r$. Therefore, $\gamma$ and $(c_p^r)_{p \geq 0}$ are recursive sequences, associated with the same initial conditions and characteristic polynomial. The required assertion is proved.

The following lemma will be used to prove the main result of this section:
Lemma 4.1. Let $A$ be as in (4) and let $z \in \mathbb{C}$ be such that $|z| > \|A\|$. Then

(10) \[ \langle (A - zI)^{-1}x, x \rangle = \frac{-1}{z} f_{\gamma} \left( \frac{-1}{z} \right) = \frac{A_r(z)}{B_r(z)}. \]

Proof.

\[ \langle (A - zI)^{-1}x, x \rangle = \frac{1}{z} \langle \left( \frac{1}{z} A - I \right)^{-1} x, x \rangle = \frac{-1}{z} \sum_{n \geq 0} \langle A^n x, x \rangle \left( \frac{1}{z} \right)^n = \frac{-1}{z} \sum_{n \geq 0} \gamma_n \left( \frac{1}{z} \right)^n = \frac{-1}{z} f_{\gamma} \left( \frac{-1}{z} \right). \]

The second equality is trivial from Corollary 4.1.

Using this lemma we obtain:

Proposition 4.2. For any entire function $f$, denote by $f(A)$ the operator defined by the Riesz functional calculus. Then

\[ \langle f(A)x, x \rangle = \sum_{z_j \in \sigma(A)} f(z_j) \frac{A_r(z_j)}{B_r'(z_j)}, \]

where $\sigma(A)$ is the spectrum of $A$.

Proof. For $R > \|A\|$, let $\Gamma_R = \{ z \in \mathbb{C} : |z| = R \}$. We have

\[ f(A) = \frac{1}{2i\pi} \int_{\Gamma_R} f(z)(A - zI)^{-1} \, dz. \]

Then

\[ \langle f(A)x, x \rangle = \frac{1}{2i\pi} \int_{\Gamma_R} \langle f(z)(A - zI)^{-1}x, x \rangle \, dz \]
\[ = \frac{1}{2i\pi} \int_{\Gamma_R} f(z)\langle (A - zI)^{-1}x, x \rangle \, dz \]
\[ = \frac{1}{2i\pi} \int_{\Gamma_R} f(z) \frac{A_r(z)}{B_r(z)} \, dz \quad \text{(by (10))} \]
\[ = \sum_{z_j \in Z(B_r) = \sigma(A)} f(z_j) \frac{A_r(z_j)}{B_r'(z_j)} \quad \text{(by the Residue Theorem)}. \]

Remark. As $\sigma(A) = Z(B_r)$ is a finite set of eigenvalues, the spectral measure associated with the self-adjoint operator $A$ is given by the orthonormal projection onto the associated eigenspaces. This fact is also derived from Proposition 4.3.
For any entire function $f$ and complex number $u$, we denote by $L_u(f)$ the holomorphic function defined as follows:

\[
L_u(f)(z) = \begin{cases} 
\frac{f(z) - f(u)}{z - u} & \text{if } z \neq u, \\
 f'(u) & \text{if } z = u.
\end{cases}
\]

(11)

Let $S_\gamma$ be the linear form defined by (3), associated with the linear moment sequence $\gamma$.

The following proposition unifies some results of [8].

**Proposition 4.3.** For any holomorphic function $f$, we have

\[ S_\gamma(L_u(fB_r)) = f(u)A_r(u). \]

**Proof.** As in the proof of Proposition 4.2, we have

\[
S_\gamma(L_u(fB_r)) = \langle (L_u(fB_r)(A)x, x) 
\]

\[
= \frac{1}{2i\pi} \int_{\Gamma_R} f(u)B_r(u) - f(z)B_r(z) \frac{A_r(z)}{B_r(z)} \frac{dz}{u - z}
\]

\[
= \frac{1}{2i\pi} \int_{\Gamma_R} f(u)B_r(u) A_r(z) \frac{dz}{u - z} - \frac{1}{2i\pi} \int_{\Gamma_R} f(z) A_r(z) \frac{dz}{u - z}
\]

\[
= f(u)A_r(u) + f(u)B_r(u) \frac{A_r(z)}{2i\pi} \int_{\Gamma_R} \frac{dz}{(u - z)B_r(z)}
\]

\[
= f(u)A_r(u) - f(u)B_r(u) \left( \frac{A_r(u)}{B_r(u)} - \sum_{z_j \in \mathbb{Z}(B_r)} \frac{A_r(z_j)}{B_r'(z_j)} \frac{1}{u - z_j} \right)
\]

\[
= f(u)A_r(u),
\]

since the expression in big parentheses on the penultimate line is 0.

We derive two corollaries:

**Corollary 4.2 ([8], Theorem 1 (17)).** *With the same notation as in Proposition 4.3, we have*

\[ S_\gamma(L_u(B_r)) = A_r(u). \]

**Proof.** $f \equiv 1$ in Proposition 4.3.

**Corollary 4.3 ([8], p. 6).** *For any polynomial $P$, we have*

\[ S_\gamma(P) = \sum_{z_j \in \mathbb{Z}(B_r)} S_\gamma(L_{z_j}(B_r)) \frac{P(z_j)}{B_r'(z_j)} = \sum_{z_j \in \mathbb{Z}(B_r)} A_r(z_j) \frac{P(z_j)}{B_r'(z_j)}. \]

**Proof.** Combine Proposition 4.2 with the preceding corollary.
5. Moment problems associated with limited continued fractions

In this section, we use the preceding section to shed some light on the moment problem arising from the terminating fraction (7).

Consider the limited Jacobi fraction

\[ \frac{1}{|u-b_0| - \frac{c_0^2}{|u-b_1|} - \frac{c_1^2}{|u-b_2|} - \cdots - \frac{c_{r-2}^2}{|u-b_{r-1}|}}, \]

where \(b_0, b_1, \ldots, b_{r-1}\) are real and \(c_0, c_1, \ldots, c_{r-2}\) are nonzero real numbers. Let

\[ J = \begin{pmatrix} b_0 & c_0 & 0 & 0 & \cdots & \cdots \\ c_0 & b_1 & c_1 & 0 & \cdots & \cdots \\ 0 & c_1 & b_2 & c_2 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & c_{r-2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & b_{r-1} \end{pmatrix} \]

be the Jacobi matrix associated with (12) and consider the operator \(A\) associated with the matrix \(J\) defined on an \(r\)-dimensional Hilbert space \(H\). For \(x \in H\) a nonzero vector, the sequence \(\gamma_n(x) = \langle A^n x, x \rangle\) is clearly a recursive moment sequence defined by its \(r\) initial conditions and the characteristic polynomial of the matrix \(J\).

Given \(\gamma_0, \ldots, \gamma_{r-1}\) some real numbers, is there \(x \in H\) such that \(\gamma_j = \gamma_j(x)\) for \(0 \leq j \leq r-1\)?

Suppose such an \(x\) exists and write

\[ x = \sum_{j=0}^{r-1} \rho_j x_j, \]

where \(\{x_j : j = 0, \ldots, r-1\}\) is the orthonormal basis of eigenvectors of \(A\) associated with the eigenvalues \(\{z_j : j = 0, \ldots, r-1\}\). Therefore,

\[ \langle A^n x, x \rangle = \sum_{j=0}^{r-1} \rho_j^2 z_j^n = \gamma_n. \]

Let \(\{\gamma_n\}_{0 \leq n \leq r-1}\) be the initial conditions of the recursive sequence \(\{\gamma_n\}_{n \geq 0}\) associated with the characteristic polynomial \(P(X) = \prod_{j=0}^{r-1} (X - z_j)\). By Theorem 3 of [8], the sequence \(\{\gamma_n\}_{n \geq 0}\) is associated with a positive linear form if and only if the Hankel matrix \(H_r = [\gamma_{i+j}]_{0 \leq i,j \leq r-1}\) is positive definite.

Let \(A_j(u)\) and \(B_j(u)\) be defined as in (8). It is known that there exists a linear functional \(S\), defined on polynomials, that orthogonalizes the \(B_j\)’s. That is,

\[ S(B_i B_j) = 0, \quad \text{for } 0 \leq i < j \leq r - 1. \]

Under the additional assumption

\[ S(B_n^2) = c_0 c_1 \cdots c_n, \quad \text{for } 0 \leq n \leq r - 1, \]

the sequence \(\{\gamma_n\}_{n \geq 0}\) is associated with a positive linear form.

\[ S(B_i B_j) = 0, \quad \text{for } 0 \leq i < j \leq r - 1. \]
$S$ is unique and satisfies the following property:

**Theorem 5.1.** There exist positive numbers $\lambda_0, \ldots, \lambda_{r-1}$ such that
\[
S(P) = \sum_{j=0}^{r-1} \lambda_j P(z_j),
\]
for every polynomial $P$.

In view of Corollary 4.3 we have $\lambda_j = A_r(z_j)/B_r'(z_j) > 0$. Thus:

**Proposition 5.1.** With the same notations, we have
\[
S(X^n) = \langle A^n x, x \rangle = \int_{\mathbb{R}} t^n d\mu(t),
\]
with
\[
x = \sum_{j=0}^{r-1} \sqrt{\frac{A_r(z_j)}{B_r'(z_j)}} x_j \quad \text{and} \quad \mu = \sum_{j=0}^{r-1} \frac{A_r(z_j)}{B_r'(z_j)} \delta_{z_j}.
\]

Moreover, $B_r$ is the characteristic polynomial of $\{S(X^n)\}_{n \geq 0}$.

In the spirit of [12], it will be of interest to write the results of this paper for multivariable truncated moment sequences. The authors are thankful to the anonymous referee for pointing out reference [12] and for his suggestions and remarks.

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A PIERI RULE FOR HERMITIAN SYMMETRIC PAIRS II

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Let $X$ be an irreducible Hermitian symmetric space of noncompact type and rank $r$. Let $p \in X$ and let $K$ be the isotropy group of $p$ in the group of biholomorphic transformations. Let $S$ denote the symmetric algebra in the holomorphic tangent space to $X$ at $p$. Then $S$ is multiplicity free as a representation of $K$ and the irreducible constituents are parametrized by $r$-tuples, $(m_1, \ldots, m_r)$ with $m_1 \geq \cdots \geq m_r \geq 0$. That is, the same parameters as the irreducible polynomial representations of $GL(r)$. Let $S[m_1, \ldots, m_r]$ be the corresponding isotypic component. In this article we show that the product in $S$, $S[m_1, \ldots, m_r]S[k, 0, \ldots, 0]$ is a direct sum of constituents following precisely the classical Pieri rule.

1. Introduction

In the classical theory of representations of $GL(r, \mathbb{C})$ the irreducible polynomial representations are parametrized by $r$-tuples $(m_1, \ldots, m_r)$ of integers with $m_1 \geq m_2 \geq \cdots \geq m_r \geq 0$. Denoting the corresponding irreducible representation by $F^{m_1, \ldots, m_r}$ then the Pieri rule $[F]$ or $[M]$ says that the tensor product $F^{m_1, \ldots, m_r} \otimes F^{k, 0, \ldots, 0}$ is a direct sum of irreducible representations with parameters $(m_1 + a_1, \ldots, m_r + a_r)$ with $a_1 + \cdots + a_r = k$ and $m_1 + a_1 \geq m_2 + a_2 \geq \cdots \geq m_r + a_r \geq m_r$, each occurring with multiplicity one. In this article we will show that there is a completely analogous formula for every irreducible Hermitian symmetric space. We will now describe the result.

Let $X$ be an irreducible Hermitian symmetric space of noncompact type. This means that $X$ is a complex manifold such that, $G$, the group of biholomorphic transformations of $X$ is simple, noncompact and acts transitively and if $K$ is the stability group of a point $p \in X$ then $K$ is a maximal compact subgroup. Thus we may think of $X = G/K$. We note that the general theory implies that Lie $(K)$ has a one dimensional center, $\mathfrak{z}$. We set $\mathfrak{g} = \text{Lie} (G) \otimes \mathbb{C}$ and $\mathfrak{k} = \text{Lie} (K) \otimes \mathbb{C} \subset \mathfrak{g}$. Then $\mathfrak{z} = \mathbb{R} i \mathfrak{h}$ and $ad H$ has eigenvalues $0, 1, -1$ with eigenspace for $0$ being $\mathfrak{k}$ and the eigenspace for $1$ identified with the holomorphic tangent space. We will use the standard notation $\mathfrak{p}^\pm = \{ X \in \mathfrak{g} | [H, X] = \pm X \}$. Thus the holomorphic tangent space is identified with $\mathfrak{p}^+$. Using the Killing form of $\mathfrak{g}$ we can identify $\mathfrak{p}^-$ with $(\mathfrak{p}^+)^\ast$. 
Thus the ring of polynomials on the holomorphic tangent space is identified with \( S(\mathfrak{p}^-) \). It is a theorem of Schmid [S] that as a \( K \)-representation \( S(\mathfrak{p}^-) \) is multiplicity free and if \( r \) is the rank of \( X \) (the dimension of a maximal abelian subalgebra of \( \mathfrak{g} \) contained in \( \mathfrak{t}^\perp \) relative to the Killing form) then the irreducible constituents are labeled (in a natural manner, see the next section for details) by \( r \)-tuples as in the classical case. We will denote the isotypic component corresponding to \((m_1, \ldots, m_r)\) by \( S(\mathfrak{p}^-)[m_1, \ldots, m_r] \).

Then our main result asserts that relative to the multiplication in the algebra \( S(\mathfrak{p}^-) \) we have \( S(\mathfrak{p}^-)[m_1, \ldots, m_r] \cdot S(\mathfrak{p}^-)[k, 0, \ldots, 0] \) is the direct sum of the \( S(\mathfrak{p}^-)[m_1 + a_1, \ldots, m_r + a_r] \) with the \( a_1, \ldots, a_r \) following the classical Pieri rule above. In the last section of this article we show how one can use the classical theory of Littlewood–Richardson combined with the Schmid result to determine that the constituents that can appear in the product are contained in the set described by the Pieri rule in the cases when \( G \) is locally isomorphic with \( SU(p, q), SO^\ast(2n) \) and \( Sp(n, \mathbb{R}) \). However, even in these cases the fact that the constituents actually occur is not a clear implication of the classical theory (since the classical theory involves tensor products and not multiplication in a symmetric algebra).

This paper is an extension of some of the results in [EHW] although it is independent of that article.

2. \( K \)-decomposition of products

We continue with the Hermitian symmetric setting and the notation of the introduction. Fix a Cartan subalgebra \( \mathfrak{h} \) of \( \mathfrak{k} \) and a system of positive roots \( \Phi^+ \) such that if \( \alpha \in \Phi^+ \) then \( \alpha(H) \geq 0 \). We set \( \Phi^+_\mathfrak{h} = \{ \alpha \in \Phi \mid \alpha(H) = 1 \} \). Let \( \Phi^+_\mathfrak{c} = \Phi^+ - \Phi^+_\mathfrak{h} \). Then \( \Phi^+_\mathfrak{c} \) is a system of positive roots for \( \mathfrak{k} \) on \( \mathfrak{h} \). Let \( \gamma_1 < \gamma_2 < \cdots < \gamma_r \) denote Harish–Chandra’s strongly orthogonal roots [HS] which are defined as follows: Let \( \gamma_1 \) denote the unique minimal element of \( \Phi^+_\mathfrak{c} \). Inductively define \( \gamma_i \), for \( 1 < i \leq r \), to be the unique minimal element in \( \{ \delta \in \Phi^+_\mathfrak{c} \mid (\delta, \gamma_j) = 0, 1 \leq j < i \} \). Schmid [S] has shown that as a \( K \)-module under the restriction of the adjoint representation \( S(\mathfrak{p}^-) \) is multiplicity free and the irreducible constituents are exactly the \( K \)-modules with highest weights of the form

\[-(n_1 \gamma_1 + \cdots + n_r \gamma_r), \quad n_1 \geq n_2 \geq \cdots \geq n_r \geq 0, \quad n_i \in \mathbb{Z}.\]

We write \( S(\mathfrak{p}^-)[n_1, \ldots, n_r] \) for the corresponding isotypic component. Note that if \( d = \sum n_i \) then \( S(\mathfrak{p}^-)[n_1, \ldots, n_r] \subset S^d(\mathfrak{p}^-) \), the homogeneous elements of degree \( d \). For each positive integer \( k \) and multi-index \( \underline{n} = (n_1, n_2, \ldots, n_r) \) with \( n_1 \geq n_2 \geq \cdots \geq n_r \geq 0 \), define a set of multi-indices: \( I_k(\underline{n}) = \{ m \mid m_1 \geq n_1 \geq m_2 \geq n_2 \geq \cdots \geq m_{r-1} \geq n_{r-1} \geq m_r \geq n_r \geq 0 \} \) and \( \sum m_i = \sum n_i + k \).

The main result of this article is:
Theorem 1. Let $n_1 \geq n_2 \geq \cdots \geq n_r \geq 0$, $n_i \in \mathbb{Z}$. Then

\[ S(p^-)[n_1, \ldots, n_r] \otimes S(p^-)[k, 0, \ldots, 0] = \bigoplus_{m \in I_k(\mathfrak{n})} S(p^-)[m], \]

with multiplication in $S(p^-)$.

We begin with some preliminary definitions and results which will be the basis for a proof by induction on the rank of $\mathfrak{g}$ and the integer $k$. First we recall four well-known results on the strongly orthogonal roots \[ \text{[Mo]} \text{ and } \text{[W1]}. \]

Let $B$ denote the Killing form of $\mathfrak{g}$. Then $B$ induces a perfect pairing between $p^+$ and $p^-$. Thus we can look upon the symmetric algebra $S(p^-)$ as polynomials on $p^+$ and $S(p^+)$ as differential operators with constant coefficients on $p^+$. Let $(\cdot, \cdot)$ denote the dual form to $B$.

(2.2) $(\gamma_i, \gamma_i) = (\gamma_j, \gamma_j)$ for all $i, j$.

(2.3) If $\alpha \in \Phi^+$ then $(\alpha, \alpha) \leq (\gamma_1, \gamma_1)$.

Let $\mathfrak{h}^-$ denote the linear span of the coroots, $\gamma_i^\vee$, of the $\gamma_i$. Then $\dim \mathfrak{h}^- = r$ and the $\gamma_i^\vee$ form an orthogonal basis of $\mathfrak{h}^-$. 

(2.4) If $\alpha \in \Phi^+_c$ and $\alpha \neq \gamma_i$ for any $i$, then $\alpha_{|\mathfrak{h}^-}$ is either of the form $\frac{1}{2}(\gamma_i + \gamma_j)$ with $i < j$ or $\frac{1}{2} \gamma_i$.

(2.5) If $\alpha \in \Phi^+_c$ then $\alpha_{|\mathfrak{h}^-}$ is either of the form $-\frac{1}{2}(\gamma_i - \gamma_j)$ with $i \leq j$ or $-\frac{1}{2} \gamma_i$.

Let $n^+_c$ denote the sum of the root spaces for the elements of $\Phi^+_c$. We choose a nonzero element $u_i$ in $S(p^-)[n^+_c] \cap S(p^-)[n_1, \ldots, n_r]$ with $n_j = 1$ for $j \leq i$ and $n_j = 0$ for $j > i$. Then one can easily see from Schmid’s result that:

(2.6) $S(p^-)[n^+_c]$ is the polynomial ring on the algebraically independent elements $u_1, \ldots, u_r$.

We will now analyze these covariants in further detail. For each $j, 1 \leq j \leq r$, let $\Phi^+_0,j$ denote the set of elements $\alpha$ of $\Phi^+$ such that $\alpha_{|\mathfrak{h}^-}$ is of the form $\frac{1}{2}(\gamma_p \pm \gamma_q)$ with $q \leq p \leq j$. Then $\Phi^+_0,j = \Phi^+_0,j \cup -\Phi^+_0,j$ is a subrootsystem of $\Phi$. Let $\mathfrak{g}_{0,j}$ denote the subalgebra of $\mathfrak{g}$ generated by the root spaces for the roots in $\Phi^+_0,j$. If $u_j$ is the sum of all ideals of $\mathfrak{g}_{0,j}$ contained in $\mathfrak{g}_{0,j} \cap \mathfrak{h}$ then $(\mathfrak{g}_{0,j}/u_j, (\mathfrak{h} \cap \mathfrak{g}_{0,j})/u_j)$ is also an irreducible symmetric pair of Hermitian type \[ \text{[W1]}. \]

Set $\mathfrak{g}_{0,j}^\pm = \mathfrak{g}_{0,j} \cap \mathfrak{p}^\pm$.

Next we introduce a flag of parabolic subalgebras needed for an induction argument. For any root $\alpha$ write $\alpha_{|\mathfrak{h}^-} = \frac{1}{2} \sum_{1 \leq i \leq r} a_i \gamma_i$. Then we call $\alpha$ an even (odd) root if the sum $\sum_{1 \leq i \leq r} a_i$ is even (odd). We say $\alpha$ has level $j, 1 \leq j \leq r$, if $j$ is the maximal integer with $a_j \neq 0$. Otherwise we say $\alpha$ has level 0. Set $m_j = \mathfrak{h} \oplus \sum \mathfrak{g}_\alpha$ with the sum taken over all even roots of level $\leq j$. Set $n_j = n_{e,j} \oplus n_{o,d}$, where $n_{e,j} = \sum \mathfrak{g}_\alpha$ with the sum taken over...
all even positive roots of level greater than \( j \) and \( n_{\text{odd}} = \sum g_\alpha \) with the sum taken over all positive odd roots. Finally set \( q_j = m_j \oplus n_j \) and let \( b \) denote the Borel subalgebra with weight spaces corresponding to \( \Phi^+ \).

**Lemma 1.** For \( 1 \leq j \leq r \), \( q_j = m_j \oplus n_j \) is a parabolic subalgebra of \( g \) containing \( b \) with Levi component \( m_j \) and nilradical \( n_j \). Moreover \( m_j = h + g_{o,j} \).

**Proof.** By parity \( m_j \) is a subalgebra of \( g \). From (2.4) and (2.5), \( n_{e,j} \) is a subalgebra and in fact an ideal in \( m_j \oplus n_{e,j} \). Next we check that \( n_{\text{odd}} \) is abelian. Split \( n_{\text{odd}} \) into a direct sum of root spaces for compact and noncompact roots \( n_{\text{odd}} = n_{c,\text{odd}} \oplus n_{n,\text{odd}} \). Since \( p^+ \) is abelian, \( n_{n,\text{odd}} \) is as well. Suppose \( \alpha \) is a positive, odd noncompact root and \( \beta \) is positive, odd and compact. But then \( \alpha + \beta \) if a root must be positive, even, noncompact and with \( (\alpha + \beta)_{||h^-} = \frac{1}{2}(\gamma_i - \gamma_l) \). This violates (2.5) and so \( [n_{n,\text{odd}}, n_{c,\text{odd}}] = 0 \). Similarly assume \( \alpha \) and \( \beta \) are both positive, odd and compact. Then if a root \( \alpha + \beta \) is positive, even, compact and with \( (\alpha + \beta)_{||h^-} = \frac{1}{2}(\gamma_i - \gamma_l) \). This violates (2.5) and so \( [n_{n,\text{odd}}, n_{c,\text{odd}}] = 0 \). Combining these we conclude that \( n_{\text{odd}} \) is abelian.

Next we show that \( n_{\text{odd}} \) is an ideal in \( m_r \oplus n_{\text{odd}} \). Suppose \( \alpha \) is a positive, even root and \( \beta \) is positive odd. Then clearly if a root, \( \alpha + \beta \) is a root, it also must be positive odd. We proceed in cases. First assume \( \alpha \) is noncompact. Then \( (-\alpha + \beta)_{||h^-} = \frac{1}{2}(\gamma_i - \gamma_l) + \beta_{||h} \). So if this is a root we must have cancellation and thus \( \beta \) must be noncompact. But then \( -\alpha + \beta \) is compact and so the form of \( (-\alpha + \beta)_{||h^-} \) implies \( -\alpha + \beta \) is positive. This proves our claim when \( \alpha \) is noncompact. Now suppose \( \alpha \) is compact, even and positive. We consider two cases for \( \beta \). First assume \( \beta \) is positive, odd and compact. If \( -\alpha + \beta \) is a root then \( (-\alpha + \beta)_{||h^-} = \frac{1}{2}(\gamma_i - \gamma_l) + \beta_{||h} = \frac{1}{2}(\gamma_i) \). Since \( -\alpha + \beta \) is compact this identity implies \( -\alpha + \beta \) is positive odd proving the claim in this case. Finally suppose \( \beta \) is positive, odd and noncompact. Then if \( -\alpha + \beta \) is a root, \( (-\alpha + \beta)_{||h^-} = \frac{1}{2}(\gamma_i - \gamma_l) + \beta_{||h} = \frac{1}{2}(\gamma_i) \). Since \( -\alpha + \beta \) is noncompact this identity implies \( -\alpha + \beta \) is positive odd proving the claim in this last case.

To check that \( n_j \) is an ideal in \( q_j \) we note:

\[
[q_j, n_j] \subset [m_j \oplus n_{e,j}, n_{e,j}] \oplus n_{\text{odd}} = n_{e,j} \oplus n_{\text{odd}} = n_j.
\]

This proves \( n_j \) is an ideal in \( q_j \). It follows that the simple roots of \( (m_j, h) \) must also be simple for \( (g, h) \). This completes the proof of the lemma.

As an immediate corollary which also follows directly from (2.4) we have:

(2.7)

Suppose \( \beta_i \in \Phi^+_n \) for \( 1 \leq i \leq j \) and \( \sum_{1 \leq i \leq j} \beta_i = \sum_{1 \leq i \leq j} \gamma_i \). Then \( \beta_i \in \Phi_{0,j} \).
(2.8) Fix \( j \) and let \( E \) and \( F \) be irreducible \( \mathfrak{m}_j \)-modules with highest weights \( \xi \) and \( \nu \). Let \( N(E) \) and \( N(F) \) denote the \( \mathfrak{g} \)-modules obtained by inducing up from \( \mathfrak{q}_j \) to \( \mathfrak{g} \); i.e., \( N(E) = U(\mathfrak{g}) \otimes U(\mathfrak{q}_j) E \) and \( N(F) = U(\mathfrak{g}) \otimes U(\mathfrak{q}_j) F \). For \( \mu \in \mathfrak{h}^* \) let a subscript \( \mu \) denote the weight space for weight \( \mu \). The following weight space identity is an easy consequence of Lemma 1:

(2.9) Suppose \( \mu = \xi + \nu + \sum_{1 \leq i \leq j} a_i \gamma_i \) with \( a_i \in \mathbb{Z} \). Then we have an identity of weight spaces:

\[
(N(E) \otimes N(F))_{\mu} = (E \otimes F)_{\mu}.
\]

From [W1] we have:

**Lemma 2.** For \( 1 \leq j \leq r \), \( u_j \in S(\mathfrak{p}_{\mathfrak{0}, j}^-) \).

Moreover, for any multi-index \( m \) with \( m_{j+1} = \cdots = m_r = 0 \), then:

\[
(S(\mathfrak{p}^- m))^n = (S(\mathfrak{p}_{\mathfrak{0}, j}^- m))^n \cap \mathfrak{g}_0.\]

By Lemma 2 the highest weight vectors although not the full \( \mathfrak{t} \)-modules are contained in \( S(\mathfrak{p}_{\mathfrak{0}, r}^-) \). As a first reduction we claim it is sufficient to prove Theorem 1 when \( \mathfrak{g} = \mathfrak{g}_{0, r} \). To this end set \( E = S(\mathfrak{p}_{\mathfrak{0}, j}^-)[n_1, \ldots, n_j] \) and \( F = S(\mathfrak{p}_{\mathfrak{0}, j}^-)[k, 0, \ldots, 0] \). If \( \mu \) is a highest weight of the form \( \mu = \sum_{1 \leq i \leq j} a_i \gamma_i \) then by (2.9), we obtain:

(2.10) \[
(S(\mathfrak{p}^-)[n_1, \ldots, n_j, 0, \ldots, 0] \cdot S(\mathfrak{p}^-)[k, 0, \ldots, 0])_{\mu} = (S(\mathfrak{p}_{\mathfrak{0}, j}^-)[n_1, \ldots, n_j] \cdot S(\mathfrak{p}_{\mathfrak{0}, j}^-)[k, 0, \ldots, 0])_{\mu},
\]

where on the right the bracket designates the \( \mathfrak{t} \cap \mathfrak{g}_{0, j} \) isotypic subspaces.

From this identity (2.10) with \( j = r \) and Lemma 2 we conclude that Theorem 1 holds for \( \mathfrak{g} \) if and only if it holds for \( \mathfrak{g}_{0, r} \).

From this point forward we assume \( \mathfrak{g} = \mathfrak{g}_{0, r} \). This reduction has several simplifying features which we now summarize. By (2.5) for any compact root \( \delta \), \( \delta|_{\mathfrak{h}_-} = \frac{1}{2} (\gamma_i - \gamma_r) \). This implies that the \( \mathfrak{t} \)-module \( S(\mathfrak{p}^-)[n, n, \ldots, n] \) is one dimensional. If \( \delta \) is positive and compact then for some \( i < j \), \( \delta|_{\mathfrak{h}_-} + \gamma_r = \frac{1}{2} (\gamma_i + \gamma_r) + 2 \gamma_r \) which contradicts (2.5). This proves \( \gamma_r \) is a maximal root with respect to \( \mathfrak{t} \). This same identity shows that the weight spaces \( \mathfrak{g}_{\gamma_r} \) and \( \mathfrak{g}_{-\gamma_r} \) are one dimensional \( \mathfrak{t} \cap \mathfrak{g}_{0, r-1} \)-modules. Since \( \mathfrak{p}^+ \) is abelian, \( \gamma_r \) is the maximal weight in \( \Phi^+ \).

From the one dimensionality of \( S(\mathfrak{p}^-)[n, n, \ldots, n] \), we obtain:

(2.11) \[
S(\mathfrak{p}^-)[n_1, \ldots, n_r] = S(\mathfrak{p}^-)[n_1 - n_r, \ldots, n_r - n_r, 0] u_r^{n_r}.
\]

**Lemma 3.** Suppose Theorem 1 holds for \( \mathfrak{g}_{0, r-1} \), \( n_r = 0 \) and \( m \) is a multi-index not in \( I_k(\mathfrak{h}) \) whose K-type occurs in the product in the left-hand side of Equation (2.1). Then \( m_r \geq 1 \).

**Proof.** Let \( E = S(\mathfrak{p}_{\mathfrak{0}, r-1}^-)[n_1, \ldots, n_r, 0] \) and \( F = S(\mathfrak{p}_{\mathfrak{0}, r-1}^-)[k, 0, \ldots, 0] \) and let \( v \) be a highest weight vector of the K-type indexed by \( m \). By hypothesis
the \( \mathfrak{f} \cap \mathfrak{g}_{0,r-1} \)-decomposition of \( E \cdot F \) shows that \( v \) is not in \( E \cdot F \). So by (2.10) with \( j = r - 1 \), we have a contradiction unless \( m_r \geq 1 \).

We denote by \( x \mapsto \overline{x} \) the conjugation of \( \mathfrak{g} \) with respect to \( \text{Lie}(G) \). Then \( \overline{\mathfrak{p}^-} = \mathfrak{p}^+ \). If \( x \in \mathfrak{p}^+ \) then we denote by \( \partial(x) \) the derivation of \( S(\mathfrak{p}^-) \) defined by \( \partial(x)y = B(x,y) \) for \( y \in \mathfrak{p}^- \). We will also denote the extension of \( \partial \) to \( S(\mathfrak{p}^+) \) by \( \partial \). In addition we will use the notation \( u \mapsto u(0) \) for the augmentation map of \( S(\mathfrak{p}^-) \) to \( \mathbb{C} \) given as the extension to a homomorphism of \( y \mapsto 0 \) for \( y \in \mathfrak{p}^- \). We define for \( u, v \in S(\mathfrak{p}^-) \), \( \langle u, v \rangle = \langle \partial(\overline{u})u(0) \rangle \rangle \). For any root \( \beta \) let \( X_\beta \) denote an element in the root space \( \mathfrak{g}_\beta \) normalized so that \( X_\beta = X_{-\beta} \).

The following observation is well-known and easily checked:

(2.12) The Hermitian form \( \langle \cdot , \cdot \rangle \) is positive definite and \( K \)-invariant. Furthermore, if \( u, v, w \in S(\mathfrak{p}^-) \) then \( \langle uv, w \rangle = \langle v, \partial(\overline{u})w \rangle \) if \( n_1 \geq \cdots \geq n_r \geq 0 \) then we note that \( u_1^{r_1-n_1} u_2^{n_2-n_3} \cdots u_r^{r-r_r} u_r^{n_r} \) is a basis of the highest weight space of \( S(\mathfrak{p}^-)[n_1, \ldots, n_r] \). Let \( D = \partial(\overline{\mathfrak{p}}) \). Then

\[
D : S(\mathfrak{p}^-)[n_1, \ldots, n_r] \to S(\mathfrak{p}^-)[n_1 - 1, \ldots, n_r - 1].
\]

Here if \( n_r < 0 \) then we write \( S(\mathfrak{p}^-)[n_1, \ldots, n_r] = \{0\} \).

The maps \( D \) and multiplication by \( u_r \) are semi-invariant maps and the product \( u_r D \) is a \( K \)-invariant map which plays an important role below. The eigenvalues of this operator are given in:

**Lemma 4.** For each multi-index \( \underline{n} \) the operator \( u_r D \) is diagonalizable and there are nonzero constants \( c \) and \( C_2 \) such that \( u_r D \) restricted to \( S(\mathfrak{p}^-)[\underline{n}] \)

\[
= C_2 \prod_{i=1}^r \left( \frac{r - i}{2} c + n_i \right) \cdot I.
\]

**Proof.** This is the first case of Theorem 3.3 in [W2].

**Remark 1.** The referee has noted that Lemma 4 is employed only to show that (2.13) is a bijective map for all \( n_r \geq 1 \). An alternative argument can be given since \( D \) is the adjoint to multiplication by \( u_r \).

**Lemma 5.** Suppose \( \underline{n} \) is a multi-index with \( n_r = 0 \), set

\[
v = u_1^{n_1-n_2} u_2^{n_2-n_3} \cdots u_r^{n_r-1}.
\]

Then

\[
D(vX_{-\gamma_r}^k) = C_1 u_1^{n_1-n_2} u_2^{n_2-n_3} \cdots u_r^{n_r-2} u_{r-1}^{n_{r-1}-1} X_{-\gamma_r}^{k-1},
\]

with \( C_1 = 0 \) if and only if \( n_{r-1} = 0 \). Moreover \( D \) is a \( K \)-intertwining operator that carries a cyclic vector for the product \( S(\mathfrak{p}^-)[n_1, \ldots, n_r - 1, 0] S(\mathfrak{p}^-)[k, 0, \ldots, 0] \) to a cyclic vector for the product \( S(\mathfrak{p}^-)[n_1 - 1, \ldots, n_r - 1 - 1, 0] S(\mathfrak{p}^-)[k - 1, 0, \ldots, 0] \).
Proof. The identity (2.15) is a consequence of calculations found in [W1] (see the proofs of Lemmas 5.7 and 5.8). Since the identity is so essential in our proof of Theorem 1 we repeat the argument here.

Expanding with respect to noncompact root vectors we choose constants \(a_{\beta_1}, \ldots, \beta_r\) with

\[
(2.16) \quad u_r = \sum a_{\beta_1}, \ldots, \beta_r X_{-\beta_1} \cdots X_{-\beta_r} \quad \text{and} \quad u_r = uX_{-\gamma_r} + w \quad \text{where} \quad w = \sum a_{\beta_1}, \ldots, \beta_r X_{-\beta_1} \cdots X_{-\beta_r},
\]

and the first sum is over all \(\beta_1 \leq \cdots \leq \beta_r\) with \(\sum 1 \leq i \leq \beta_i = \sum 1 \leq i \leq \gamma_i\) and the second sum is over the subset where \(\beta_i \neq \gamma_i\) for all \(i\). The identity (2.7) implies that \(u \in S(p_{0,r-1})\). Moreover from our reduction \(g = g_{0,r}\) we know \(u\) is a semi-invariant for \(\mathfrak{g} \cap g_{0,r-1}\). This implies that \(u\) is a scalar multiple of \(u_{r-1}\). Computing the action of \(D = \partial p_r\) on \(vX_{-\gamma_r}\), we obtain:

\[
D(vX_{-\gamma_r}) = k\partial p(v)X_{-\gamma_r}^{k-1}. \quad \text{Since} \quad u \quad \text{is a scalar multiple of the} \quad \mathfrak{g} \cap g_{0,r-1} \quad \text{semi-invariant} \quad u_{r-1} \in S(p_{0,r-1}), \quad \text{we may apply (2.13) with} \quad r \quad \text{replaced with} \quad r - 1 \quad \text{to obtain the identity (2.15)}.
\]

The highest weight space times the lowest weight space is cyclic for the tensor product of any two irreducible finite dimensional representations. So the second assertion follows from the identity.

Proof of Theorem 1. We will prove the theorem by induction on \(r\) and then by induction on \(k\). If \(r = 1\) then by an earlier reduction \(g = g_{0,1} \cong sl(2)\) and the result is trivial in this case. Now assume the theorem for \(r - 1 \geq 1\). If \(k = 0\) there is nothing to prove so assume the theorem for \(k - 1 \geq 0\). We now prove the result for \(k\).

As above we may assume \(g = g_{0,r}\). By identity (2.11) we may assume \(n_r = 0\). If \(n_r - 1 = 0\) as well we restrict our attention to \(g_{0,r-1}\). By the induction hypothesis the theorem holds for \(g_{0,r-1}\) so we obtain by Lemma 2 the inclusion

\[
S(p^-)[n_1, \ldots, n_r] \quad S(p^-)[k, 0, \ldots, 0] \supset \bigoplus_{m \in I_k(n)} S(p^-)[m].
\]

Now we prove equality here. Suppose not. Then by Lemma 3, there is an index \(m \notin I_k(n)\) with \(m_r \geq 1\) whose \(K\)-type contains a vector \(z\) occurring on the left side. By Lemma 4, \(z\) is an eigenvector for \(u_rD\) with nonzero eigenvalue. However by Lemma 5 and especially the cyclicity of \(vX_{-\gamma_r}\), \(D\) acts by zero on the left side of (2.1) in Theorem 1, which is a contradiction. This proves the case \(n_r - 1 = n_r = 0\).

Now assume \(n_r - 1 > 0, n_r = 0\). Set \(m' = (n_1 - 1, n_2 - 1, \ldots, n_{r-2} - 1, n_{r-1} - 1, 0)\). By Lemma 5 we obtain:

\[
(2.17) \quad D(S(p^-)[n]S(p^-)[k, 0, \ldots, 0]) = S(p^-)[m']S(p^-)[k - 1, 0, \ldots, 0].
\]
Also by the induction hypothesis for $k$,

\[(2.18) \quad S(p)_{[m']} S(p_{k - 1, 0, \ldots, 0}] \cong \sum_{m \in I_{k - 1}(m')} S(p)[m].\]

Now combining these identities and multiplying by $u_r$, we conclude:

\[(2.19) \quad S(p)_{[n]} S(p_{k, 0, \ldots, 0}] \supset u_r D(S(p)_{[n]} S(p_{k, 0, \ldots, 0}]) \cong \bigoplus_{m \in I_k(n), m_r \geq 0} S(p)[m].\]

Since $n_r = 0$ we can restrict to $g_{0, r - 1}$. From Lemma 2 we have:

\[(2.20) \quad S(p)_{[n]} S(p_{k, 0, \ldots, 0}] \supset \bigoplus_{m \in I_k(n)} S(p)[m].\]

Combining these two inclusions we get:

\[(2.21) \quad S(p)_{[n]} S(p_{k, 0, \ldots, 0}] \supset \bigoplus_{m \in I_k(n)} S(p)[m].\]

We now prove equality. Suppose not and choose $m$ with the corresponding $K$-type containing a vector $z$ occurring on the left but not on the right in (2.1) in Theorem 1. Applying Lemma 3 we find $m_r \geq 1$. This inequality implies that $u_r Dz$ is a nonzero multiple of $z$. So $z$ is contained in the $K$-representation generated by $D(vX^k_{\gamma_r})$. By (2.19) $m \in I_k(n)$, a contradiction. This completes the proof of the theorem.

3. Connections with the classical Pieri rule

For the classical cases of $Sp(n, \mathbb{R}), SO^*(2n)$ and $U(p, q)$, the decomposition of products given by Theorem 1 is related to the Pieri rule for $Gl(n)$ for the first and third cases and to the Littlewood–Richardson rule for $Gl(n)$ in the case of $SO^*(2n)$. We handle each case separately and keep in place the earlier notation. For these classical cases we use the standard Euclidean coordinates $e_i$ as in Bourbaki.

Case 1, $Sp(n, \mathbb{R})$. Here $\mathfrak{g} \cong u(n)$ and for $1 \leq i \leq n$, $\gamma_i = 2e_{n+1-i}$. Set $E = S(p^*)_{[m_1, \ldots, m_n]}$ and $F = S(p^-)[k, 0, \ldots, 0]$. Let $^*$ denote the dual representation. Then $E^*$ and $F^*$ have highest weights $(2m_1, \ldots, 2m_n)$ and $(2k, 0, 0, \ldots, 0)$ respectively. Then applying the Pieri rule, we obtain:

\[E^* \otimes F^* \cong \sum E_b,\]

where the sum is over all $n$-tuples $b = (b_1, b_2, \ldots, b_n)$ with $b_1 \geq 2m_1 \geq b_2 \geq 2m_2 \geq \cdots \geq b_n \geq 2m_n$ and $\sum b_i - 2m_i = 2k$. From Schmid’s result combined with this identity we obtain the inclusion in $S(p^-)$:

\[E \cdot F \subset \sum E_b,\]
where the sum is over all $n$-tuples $b = (b_1, b_2, \ldots, b_n)$ with even entries, $b_1 \geq 2m_1 \geq b_2 \geq 2m_2 \geq \cdots \geq b_n \geq 2m_n$ and $\sum b_i - 2m_i = 2k$. Applying Theorem 1 in this setting we conclude that the inclusion is an equality. So the multiplicative product is as large as it could be.

Case 2, $SO^\ast(2n)$. Here $\mathfrak{t} \cong u(n)$ and for $1 \leq i \leq r = \left\lfloor \frac{n}{2} \right\rfloor$, $\gamma_i = e_{n+1-2i} + e_{n+2-2i}$. Set $E = S(p^-)[m_1, m_2, \ldots, m_r]$ and $F = S(p^-)[k, 0, \ldots, 0]$. Then $E^*$ and $F^*$ have highest weights $(m_1, m_1, m_2, m_2, \ldots)$ and $(k, k, 0, \ldots, 0)$ respectively. The Littlewood–Richardson rule $[\mathcal{F}]$ or $[\mathcal{M}]$ gives the decomposition of $E^* \otimes F^*$.

$$E^* \otimes F^* \cong \sum m_b E_b,$$

where the multiplicity of $E_b$, denoted $m_b$, equals the number of ways the diagram $m = (m_1, m_1, m_2, m_2, \ldots)$ can be expanded to the diagram $b$ by a strict $(k, k, 0, \ldots, 0)$-expansion. Such an expansion is determined by augmenting the diagram $m$ by adding $k$ boxes each labeled with a 1 and then adding $k$ boxes each labeled with a 2 so that the following three conditions hold:

(i) The labels in each row of are nondecreasing,

(ii) the labels in each column of are strictly increasing and

(iii) if we read the labels from top right to bottom left at every stage the number 1 must have occurred at least as many times as the number 2.

As an example, suppose $m = (3, 3, 1, 1)$ and $k = (2, 2, 0, 0)$. Then all the multiplicities are one and the $\mathfrak{t}$-modules which occur are: $(5, 5, 1, 1) (5, 4, 2, 1) (5, 3, 3, 1) (4, 4, 2, 2) (4, 3, 3, 2) (3, 3, 3, 3)$.

From Schmid’s theorem, the $\mathfrak{t}$-modules in $S(p^-)$ have duals with highest weights whose diagrams have columns of even length. Since $m$ has columns of even length, the rules (i), (ii), and (iii) imply that the only placements of $k1$’s and $k2$’s which yield a diagram with columns of even length will have the form $b = (b_1, b_1, b_2, b_2, \ldots)$ with $\sum b_i - m_i = k$ and $b_1 \geq m_1 \geq b_2 \geq m_2 \geq \cdots \geq b_r \geq m_r$. So here the Littlewood–Richardson rule and Schmid’s Theorem combine to give the inclusion:

$$E^* \cdot F^* \subset \sum E_b,$$

where the sum is over all $b$ with $b = (b_1, b_1, b_2, b_2, \ldots)$, $\sum b_i - m_i = k$ and $b_1 \geq m_1 \geq b_2 \geq m_2 \geq \cdots \geq b_r \geq m_r$. Our Theorem 1 asserts that this inclusion is equality.

Case 3, $U(p, q)$. Set $n = p + q$ and assume $p \leq q$. Here $\mathfrak{t} \cong u(p) \oplus u(q)$ and for $1 \leq i \leq p$, $\gamma_i = e_{p+1-i} - e_{p+i}$. Set $E = S(p^-)[m_1, m_2, \ldots, m_p]$ and $F = S(p^-)[k, 0, \ldots, 0]$. Since $\mathfrak{t}$ is a product of ideals $\mathfrak{t} = \mathfrak{t}^{(1)} \oplus \mathfrak{t}^{(2)}$ and every irreducible $\mathfrak{t}$-module splits into a tensor product. We write $E \cong E^{(1)*} \otimes E^{(2)}, F \cong F^{(1)*} \otimes F^{(2)}$ where the highest weights of $E^{(1)}, E^{(2)}, F^{(1)}, F^{(2)}$ are
respectively:

\[ m^{(1)} = (m_1, m_2, \ldots, m_p), \quad m^{(2)} = (m_1, m_2, \ldots, m_p, 0, \ldots, 0), \]

\[ (k, 0, \ldots, 0), (k, 0, \ldots, 0). \]

Here the first and third are \( p \)-tuples and the second and fourth are \( q \)-tuples.

Applying the Pieri rule for each factor we obtain:

\[ (E^{(1)} \otimes E^{(2)}) \otimes (F^{(1)} \otimes F^{(2)}) \cong \sum E^{(1)}_b \otimes E^{(2)}_c, \]

where the sum is over all \( p \)-tuples \( b \) and all \( q \)-tuples \( c \) which satisfy the conditions:

\[ \sum_{1 \leq i \leq p} b_i - m_i = k, \quad b_1 \geq m_1 \geq b_2 \geq m_2 \geq \cdots \geq b_p \geq m_p, \quad \text{and} \]

\[ \sum_{1 \leq i \leq q} c_i - m_i = k, \quad c_1 \geq m_1 \geq c_2 \geq m_2 \geq \cdots \geq c_p \geq m_p \geq c_{p+1} \geq 0, \]

with \( 0 = c_{p+2} = \cdots = c_q \).

By Schmid’s Theorem the only summands which occur in \( S(p^-) \) are those \( E_b^{(1)*} \otimes E_c^{(2)*} \) for which \( b_i = c_i, \ 1 \leq i \leq p \) and \( c_i = 0, \ p+1 \leq i \leq q \). So taking the product in \( S(p^-) \) we obtain the inclusion:

\[ (E^{(1)*} \otimes E^{(2)*}) \cdot (F^{(1)*} \otimes F^{(2)*}) \subset \sum E^{(1)*}_b \otimes E^{(2)*}_c, \]

where \( \sum_{1 \leq i \leq p} b_i - m_i = k, \ b_1 \geq m_1 \geq b_2 \geq m_2 \geq \cdots \geq b_p \geq m_p, \ b_i = c_i, \ 1 \leq i \leq p \) and \( c_i = 0, \ p+1 \leq i \leq q \). Finally as before Theorem 1 asserts the inclusion is equality.

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RANDOM ATTRACTOR FOR A DAMPED SINE-GORDON EQUATION WITH WHITE NOISE

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We prove the existence of a compact random attractor for the random dynamical system generated by a damped sine-Gordon with white noise. And we obtain a precise estimate of the upper bound of the Hausdorff dimension of the random attractor, which decreases as the damping grows and shows that the dimension is uniformly bounded for the damping. In particular, under certain conditions, the dimension is zero.

1. Introduction

This article is devoted to the existence and estimate of Hausdorff dimension of the random attractor for a damped sine-Gordon equation with homogeneous Dirichlet boundary condition when there is a random term.

Let $\Omega$ be an open bounded set of $\mathbb{R}^n$ with a smooth boundary $\partial \Omega$. We consider the equation

\begin{equation}
\begin{aligned}
&u_{tt} + \alpha u_t - \Delta u + \beta \sin u = q(x)\dot{W}, \quad \text{in } \Omega \times [\tau, +\infty), \quad \tau \in \mathbb{R}, \\
&u(x, t)|_{x \in \partial \Omega} = 0, \quad t \geq \tau, \\
&u(x, \tau) = u_0(x) \in H_0^1(\Omega), \quad u_t(x, \tau) = u_1(x) \in L^2(\Omega).
\end{aligned}
\end{equation}

where $u = u(x, t)$ is a real-valued function on $\Omega \times [\tau, +\infty)$, for $\tau \in \mathbb{R}$; $\alpha > 0$ is called the damping; $q(x) \in H^2(\Omega) \cap H_0^1(\Omega)$, $\dot{W}(t)$ is the derivative of a one-dimensional two-sided Wiener process $W(t)$; and $q(x)\dot{W}$ formally describes white noise.

A random attractor of a random dynamical system (RDS) is a measurable and compact invariant random set attracting all the orbits. When such an attracting exists, it is the smallest attracting compact set and the largest invariant set [3]. This seems to be a good generalization of the now classical concept of a global attractor for deterministic dynamical systems. The notion of a random attractor is very useful for many infinite-dimensional random dynamical systems [4, 3].

Many authors [8, 10] have studied and estimated the Hausdorff dimension of the global attractor for a deterministic damped sine-Gordon equation. H. Crauel and F. Flandoli [4] introduced the notion of a random attractor and obtained a general theorem on the existence of a random attractor for an
RDS. Their theorem has been successfully applied to the stochastic reaction-diffusion equation and the stochastic Navier–Stokes equation. H. Crauel et al. [3] generalized the notion of an attractor for the stochastic dynamical system introduced previously and considered a stochastic nonlinear wave equation. A. Debussche [6, 7] provided a general way to obtain the Hausdorff dimension of a random invariant set or random attractor and applied it to the random attractor for a stochastic reaction-diffusion equation. In this paper, we use the notion and framework in [6, 7, 3] to study a damped sine-Gordon equation with white noise. We prove the existence of the random attractor of the equation and estimate its Hausdorff dimension. The upper bound of the Hausdorff dimension decreases as the damping grows, and the dimension is uniformly bounded for the damping. In particular, under certain conditions, the dimension is zero.

2. Random dynamical systems; existence and uniqueness of solutions

Let $(\Theta, F, P)$ be a probability space and $\{\theta_t: \Theta \rightarrow \Theta, t \in \mathbb{R}\}$ a family of measure preserving transformations such that $(t, \omega) \mapsto \theta_t \omega$ is measurable, $\theta_0 = \text{id}$, and $\theta_{t+s} = \theta_t \theta_s$ for all $s, t \in \mathbb{R}$. The flow $\theta_t$ together with the probability space $(\Theta, F, P, (\theta_t)_{t \in \mathbb{R}})$ is called as a (measurable) dynamical system.

An RDS is continuous or differentiable if $\phi(t, \omega): X \rightarrow X$ is continuous or differentiable.

A map $D : \Theta \rightarrow 2^X$ is said to be a closed (compact) random set if $D(\omega)$ is closed (compact) for $P$-a.s. $\omega \in \Theta$ and if $\omega \mapsto d(x, D(\omega))$ is $P$-a.s. measurable for all $x \in X$.

It is well-known that the operator $A = -\Delta : D(A) = H^1_0(\Omega) \cap H^2(\Omega) \rightarrow L^2(\Omega)$ is self-adjoint, positive and linear, and its eigenvalues $\{\lambda_i\}_{i \in \mathbb{N}}$ (with $\lambda_i \leq \lambda_j$ for $i < j$) are positive and satisfy $\lambda_m \rightarrow +\infty$ as $m \rightarrow +\infty$. Consider $L^2(\Omega)$, $H^1_0(\Omega)$ and $E = H^1_0(\Omega) \times L^2(\Omega)$ with the usual inner products and norms:

$$(u, v) = \int_{\Omega} uv \, dx, \quad \|u\|_0 = (u, u)^{1/2} \quad \text{for all } u, v \in L^2(\Omega),$$
\[ (u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx, \quad \|u\| = ((u, u))^{1/2} \text{ for all } u, v \in H^1_0(\Omega), \]
\[ (y_1, y_2)_E = ((u_1, u_2)) + (v_1, v_2), \quad \|y\|_E = (y, y)_E^{1/2}, \]
for all \( y_i = (u_i, v_i)^T, \quad y = (u, v)^T \in E, \quad i = 1, 2. \)

It is convenient to reduce (1) to an evolution equation of the first-order in time:

\[
\begin{cases}
  u_t = v, \\
  v_t = -\alpha v + \Delta u - \beta \sin u + q(x) \dot{W}, \\
  u(x, \tau) = u_0(x), \quad v(x, \tau) = u_1(x), \quad x \in \Omega,
\end{cases}
\]

whose equivalent Itô equation is

\[
\begin{cases}
  du = v \, dt, \\
  dv = -\alpha v \, dt + \Delta u \, dt - \beta \sin u \, dt + q(x) \, dW, \\
  u(x, \tau) = u_0(x), \quad v(x, \tau) = u_1(x), \quad x \in \Omega.
\end{cases}
\]

\( W(t) \) is a one-dimensional two-sided Wiener process with path \( \omega(\cdot) \) in the space \( C(R, R) \) of continuous functions on \( R \), \( \omega(0) = 0 \). We can define a family of measure-preserving and ergodic transformations (a flow) \( \{\theta_t\}_{t \in \mathbb{R}} \) by

\[ \theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t). \]

Let \( z = v - q(x)W \), then \( v = z + q(x)W \). We consider the random partial differential equation equivalent to (3)

\[
\begin{cases}
  \frac{du}{dt} = z + q(x)W, \\
  \frac{dz}{dt} = -\alpha z + \Delta u - \beta \sin u - \alpha q(x)W, \\
  u(x, \tau) = u_0(x), \quad z(x, \tau, \omega) = z(x, \tau, \omega) = u_1(x) - q(x)W(\tau), \quad x \in \Omega.
\end{cases}
\]

In contrast to the stochastic differential equation (2), no stochastic differential appears here. Let

\[ \varphi = \begin{pmatrix} u \\ z \end{pmatrix}, \quad L = \begin{pmatrix} 0 & I \\ -A & -\alpha I \end{pmatrix}, \quad F(\varphi, \omega) = \begin{pmatrix} q(x)W \\ -\beta \sin u - \alpha q(x)W \end{pmatrix}. \]

Then (4) can be written as

\[
\dot{\varphi} = L \varphi + F(\varphi, \omega), \quad \varphi(\tau, \omega) = (u_0, z(\tau, \omega))^T.
\]

We know that \( L \) is the infinitesimal generator of a \( C_0 \)-semigroup \( e^{Lt} \) on \( E \) with the exponential dichotomy from [9]. It is easy to check that the function \( F(\cdot, \omega) : E \rightarrow E \) is globally Lipschitz continuous with respect to \( \varphi \) and bounded for every \( \omega \in \Theta \). By the classical semigroup theory of existence and uniqueness of the solutions of evolution differential equations [9], the unique solution of (5) can be interpreted in a mild sense:

\[ \varphi(t, \omega) = e^{L(t-\tau)} \varphi(\tau, \omega) + \int_{\tau}^{t} e^{L(t-s)} F(\varphi(s), \omega) \, ds, \]
surely for any \( \varphi(\tau, \omega) \in E \). One can show that for \( P \)-a.s. every \( \omega \in \Theta \) the following statements hold for all \( T > 0 \):

(i) If \( \varphi(\tau, \omega) \in E \) then \( \varphi(t, \omega) \) lies in

\[
C([\tau, \tau + T]; H^1_0(\Omega)) \times C([\tau, \tau + T]; L^2(\Omega)).
\]

(ii) \( \varphi(t, \varphi(\tau, \omega)) \) is jointly continuous in \( t \) and \( \varphi(\tau, \omega) \).

(iii) The solution mapping of (5) satisfies the properties of an RDS.

This equation has a unique solution for every \( \omega \in \Theta \). No exceptional sets appear. Hence the solution mapping

\[
\hat{S}(t, \omega) : \varphi(\tau, \omega) \mapsto \varphi(t, \omega)
\]
generates a random dynamical system. So the transformation

\[
S(t, \omega) : \varphi(\tau, \omega) + (0, q(x)W(\tau))^T \mapsto \varphi(t, \omega) + (0, q(x)W(t))^T
\]
also determines a random dynamical system corresponding to problem (2).

We will prove the existence of a nonempty compact random attractor for the random dynamical system \( S(t, \omega) \) and estimate the Hausdorff dimension of the random attractor.

### 3. Existence of a random attractor

A random set \( K(\omega) \) is said to absorb the set \( B \subset X \) for an RDS \( \varphi \) if \( P \)-a.s. there exists \( t_B(\omega) \) such that

\[
\varphi(t, \theta_{-t}(\omega))B \subset K(\omega) \quad \text{for all } t \geq t_B(\omega).
\]

A random set \( A(\omega) \) is said to be a random attractor associated to the RDS \( \varphi \) if \( P \)-a.s.:

(i) \( A(\omega) \) is a random compact set, that is, \( P \)-a.s. \( \omega \in \Theta, A(\omega) \) is compact and for all \( x \in X \) and \( P \)-a.s. the map \( x \mapsto \text{dis}(x, A(\omega)) \) is measurable.

(ii) \( \varphi(t, \omega)A(\omega) = A(\theta_t(\omega)) \) for all \( t \geq 0 \) (invariance).

(iii) For all bounded (and nonrandom) \( B \subset X \),

\[
\lim_{t \to \infty} \text{dis} (\varphi(t, \theta_{-t}(\omega))B, A(\omega)) = 0
\]

where \( \text{dis}(\cdot, \cdot) \) denotes the Hausdorff semidistance:

\[
\text{dis}(A, B) = \sup_{x \in A} \inf_{y \in B} d(x, y), \quad A, B \in X.
\]

Note that \( \varphi(t, \theta_{-t}(\omega))x \) can be interpreted as the position at \( t = 0 \) of the trajectory which was in \( x \) at time \( -t \). Thus, the attraction property holds from \( t = -\infty \).
**Theorem 1** (Existence of a random attractor). Let $\phi$ be an RDS on a Polish space $(X,d)$ with Borel $\sigma$-algebra $\mathcal{B}$ over the flow $\{\theta_t\}_{t \in \mathbb{R}}$ on a probability space $(\Theta, F, P)$. Suppose there exists a random compact set $K(\omega)$ such that for any bounded nonrandom set $B \subset X$ $P$-a.s.

\begin{equation}
\text{dis}(\varphi(t, \theta_{-t}\omega)B, K(\omega)) \to 0 \quad \text{as} \quad t \to +\infty.
\end{equation}

Then the set

\[ A(\omega) = \bigcup_{B \subset X} \Lambda(\omega) \]

is a random attractor for $\phi$, where the union is taken over all bounded $B \subset X$, and $\Lambda_B(\omega)$ is the omega-limit set of $B$ given by

\[ \Lambda_B(\omega) = \bigcap_{s \geq 0} \bigcup_{t \geq s} \phi(t, \theta_{-t}\omega)B. \]

Moreover, the random attractor is unique.

**Proof.** Since $K(\omega)$ is a random compact set, then by (8), $\Lambda_B(\omega)$ is also random compact and nonempty. By the proof of Theorem 3.11 in [4], $A(\omega)$ is a random attractor for $\phi$ and it is unique.

**Remark 1.** Theorem 1 can be regarded as an analog of [2, Theorem 2.2]. As in [2], the RDS $\phi$ can be also said to be uniformly asymptotically compact.

We show the existence of a random attractor for the RDS (7) in the space $E$. Let $\psi = (u, z)^T$, $z = z + \varepsilon u$, where

\begin{equation}
\varepsilon = \frac{\alpha \lambda_1}{2 \alpha^2 + 3 \lambda_1}.
\end{equation}

Then the system (4) can be written as

\begin{equation}
\psi_t + Q\psi = F(\psi, \omega), \quad \psi(\tau, \omega) = (u_0, z(\tau, \omega) + \varepsilon u_0)^T, \quad t \geq \tau,
\end{equation}

where

\[ Q = \begin{pmatrix}
\varepsilon I & -I \\
A - \varepsilon(\alpha - \varepsilon)I & (\alpha - \varepsilon)I
\end{pmatrix},
\]

\[ F(\psi, \omega) = \begin{pmatrix}
q(x)W \\
-\beta \sin u - (\alpha - \varepsilon)q(x)W
\end{pmatrix}.
\]

The mapping

\begin{equation}
\hat{S}_\varepsilon(t, \omega) : (u_0, z(\tau, \omega) + \varepsilon u_0)^T \mapsto (u(t), z(t) + \varepsilon u(t))^T, \quad E \mapsto E, \quad t \geq \tau
\end{equation}

defined by (10) has the following relation with $\hat{S}(t, \omega)$:

\begin{equation}
\hat{S}_\varepsilon(t, \omega) = R_\varepsilon \hat{S}(t, \omega) R_{-\varepsilon}
\end{equation}
where \( R_\varepsilon : (u, z)^T \mapsto (u, z + \varepsilon u)^T \) is an isomorphism of \( E \). So, for the RDS (7) we only need consider the equivalent random dynamical system 
\[ S_\varepsilon(t, \omega) = R_\varepsilon S(t, \omega) R_{-\varepsilon}, \]
where \( S_\varepsilon(t, \omega) \) is decided by
\[ \dot{\vartheta}_t + Q \vartheta = G(\vartheta, \omega), \quad \vartheta(\tau) = (u_0, u_1 + \varepsilon u_0)^T, \quad t \geq \tau, \]
where \( \dot{\vartheta}(t) = (u(t), u(t) + \varepsilon u(t))^T \) and
\[ G(\vartheta, \omega) = \begin{pmatrix} 0 \\ -\beta \sin u + q(x) \dot{W} \end{pmatrix}. \]

First we present a positivity property of the operator \( Q \) in \( E \) that plays an important role in this article.

**Lemma 1.** For any \( \varphi = (u, v)^T \in E \),
\[ (Q \varphi, \varphi)_E \geq \frac{\varepsilon}{2} \|\varphi\|_E^2 + \frac{\varepsilon}{4} \|u\|^2 + \frac{\alpha}{2} \|v\|^2. \]

**Proof.** This is easily obtained after simple computations.

**Lemma 2.** There exist a random variable \( r_1(\omega) > 0 \) and a bounded ball \( B_0 \) of \( E \) centered at 0 with random radius \( r_0(\omega) \) such that for any bounded nonrandom set \( B \) of \( E \), there exists a deterministic \( T(B) \leq -1 \) such that the solution \( \psi(t, \omega; \psi(\tau, \omega)) = (u(t, \omega), v(t, \omega))^T \) of (10) with initial value \( (u_0, u_1 + \varepsilon u_0)^T \in B \) satisfies for \( \Theta \)-a.s. \( \omega \in \Theta \)
\[ \|\psi(-1, \omega; \psi(\tau, \omega))\|_E \leq r_0(\omega), \quad \tau \leq T(B), \]
and for \( \tau \leq t \leq 0 \),
\[ (14) \quad \|\psi(t, \omega; \psi(\tau, \omega))\|_E^2 \leq 2 \left( e^{-\varepsilon(t-\tau)} (\|u_0\|^2 + \|u_1 + \varepsilon u_0\|^2 + \|q\|^2 \|W(\tau)\|^2) + r_1^2(\omega) \right), \]
where \( \bar{z}(t, w) = u_t(t) + \varepsilon u(t) - q(x) W(t) \).

Of course one can deduce a similar absorption result for
\[ \vartheta(-1) = (u(-1), u(-1) + \varepsilon u(-1))^T \]
instead of \( \psi(-1) \).

**Proof.** Taking the inner product \( (\cdot, \cdot)_E \) of (10) with \( \psi = (u, v)^T \), in which \( v = u_t + \varepsilon u - qW \), we obtain
\[ (15) \quad \frac{1}{2} \frac{d}{dt} \|\psi\|_E^2 + (Q \psi, \psi)_E \]
\[ = (-\beta \sin u, v) - (\alpha - \varepsilon)(q(x), v) W(t) + ((q(x), u)) W(t), \quad t \geq \tau. \]

By Young’s inequality and Lemma 1
\[ \frac{d}{dt} \|\psi\|_E^2 + \varepsilon \|\psi\|_E^2 \leq 2 \left( \frac{\beta^2}{\alpha} + \alpha \|q\|^2 \|W(t)\|^2 + \frac{\|q\|^2}{\varepsilon} \|W(t)\|^2 \right), \quad t \geq \tau. \]
By the Gronwall lemma

\begin{equation}
\| \psi(t, \omega; \psi(\tau, \omega)) \|_E^2 \\
\leq e^{-\varepsilon(t-\tau)} \| \psi(\tau, \omega) \|_E \\
+ 2 \int_{\tau}^{t} e^{-\varepsilon(t-s)} \left( \frac{\beta^2}{\alpha} + \alpha \| q \|_0^2 \| W(s) \|^2 + \frac{\| q \|_0^2}{\varepsilon} \| W(s) \|^2 \right) ds \\
\leq 2e^{-\varepsilon(t-\tau)} (\| u_0 \|^2 + \| u_1 + \varepsilon u_0 \|_0^2 + \| q \|_0^2 \| W(\tau) \|^2) \\
+ 2 \int_{\tau}^{t} e^{-\varepsilon(t-s)} \left( \frac{\beta^2}{\alpha} + \left( \alpha \| q \|_0^2 + \frac{\| q \|_0^2}{\varepsilon} \right) \| W(s) \|^2 \right) ds.
\end{equation}

Put

\[ r_0^2(\omega) = 2 \left( 1 + \frac{\beta^2}{\varepsilon \alpha} + \sup_{\tau \leq 1} e^{\varepsilon s} \| q \|_0^2 \| W(\tau) \|^2 \\
\left. \right. + \left( \alpha \| q \|_0^2 + \frac{\| q \|_0^2}{\varepsilon} \right) \int_{-\infty}^{-1} e^{-\varepsilon(-1-s)} \| W(s) \|^2 ds \right) \]

and

\[ r_1^2(\omega) = \frac{\beta^2}{\varepsilon \alpha} + \left( \alpha \| q \|_0^2 + \frac{\| q \|_0^2}{\varepsilon} \right) \int_{-\infty}^{0} e^{\varepsilon s} \| W(s) \|^2 ds. \]

Since \( \lim_{t \to -\infty} W(t)/t = 0 \), the quantities \( r_0^2(\omega) \) and \( r_1^2(\omega) \) are finite \( P \)-a.s.

Given a bounded set \( B \) of \( E \), choose \( T(B) \leq -1 \) such that

\[ e^{-\varepsilon(-1-\tau)} \left( \| u_0 \|^2 + \| u_1 + \varepsilon u_0 \|_0^2 \right) \leq 1 \quad \text{for all} \quad (u_0, u_1 + \varepsilon u_0)^T \in B \]

and

\begin{equation}
-\tau e^{\varepsilon \tau} \left( \| u_0 \|^2 + \| u_1 + \varepsilon u_0 \|_0^2 \right) \leq 1 \quad \text{for all} \quad (u_0, u_1 + \varepsilon u_0)^T \in B \quad \text{for all} \quad \tau \leq T(B). \end{equation}

The proof is completed from (16).

Let \( u(t) \) be a solution of system (1) with initial value \((u_0, u_1 + \varepsilon u_0)^T \in B \).

We make the decomposition \( u(t) = y_1(t) + y_2(t) \), where \( y_1 \) and \( y_2 \) satisfy

\begin{equation}
\begin{cases}
y_{1tt} + \alpha y_{1t} - \Delta y_1 = 0 & \text{in} \quad \Omega \times [\tau, +\infty), \\
y_1(x, t) |_{x \in \partial \Omega} = 0, & \quad t \geq \tau, \\
y_1(\omega, \tau) = u_0(x), & \quad y_{1t}(x, \tau) = u_1(x), \quad x \in \Omega,
\end{cases}
\end{equation}

and

\begin{equation}
\begin{cases}
y_{2tt} + \alpha y_{2t} - \Delta y_2 + \beta \sin u = q(x) \dot{W} & \text{in} \quad \Omega \times [\tau, +\infty), \\
y_2(x, t) |_{x \in \partial \Omega} = 0, & \quad t \geq \tau, \\
y_2(x, \tau) = y_{2t}(x, \tau) = 0, & \quad x \in \Omega.
\end{cases}
\end{equation}

**Lemma 3.** Let \( B \) be a bounded nonrandom subset of \( E \). We have, for any \((u_0, u_1 + \varepsilon u_0)^T \in B \),

\begin{equation}
\| Y_1(0) \|^2_E = \| y_1(0) \|^2 + \| y_{1t}(0) + \varepsilon y_1(0) \|_0^2 \leq (\| u_0 \|^2 + \| u_0 + \varepsilon u_1 \|_0^2) e^{\alpha \tau},
\end{equation}
and there exists a random radius $r_2(\omega)$ such that for $P$-a.s. $\omega \in \Theta$,
\begin{equation}
\|A^{1/2}Y_2(0, \omega; Y_2(\tau, \omega))\|_E^2 \leq r_2^2(\omega),
\end{equation}
where $Y_1 = (y_1, y_{1t} + \varepsilon y_1)^T$ and $Y_2 = (y_2, y_{2t} + \varepsilon y_2 - q(x)W)^T$ satisfy (18), (19).

**Proof.** Take the inner product of (18) in $L^2(\Omega)$ with $y_{1t} + \varepsilon y_1$ whose initial value is $(u_0, u_1 + \varepsilon u_0)^T$. After a simple computation using Lemma 1, one obtains (20).

Set $Y_2 = (y_2, y_{2t} + \varepsilon y_2 - q(x)W)^T$. Equation (19) can be reduced to
\begin{equation}
Y_{2t} + QY_2 = H(Y_2, \omega), \quad Y_2(\tau) = (0, -q(x)W(\tau)), \quad t \geq \tau
\end{equation}
where
\[
H(Y_2, \omega) = \begin{pmatrix} q(x)W(t) \\ -\beta \sin u - (\alpha - \varepsilon)q(x)W(t) \end{pmatrix}.
\]

Taking the inner product of (22) in $E$ with $AY_2$ and using Young’s inequality, Lemma 1 and (14), we obtain for $t \leq T(B)$
\[
\frac{d}{dt} \|A^{1/2}Y_2\|_E^2 + \varepsilon \|A^{1/2}Y_2\|_E^2 \\
\leq \frac{4\beta^2}{\alpha} \left( e^{-\varepsilon(t-\tau)} \left( \|u_0\|^2 + \|u_1 + \varepsilon u_0\|^2_0 + \|q\|^2_0|W(\tau)|^2 \right) + r_1^2(\omega) \right)
\]
\[
+ 2 \left( \alpha \|q\|^2 + \|Aq\|_0^2 \varepsilon \right) |W(t)|^2, \quad \tau \leq t \leq 0.
\]

By the Gronwall lemma,
\begin{equation}
\|A^{1/2}Y_2(0, \omega; Y_2(\tau, \omega))\|_E^2 \\
\leq \frac{4\beta^2}{\varepsilon \alpha} \left( (-\tau)e^{\varepsilon\tau} \left( \|u_0\|^2 + \|u_1 + \varepsilon u_0\|^2_0 + \|q\|^2_0|W(\tau)|^2 \right) + r_1^2(\omega) \right)
\]
\[
+ e^{\varepsilon\tau}\|q\|^2|W(\tau)|^2 + 2 \left( \alpha \|q\|^2 + \|Aq\|_0^2 \varepsilon \right) \int_\tau^0 e^{\varepsilon s}|W(s)|^2 ds.
\]

Put
\[
r_2^2(\omega) = \frac{4\beta^2}{\varepsilon \alpha} \left( 1 + \|q\|^2_0 \sup_{\tau \leq 0} (-\tau)e^{\varepsilon\tau}|W(\tau)|^2 + r_1^2(\omega) \right)
\]
\[
+ \|q\|^2 \sup_{\tau \leq 0} e^{\varepsilon\tau}|W(\tau)|^2 + 2 \left( \alpha \|q\|^2 + \|Aq\|_0^2 \varepsilon \right) \int_{-\infty}^0 e^{\varepsilon s}|W(s)|^2 ds.
\]

Since $\lim_{t \to \infty} W(t)/t = 0$, the quantity $r_2^2(\omega)$ is finite $P$-a.s. By (17) and (23), we have
\[
\|A^{1/2}Y_2(0, \omega; Y_2(\tau, \omega))\|_E^2 \leq r_2^2(\omega) \quad \text{for all } (u_0, u_1 + \varepsilon u_0)^T \in B, \quad \tau \leq T(B).
\]
The proof is complete.
Let $B_{1/2}(w)$ be the ball of $D(A) \times H^1_0(\Omega)$ of radius $r_1(\omega)$. From the compact embedding $D(A) \times H^1_0(\Omega) \hookrightarrow E$ we see that $B_{1/2}(w)$ is compact. For every bounded nonrandom set $B$ of $E$, pick any $\psi(0) \in \hat{S}_\varepsilon(t, \theta_{-\varepsilon})B$. From Lemma 3, we have $Y_2(0) = \psi(0) - Y_1(0) \in B_{1/2}(w)$ where $Y_2(t, \omega)$ is given by (22). Therefore, again by Lemma 3,

$$\inf_{\ell(0) \in B_{1/2}(w)} \|\psi(0) - \ell(0)\|_E^2 \leq \|Y_1(0)\|_E^2 \leq (\|u_0\|^2 + \|u_0 + \varepsilon u_1\|^2_0) e^{\varepsilon \tau}, \quad \tau \leq 0.$$ 

So $\text{dis}(\hat{S}_\varepsilon(t, \theta_{-\varepsilon})B, B_{1/2}(w)) \leq (\|u_0\|^2 + \|u_0 + \varepsilon u_1\|^2_0) e^{-\varepsilon t}$, for all $t \geq 0$.

From the relation (12) between $S_\varepsilon(t, \omega)$ and $\hat{S}_\varepsilon(t, \omega)$, one can easily obtain that for any nonrandom bounded $B \subset E$ P-a.s.

$$\text{dis}(S_\varepsilon(t, \theta_{-\varepsilon})B, B_{1/2}(w)) \to 0 \quad \text{as} \quad t \to +\infty.$$ 

**Corollary 1.** The RDS $S_\varepsilon(t, \omega)$ associated with (7) possesses a uniformly attracting compact set $B_{1/2}(w) \subset E$. So the RDS $S_\varepsilon(t, \omega)$ is uniformly asymptotically compact in $E$.

**Theorem 2.** The RDS $S_\varepsilon(t, \omega)$ has a nonempty compact random attractor $A(\omega)$.

**Proof.** This follows from Lemmas 2 and 3 and Corollary 1.

## 4. Hausdorff dimension of the random attractor

To bound the attractor’s dimension we use the following result of Debussche [6, 7]. He treats the case of a random attractor $A(\omega)$ invariant under a random map $S(t, \omega)$: for some measure-preserving ergodic transformation $\theta$ on $(\Theta, \mathcal{F}, \mathcal{P})$ we have

$$S(\omega)A(\omega) = A(\theta \omega).$$

One must make some assumptions about the map $S(t, \omega)$: first, we need $S(\omega)$ to be almost surely uniformly differentiable on $A(\omega)$, which means that $P$-almost surely, for every $u \in A(\omega)$, there exists a bounded linear operator $DS(u, \omega) : X \to X$ such that $u + h \in A(\omega)$ implies

$$|S(\omega)(u + h) - S(\omega)u - DS(u, \omega)h| \leq K(\omega)|h|^{1+\delta},$$

where $\delta > 0$ and $K(\omega)$ is a random variable with $K(\omega) \geq 1$ and $E(\log K) < \infty$. Given a bounded linear operator $L$ on $X$ and $n \in N$, we set

$$\alpha_n(L) = \sup_{G \subset X} \inf_{\dim G \leq n} \sup_{|\phi| = 1} |L\phi|$$

and

$$\epsilon_n(L) = \alpha_1(L) \cdots \alpha_n(L).$$

Assume that

$$\epsilon_a(DS(u, \omega)) \leq \varepsilon(\omega),$$
where $\bar{\tau}(\omega)$ is a random variable satisfying $E(\log \bar{\tau}(\omega)) < 0$ and the additional (relatively easy) condition that, for some random variable $\tilde{\alpha}(\omega) \geq 1$, we have

\begin{equation}
\alpha_1(DS(u, \omega)) \leq \tilde{\alpha}(\omega) \quad \text{with } E(\log \tilde{\alpha}(\omega)) < \infty.
\end{equation}

Under these assumptions, the Hausdorff dimension satisfies $d_H(A(\omega)) < d$ almost surely.

**Lemma 4.** Consider the linearized equation of (13) with initial boundary conditions

\begin{equation}
\Phi_t + Q\Phi = G'_\vartheta(\vartheta, \omega)\Phi, \quad \Phi(0) = (\xi, \eta)^T, \quad t \geq 0,
\end{equation}

where $\Phi = (U, V)^T \in E$ and $\vartheta(t) = (u(t), u_t(t) + \varepsilon u(t))^T \in E$, $t \geq 0$ is the solution of (13) with initial value $\vartheta(0) = (u_0, u_1 + \varepsilon u_0)^T$, and

\begin{equation}
G'_\vartheta(\vartheta, \omega) = \begin{pmatrix} 0 & 0 \\ -\beta \cos u & 0 \end{pmatrix}.
\end{equation}

Then (25) is a $P$-a.s. well-posed problem in $E$ and $S_\varepsilon(t, \omega)$ is uniformly differentiable for $P$-a.s. $\omega \in \Theta$ on the random attractor $A(\omega)$, with differential $DS_\varepsilon(\vartheta(0), t, \omega)(\xi, \eta)^T = \Phi(t, \omega) : E \mapsto E$ a bounded linear operator satisfying $P$-a.s.

\begin{equation}
\|S_\varepsilon(t, \omega)(\vartheta(0) + (\xi, \eta)^T) - S_\varepsilon(t, \omega)\vartheta(0) - DS_\varepsilon(\vartheta(0), t, \omega)(\xi, \eta)^T\|_E \leq k(t)\|((\xi, \eta)^T\|^2_E,
\end{equation}

where $K(t) \geq 1$ is independent of $\omega$, $t \geq 0$.

**Proof.** It is clear that the problem (25) is $P$-a.s. well-posed in $E$.

We consider the Lipschitz property of $S_\varepsilon(t, \omega)$. Set

\begin{align*}
\tilde{\vartheta}(t) &= S_\varepsilon(t, \omega)\vartheta_0 = (u(t), u_t(t) + \varepsilon u(t))^T, \\
\tilde{\Phi}(t) &= S_\varepsilon(t, \omega)(\vartheta(0) + (\xi, \eta)^T) = (\tilde{u}(t), \tilde{u}_t(t) + \varepsilon \tilde{u}(t))^T.
\end{align*}

Let $\tilde{\vartheta}(t) = \tilde{\Phi}(t) - \vartheta(t)$, which satisfies

\begin{equation}
\tilde{\vartheta}_t + Q\tilde{\Phi} = \begin{pmatrix} 0 \\ -\beta(\sin \tilde{u} - \sin u) \end{pmatrix}, \quad \tilde{\Phi}(0) = (\xi, \eta)^T, \quad t \geq 0.
\end{equation}

Taking the inner product of (27) with $\tilde{\Phi}$ in $E$ we have, after a simple computation,

\begin{equation}
\|\tilde{\vartheta}(t)\|_E = \|\tilde{\Phi}(t) - \vartheta(t)\|_E \leq \|((\xi, \eta)^T\|_E e^{\beta|t|}.
\end{equation}

Next we show the differentiability of the RDS $S_\varepsilon(t, \omega)$. Let $Z(t) = \tilde{\Phi}(t) - \vartheta(t) - \Phi(t)$. Then $Z(t)$ satisfies

\begin{equation}
Z_t + QZ = \begin{pmatrix} 0 \\ -\beta(\sin \tilde{u} - \sin u - \cos uU) \end{pmatrix}, \quad Z(0) = 0, \quad t \geq 0.
\end{equation}
It is easily checked that there exists a deterministic constant $\delta > 0$ such that
\[ |\sin \bar{u} - \sin u - \cos u U| \leq |\bar{u} - u|^2 + |\bar{u} - u - U|. \]
Taking the inner product of (29) with $Z$ in $E$, by the preceding inequality
and Young’s inequality we see that there exists a deterministic constant $\gamma_1 > 0$ such that
\[ \frac{d}{dt} \|Z(t)\|^2_E \leq \gamma_1 \|Z(t)\|^2_E + \gamma_1 \|\vec{\vartheta}(t)\|^4_E. \]
From (28),
\[ \frac{d}{dt} \|Z(t)\|^2_E \leq \gamma_1 \|Z(t)\|^2_E + \gamma_1 \|\vec{\vartheta}(t)\|^4_E. \]
By the Gronwall lemma and zero initial value at $t = \tau$, there exist deterministic constants $\gamma_2, \gamma_3 > 0$ such that
\[ \|\vartheta(t) - \vartheta(t) - \Phi(t)\|^2_E \leq \gamma_2 e^{\gamma_3 \alpha_2 \lambda_1 (2\alpha_2^2 + 3\lambda_1)} \|\vec{\vartheta}(t)\|^4_E. \]
The proof is complete.

**Lemma 5.** Let $\{(\xi_j, \omega_j)^T\}_{j=1}^m$ be an orthonormal family of elements of $(E, \|\cdot\|_E)$. We have
\[ \sum_{j=1}^m \|\xi_j\|^2_0 \leq \sum_{j=1}^m \lambda_j^{-1}. \]
**Proof.** This is a direct consequence of [10, Lemma VI.6.3].

**Theorem 3.** If
\[ \beta^2 > \frac{\alpha^2 \lambda_1}{2(2\alpha^2 + 3\lambda_1)}, \]
the Hausdorff dimension of the random attractor $A(\omega)$ for the RDS (7) satisfies
\[ d_H(A(\omega)) \leq \min \left\{ m \in \mathbb{N} \left| \frac{1}{m} \sum_{j=1}^m \lambda_j^{-1} < \frac{2\alpha^2 \lambda_1}{2\beta^2 (2\alpha^2 + 3\lambda_1) - \alpha^2 \lambda_1} \right. \right\}. \]
Otherwise $d_H(A(\omega)) = 0$.

**Proof.** We apply Debussche’s result. Let $DS_\varepsilon(\vartheta(0), \omega) = DS_\varepsilon(\vartheta(0), 1, \omega)$. Firstly, check that there exists a deterministic constant $\bar{\alpha} \geq 1$ such that $\alpha_1(DS_\varepsilon(\vartheta(0), \omega)) \leq \bar{\alpha}$ from (26).

To find a $d$ such that $\varepsilon_d(DS_\varepsilon(\vartheta(0), \omega)) < 1$, we use the trace formula (see also Témam [10], Chapter V). This allows us to write $\varepsilon_d$ in another way more dependent on the dynamics. Since $DS_\varepsilon(\vartheta(0), \omega)(\xi, \eta)^T$ is the solution of the linear equation
\[ \frac{d\Phi}{dt} = M(t, \vartheta(t))\Phi, \quad \Phi(0) = (\xi, \eta)^T, \]
where
\[ M(t, \vartheta(t)) = -Q + G'_\vartheta(\vartheta, \omega), \]
and \( \vartheta(t) \) is the solution of (13) with \( \vartheta(0) = (u_0, u_1 + \varepsilon u_0)^T \), we can write
\[ DS_\varepsilon(\vartheta(0), \omega) = \exp \left( \int_0^1 M(s, \vartheta(s)) \, ds \right). \]

Let \( \{ \Phi_j \}_{j=1}^m \) be \( m \) solutions of (25) with initial values \( \Phi_j(0), \ j = 1, 2, \ldots, m \). Let \( P(s) \) be an orthogonal projector of rank \( m \) at the time \( t = s \), onto the space spanned by \( \{ \Phi_j \}_{j=1}^m \) in \( E \). Then
\[ \epsilon_d(DS_\varepsilon(\vartheta(0), \omega)) = \sup_{P(0)} \exp \left( \text{Tr} \int_0^1 M(s, \vartheta(s)) P(s) \, ds \right). \]

Let \( \{ \Psi_j = (\xi_j, \eta_j)^T \}_{j=1}^m \) be a standard orthonormal basis of the space spanned by \( \{ \Phi_j \}_{j=1}^m \). By Young's inequality and Lemma 1, We have
\[ (M(s, \vartheta(s))\Psi_j, \Psi_j)_E = -(Q\Psi_j, \Psi_j)_E - (G'_\vartheta(\vartheta, \omega)\Psi_j, \Psi_j)_E \leq -\frac{\varepsilon}{2} + \left( \frac{\beta^2}{2\alpha} - \frac{\varepsilon}{4} \right) \| \xi_j \|^2. \]

If \( \beta^2 > \frac{\alpha^2 \lambda_1}{2(2\alpha^2 + 3\lambda_1)} \), by (33) and (30) of Lemma 5,
\[ \text{Tr} (M(s, \vartheta(s)) P(s)) = \sum_{j=1}^m (M(s, \varphi(s)) \Phi_j, \Phi_j)_E \leq -\frac{\varepsilon m}{2} + \left( \frac{\beta^2}{2\alpha} - \frac{\varepsilon}{4} \right) \sum_{j=1}^m \lambda_j^{-1}. \]

If there exists a number \( m \in N \) such that
\[ \frac{1}{m} \sum_{j=1}^m \lambda_j^{-1} < \frac{2\alpha^2 \lambda_1}{2\beta^2(2\alpha^2 + 3\lambda_1) - \alpha^2 \lambda_1}, \]
by (32) and (34), we obtain \( \text{E}(\log \epsilon_d(DS_\varepsilon(\vartheta(0), \omega))) < 0 \), then \( d_H(A(\omega)) \leq m \). If \( \beta^2 \leq \frac{\alpha^2 \lambda_1}{2(2\alpha^2 + 3\lambda_1)} \), then we deduce from (33) that
\[ (M(s, \vartheta(s))\Psi_j, \Psi_j)_E < 0, \quad j = 1, 2, \ldots, m. \]

Then \( d_H(A(\omega)) = 0 \).

**Corollary 2.** If
\[ \frac{\alpha^2 \lambda_1}{2(2\alpha^2 + 3\lambda_1)} < \beta^2 \leq \frac{\alpha^2 \lambda_1(2\lambda_1 + 1)}{2(2\alpha^2 + 3\lambda_1)}, \]
the Hausdorff dimension of the random attractor \( A(\omega) \) for the RDS (7) satisfies \( d_H(A(\omega)) = 0 \).
Remark 2. The upper bound in the right side of (31) decreases as $\alpha$ grows because the function $1/m \sum_{j=1}^{m} \lambda_j^{-1}$ is decreasing in $m$ and tends to zero as $m \to \infty$, while the function
\[
\frac{2\alpha^2 \lambda_1}{2\beta^2 (2\alpha^2 + 3\lambda_1) - \alpha^2 \lambda_1}
\]
is increasing and uniformly bounded in $\alpha$. So, the dimension $d_H(A(\omega))$ is uniformly bounded for the damping $\alpha$.

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THE MONGE–AMPÈRE EQUATION WITH INFINITE BOUNDARY VALUE

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This article concerns the Monge–Ampère equations with infinite boundary value in convex domains in Euclidean space. We were able to characterize the growth rate conditions, which are nearly optimal, for the existence/nonexistence of solutions to the problem.

1. Introduction

Let $\Omega$ be a domain in $\mathbb{R}^n$ and $\psi$ a positive function defined on $\Omega \times \mathbb{R} \times \mathbb{R}^n$. In this paper we study the Dirichlet problem for the Monge–Ampère equation

$$\det D^2 u = \psi(x, u, Du) > 0 \quad \text{in } \Omega,$$

with the infinite boundary value condition

$$u = +\infty \quad \text{on } \partial \Omega.$$  

We will look for strictly convex solutions in $C^\infty(\Omega)$; it is necessary to assume the underlying domain $\Omega$ to be convex for such solutions to exist.

This problem was first considered by Cheng and Yau ([5], [6]) for $\psi(x, u) = e^{Ku}f(x)$ in bounded convex domains and for $\psi(u) = e^{2u}$ in unbounded domains. More recently, Matero [11] treated the case $\psi = \psi(x, u)$ for bounded strictly convex domains, generalizing a result of Keller [8] and Osserman [12] for the Laplace operator; his results were further extended by Salani [13] to some Hessian equations. (See also [9], where problem (1.1)–(1.2) was studied for $\psi(x, u) = e^uf(x)$ and $\psi(x, u) = u^p f(x)$.) For the complex Monge–Ampère equation with $\psi(z, u) = e^{Ku}f(z)$ the corresponding problem was also treated in [5] in connection with the problem of finding complete Kähler–Einstein metrics on pseudoconvex domains. In this article we will consider more general cases, including allowing domains that are unbounded and not strictly convex when $\psi = \psi(x, u)$. Our main results are stated as follows:

**Theorem 1.1.** Let $\Omega$ be a bounded strictly convex domain. Suppose that $\psi \in C^\infty(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ satisfies $\psi > 0$,

$$M(z^+)^p \leq \psi(x, z, p), \quad \forall (x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n,$$
where \( p > n, \ M > 0, \ z^+ = \max \{z, 0\}, \) and finally
\[
\psi(x, z, p) \leq \Psi(z)(1 + |p|^n), \quad \forall \ (x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n,
\]
where \( \Psi \) is a smooth positive function and
\[
\sup_{z \leq 0} e^{-\varepsilon z} \Psi(z) < +\infty
\]
for some \( \varepsilon > 0. \) Then there exists a strictly convex solution \( u \in C^\infty(\Omega) \) to (1.1)–(1.2). Moreover, there exist functions \( \underline{h}, \overline{h} \in C(\mathbb{R}^+) \) with \( \underline{h}(r), \overline{h}(r) \to \infty \) as \( r \to 0, \) such that
\[
\underline{h}(d(x)) \leq u(x) \leq \overline{h}(d(x)), \quad \forall x \in \Omega,
\]
where \( d \) is the distance function to \( \partial \Omega. \)

When \( \psi \) does not depend on \( Du, \) Theorem 1.1 holds under weaker conditions. In particular, \( \Omega \) need not be bounded or strictly convex:

**Theorem 1.2.** Let \( \Omega \subset \mathbb{R}^n \) be a convex domain not containing a straight line. Suppose \( \psi \in C^\infty(\overline{\Omega} \times \mathbb{R}) \) satisfies \( \psi > 0, \) as well as (1.3) for some \( p > n \) and also
\[
\sup_{x \in \Omega, z \leq 0} e^{-\varepsilon z} \psi(x, z) < +\infty
\]
for some \( \varepsilon > 0. \) Then (1.1)–(1.2) has a strictly convex solution \( u \in C^\infty(\Omega) \) that satisfies (1.6). In addition, when \( \Omega \) is bounded, assumption (1.7) can be weakened to allow \( \varepsilon = 0. \)

**Remark 1.3.** Suppose \( \psi \geq 0 \) and there exists a convex supersolution \( \overline{u} \in C^2(\Omega) \) satisfying
\[
\det D^2 \overline{u} \leq \psi(x, \overline{u}, D\overline{u}) \quad \text{in} \quad \Omega,
\]
\[
\overline{u} = \infty \quad \text{on} \quad \partial \Omega.
\]
Theorems 1.1 and 1.2 then remain valid, with \( \overline{u}(x) \) in place of the function \( \overline{h}(d(x)) \) in (1.6), without assumption (1.3).

The following nonexistence results complement Theorems 1.1 and 1.2 and indicate that the growth conditions in Theorems 1.1 and 1.2 are nearly optimal.

**Theorem 1.4.** Let \( \Omega \) be a bounded convex domain in \( \mathbb{R}^n. \) If
\[
0 \leq \psi(x, z, p) \leq M \left(1 + (z^+)^p\right) (1 + |p|^n), \quad \forall \ (x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n,
\]
for some \( p, q \geq 0, \ p + q \leq n, \) there exists no convex solution to (1.1)–(1.2).

**Theorem 1.5.** Let \( \Omega \) be a convex domain in \( \mathbb{R}^n. \) If
\[
\psi(x, z, p) \geq M (1 + |p|^n)^\alpha, \quad \forall \ (x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n,
\]
where \( \alpha > 1 \) and \( M > 0, \) and \( \Omega \) contains a ball of radius \( a > (M(\alpha-1))^{-1/n}, \) there exists no convex solution to (1.1)–(1.2).
Note that $\Omega$ is not assumed to be bounded in Theorem 1.5.

**Theorem 1.6.** Assume $\Omega$ is an unbounded convex domain that contains a straight line. If $\psi > 0$ satisfies (1.3), where $p > n$, there is no convex solution to (1.1)–(1.2) in $C^2(\Omega)$.

The article is organized as follows: we start with some comparison principle and uniqueness results in Section 2. In Section 3 we construct some radially symmetric functions that will be used as barriers in proving our theorems. Section 3 also contains the proofs of Theorems 1.4–1.6, while Theorems 1.1 and 1.2 are proved in Sections 4 and 5.

2. The comparison principle and uniqueness

Throughout this section $\Omega \subset \mathbb{R}^n$ is assumed to be a bounded convex domain and $u, v \in C^2(\Omega)$ are convex functions satisfying

$$\det D^2 u \geq \psi(x, u, Du) \quad \text{and} \quad \det D^2 v \leq \phi(x, v, Dv) \quad \text{in} \quad \Omega,$$

where $\psi, \phi \in C^2(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ and

$$\psi(x, z, p) \geq \phi(x, z, p) \geq 0, \forall (x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n.$$  

For later reference we recall the following comparison principle, which will be used repeatedly:

**Lemma 2.1.** Assume $u, v \in C(\overline{\Omega})$ and $u \leq v$ on $\partial \Omega$. If either $\psi_z(x, z, p) > 0$ or $\phi_z(x, z, p) > 0$ for any $(x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, then $u \leq v$ in $\Omega$.

**Proof.** Assume that $u(y) - v(y) = \max_{\Omega} (u - v) > 0$

for some $y \in \Omega$. Then $\det D^2 u(y) \leq \det D^2 v(y)$, as the Hessian $D^2 (v - u)$ is positive semidefinite at $y$. On the other hand, we have

$$\det D^2 v(y) \leq \phi(y, v(y), Dv(y)) < \psi(y, u(y), Du(y)) \leq \det D^2 u(y),$$

since $u(y) > v(y)$ and $Du(y) = Dv(y)$. This contradiction shows $u \leq v$ in $\Omega$. $\Box$

We have the following comparison principle and uniqueness for solutions of problem (1.1)–(1.2):

**Theorem 2.2.** Assume $u = +\infty$, $v = +\infty$ on $\partial \Omega$ and $v$ is strictly convex in $\Omega$. Suppose $\Omega$ contains the origin in $\mathbb{R}^n$ and $\psi$ satisfies

$$x \cdot D_x \psi(x, z, p) \leq 0, \quad p \cdot D_p \psi(x, z, p) \geq 0, \quad \forall (x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n.$$

If, in addition, either

$$\psi(x, \lambda z + p) \geq \lambda \psi(x, z, p), \quad \forall \lambda \geq 1, \quad (x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n,$$
where $p > n$, or there exists $\varepsilon > 0$ such that
\begin{equation}
\psi_z(x, z, p) \geq \varepsilon \psi(x, z, p), \quad \forall (x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n,
\end{equation}
then $u \leq v$ in $\Omega$. In particular, problem (1.1)–(1.2) admits at most one strictly convex solution in $C^2(\Omega)$.

**Remark 2.3.** Assumption (2.4) implies that $\psi = 0$ where $z \leq 0$. Thus any strictly convex solution of (1.1)–(1.2) must be positive when (2.4) is satisfied. Note also that $\psi_z > 0$ wherever $\psi > 0$ if either (2.4) or (2.5) holds.

**Remark 2.4.** If $\psi_z(x, z, p) \geq 0$, then $(z^+)^p \psi(x, z, p)$ satisfies (2.4) and $\varepsilon z \psi(x, z, p)$ satisfies (2.5).

**Proof of Theorem 2.2.** Let $u \in C^2(\Omega)$ be a convex solution of (1.1)–(1.2). Consider for $0 < \lambda \leq 1$,
\[ u_\lambda(x) := \lambda^\alpha u(\lambda x) - a, \quad x \in \Omega_\lambda, \]
where $\Omega_\lambda = \{ x \in \mathbb{R}^n : \lambda x \in \Omega \}$ and
\[
\begin{cases}
a = 0, & \alpha = 2n/(p - n), \quad \text{if (2.4) holds,} \\
\alpha = 0, & a = \varepsilon^{-1} \lambda^{-(2+\alpha)n}, \quad \text{if (2.5) holds.}
\end{cases}
\]
We calculate
\begin{equation}
\det D^2 u_\lambda(x) = \lambda^{(2+\alpha)n} \det D^2 u(\lambda x)
= \lambda^{(2+\alpha)n} \psi(\lambda x, u(\lambda x), Du(\lambda x))
= \lambda^{(2+\alpha)n} \psi(\lambda x, \lambda^{-\alpha}(u_\lambda(x) + a), \lambda^{-(1+\alpha)} Du_\lambda(x))
\geq \lambda^{(2+\alpha)n} \psi(x, \lambda^{-\alpha}(u_\lambda(x) + a), Du_\lambda(x))
\geq \psi(x, u_\lambda(x), Du_\lambda(x))
\end{equation}
by assumption (2.3), when either (2.4) or (2.5) holds.

Now note that $\overline{\Omega} \subset \Omega_\lambda$ and $v - u_\lambda = +\infty$ on $\partial \Omega$ for all $0 < \lambda < 1$. We claim that $v \geq u_\lambda$ on $\Omega$ for all $0 < \lambda < 1$. Indeed, assume that
\[ u_\lambda(y) - v(y) = \max_{\Omega}(u_\lambda - v) > 0 \]
for some $y \in \Omega$. Then
\[ \psi(y, u_\lambda(y), Du_\lambda(y)) \geq \psi(y, v(y), Dv(y)) \]
\[ \geq \phi(y, v(y), Dv(y)) \geq \det D^2 v > 0. \]
It follows that $\psi_z(y, u_\lambda(y), Du_\lambda)(y) > 0$; see Remark 2.3. Consequently, we obtain a contradiction as in the proof of Lemma 2.1. This proves our claim, that is, $v \geq u_\lambda$ on $\Omega$ for all $0 < \lambda < 1$. Letting $\lambda \to 1$ we obtain $v \geq u$. \qed
3. Barriers

The main purpose of this section is to construct some radially symmetric strictly convex functions that will be used as barriers in proving our main results. Using these barriers we present proofs of Theorems 1.4–1.6 at the end of this section.

Let $u(x) = u(|x|)$ be a radially symmetric function. A straightforward calculation shows that

$$\det D^2 u = \left(\frac{u'}{r}\right)^{n-1} u'', \quad r = |x|. \quad (3.1)$$

Thus Equation (1.1) takes the form

$$\left(\frac{u'}{r}\right)^{n-1} u'' = r^{n-1} \psi(x, u, u') \quad (3.2)$$

for radially symmetric functions.

**Lemma 3.1.** Let $\eta \in C^1(\mathbb{R})$ satisfy $\eta(z) > 0$, $\eta'(z) \geq 0$ for all $z \in \mathbb{R}$. Then, for any $a > 0$, there exists a strictly convex radially symmetric function $v \in C^2(B_a(0))$ satisfying

$$\det D^2 v \geq e^v \eta(v)(1 + |Dv|^n) \quad \text{in } B_a(0),$$

$$v = +\infty \quad \text{on } \partial B_a(0). \quad (3.3)$$

**Proof.** Consider the initial value problem

$$\varphi' = \left(\exp\left(r^n e^{\varphi} \eta(\varphi)\right) - 1\right)^{1/n}, \quad r > 0$$

$$\varphi(0) = 0. \quad (3.4)$$

Let $[0, R)$ be the maximal interval on which the solution to (3.4) exists. We claim that $R$ is finite. Indeed, by (3.4) we have

$$\varphi'(r) \geq r \left(\exp\left(r^n e^{\varphi} \eta(\varphi(r))\right)\right)^{1/n} \geq r \left(\exp\left(r^n e^{\varphi(0)} \eta(0)\right)\right)^{1/n}, \quad 0 < r < R,$$

since $\eta' \geq 0$, $\varphi' \geq 0$ and $\varphi(0) = 0$. It follows that

$$n \geq n(1 - e^{-\varphi(\rho)/n}) \geq \int_0^\rho \varphi'(r) e^{-\varphi(r)/n} dr \geq (\eta(0))^{1/n} \int_0^\rho r dr \geq \frac{1}{2} (\eta(0))^{1/n} \rho^2$$

for any $\rho < R$. This proves that $R < \infty$. Moreover, by the theory of ordinary differential equations we see that $\varphi \in C^2(0, R)$ and $\varphi(R) = +\infty$ as $\varphi$ is strictly increasing. Rewriting (3.4) in the form

$$\log\left(1 + \frac{\varphi'}{\varphi}\right) = e^{\varphi} \eta(\varphi),$$

we obtain by differentiation

$$\frac{n(\varphi')^{n-1} \varphi''}{1 + (\varphi')^n} = \left(r^n e^{\varphi} \eta(\varphi)\right)' \geq nr^{n-1} e^{\varphi} \eta(\varphi), \quad 0 < r < R. \quad (3.5)$$

In particular, $\varphi''(r) > 0$ for $0 < r < R$. 


For given $a > 0$, let $v$ be defined by
\[
v(x) := \varphi(\lambda|x|) - 2n(-\log \lambda)^+, \quad x \in B_a(0),
\]
where $\lambda = R/a$. Note that $\varphi'(0) = 0$. We see that $v$ lies in $C^2(B_a(0))$ and is strictly convex since $\varphi \in C^2[0, R)$ and $\varphi'' > 0$. By (3.1) and (3.5) we obtain in $B_a(0)$
\[
\det D^2v(x) = \lambda^{2n} \frac{(\varphi'(\lambda|x|))^{n-1}\varphi''(\lambda|x|)}{(|\lambda|x|)^{n-1}}
\]
\[
\geq \lambda^{2n}e^{\varphi(\lambda|x|)}(\varphi(\lambda|x|))(1 + (\varphi'(\lambda|x|))^n)
\]
\[
\geq \lambda^{2n}e^{\varphi(x)+2n(-\log \lambda)^+} \eta(v(x) + 2n(-\log \lambda)^+)(1 + \lambda^{-n}|Dv(x)|^n)
\]
\[
\geq e^{v(x)} \eta(v(x))(1 + |Dv(x)|^n).
\]
In the last inequality we used the fact that $\eta$ is nondecreasing. The proof of Lemma 3.1 is complete.

\begin{remark}{Remark 3.2.}\end{remark}
In the sequel we will denote the function $v \in C^2(B_a(0))$ in Lemma 3.1 by $v^{a,\eta}$. We will also write $v^{a,\eta}(x) = v^{a,\eta}(|x|)$, since it is radially symmetric.

By Lemma 2.1 we have:

\begin{lemma}{Lemma 3.3.}\end{lemma}
Let $u \in C^2(\Omega)$ be a strictly convex solution of (1.1)–(1.2), where $\Omega$ is a bounded convex domain contained in a ball $B_a(x_0)$. Suppose
\[
\psi(x, z, p) \leq e^{\gamma}(z)(1 + |p|^n), \quad \forall(x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n,
\]
where $\gamma \in C^1(\mathbb{R})$ satisfies $\gamma(z) > 0$ and $\gamma'(z) \geq 0$. Then $u(x) \geq v^{a,\eta}(x - x_0)$ for all $x \in \Omega$.

\begin{proof}{Proof.}\end{proof}
We may assume $x_0 = 0$. For any $r > a$, note that $u - v^{r,\eta} = +\infty$ on $\partial \Omega$. Lemma 2.1 implies that $u \geq v^{r,\eta}$ in $\Omega$. Letting $r \to a$ we obtain $u \geq v^{a,\eta}$.

We next construct a function on $B_a(0)$ that will serve as an upper barrier when $\psi$ satisfies (1.3) with $p > n$. A straightforward calculation shows that when $p > n$ the function
\[
w(x) := (1 - |x|^2)^{(n+1)/(n-p)}
\]
is strictly convex and satisfies the inequality
\[
\det D^2w \leq C(n, p) w^p \quad \text{in} \quad B_1(0),
\]
where $C(n, p) = p[2(n + 1)/(p - n)]^{n+1}$. By rescaling, we have:

\begin{lemma}{Lemma 3.4.}\end{lemma}
Let $a, M > 0$ and $p > n$ and define $w^{a,M} \in C^\infty(B_a(0))$ by
\[
w^{a,M}(x) := \lambda w\left(\frac{x}{a}\right), \quad x \in B_a(0),
\]
where $\lambda > 0$.

\end{lemma}
where
\[ \lambda = \left( \frac{C(n,p)}{a^{2n}M} \right)^{1/(p-n)}. \]

Then
\[ \det D^2 w^{a,M} \leq M(w^{a,M})^p \text{ in } B_a(0). \]

**Proof.** One calculates directly
\[ \det D^2 w^{a,M}(x) = \frac{\lambda^n}{a^{2n}} \det D^2 w \left( \frac{x}{a} \right) \]
\[ \leq \frac{\lambda^n C(n,p)}{a^{2n}} \left( w \left( \frac{x}{a} \right) \right)^p = M \left( w^{a,M}(x) \right)^p. \]

This completes the proof. \( \square \)

From Lemmas 3.4 and 2.1 we derive the following comparison lemma:

**Lemma 3.5.** Let \( u \in C^2(\Omega) \) be a strictly convex solution of (1.1). Suppose that \( \psi \) satisfies (1.3) with \( p > n \) and that \( \Omega \) contains a ball \( B_a(x_0) \). Then
\[ u(x) \leq w^{a,M}(x-x_0) \text{ for all } x \in \Omega. \]

The second inequality in (1.6) now follows from Lemma 3.5:

**Corollary 3.6.** Let \( u \in C^2(\Omega) \) be a strictly convex solution of (1.1) where \( \Omega \) is a convex (not necessarily bounded) domain in \( \mathbb{R}^n \). Suppose \( \psi \) satisfies (1.3) with \( p > n \) and \( M > 0 \). Then
\[ u(x) \leq h(d(x)), \forall x \in \Omega, \]
where \( d \) is the distance function to \( \partial \Omega \) and \( h \in C^\infty(\mathbb{R}^+) \) is given by
\[ h(r) := w^{r,M}(0), \quad r > 0. \]

We next construct subsolutions to (1.1) defined on the whole space \( \mathbb{R}^n \) when \( \psi \) satisfies (1.9) with \( p + q \leq n \).

**Lemma 3.7.** Assume \( p,q \geq 0, p + q \leq n \) and \( M > 0 \). Then there exists a strictly convex radially symmetric positive function \( \tilde{u} \in C^\infty(\mathbb{R}^n) \) satisfying
\[ \det D^2 \tilde{u}(x) \geq M \left( 1 + (\tilde{u}(x))^p \right) \left( 1 + |D\tilde{u}(x)|^q \right), \forall x \in \mathbb{R}^n. \]

**Proof.** Without loss of generality we may assume \( M = 1 \). Let us consider separately three cases: \( q = 0 \), \( q = n \) and \( 0 < q < n \).

**Case i:** \( q = 0 \). Consider the initial value problem
\[ \varphi' = r \left( 1 + \varphi^p \right)^{1/n}, \quad r > 0, \]
\[ \varphi(0) = 1. \]
It is easy to see that when $p \leq n$ there exists a unique smooth solution $\varphi$ to (3.8) defined for all $r \geq 0$ and strictly increasing. Indeed, suppose $\varphi$ is defined on $[0, R)$. For any $\rho < R$, by (3.8) we have

$$
\rho^2 = 2 \int_0^\rho r dr = 2 \int_0^\rho \frac{\varphi'(r)}{(1 + (\varphi(r))^p)^{1/n}} dr 
\geq \int_0^\rho \frac{\varphi'(r)}{(\varphi(r))^{p/n}} dr = \begin{cases} \log \varphi(\rho), & p = n; \\ \frac{n}{n-p} (\varphi(\rho))^{(n-p)/n}, & p < n. \end{cases}
$$

It follows that $\lim_{\rho \to R} \varphi(\rho) = +\infty$ if and only if $R = +\infty$. We rewrite (3.8) in the form

$$
(\varphi')^n = r^n (1 + \varphi^p)
$$
and take derivatives of both sides to obtain

$$
(\varphi')^{n-1} \varphi'' \geq r^{n-1} (1 + \varphi^p).
$$

By (3.1) we see that the function $\tilde{u}(x) := \varphi(|x|)$ is strictly convex and

$$
\det D^2 \tilde{u} \geq (1 + \tilde{u}^p) \text{ in } \mathbb{R}^n.
$$

**Case ii:** $q = n$. In this case $p = 0$. Let $\varphi \in C^{\infty}(\mathbb{R}^+)$ be given by

$$
\varphi(r) := \int_0^r \left( \exp(r^n) - 1 \right)^{1/n} dr, \quad r \geq 0.
$$

Then

$$
(3.9) \quad \varphi'(r) = (\exp(r^n) - 1)^{1/n} > 0, \quad r > 0.
$$

Moreover $\varphi$ is strictly convex and $\varphi(0) = \varphi'(0) = 0$. Rewriting (3.9) as

$$
\log (1 + (\varphi')^n) = r^n
$$
and taking derivatives, we obtain

$$
(\varphi')^{n-1} \varphi'' = r^{n-1} (1 + (\varphi')^n).
$$

Consequently, the function $\tilde{u}(x) := 1 + \varphi(|x|), \ x \in \mathbb{R}^n$, which is smooth and strictly convex, satisfies

$$
\det D^2 \tilde{u} = (1 + |D\tilde{u}|^n) \text{ in } \mathbb{R}^n.
$$

**Case iii:** $0 < q < n$. Let $\varphi$ be the solution defined in some interval $[0, R)$ of the initial value problem

$$
(3.10) \quad \varphi' = \left( (1 + r^n (1 + \varphi^p))^{n/n-q} - 1 \right)^{1/n}, \quad r > 0
$$

$$
\varphi(0) = 1.
$$
Lemma

Proof of Theorem

contains the line

\[ \text{C > } \]

since \( \Omega \) is bounded and \( \psi \)

Then

\[ \varphi'(0) > 0 \] and \( \varphi'(r) > 0 \) for \( r > 0 \). Moreover,

\[ \varphi' \leq (1 + r^n(1 + \varphi^p))^{1/(n-q)} \]

\[ \leq (1 + \varphi^p)^{1/(n-q)}(1 + r^n)^{1/(n-q)} \leq c_\varphi p/(n-q)(1 + r^n)^{1/(n-q)}. \]

Since \( p + q \leq n \), we see that \( \varphi \) is defined for all \( r \geq 0 \). Rewriting (3.10) as

\[ (1 + (\varphi')^n)^{(n-q)/n} = 1 + r^n(1 + \varphi^p) \]

we obtain by differentiation

\[ (n-q)(\varphi')^{n-1} \varphi'' = r^{n-1}(1 + \varphi^p)(1 + (\varphi')^n)^{q/n} \]

\[ \geq \frac{1}{2} r^{n-1}(1 + \varphi^p)(1 + (\varphi')^q). \]

Consequently the function \( \tilde{u}(x) := c_\varphi |x| \), where \( c \) is a constant, is smooth, strictly convex and satisfies (3.7) when \( c \) is large enough. \( \square \)

We conclude this section with proofs of Theorems 1.4–1.6.

Proof of Theorem 1.4. Let \( u \in C^2(\Omega) \) be a convex solution of (1.1)–(1.2), where \( \Omega \) is bounded and \( \psi \) satisfies (1.9). Let \( \tilde{u} \in C^\infty(\mathbb{R}^n) \) satisfy (3.7) in Lemma 3.7. Note that \( u - C\tilde{u} = \infty \) on \( \partial\Omega \) for any \( C > 0 \). Since \( \tilde{u} > 0 \), we can choose \( C > 1 \) such that

\[ u(y) - C\tilde{u}(y) = \min_{\Omega}(u - C\tilde{u}) < 0 \]

for some \( y \in \Omega \).

By (1.9) we have

\[ \det D^2 u(y) \leq M(1 + (u^+(y))^p)(1 + |Du(y)|^q) \]

\[ \leq M(1 + (C\tilde{u}(y))^p)(1 + C^q|D\tilde{u}(y)|^q) \]

\[ < C^nM(1 + (\tilde{u}(y))^p)(1 + |D\tilde{u}(y)|^q) \leq C^n \det D^2 \tilde{u}(y), \]

since \( C > 1 \), \( p + q \leq n \) and \( Du(y) = D\tilde{u}(y) \), contradicting the fact that \( D^2(u - C\tilde{u})(y) \) is a positive semidefinite matrix. The proof is complete. \( \square \)

Proof of Theorem 1.6. We follow an idea of Cheng and Yau [6]. Assume \( \Omega \) contains the line

\[ L : x_1 = \cdots = x_{n-1} = 0. \]

Since \( \Omega \) is convex, it contains a solid cylinder \( \{ x := (x', x_n) \in \mathbb{R}^n : |x'| < \delta \} \), for some \( \delta > 0 \), where \( x' = (x_1, \ldots, x_{n-1}) \). For any \( \lambda > 0 \), let \( E_\lambda \) be the ellipsoid

\[ \frac{|x'|^2}{\delta^2} + \frac{x_n^2}{(\delta\lambda)^2} \leq 1 \]

and consider the function

\[ w_\lambda(x) := \lambda^a w^{\delta,M}(x', \lambda^{-1} x_n), \ x \in E_\lambda \]
where \( \alpha = 2/(n - p) \) and \( w^{\delta, M} \) is as in Lemma 3.4. We have

\[
\det D^2 w_\lambda(x) = \lambda^{n\alpha-2} \det D^2 w^{\delta, M}(x', \lambda^{-1} x_n) \leq M(\lambda^{\alpha} w^{\delta, M}(x', \lambda^{-1} x_n))^p = M(w_\lambda(x))^p, \quad \forall x \in E_\lambda.
\]

Now assume that \( u \in C^2(\Omega) \) is a convex solution of \((1.1)-(1.2)\) in \( \Omega \), where \( \psi \) satisfies \((1.3)\). Since \( w_\lambda = +\infty \) on \( \partial E_\lambda \subset \Omega \), Lemma 2.1 yields

\[
w_\lambda \geq u \quad \text{in} \quad E_\lambda.
\]

Note that \( \alpha < 0 \). Letting \( \lambda \to \infty \), we see that \( u = 0 \) on \( L \). It follows that \( u_{x_n x_n} = 0 \) on \( L \), contradicting the positivity of \( \det D^2 u \) everywhere in \( \Omega \).

Finally, Theorem 1.5 follows from the comparison principle (Lemma 2.1) and the following lemma:

**Lemma 3.8.** Let \( \alpha > 1 \) and \( a > 0 \). There exists a strictly convex radially symmetric function \( \bar{u} \in C^2(B_a(0)) \) satisfying

\[
det D^2 \bar{u} = \frac{1}{a^n(\alpha-1)}(1 + |D\bar{u}|^n)^\alpha \quad \text{in} \quad B_a(0),
\]

\[
\frac{\partial \bar{u}}{\partial \nu} = +\infty \quad \text{on} \quad \partial B_a(0),
\]

where \( \nu \) is the unit normal to \( \partial B_a(0) \). Moreover, if \( \alpha > (n+1)/n \) then \( \bar{u} \in C^0(B_a(0)). \)

**Proof.** Let \( \beta > 0 \) and consider the function \( \varphi \) defined by

\[
\varphi(r) := \int_0^r ((1 - r^n)^{-\beta} - 1)^{1/n} dr, \quad 0 \leq r < 1.
\]

Then

\[
1 + (\varphi')^n = \frac{1}{(1-r^n)^\beta}
\]

and

\[
(\varphi')^{n-1}\varphi'' = \beta r^{n-1}(1 + (\varphi')^n)^{(\beta+1)/\beta}.
\]

We see that \( \varphi(0) = \varphi'(0) = 0 \), that \( \varphi''(r) > 0 \) for all \( 0 \leq r < 1 \) and that \( \lim_{r \to 1} \varphi'(r) = \infty \). Note also that if \( \beta < n \),

\[
\varphi(r) \leq \int_0^r (1 - r^n)^{-\beta/n} dr \leq \int_0^r (1 - r)^{-\beta/n} dr \leq \frac{n}{n-\beta}, \quad \forall r < 1.
\]

Taking \( \beta = 1/(\alpha - 1) \), we obtain the desired function \( \bar{u}(x) := a\varphi(a^{-1}|x|), \) for \( x \in B_a(0) \). \( \square \)
4. Proof of Theorem 1.1

By assumption (1.5) we may find a positive nondecreasing function $\eta$ in $C^\infty(\mathbb{R}^n)$ satisfying
\begin{equation}
e^{\varepsilon z} \eta(z) \geq \max_{y \leq z} \Psi(y), \quad \forall \ z \in \mathbb{R}.
\end{equation}
For simplicity we will assume throughout this section that $\varepsilon = 1$; this may be achieved by rescaling.

We first assume $\Omega$ to be smooth. For each integer $k \geq 1$, consider the Dirichlet problem
\begin{equation}
det D^2 u = \psi(x, u, Du) > 0 \text{ in } \Omega,
\end{equation}
\begin{equation}
u_{r, \eta} \leq u \leq k \text{ on } \partial \Omega.
\end{equation}

Since $\Omega$ is bounded, we may choose $r > 0$ sufficiently large that $\Omega \subset B_r(0)$ and $v_{r, \eta} \leq 1$ on $\partial \Omega$. It follows from Lemma 2.1 that $v_{r, \eta} \leq u \leq k$ in $\Omega$, so
\begin{equation*}
|u| \leq C_k
\end{equation*}
for any convex solution $u$ of (4.2), where $C_k$ is a constant depending on $k$. By a result of Lions [10] (see also [4]), there exists for each $k$ a strictly convex function $u_k \in C^2(\Omega)$ satisfying
\begin{equation}
det D^2 u_k \geq \Psi(C_k)(1 + |Du_k|^n) \text{ in } \Omega,
\end{equation}
\begin{equation*}
u_k = k \text{ on } \partial \Omega.
\end{equation*}

Note that $u_k$ is a subsolution of of (4.2). By a theorem of Caffarelli–Nirenberg–Spruck [4] there exists a strictly convex solution $u_k \in C^\infty(\overline{\Omega})$ of (4.2) satisfying $u_k \geq u_k$ in $\overline{\Omega}$ for each $k \geq 1$. Moreover, $u_k$ satisfies the a priori estimate
\begin{equation}
\|u_k\|_{C^{2, \alpha}(\overline{\Omega})} \leq C(k), \quad k \geq 1,
\end{equation}
where $C(k) > 0$ depends on $k$. We next need to derive a priori interior estimates which are independent of $k$.

**Proposition 4.1.** For an arbitrary compact subset $K$ of $\Omega$, there exists a constant $C$ independent of $k$ such that
\begin{equation}
\|u_k\|_{C^{2, \alpha}(K)} \leq C, \quad \forall \ k \geq 1.
\end{equation}

The proof of this estimate is based on the following lemma and some well-known results in the theory of Monge–Ampère and more general fully nonlinear elliptic equations.

**Lemma 4.2.** There exists $a > 0$ depending only on $\Omega$ and a decreasing sequence $a_k \to a$ ($k \to \infty$) such that
\begin{equation}
v^{a_k, \eta}(a - d(x)) \leq u_k(x) \leq \tilde{h}(d(x)), \quad \forall \ x \in \Omega, \ k \geq 1,
\end{equation}
where \( d \) is the distance function to the boundary of \( \Omega \). (For the definitions of \( \mathring{h} \) and \( v^{a_k,\eta} \), see (3.6) and Remark 3.2.)

Proof. The second inequality follows from Corollary 3.6. Next, let \( a > 0 \) be the smallest number such that for any point \( \bar{x} \in \partial \Omega \) there is a ball \( B_a(x_0) \) of radius \( a \) with \( \Omega \subset B_a(x_0) \) and \( \Omega \cap \partial B_a(x_0) = \{ \bar{x} \} \); such a number exists since \( \Omega \) is bounded and strictly convex. Choose \( a_1 > a_2 > \cdots > a_k > \cdots \), \( a_k \to a \) as \( k \to \infty \), such that \( v^{a_k,\eta}(a) = k \) for each \( k \geq 1 \). For \( x \in \Omega \), choose \( \bar{x} \in \partial \Omega \) and a ball \( B_a(x_0) \) such that \( d(x) = |x - \bar{x}|, \) \( \Omega \subset B_a(x_0) \) and \( \Omega \cap \partial B_a(x_0) = \{ \bar{x} \} \). Since \( v^{a_k,\eta}(x - x_0) \leq u_k(x) \) for all \( x \in \partial \Omega \), Lemma 2.1 gives

\[
v^{a_k,\eta}(x - x_0) \leq u_k(x), \quad \forall \ x \in \Omega.
\]

This proves the first inequality in (4.6). \( \Box \)

For convenience let us now introduce some notation. Let \( h, v_k \) denote the functions defined in \( \Omega \) by

\[
h(x) := \mathring{h}(d(x)), \quad v_k(x) := v^{a_k,\eta}(a - d(x)), \quad x \in \Omega.
\]

For \( l > 0 \) and \( k \geq 1 \) write

\[
H_l := \{ x \in \Omega : h(x) < l \},
\]

\[
U_{k,l} := \{ x \in \Omega : u_k(x) < l \},
\]

\[
V_{k,l} := \{ x \in \Omega : v_k(x) < l \}.
\]

By (4.6) we have \( H_l \subset U_{k,l} \subset V_{k,l} \) for each \( k \geq 1 \).

Proof of Proposition 4.1. Let \( K \) be a compact subset of \( \Omega \). We may choose \( l > 0 \) and then \( k_0 \) sufficiently large so that \( K \subset H_{l/2} \) and \( \nabla v_{k_0,\eta} \subset \Omega \). From (4.6) we see that

\[
|u_k| \leq C_0 \quad \text{in} \quad U_{k,2l}, \quad \forall \ k \geq k_0,
\]

where \( C_0 \) is independent of \( k \). Moreover, by the strict convexity of \( u_k \),

\[
\max_{U_{k,2l}} |D u_k| = \max_{\partial U_{k,2l}} |D u_k| \leq \max_{x \in \partial V_{k_0,\eta}} \frac{u_k(x) - 2l}{\text{dist}(U_{k,2l}, \partial V_{k_0,\eta})} \leq \max_{x \in \partial V_{k_0,\eta}} \frac{h(x) - 2l}{\text{dist}(V_{k,2l}, \partial V_{k_0,\eta})} \leq \max_{x \in \partial V_{k_0,\eta}} \frac{h(x) - 2l}{\text{dist}(V_{k_0,2l}, \partial V_{k_0,\eta})} = C_1
\]

for all \( k \geq k_0 \), where the last two inequalities follow from the relations \( u_k < h \) and \( U_{k,2l} \subset V_{k,2l} \subset V_{k_0,2l} \), since \( v_k \geq v_{k_0} \) for \( k \geq k_0 \).

Next, applying Pogorelov’s interior estimates (see [7]) we obtain

\[
|D^2 u_k(x)| \leq \frac{C_2}{\text{dist}(x, \partial U_{k,2l})}, \quad \forall \ x \in U_{k,2l}, \ k \geq k_0,
\]
where $C_2$ depends on $C_0, C_1$ and the $C^2$ norm of $\psi$, as well as $\min \psi$, in $\Omega \times \{z \leq C_0\} \times \{|p| \leq C_1\}$, but is independent of $k$. Since $H_t \subset H_{2t} \subset U_{k,2t}$, 
$$\text{dist}(H_t, \partial U_{k,2t}) \geq \text{dist}(H_t, \partial H_{2t}).$$

It follows from (4.9) that

$$\|D^2 u_k\|_{C^0(\overline{H_t})} \leq \frac{C_2}{\text{dist}(H_t, \partial H_{2t})}. \tag{4.10}$$

Finally, by the Evans–Krylov theorem (see [3]) we have

$$\|D^2 u_k\|_{C^{\alpha}(\overline{H_{1/2}})} \leq C_3, \quad \forall \, k \geq k_0 \tag{4.11}$$

where $C_3$ is independent of $k$. Now (4.5) follows from (4.7), (4.8), (4.10) and (4.11), combining with (4.4) for $k \leq k_0$.

By Proposition 4.1, there exists a subsequence $\{u_{k_j}\}$ and $u \in C^{2,\alpha}(\Omega)$ such that

$$\lim_{j \to \infty} \|u_{k_j} - u\|_{C^{2,\alpha}(K)} = 0$$

for any compact subset $K$ of $\Omega$. We see that $u$ is strictly convex and solves (1.1). From (4.6) we obtain

$$h_t(d(x) := v^{a,\eta}(a - d(x)) \leq u(x) \leq \overline{h_t}(d(x)), \quad \forall \, x \in \Omega. \tag{4.12}$$

Consequently, $u = +\infty$ on $\partial \Omega$. This completes the proof of Theorem 1.1 when $\Omega$ is smooth.

Suppose now that $\Omega$ is not smooth. We choose a sequence of smooth strictly convex domains

$$\Omega_1 \subseteq \cdots \subseteq \Omega_k \subseteq \Omega_{k+1} \subseteq \cdots \subseteq \Omega$$

such that

$$\Omega = \bigcup_{k=1}^{\infty} \Omega_k.$$ 

For each $k \geq 1$, let $u_k \in C^\infty(\Omega_k)$ be a strictly convex solution of the problem

$$\text{det} \, D^2 u = \psi(x,u,Du) \quad \text{in} \quad \Omega_k,$$

$$u = +\infty \quad \text{on} \quad \partial \Omega_k.$$ 

We have

$$v^{a,\eta}(a - d(x)) \leq u_k(x) \leq \overline{h_t}(d_k(x)), \quad \forall \, x \in \Omega_k, \tag{4.14}$$

where $a$ is as in (4.12) and $d_k$ is the distance function to $\partial \Omega_k$. Using this in place of Lemma 4.2 we can derive the estimate (4.5) as before, and therefore obtain a subsequence that converges to a solution $u \in C^{2,\alpha}(\Omega)$ of (1.1)–(1.2) satisfying (4.12). That $u$ lies in $C^\infty(\Omega)$ follows from elliptic regularity theory. The proof of Theorem 1.1 is complete.
Remark 4.3. As an alternative approach, one may first prove the existence of a convex weak solution and then apply the strict convexity and regularity theorems of Caffarelli [1], [2] to prove Theorem 1.1.

5. Proof of Theorem 1.2

The proof of Theorem 1.2 follows that of Theorem 1.1, except that we have to reconstruct lower barriers when $\Omega$ is unbounded or not strictly convex. To this end we consider the equation

$$\det D^2 u = F(u) \text{ in } \Gamma^+ := \{ x \in \mathbb{R}^n : x_i > 0 \},$$

where $F$ is a positive nondecreasing function. When $F(u) = e^{2u}$, Cheng and Yau [6] observed that $u(x) := -\log(x_1 \cdots x_n)$ is a strictly convex solution of (5.1) in $\Gamma^+$. Inspired by this we look for solutions to (5.1) of the form

$$u(x) = \varphi(a - \log(x_1 \cdots x_n)), \ x = (x_1, \ldots, x_n) \in \Gamma^+,$$

for some function $\varphi$, where $a$ is a constant. We calculate

$$u_{x_i} = -\frac{\varphi'}{x_i}, \ u_{x_i x_j} = \frac{1}{x_i x_j} (\varphi'' + \varphi' \delta_{ij}).$$

It follows that

$$\det D^2 u = \frac{1}{(x_1 \cdots x_n)^2} (\varphi')^{n-1} (n \varphi'' + \varphi').$$

Equation (5.1) thus reduces to

$$\left(\varphi'\right)^{n-1} (n \varphi'' + \varphi') = e^{2(a-t)} F(\varphi).$$

Lemma 5.1. Let $a > 0$ and $F \in C^\infty(\mathbb{R})$ satisfy $F > 0$, $F' \geq 0$ and

$$F(z) \geq M(z^+)^p, \ \forall \ z \in \mathbb{R},$$

where $p > n$. There exists a strictly increasing function $\varphi \in C^\infty(\mathbb{R}^+)$ with

$$\left(\varphi'\right)^{n-1} (n \varphi'' + \varphi') \geq e^{2(a-t)} F(\varphi(t)), \ \forall \ t \geq 0,$$

and

$$\lim_{t \to +\infty} \varphi(t) = +\infty.$$

Proof. We construct $\varphi$ from $F$. For convenience we write $f := A(e^{2a} F)^{1/n}$, where $A$ is an undetermined constant, and define

$$g(z) := \int_0^z \frac{dz}{f(z)}.$$  

We see that $g$ is a strictly increasing function defined for all $z \in \mathbb{R}$. Let $g^{-1}$ denote the inverse function of $g$ and define

$$\varphi(t) := g^{-1}(B - e^{-\beta t}),$$

where $B$ is a constant to be determined. It follows that

$$\left(\varphi'\right)^{n-1} (n \varphi'' + \varphi') = e^{2(a-t)} F(\varphi(t)).$$

This completes the proof of Lemma 5.1.
where $\beta$ is a constant to be determined and
\[ B := \int_0^\infty \frac{dz}{f(z)} < \infty \]
by assumption (5.3). It is clear that $\varphi$ satisfies (5.5). We calculate
\[ \varphi'(t) = \frac{\beta e^{-\beta t}}{g'(\varphi(t))} = \beta e^{-\beta t} f(\varphi(t)) > 0, \quad \forall t \in \mathbb{R}, \]
and
\[ \varphi''(t) = \beta e^{-\beta t} (f'(\varphi(t))\varphi'(t) - \beta f(\varphi(t))) = \beta^2 e^{-\beta t} f(\varphi(t)) (e^{-\beta t} f'(\varphi(t)) - 1) \geq -\beta^2 e^{-\beta t} f(\varphi(t)), \]
since $f'(\varphi(t)) \geq 0$. It follows that
\[ (\varphi')^n (n\varphi'' + \varphi') \geq \beta^n (1 - n\beta) e^{-n\beta t} (f(\varphi(t)))^n. \]
Taking $\beta < 1/n$ and $A = \beta^{-1}(1 - n\beta)^{-1/n}$ we obtain (5.4).

A slight modification of this proof yields the following:

**Lemma 5.2.** Let $a > 0$ and $F(z) = e^{\varepsilon z} \eta(z)$, where $\varepsilon > 0$ and $\eta \in C^\infty(\mathbb{R})$ is a positive nondecreasing function. There exists a strictly increasing function $\varphi \in C^\infty(\mathbb{R}^+)$ satisfying (5.4) for all $t \in \mathbb{R}$ and (5.5). Moreover, $\varphi$ is a convex function.

**Proof.** As in the proof of Lemma 5.1 we define $\varphi$ by (5.7). Note that here we still have $B := \int_0^\infty dz/f(z) < +\infty$. Write $s = \varphi(t)$. By (5.7) we have
\[ e^{-\beta t} = B - g(s) = \int_s^\infty \frac{dz}{f(z)} \geq \frac{1}{A(e^{2\alpha} \eta(s))^{1/n}} \int_s^\infty e^{-\varepsilon z/n} dz = \frac{n}{\varepsilon f(s)}, \]
since $\eta$ is nondecreasing. Next,
\[ f'(s) = \frac{A^n e^{2\alpha} F'(s)}{n f(s)^{n-1}} \geq \frac{\varepsilon A^n e^{2\alpha} F(s)}{n f(s)^{n-1}} = \frac{\varepsilon f(s)}{n}, \]
since $\eta' \geq 0$. Consequently, $\varphi''(t) \geq 0$ by (5.8). Finally, taking $\beta = A^{-1} = \frac{2}{n}$ we have
\[ (\varphi')^{n-1} (n\varphi'' + \varphi') \geq (\varphi'(t))^{n-1} = \beta^n e^{-n\beta t} (f(\varphi(t)))^n = e^{2(a-\ell)t} F(\varphi(t)) \]
for all $t \in \mathbb{R}$. \hfill $\square$

**Remark 5.3.** Let $\varphi$ be the unique solution of (5.2) satisfying the initial data
\[ \varphi(0) = \varphi(0), \quad \varphi'(0) = \varphi'(0). \]
We have \( \varphi(t) \leq \varphi(t) \) for all \( t > 0 \) where \( \varphi(t) \) is defined. Equation (5.2) can be recast as
\[
(e^t(\varphi')^n)' = e^{2a-t} F(\varphi),
\]
so \( 0 < \varphi'(t) \leq \varphi'(t) \) for all \( t > 0 \) where \( \varphi(t) \) is defined. By the extension theorem we see that \( \varphi \) is defined for all \( t > 0 \). However, \( \varphi \) may be bounded above on all of \( \mathbb{R}^+ \), so we cannot replace \( \varphi \) by \( \varphi \) in the construction below.

**Proof of Theorem 1.2.** As in the last part of the proof of Theorem 1.1 we choose a sequence of bounded smooth strictly convex domains \( \Omega_1 \subseteq \Omega_2 \subseteq \cdots \subseteq \Omega_k \subseteq \cdots \subseteq \Omega \) such that \( \Omega = \bigcup \Omega_k \) and we consider, for each \( k \),
\[
\det D^2 u = \psi(x, u) \text{ in } \Omega_k, \quad u = k \text{ on } \partial \Omega_k.
\]
Let \( u_k \in C^\infty(\Omega_k) \) be a strictly convex solution of (5.12); the existence of \( u_k \) follows from [4]. By assumption (1.3) and Corollary 3.6 we have
\[
u_k(x) \leq \overline{h}(d_k(x)), \quad \forall x \in \Omega_k,
\]
where \( d_k \) is the distance function to \( \partial \Omega_k \). We need an a priori lower bound for \( u_k \), which is derived below (Lemma 5.4). With the aid of such estimates, the rest of proof proceeds as that of Theorem 1.1. \( \Box \)

**Lemma 5.4.** There exists an increasing sequence of functions \( \overline{h}_k \in C(\mathbb{R}^+) \) such that
\[
\lim_{k \to \infty} \lim_{r \to 0} \overline{h}_k(r) = +\infty
\]
and
\[
u_k(x) \geq \overline{h}_k(d(x)), \quad \forall x \in \Omega_k,
\]
for all \( k \) sufficiently large, where \( d(x) = \text{dist}(x, \partial \Omega) \).

**Proof.** By assumption (1.7) we may find a function \( \eta \in C^\infty(\mathbb{R}) \) with \( \eta > 0 \), \( \eta' \geq 0 \) and \( F(z) := e^{\varepsilon z} \eta(z) \geq \psi(x, z) \) for all \( (x, z) \in \overline{\Omega} \times \mathbb{R} \), where \( \varepsilon \geq 0 \) as in Theorem 1.2. We consider two cases.

**Case i:** \( \varepsilon > 0 \). We apply Lemma 5.2 with \( a = 0 \) to obtain \( \varphi \in C^\infty(\mathbb{R}) \) satisfying (5.4) and (5.5). By the assumption that \( \Omega \) contains no straight lines we may assume \( \Omega \subset \Gamma^+ = \{ x \in \mathbb{R}^n : x_i > 0 \} \). For a fixed point \( x_0 \in \Omega \) let \( \bar{x} \) be a point on \( \partial \Omega \) such that \( d(x_0) = \text{dist}(x_0, \bar{x}) \). We may assume \( \bar{x} \) lies on the hyperplane \( x_1 = 0 \). For each integer \( k \geq 1 \) let
\[
u_k(x) := \varphi(-n \log \left( (x_1 + b_k) \cdots (x_n + b_k) \right)), \quad x \in \Gamma^+,
\]
where \( b_k \) satisfies \( \varphi(-n \log b_k) = k \). Then \( \nu_k \in C^\infty(\Gamma^+) \) is strictly convex and
\[
\det D^2 \nu_k(x) \geq F(\nu_k(x)) \geq \psi(x, \nu_k(x)), \quad x \in \Omega.
\]
Note that \( u_k \leq u \) on \( \partial \Omega_k \). By Lemma 2.1 we obtain
\[
(5.16) \quad u_k \leq u \quad \text{in } \Omega.
\]
In particular, \( u_k(x_0) \leq u_k(x_0) \) if \( k \) is sufficiently large and \( x_0 \in \Omega_k \). The function
\[
h_k(r) := \min_{|x-x_0|=r, \, x \in \Gamma^+} u_k(x)
\]
then has the desired properties.

Case ii: \( \varepsilon = 0 \) and \( \Omega \) is bounded. We may assume that
\[
\Omega \subseteq Q := \{ x \in \mathbb{R}^n : 0 < x_i < \rho, \, 1 \leq i \leq n \} \subset \mathbb{R}^n
\]
and \( \tau = (0, \frac{1}{2}\rho, \ldots, \frac{1}{2}\rho) \), where \( \rho \) is the diameter of \( \Omega \). Applying Lemma 5.1 to \( F \) with \( a = a_k := n \log(\rho + b_k) \), where \( b_k > 0 \) is to be determined, we obtain \( \varphi_k \in C^\infty(\mathbb{R}^+) \) satisfying (5.4) for \( t \geq 0 \) and (5.5). Let
\[
u_k(x) := \varphi_k(a_k - n \log((x_1 + b_k) \cdots (x_n + b_k))), \quad x \in Q,
\]
and choose a decreasing sequence \( b_k \) such that \( \varphi_k(a_k - n \log b_k) \leq k \) for all \( k \) sufficiently large. We now can proceed as in the previous case. \( \square \)

This completes the proof of Theorem 1.2. Finally, it is clear that with minor modifications the proof yields Theorems 1.1 and 1.2 with assumption (1.8) in place of (1.3) when \( \psi_z \geq 0 \). (See Remark 1.3.)

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CÓRDOBA–FEFFERMAN COLLECTIONS IN HARMONIC ANALYSIS

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Córdoba–Fefferman collections are defined and used to characterize functions whose corresponding maximal functions are locally integrable. Córdoba–Fefferman collections are also used to show that, if $M_x$ and $M_y$ respectively denote the one-dimensional Hardy–Littlewood maximal operators in the horizontal and vertical directions in $\mathbb{R}^2$, $M_{HL}$ denotes the standard Hardy–Littlewood maximal operator in $\mathbb{R}^2$, and $f$ is a measurable function supported in the unit square $Q = [0,1] \times [0,1]$, then $\int_Q M_{HL}f \sim \int_Q M_x f + \int_Q M_y f$.

We begin by introducing the following definition:

**Definition 1.** Let $\beta$ be a countable collection of Lebesgue measurable subsets of the unit $n$-cube $I^n$ in $\mathbb{R}^n$ of positive measure. A (possibly finite) subset $\{R_i\}$ of $\beta$ is said to be a Córdoba–Fefferman collection with respect to $\beta$ (denoted by $\{R_i\} \in \text{CFC}(\beta)$) if and only if there exists an enumeration $\tilde{R}_1, \tilde{R}_2, \tilde{R}_3, \ldots$ of the elements of $\{R_i\}$ such that $|\tilde{R}_i \cap \bigcup_{j<i} \tilde{R}_j| \leq \frac{1}{2} |\tilde{R}_i|$ for each $i = 2, 3, 4, \ldots$.

A. Córdoba and R. Fefferman used what we are now calling Córdoba–Fefferman collections in [1] to characterize geometric maximal operators that are of weak type $(p,p)$ for $p > 1$. The purpose of this paper is to show that Córdoba–Fefferman collections may also be used to estimate the integrals of maximal functions. The primary result in this regard is the following:

**Theorem 2.** Let $\beta$ be a countable collection of Lebesgue measurable subsets of the unit $n$-cube $I^n$ in $\mathbb{R}^n$ of positive measure. Let $\beta$ be such that for any point $x$ in $I^n$, $x \in R$ for some $R \in \beta$. Define the maximal operator $M_{\beta}$ on $L^1(I^n)$ by

$$M_{\beta}f(x) = \sup_{x \in R \in \beta} \frac{1}{|R|} \int_R |f(y)| \, dy.$$  \hspace{1cm} (1)

Suppose $M_{\beta}$ satisfies the (Tauberian) condition

$$|\{x \in I^n : M_{\beta} \chi_E(x) \geq \frac{1}{2}\}| \leq C_{\beta}|E|$$  \hspace{1cm} (2)
for all measurable sets $E \subset I^n$. Then if $f \in L^1(I^n)$,
\begin{equation}
\int_{I^n} M_\beta f \sim \sup_{\{R_i\} \in \text{CFC}(\beta)} \int_{I^n} |f| \sum_i \chi_{R_i}.
\end{equation}

In particular,
\[
\frac{1}{2} \sup_{\{R_i\} \in \text{CFC}(\beta)} \int_{I^n} |f| \sum_i \chi_{R_i} \leq \int_{I^n} M_\beta f \leq 4 C_\beta \sup_{\{R_i\} \in \text{CFC}(\beta)} \int_{I^n} |f| \sum_i \chi_{R_i}.
\]

**Proof.** We assume without loss of generality that $f \in L^\infty(I^n)$, $f \not\equiv 0$. We
begin by showing
\[
\int_{I^n} M_\beta f \leq 4 C_\beta \sup_{\{R_i\} \in \text{CFC}(\beta)} \int_{I^n} |f| \sum_i \chi_{R_i}.
\]

Let $\epsilon > 0$. It suffices to show there exists $\{R_i\} \in \text{CFC}(\beta)$ such that
\[
4 C_\beta \int_{I^n} |f| \sum_i \chi_{R_i} \geq \int_{I^n} M_\beta f - \epsilon.
\]

Let $m$ be a positive integer such that
\[
0 \leq \int_{\{x \in I^n : M_\beta f(x) > 2^m\}} M_\beta f < \frac{\epsilon}{3}.
\]

Let $R_1, R_2, R_3, \ldots$ be an enumeration of the elements of $\beta$. Let $R_{1,1}$ be the first element on the list of the $R_i$ such that
\[
\frac{1}{|R_{1,1}|} \int_{R_{1,1}} |f| > 2^m.
\]

Assuming $R_{1,1}, R_{1,2}, \ldots, R_{1,k}$ have been chosen, let $R_{1,k+1}$ be the first element on the list of the $R_i$ such that
\[
|R_{1,k+1} \cap \bigcup_{i=1}^k R_{1,i}| \leq \frac{1}{2} |R_{1,k+1}|
\]
and
\[
\frac{1}{|R_{1,k+1}|} \int_{R_{1,k+1}} |f| > 2^m.
\]

(If such an element of $\beta$ does not exist, we stop the selection procedure at this point.) In this manner, a (possibly finite) sequence $R_{1,1}, R_{1,2}, \ldots$ is attained.

Let $j_1$ be an integer such that
\[
|\bigcup_{i=1}^{j_1} R_{1,i}| > \frac{1}{2} |\bigcup_{i=1}^\infty R_{1,i}|.
\]
We renumerate $\beta$ (allowing for multiple counting of individual elements) as

\[ R_{1,1}, R_{1,2}, \ldots, R_{1,j_1}, R_{1,1}, R_{2,1}, R_{2,2}, R_{2,3}, \ldots. \] (*2)

Now let $R_{2,1} = R_{1,1}$. Let $R_{2,2}$ be the first element on the list (*2) such that $|R_{2,2} \cap R_{2,1}| \leq \frac{1}{2} |R_{2,2}|$ and

\[ \frac{1}{|R_{2,2}|} \int_{R_{2,2}} |f| > 2^{m-1}. \]

Assuming $R_{2,1}, R_{2,2}, R_{2,k}$ have been selected, let $R_{2,k+1}$ be the first element on the list (*2) such that

\[ |R_{2,k+1} \cap \bigcup_{i=1}^{k} R_{2,i}| \leq \frac{1}{2} |R_{2,k+1}| \]

and

\[ \frac{1}{|R_{2,k+1}|} \int_{R_{2,k+1}} |f| > 2^{m-1}. \]

In this manner the sequence $R_{2,1}, R_{2,2}, \ldots$ is generated.

Let $j_2 \geq j_1$ be an integer such that

\[ \left| \bigcup_{i=1}^{j_2} R_{2,i} \right| > \frac{1}{2} \left| \bigcup_{i=1}^{\infty} R_{2,i} \right| . \]

Note that $R_{1,1} = R_{2,1}, R_{1,2} = R_{2,2}, \ldots, R_{1,j_1} = R_{2,j_1}$.

We continue inductively. Assume that $R_{n,1}, R_{n,2}, \ldots, R_{n,j_n}$ have been selected. We renumerate $\beta$ as

\[ R_{n,1}, R_{n,2}, \ldots, R_{n,j_n}, R_{1,1}, R_{2,1}, R_{3,1}, \ldots. \] (*n+1)

Let $R_{n+1,1} = R_{n,1}$. Let $R_{n+1,2}$ be the first element on the list (*n+1) such that $|R_{n+1,2} \cap R_{n+1,1}| \leq \frac{1}{2} |R_{n+1,2}|$ and

\[ \frac{1}{|R_{n+1,2}|} \int_{R_{n+1,2}} |f| > 2^{m-n}. \]

Assuming $R_{n+1,1}, \ldots, R_{n+1,k}$ have been selected, let $R_{n+1,k+1}$ be the first element on the list (*n+1) such that

\[ |R_{n+1,k+1} \cap \bigcup_{i=1}^{k} R_{n+1,i}| \leq \frac{1}{2} |R_{n+1,k+1}| \]

and

\[ \frac{1}{|R_{n+1,k+1}|} \int_{R_{n+1,k+1}} |f| > 2^{m-n}. \]

In this manner, a sequence $R_{n+1,1}, R_{n+1,2}, \ldots$ is selected. Let $j_{n+1} \geq j_n$ be an integer such that

\[ \left| \bigcup_{i=1}^{j_{n+1}} R_{n+1,i} \right| > \frac{1}{2} \left| \bigcup_{i=1}^{\infty} R_{n+1,i} \right| . \]
Note that \( R_{n,1} = R_{n+1,1}, \ R_{n,2} = R_{n+1,2}, \ldots, \ R_{n,j_n} = R_{n+1,j_n} \). This is clear, as \( R_{n,1}, R_{n,2}, \ldots, R_{n,j_n} \) are the first \( j_n \) elements of \( \beta \) chosen in the procedure for selecting the \( R_{n+1,i} \).

We now relate \(|\bigcup_{i=1}^{j_n+1} R_{n+1,i}\)| to \(|\{x \in I^n : M_\beta f(x) > 2^{m-n}\}|\). Suppose for some \( p \in I^n \) that \( M_\beta f(p) > 2^{m-n} \). Then \( \frac{1}{|R|} \int_R |f| > 2^{m-n} \) for some \( R \in \beta \). Then \( |R \cap \bigcup_{i=1}^{j_n+1} R_{n+1,i}| \geq \frac{1}{2} |R| \). Hence

\[
M_\beta(\chi_{\bigcup_{i=1}^{j_n+1} R_{n+1,i}})(p) \geq \frac{1}{2}.
\]

Since

\[
|\{x \in I^n : M_\beta \chi_E(x) \geq \frac{1}{2}\}| \leq C_\beta |E|
\]

for all measurable sets \( E \subset I^n \), we see that

(4) \[
|\{x \in I^n : M_\beta f(x) > 2^{m-n}\}| \leq C_\beta \bigg| \bigcup_{i=1}^{j_n+1} R_{n+1,i} \bigg| \leq 2 C_\beta \bigg| \bigcup_{i=1}^{j_n+1} R_{n+1,i} \bigg|.
\]

We now let \( l \) be a positive integer such that \( 2^{m-l} < \epsilon/3 \). Then

\[
\left| \int_{\{x \in I^n : 2^{m-l} < M_\beta f(x) < 2^m\}} M_\beta f - \int_{I^n} M_\beta f \right| < \frac{2\epsilon}{3}.
\]

We now compare \( \int_{\{x \in I^n : 2^{m-l} < M_\beta f(x) < 2^m\}} M_\beta f \) to \( \int_{I^n} |f| \sum_{i=1}^{j_n+1} \chi_{R_{n+1,i}} \). Set

\[
\lambda(\alpha) = \left| \{x \in I^n : M_\beta f(x) > \alpha \} \right|,
\]

\[
\mu(\alpha) = \left| \{x \in I^n : |f(x)| \cdot \sum_{i=1}^{j_n+1} \chi_{R_{n+1,i}}(x) > \alpha \} \right|,
\]

\[
\omega(\alpha) = \left| \{x \in I^n : \sum_{i=1}^{j_n+1} \left( \frac{1}{|R_{n+1,i}|} \int_{R_{n+1,i}} |f| \right) \cdot \chi_{R_{n+1,i}}(x) > \alpha \} \right|.
\]

Suppose \( 2^{m-l} \leq \alpha \leq 2^m \). Let \( r \) be the largest integer such that \( 2^r \leq \alpha \). Clearly \( m-l \leq r \leq m \). Now

\[
R_{l+1,1} = R_{m-r+1,1}, \ R_{l+1,2} = R_{m-r+1,2}, \ldots, \ R_{l+1,j_m-r+1} = R_{m-r+1,j_m-r+1}.
\]

Also, by (4) we have

\[
\lambda(2^r) \leq 2 C_\beta \bigg| \bigcup_{i=1}^{j_m-r+1} R_{m-r+1,i} \bigg|.
\]

Since

\[
\frac{1}{|R_{m-r+1,i}|} \int_{R_{m-r+1,i}} |f| > 2^r \quad \text{for } i = 1, 2, \ldots, j_m-r+1,
\]
we get \( \lambda(2^r) \leq 2 C_\beta \omega(2^r) \). Hence \( \lambda(\alpha) \leq 2 C_\beta \omega \left( \frac{\alpha}{2} \right) \) for \( 2^{m-l} \leq \alpha \leq 2^m \). So

\[
\int_{\{x \in I^n: 2^{m-l} < M_\beta f(x) < 2^m\}} M_\beta f \leq \int_{2^{m-l}}^{2^m} \lambda(\alpha) \, d\alpha + \frac{\epsilon}{3}
\]

\[
\leq 2 C_\beta \int_{2^{m-l}}^{2^m} \omega \left( \frac{\alpha}{2} \right) \, d\alpha + \frac{\epsilon}{3}
\]

\[
\leq 4 C_\beta \int_{0}^{\infty} \omega(\alpha) \, d\alpha + \frac{\epsilon}{3}
\]

\[
= 4 C_\beta \int_{0}^{\infty} \mu(\alpha) \, d\alpha + \frac{\epsilon}{3}
\]

\[
= 4 C_\beta \int_{I^n} |f| \cdot \sum_{i=1}^{j_{i+1}} \chi_{R_{i+1, i}} \, d\alpha + \frac{\epsilon}{3}.
\]

Hence

\[
\int_{I^n} M_\beta f \leq 4 C_\beta \int_{I^n} |f| \cdot \sum_{i=1}^{j_{i+1}} \chi_{R_{i+1, i}} + \epsilon.
\]

As \( \epsilon \) is an arbitrary positive real number and \( \{R_{i+1, i}\} \in \text{CFC}(\beta) \), we see that

\[
\int_{I^n} M_\beta f \leq 4 C_\beta \sup_{\{R_i\} \in \text{CFC}(\beta)} \int_{I^n} |f| \cdot \sum_i \chi_{R_i},
\]

as desired.

We now show that

\[
\int_{I^n} M_\beta f \geq \frac{1}{2} \int_{I^n} |f| \cdot \sum_i \chi_{R_i},
\]

Let \( \{R_i\} \in \text{CFC}(\beta) \). Without loss of generality, we assume

\[
|R_i \cap \left( \bigcup_{j=1}^{i-1} R_j \right)| \leq \frac{1}{2} |R_i| \quad \text{for } i = 2, 3, \ldots.
\]

It suffices to show that

\[
\int_{I^n} M_\beta f \geq \frac{1}{2} \int_{I^n} |f| \cdot \sum_i \chi_{R_i}.
\]

Let \( E_1 = R_1 \) and \( E_k = R_k \setminus \bigcup_{i=1}^{k-1} R_i \) for \( k = 2, 3, \ldots \). Let

\[
Tf(x) = \sum_k \left( \frac{1}{|R_k|} \int_{R_k} |f(y)| \, dy \right) \chi_{E_k}(x).
\]
Clearly $Tf(x) \leq M_\beta f(x)$. Also,

$$
\int_{I^n} Tf = \int_{I^n} \sum_k \left( \frac{1}{|R_k|} \int_{R_k} |f(y)| \, dy \right) \chi_{E_k}(x) \, dx \\
= \sum_k \int_{I^n} \left( \frac{1}{|R_k|} \int_{R_k} |f(y)| \, dy \right) \chi_{E_k}(x) \, dx \\
\geq \frac{1}{2} \sum_k \int_{I^n} \left( \frac{1}{|R_k|} \int_{R_k} |f(y)| \, dy \right) \chi_{R_k}(x) \, dx \\
\quad \text{(since $|E_k| \geq \frac{1}{2} |R_k|$)} \\
= \frac{1}{2} \sum_k \int_{R_k} |f(y)| \, dy \\
= \frac{1}{2} \int_{I^n} |f| \sum_k \chi_{R_k}.
$$

So

$$
\int_{I^n} |f| \sum_i \chi_{R_i} \leq 2 \int_{I^n} M_\beta f,
$$
as desired.

To illustrate the role of the Tauberian condition in the above theorem, we consider the following example:

For $0 < \delta < \frac{1}{10}$, define $\beta_\delta$ by

$$
\beta_\delta = \left\{ A \subset [0,1] : A = [0,\delta] \cup [x, x + \delta^2] \text{ for some } x \in [0, 1 - \delta^2] \right\}.
$$

Note that $C_{\beta_\delta} \gtrsim \delta^{-1}$, since $M_{\beta_\delta}(\chi_{[0,\delta]})(x) > \frac{1}{2}$ for all $x \in [0,1]$. If $\{R_i\} \in \text{CFC}(\beta_\delta)$, then $\{R_i\}$ has only one element, say $R_1 = [0,\delta] \cup [x, x + \delta^2]$ for some $x \in [0, 1 - \delta^2]$. So

$$
\int_0^1 \chi_{[0,1]} \cdot \chi_{R_1} \leq 2 \delta.
$$

However,

$$
\int_0^1 M_{\beta_\delta} \chi_{[0,1]} = 1.
$$

Note that although the ratio of

$$
\int_0^1 M_{\beta_\delta} \chi_{[0,1]} \text{ to } \sup_{\{R_i\} \in \text{CFC}(\beta_\delta)} \int_Q \chi_{[0,1]} \cdot \sum_i \chi_{R_i}
$$

may be arbitrarily large (depending on the value of $\delta$), the ratio is still bounded by $4C_{\beta_\delta}$.

Before indicating applications of the preceding theorem, we list some basic definitions.
**Definition 3** (Hardy–Littlewood maximal function). Let $f$ be a measurable function defined on $\mathbb{R}^n$. Let $B(p,r)$ denote the Euclidean ball of radius $r$ in $\mathbb{R}^n$ centered at $p$, and let $|B(p,r)|$ denote the Lebesgue measure of $B(p,r)$. The Hardy–Littlewood maximal function of $f$ is defined on $\mathbb{R}^n$ by

$$M_{\text{HL}}f(p) = \sup_{r>0} \frac{1}{|B(p,r)|} \int_{B(p,r)} |f(z)| \, dz.$$  

**Definition 4** (Strong maximal function). Let $f$ be a measurable function defined on $\mathbb{R}^2$. The strong maximal function of $f$ is defined on $\mathbb{R}^2$ by

$$M_Sf(x,y) = \sup_{x_1<x<x_2, y_1<y<y_2} \frac{1}{(x_2-x_1)(y_2-y_1)} \int_{x_1}^{x_2} \int_{y_1}^{y_2} |f(u,v)| \, dv \, du.$$  

**Definition 5** (Horizontal maximal function). Let $f$ be a measurable function defined on $\mathbb{R}^2$. The horizontal maximal function of $f$ is defined on $\mathbb{R}^2$ by

$$M_xf(u,v) = \sup_{u_1<u<u_2} \frac{1}{u_2-u_1} \int_{u_1}^{u_2} |f(w,v)| \, dw.$$  

**Definition 6** (Vertical maximal function). Let $f$ be a measurable function defined on $\mathbb{R}^2$. The vertical maximal function of $f$ is defined on $\mathbb{R}^2$ by

$$M_yf(u,v) = \sup_{v_1<v<v_2} \frac{1}{v_2-v_1} \int_{v_1}^{v_2} |f(u,w)| \, dw.$$  

We now turn to one of the most useful applications of Córdoba–Fefferman collection theory. In this discussion we will denote the unit square $I^2$ in $\mathbb{R}^2$ by $Q$. Also, if a given maximal operator $M_\beta$ is naturally associated to a collection $\beta$, as, say, $M_S$ is associated to the set of rectangles with sides parallel to the axes, we will frequently denote the Córdoba–Fefferman collection $\text{CFC}(\beta)$ by $\text{CFC}(M_\beta)$.

Suppose we are given the maximal operators $M_\alpha$, $M_\beta$, and $M_\gamma$, all of which satisfy the desired Tauberian condition. Suppose also we want to show that $\int_Q M_\alpha f \lesssim \int_Q M_\beta f + \int_Q M_\gamma f$ for some measurable function $f$ supported on $Q$. One strategy for doing this would be to show that, given an arbitrary collection $\{A_i\} \in \text{CFC}(M_\alpha)$, one can produce $\{B_i\} \in \text{CFC}(M_\beta)$ and $\{C_i\} \in \text{CFC}(M_\gamma)$ such that

$$\sum_i \chi_{A_i}(p) \lesssim \sum_i \chi_{B_i}(p) + \sum_i \chi_{C_i}(p)$$

for almost every $p$ in $Q$. Theorem 2 would then yield the desired result. The primary difficulty in applying this strategy is that the production of the $\{B_i\} \in \text{CFC}(M_\beta)$ and $\{C_i\} \in \text{CFC}(M_\gamma)$ can be a complicated matter and in some situations may not be possible. However, in many cases one can modify this strategy by using the geometry of $f$ to
choose a particular \( \{A_i\} \in \text{CFC}(M_\alpha) \) in such a manner that not only 
\( \int_Q M_\alpha f \sim \int_Q |f| \sum_i \chi_{A_i} \), but also the production of the \( \{B_i\} \in \text{CFC}(M_\beta) \)
and \( \{C_i\} \in \text{CFC}(M_\gamma) \) can follow in a geometrically intuitive fashion. Actually proving that 
\( \int_Q M_\alpha f \sim \int_Q |f| \sum_i \chi_{A_i} \) often requires a duplication of 
large parts of the proof of Theorem 2 in the special case determined by 
the geometry of \( f \) and the desired properties of the collection \( \{A_i\} \). We illustrate these ideas in the proof of the following lemma:

**Lemma 7.** Let \( f \) be a nonnegative measurable function supported on \( Q \) such 
that \( f(x_1, y_1) \geq f(x_2, y_2) \) whenever \( 0 \leq x_1 \leq x_2, \ 0 \leq y_1 \leq y_2 \). Then 
\[
\int_Q M_{\text{HL}} f \leq C \left( \int_Q M_x f + \int_Q M_y f \right).
\]

for some universal constant \( C \).

**Proof.** Since \( M_{\text{HL}} \) and the operators \( M_x, M_y \) are bounded on \( L^2(Q) \), we 
may assume without loss of generality that \( f \) is smooth. Hence without loss 
of generality we may assume \( f \in L^\infty(Q) \). Let \( m \) be the largest integer such 
that \( 2^m \leq \|f\|_{L^\infty(Q)} \). Let
\[
E(2^{m-j+1}) = \{x \in Q : M_{\text{HL}} f(x) > 2^{m-j+1}\}.
\]

For each positive integer \( j \) and to each \( p \in E(2^{m-j+1}) \) associate a square 
\( Q_{p,m-j+1} \) containing \( p \) such that 
\[
\frac{1}{|Q_{p,m-j+1}|} \int_{Q_{p,m-j+1}} f > 2^{m-j+1}
\]

and such that one of the edges of \( Q_{p,m-j+1} \) is contained in one of the 
coordinate axes. Note that such a square exists, since \( f \) is nonincreasing in 
each variable separately. Now associate to \( Q_{p,m-j+1} \) a dyadic subsquare 
\( Q'_{p,m-j+1} \) contained in \( Q_{p,m-j+1} \) which has an edge contained in one of the 
coordinate axes, such that \( |Q'_{p,m-j+1}| \geq \frac{1}{16} |Q_{p,m-j+1}| \), and such that no 
dyadic subsquare of \( Q_{p,m-j+1} \) of the same size contains a point closer to the 
origin than any point of \( Q'_{p,m-j+1} \).

If \( Q'_{p,m-j+1} \) intersects the origin, let \( Q''_{p,m-j+1} \) be \( Q'_{p,m-j+1} \). Otherwise, 
let \( Q''_{p,m-j+1} \) be the dyadic square with the same area as \( Q'_{p,m-j+1} \) such 
that \( Q''_{p,m-j+1} \) and \( Q'_{p,m-j+1} \) share an edge, \( Q''_{p,m-j+1} \) has an edge 
contained in one of the coordinate axes, and such that \( Q''_{p,m-j+1} \) contains a point \( q \) which is closer to the origin than any point of \( Q'_{p,m-j+1} \). Note that
\[
\frac{1}{|Q''_{p,m-j+1}|} \int_{Q''_{p,m-j+1}} f > 2^{m-j+1}.
\]

Let \( 16Q''_{p,m-j+1} \) be the square concentric to \( Q''_{p,m-j+1} \) whose sides are 
parallel to the axes and whose volume is \( 16^2 \) times that of \( Q''_{p,m-j+1} \). Note 
that \( Q_{p,m-j+1} \subset 16Q''_{p,m-j+1} \).
For each positive integer $j$, let $Q_{1,m-j+1}, Q_{2,m-j+1}, \ldots$ enumerate the squares in \( \{Q''_{p,m-j+1} : p \in E(2^{m-j+1})\} \) that are not properly contained in any of the other squares in \( \{Q''_{p,m-j+1} : p \in E(2^{m-j+1})\} \). Since the $Q_{i,m-j+1}$ are dyadic, the interiors of the squares $Q_{1,m-j+1}, Q_{2,m-j+1}, \ldots$ are disjoint. Also

\begin{equation}
\left| \bigcup_{i=1}^{\infty} Q_{i,m-j+1} \right| \geq 2^{-8}|E(2^{m-j+1})|,
\end{equation}

since $E(2^{m-j+1}) \subset \bigcup_{i=1}^{\infty} 16Q_{i,m-j+1}$. Also, by (9), each $Q_{i,m-j+1}$ satisfies

\begin{equation}
\frac{1}{|Q_{i,m-j+1}|} \int_{Q_{i,m-j+1}} f > 2^{m-j+1}.
\end{equation}

For each positive integer $j$, let $\rho_{m-j+1}$ be a positive integer such that

\begin{equation}
\left| \bigcup_{i=1}^{\rho_{m-j+1}} Q_{i,m-j+1} \right| \geq \frac{1}{2} \left| \bigcup_{i=1}^{\infty} Q_{i,m-j+1} \right|.
\end{equation}

We now form the following sequence of dyadic squares: let $Q_1 = Q_{1,m}$, $Q_2 = Q_{2,m}$, \ldots, $Q_{\rho_m} = Q_{\rho_m,m}$, $Q_{\rho_m+1} = Q_{1,m-1}$, $Q_{\rho_m+2} = Q_{2,m-1}$, \ldots, $Q_{\rho_m+\rho_{m-1}} = Q_{\rho_{m-1},m-1}$, $Q_{\rho_m+\rho_{m-1}+1} = Q_{1,m-2}$, \ldots.

Let now $\tilde{Q}_1 = Q_1$. Let $\tilde{Q}_2$ be the first $Q$ on the list $Q_2, Q_3, \ldots$ such that $|Q \cap \tilde{Q}_1| \leq \frac{1}{2}|Q|$. Assuming $\tilde{Q}_1, \ldots, \tilde{Q}_k$ have been chosen, let $\tilde{Q}_{k+1}$ be the first $Q$ on the list $Q_1, Q_2, Q_3, \ldots$ such that $|Q \cap \bigcup_{i=1}^{k} \tilde{Q}_i| \leq \frac{1}{2}|Q|$. In this manner the sequence \( \{\tilde{Q}_i\} \) is generated.

Now, let $j_1 \leq j_2 \leq j_3 \leq \ldots$ be such that $Q \in \{\tilde{Q}_1, \tilde{Q}_2, \ldots, \tilde{Q}_{j_k}\}$ implies $(1/|Q|) \int_{Q} |f| > 2^{m-k+1}$ and

\[ \left| \bigcup_{i=1}^{j_k} \tilde{Q}_i \right| \geq \frac{1}{2} \left| \bigcup_{i=1}^{\infty} Q_{i,m-k+1} \right|.
\]

This is possible via (12) and the selection rule for the $\tilde{Q}_i$.

Note that (10) implies

\begin{equation}
\left| \bigcup_{i=1}^{j_k} \tilde{Q}_i \right| \geq \frac{1}{4} \cdot 2^{-8}|E(2^{m-k+1})|.
\end{equation}

Pick $\epsilon > 0$. Let $\ell > 1$ be an integer such that $2^{m-\ell+1} < \epsilon$. Then

\[ \left| \int_{\{x \in Q : 2^{m-\ell+1} < M_{HL}f(x) < 2^{m+1}\}} M_{HL}f - \int_{Q} M_{HL}f \right| < \epsilon. \]
We compare \( \int_{\{x \in Q: 2^{m-\ell+1} < M_{HL}f(x) < 2^{m+1}\}} M_{HL}f \) to \( \int_Q f \sum_{i=1}^{j_x} \chi_{\tilde{Q}_i} \). Let \\
\[ \lambda(\alpha) = \left| \{ x \in Q : M_{HL}f(x) > \alpha \} \right|, \]
\[ \mu(\alpha) = \left| \{ x \in Q : f(x) \sum_{i=1}^{j_x} \chi_{\tilde{Q}_i}(x) > \alpha \} \right|, \]
\[ \omega(\alpha) = \left| \{ x \in Q : \sum_{i=1}^{j_x} \left( \frac{1}{|\tilde{Q}_i|} \int_{\tilde{Q}_i} f \right) \chi_{\tilde{Q}_i}(x) > \alpha \} \right|. \]

Now, suppose \( 2^{m-\ell+1} \leq \alpha < 2^{m+1} \). Let \( r \) be the largest integer such that \( 2^r \leq \alpha \). Hence \( m-\ell+1 \leq r \leq m \). By (13) and the remarks preceding (13) we see that \( \lambda(2^r) \leq 4 \cdot 2^k \omega(2^r) \). Hence \( \lambda(\alpha) \leq 2^{11} \omega \left( \frac{\alpha}{2} \right) \) for \( 2^{m-\ell+1} \leq \alpha < 2^{m+1} \). So \\
\[ \int_{\{x \in Q: 2^{m-\ell+1} < M_{HL}f(x) < 2^{m+1}\}} M_{HL}f \]
\[ \leq \int_{2^{m-\ell+1}}^{2^{m+1}} \lambda(\alpha) \, d\alpha + \epsilon \leq 2^{11} \int_{2^{m-\ell+1}}^{2^{m+1}} \omega \left( \frac{\alpha}{2} \right) \, d\alpha + \epsilon \leq 2^{11} \int_{0}^{\infty} \omega(\alpha) \, d\alpha + \epsilon \]
\[ = 2^{11} \int_{0}^{\infty} \mu(\alpha) \, d\alpha + \epsilon = 2^{11} \int_Q f \left( \sum_{i=1}^{j_x} \chi_{\tilde{Q}_i} \right) + \epsilon. \]

Hence

(14) \[ \int_Q M_{HL}f \leq 2^{11} \int_Q f \left( \sum_{i=1}^{j_x} \chi_{\tilde{Q}_i} \right) + 2\epsilon. \]

For \( i = 1, 2 \) we generate a finite sequence \( \{\tilde{Q}_{i,j}\} \) as follows: Let \( \tilde{Q}_{i,1} \) be the first square \( Q \) on the list \( \tilde{Q}_1, \tilde{Q}_2, \ldots, \tilde{Q}_{j_x} \) which contains an element whose \( i \)-th component is 0 (i.e., there is an element \( p = (p_1, p_2) \in Q \) with \( p_i = 0 \)). For each positive integer \( j \), let \( \tilde{Q}_{i,j} \) be the \( j \)-th square on the list with this property (if such a square exists). Since each of the \( \tilde{Q}_i \) intersect one of the coordinate axes, each cube \( \tilde{Q}_i \) will be an element in at least one of the two sequences. Suppose now that for each \( i = 1, 2 \) the sequence \( \{\tilde{Q}_{i,j}\} \) has \( q_i \) squares. Then by (14),

(15) \[ \int_Q M_{HL}f \leq 2^{11} \int_Q f \left( \sum_{i=1}^{q_1} \chi_{\tilde{Q}_{1,i}} + \sum_{i=1}^{q_2} \chi_{\tilde{Q}_{2,i}} \right) + 2\epsilon. \]

For \( i = 1, 2 \) and each positive integer \( j \), define \( R_{i,j} \) as the collection of \( R \subseteq Q \) such that \( R \) is a rectangle with sides parallel to the axes, one of the edges of \( R \) with smallest length is parallel to the line \( x_i = 0 \), and one of the edges of \( R \) with smallest length has length \( 2^{-j} \). For example, an element of
 would be the rectangle with corners at the points \((0, \frac{1}{8}), (0, \frac{1}{4}), (\frac{1}{2}, \frac{1}{8})\), and \((\frac{1}{2}, \frac{1}{4})\). Define the maximal operators \(M_{i,j}\) by

\[
M_{i,j} f(x) = \sup_{x \in R \subseteq R_{i,j}} \frac{1}{|R|} \int_{R} |f(y)| \, dy.
\]

Since \(f \in C^\infty(Q)\), it is clear that

\[
\lim_{j \to \infty} \int_{Q} M_{i,j} f = \int_{Q} M_{x} f \quad \text{and} \quad \lim_{j \to \infty} \int_{Q} M_{2,j} f = \int_{Q} M_{y} f.
\]

For convenience, we will frequently denote \(M_{x} f\) by \(M_{1}\) and \(M_{y} f\) by \(M_{2}\). Hence the equalities above become

\[
\lim_{j \to \infty} \int_{Q} M_{i,j} f = \int_{Q} M_{i} f.
\]

It follows that there exists an integer, designated by \(j_{0}\), such that

\[
\left| \int_{Q} M_{i,j_{0}} f - \int_{Q} M_{i,j} f \right| < \frac{\epsilon}{2^{13}} \quad \text{for} \quad i = 1, 2.
\]

We assume without loss of generality that

\[
2^{-j_{0}} \leq \inf \{|\tilde{Q}_{1}|^{1/2}, |\tilde{Q}_{2}|^{1/2}, \ldots, |\tilde{Q}_{\ell}|^{1/2}\}.
\]

We now show that for each \(i, j = 1, 2\),

\[
\int_{Q} f \sum_{j=1}^{q_{i}} \chi_{\tilde{Q}_{i,j}} \leq 2 \int_{Q} M_{i,j_{0}} f.
\]

Let \(\tilde{R}_{i,1}, \ldots, \tilde{R}_{i,2^{j_{0}} |\tilde{Q}_{i,j}|^{1/2}}\) be the \(2^{j_{0}} |\tilde{Q}_{i,j}|^{1/2}\) disjoint rectangles in \(R_{i,j_{0}}\) of equal area whose union is \(\tilde{Q}_{i,j}\). Let \(\gamma_{ij} = 2^{j_{0}} |\tilde{Q}_{i,j}|^{1/2}\). For \(i = 1, 2\), let \(\tilde{R}_{i,1} = \tilde{R}_{i,1,1}, \tilde{R}_{i,2} = \tilde{R}_{i,1,2}, \ldots, \tilde{R}_{i,\gamma_{1}+1} = \tilde{R}_{i,1,\gamma_{1}}, \tilde{R}_{i,\gamma_{1}+1} = \tilde{R}_{i,2,1}, \ldots, \tilde{R}_{i,\gamma_{1}+\gamma_{2}} = \tilde{R}_{i,2,\gamma_{2}}, \tilde{R}_{i,\gamma_{1}+\gamma_{2}+1} = \tilde{R}_{i,3,1}, \ldots, \tilde{R}_{i,\gamma_{1}+\gamma_{2}+\gamma_{3}+\gamma_{i_{1}}} = \tilde{R}_{i,q_{i},\gamma_{i_{1}}}\).

Since for \(i = 1, 2\) the squares \(\tilde{Q}_{i,j}\) for \(j = 1, \ldots, q_{i}\) are all dyadic and intersect the line \(\{x_{i} = 0\}\), the selection rule for the \(R_{k}\) yields

\[
\left| \tilde{R}_{i,j} \cap \bigcup_{k=1}^{j_{0}-1} \tilde{R}_{i,k} \right| \leq \frac{1}{2} |\tilde{R}_{i,j}|
\]

for \(j = 2, 3, \ldots, \gamma_{1}+\cdots+\gamma_{q_{i}}\).

Let \(\gamma_{i} = \gamma_{1}+\cdots+\gamma_{q_{i}}\). Now, (18) implies that \(\{R_{i,1,1}, R_{i,2,1}, \ldots, R_{i,\gamma_{1}}\} \subseteq \text{CFC} (R_{i,j_{0}})\). Hence Theorem 2 tells us that

\[
\int_{Q} |f| \sum_{j=1}^{q_{i}} \chi_{\tilde{R}_{i,j}} \leq 2 \int_{Q} M_{i,j_{0}} f.
\]
Now by the construction of the $\tilde{R}_{i,j}$ it is clear that

$$\sum_{j=1}^{q_i} \chi_{\tilde{R}_{i,j}} = \sum_{j=1}^{q_i} \chi_{\tilde{Q}_{i,j}}. \tag{20}$$

(19) and (20) then yield

$$\int_Q |f| \sum_{j=1}^{q_i} \chi_{\tilde{Q}_{i,j}} \leq 2 \int_Q M_{i,0} f. \tag{21}$$

(15), (17), and (21) yield

$$\int_Q M_{\text{HL}} f \leq 2^{12} \left( \int_Q M_1 f + \int_Q M_2 f \right) + 3\epsilon. \tag{22}$$

As $\epsilon$ is arbitrarily small, we see that

$$\int_Q M_{\text{HL}} f \leq C \left( \int_Q M_1 f + \int_Q M_2 f \right) \tag{23}$$

for some universal constant $C$, completing the proof. $\square$

The preceding lemma and the following rearrangement result will enable us to prove that if $f$ is a measurable function supported on $Q$, then $\int_Q M_{\text{HL}} f \sim \int_Q M_x f + \int_Q M_y f$.

**Lemma 8.** Let $f$ be a nonnegative measurable function supported on $Q$. Let $f$ be the function supported on $Q$ which is nonincreasing in $x$ (i.e., $f(x_1, y) \geq f(x_2, y)$ whenever $0 \leq x_1 \leq x_2 \leq 1$, $0 \leq y \leq 1$) and such that, for each $y \in [0, 1]$, $f(\cdot, y)$ and $f(\cdot, y)$ are equidistributed. Then

$$\int_Q M_y f \leq c \int_Q M_y f,$$

where $c$ is a universal constant.

**Proof.** Let $\alpha > 0$. Let $\lambda(\alpha) = |\{(u, v) \in Q : M_y f(u, v) > \alpha\}|$. Define $\tilde{\lambda}(\alpha)$ similarly. It suffices to show that $\tilde{\lambda}(\alpha) \leq 400 \lambda(\alpha/64)$.

Without loss of generality, assume $f$ is smooth on $Q$. Take the Calderón–Zygmund decomposition of $f$ with respect to $\alpha$ on each vertical segment in $\{s \times [0, 1], s \in [0, 1]\}$ of $Q$, yielding for each $x \in [0, 1]$ disjoint sets $Q_{x,j,\alpha} \subseteq [0, 1]$ such that

$$\alpha < \frac{1}{|Q_{x,j,\alpha}|} \int_{Q_{x,j,\alpha}} f(x, z) \, dz \leq 2\alpha.$$

(In the case that $\int_0^1 f(x, z) \, dz > 2\alpha$, set $Q_{x,1,\alpha} = [0, 1]$.) Note that $f(p) \leq \alpha$ for almost every $p$ in the complement of

$$\bigcup_{x \in [0, 1]} (x \times Q_{x,j,\alpha}).$$
For \( \tilde{f} \) one may produce the associated sets \( \tilde{Q}_{x,j,\alpha} \) in a similar fashion.

Let \( E_\alpha = \{ (x, y) \in Q : y \in \bigcup_{x \in [0,1], j \in \mathbb{Z}} Q_{x,j,\alpha} \} \). Define \( \tilde{E}_\alpha \) similarly. It suffices to show that \( |\tilde{E}_{4\alpha}| \leq 2|E_\alpha| \). Now, it is easily seen that if \( g \) is a measurable function supported on \([0, 1]\), \( g \geq 0 \), \( \int_0^1 g \leq 2\alpha \), \( \lambda_{HL}(\alpha) = |\{ x \in [0, 1] : MHLg(x) > \alpha \}| \), and \( E_{HL,\alpha} = \bigcup Q_{j,\alpha} \), where the \( Q_{j,\alpha} \) are the intervals obtained by taking the Calderón–Zygmund decomposition of \( g \) with respect to \( \alpha \), then \( |E_{HL,\alpha}| \leq \lambda_{HL}(\alpha/2) \leq 200|E_{HL,\alpha}/8| \). From this we readily conclude that \( |\tilde{E}_{4\alpha}| \leq 2|E_\alpha| \) implies \( \lambda(\alpha) \leq 200|\tilde{E}_\alpha/8| \leq 400|E_{\alpha/32}| \leq 400 \lambda(\alpha/64) \), as desired.

To show that \( |\tilde{E}_{4\alpha}| \leq 2|E_\alpha| \) we proceed as follows. First consider the special case in which \( \int_0^1 f(x, y) \, dy \leq \alpha \) for any \( x \in [0, 1] \). Having taken the Calderón–Zygmund decomposition of \( f \) with respect to \( \alpha \) described above, we have the disjoint sets \( Q_{x,j,\alpha} \subset [0, 1] \) for each \( x \in [0, 1] \) and the associated set \( E_\alpha \). Now, \( f(p) \leq \alpha \) for almost every \( p \) in the complement of \( E_\alpha \). So if \( S \) is a measurable subset of \( Q \) and \( |S| > |E_\alpha| \), then \( \int_S f < 2\alpha |S| \). Now let \( \phi : Q \to Q \) be a measure-preserving bijection such that \( f(\phi(p)) = f(p) \) for any \( p \in Q \). Using \( \phi \) we see that \( |\tilde{E}_{4\alpha}| \leq 2|E_\alpha| \). Otherwise, if \( |\tilde{E}_{4\alpha}| > 2|E_\alpha| \), we would have

\[
\frac{1}{|\tilde{E}_{4\alpha}|} \int_{\tilde{E}_{4\alpha}} \tilde{f} = \frac{1}{|E_\alpha|} \int_{\phi^{-1}(\tilde{E}_{4\alpha})} f \leq 2\alpha
\]

by the above; but the left-hand side is greater than \( 4\alpha \) by the construction of \( \tilde{E}_{4\alpha} \). So \( |\tilde{E}_{4\alpha}| \leq 2|E_\alpha| \) if \( \int_0^1 f(x, y) \, dy \leq \alpha \) for all \( x \in [0, 1] \).

Now we let \( f \) be an arbitrary nonnegative smooth function on \( Q \). Without loss of generality assume there exists \( c \in (0, 1) \) such that \( \int_0^1 f(x, y) \, dy > \alpha \) if \( x < c \), and \( \int_0^1 f(x, y) \, dy \leq \alpha \) if \( x \geq c \). Form the Calderón–Zygmund decomposition of \( f \) with respect to \( \alpha \) as before, obtaining the \( Q_{x,j,\alpha} \) and \( E_\alpha \). Note that \( Q_{x,1,\alpha} = [0, 1] \) if \( x < c \).

For each \( y \in [0, 1] \) we define the functions \( f_y(x) \) on \([0, 1]\) by \( f_y(x) = f(x, y) \). We construct a function \( f'_y(x) \) on \([0, 1]\) equidistributed to \( f_y(x) \) such that \( f'_y(x_2) \leq f'_y(x_1) \) if \( x_2 \geq x_1 \) and \( f'_y(x) \leq f_y(x) \) if \( x \geq c \) as follows:

Let \( B_y = \{ x \in [0, c) : f_y(x) < \tilde{f}_y(c) \} \).

Let \( A_y \subset \{ x \in [c, 1] : f_y(x) \geq \tilde{f}_y(c) \} \) be such that the measure of its interior is equal to \( |B_y| \). Let \( A^\circ_y \) and \( B^\circ_y \) denote the interiors of \( A_y \) and \( B_y \), respectively. Let \( \phi_y : A^\circ_y \to B^\circ_y \) be a measure-preserving bijection such that \( \{ b \in B^\circ_y : b < \phi_y(x) \} = \{ a \in A^\circ_y : a < x \} \) if \( x \in A^\circ_y \). Define \( f'_y(x) \) by

\[
f'_y(x) = \begin{cases} f_y(x) & \text{if } x \notin A^\circ_y \cup B^\circ_y, \\ f_y(\phi_y^{-1}(x)) & \text{if } x \in B^\circ_y, \\ f_y(\phi_y(x)) & \text{if } x \in A^\circ_y. \end{cases}
\]
Note that $f'_y(x) \leq f_y(x)$ if $x > c$. Define a function $f'$ on $Q$ by $f'(x, y) = f'_y(x)$. Form the Calderón–Zygmund decomposition of $f'$ with respect to $\alpha$ as above, obtaining the associated sets $Q'_{x,j,\alpha}$, $E'_\alpha$. Note that $E_\alpha \supseteq E'_\alpha$, so without loss of generality we may assume $f = f'$. Hence, without loss of generality, $f(x_1, y_1) \geq f(x_2, y_2)$ if $0 \leq x_1 < c \leq x_2 \leq 1$, and $\int_0^1 f(x, y) \, dy > \alpha$ if and only if $x < c$.

Let $f_1 = f \chi_{[0 \leq x < c]}$, $f_2 = f \chi_{[c \leq x \leq 1]}$. So $f = f_1 + f_2$. Let $\tilde{f}_1$ be a rearrangement of $f_1$ such that, for each $y \in [0, 1]$, $f_1(\cdot, y)$ and $\tilde{f}_1(\cdot, y)$ are equidistributed and $\tilde{f}_1(x, y)$ is nonincreasing in $x$. Define $\tilde{f}_2$ to be the rearrangement of $f_2$ within $\{Q \cap \{(x, y) : c \leq x \leq 1\}\}$ such that, for each $y \in [0, 1]$, the functions $f_2(\cdot, y)$ and $\tilde{f}_2(\cdot, y)$ are equidistributed and such that $\tilde{f}_2(x, y)$ is nonincreasing in $x$ in $\{x : c \leq x \leq 1\}$. Now $f = \tilde{f}_1 + \tilde{f}_2$. Let

$$E_{1,\alpha} = \bigcup_{x \in [0, c)} \bigcup_{j \in \mathbb{Z}_+} (x \times Q_{x,j,\alpha}), \quad E_{2,\alpha} = \bigcup_{x \in [c, 1]} \bigcup_{j \in \mathbb{Z}_+} (x \times Q_{x,j,\alpha}).$$

Define $\tilde{E}_{1,\alpha}$ and $\tilde{E}_{2,\alpha}$ similarly. Note that $|\tilde{E}_{1,4\alpha}| \leq 2|E_{1,\alpha}|$ trivially (since $Q_{x,1,\alpha} = [0, 1]$ if $x < c$) and $|\tilde{E}_{2,4\alpha}| \leq 2|E_{2,\alpha}|$ by the special case argument, since $\int_0^1 f(x, y) \, dy \leq \alpha$ for $x \geq c$. Since $|E_\alpha| = |E_{1,\alpha}| + |E_{2,\alpha}|$ and $|\hat{E}_\alpha| = |\tilde{E}_{1,\alpha}| + |\tilde{E}_{2,\alpha}|$, we see that $|\hat{E}_{4\alpha}| \leq 2|E_\alpha|$, as desired.

**Theorem 9.** Suppose $f$ is a measurable function supported on $Q$. Then

$$c \int_Q M_{HL} f \leq \int_Q M_x f + \int_Q M_y f \leq C \int_Q M_{HL} f$$

for universal constants $0 < c, C < \infty$.

**Proof.** From [4] we see that $\int_Q M_x f \lesssim \int_Q M_{HL} f$ and $\int_Q M_y f \lesssim \int_Q M_{HL} f$. Hence it suffices to show that $\int_Q M_{HL} f \lesssim \int_Q M_x f + \int_Q M_y f$. We may assume without loss of generality that $f$ is nonnegative. Let $\tilde{f}(x, y)$ be the function supported on $Q$ which is nonincreasing in $x$ and such that $\tilde{f}(\cdot, y)$ and $f(\cdot, y)$ are equidistributed for each $y \in [0, 1]$. Let $f^*(x, y)$ be the function supported on $Q$ which is nonincreasing in $y$ and such that $f^*(x, \cdot)$ and $\tilde{f}(x, \cdot)$ are equidistributed for each $x \in [0, 1]$. Note that $f^*(x_1, y_1) \geq f^*(x_2, y_2)$ whenever $0 \leq x_1 \leq x_2 \leq 1$, $0 \leq y_1 \leq y_2 \leq 1$. As $\int_Q M_{HL} f^* \sim \int_Q M_{HL} f$, $\int_Q M_y f^* \sim \int_Q M_y f$, and $\int_Q M_x f \sim \int_Q M_x f$ by Stein’s $L \log L$
result [5], we see that
\[ \int_Q M_{HL} f \sim \int_Q M_{HL} f^* \]
\[ \lesssim \int_Q M_x f^* + \int_Q M_y f^* \quad (\text{Lemma 7}) \]
\[ \lesssim \int_Q M_x \tilde{f} + \int_Q M_y \tilde{f} \quad (\text{Lemma 8}) \]
\[ \lesssim \int_Q M_x f + \int_Q M_y f \quad (\text{Lemma 8}), \]
as desired. \qed

Building upon these ideas, more sophisticated applications of Córdoa–Fefferman collections are used collectively in [2] and [3] to prove that if \( f \) is a function supported on \( Q \) such that \( \int_Q M_y M_x f < \infty \) but \( \int_Q M_x M_y f = \infty \), there exists a set \( A \) of finite measure in \( \mathbb{R}^2 \) such that \( \int_A M_S f = \infty \).

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REARRANGEMENTS AND THE LOCAL INTEGRABILITY OF MAXIMAL FUNCTIONS

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Let $M_{HL}$ and $M_S$ respectively denote the Hardy–Littlewood and strong maximal operators, and let $M_x$ and $M_y$ respectively denote the one-dimensional Hardy–Littlewood maximal operators in the horizontal and vertical directions in $\mathbb{R}^2$. It is well known that if $f$ and $\tilde{f}$ are equidistributed functions supported on $Q = [0, 1] \times [0, 1]$, then $\int_Q M_{HL}f \sim \int_Q M_{HL}\tilde{f}$. This article examines the relationships between $\int_Q M_yf$, $\int_Q M_yM_xf$, $\int_Q M_yM_xf$, and $\int_Q M_Sf$ in the scenario in which $\tilde{f}$ and $f$ are horizontal rearrangements of one another, meaning that $\tilde{f}(\cdot, y)$ and $f(\cdot, y)$ are equidistributed on $[0, 1]$ for any value of $y$.

The rearrangement results provided are not only of intrinsic interest, but also yield tools for more detailed examinations involving the local integrability of maximal functions. They are used in a companion paper to prove that if $f$ is supported on $Q$, $\int_Q M_yM_xf < \infty$, and $\int_Q M_yM_xf = \infty$, then there exists a set $A$ of finite measure in $\mathbb{R}^2$ such that $\int_A M_Sf = \infty$.

We begin with the following definitions:

**Definition 1** (Hardy–Littlewood maximal function). Let $f$ be a measurable function defined on $\mathbb{R}^n$. Let $B(p, r)$ denote the Euclidean ball in $\mathbb{R}^n$ centered at $p$ of radius $r$, and let $|B(p, r)|$ denote the Lebesgue measure of $B(p, r)$. The Hardy–Littlewood maximal function of $f$ is defined on $\mathbb{R}^n$ by

\[
M_{HL}f(p) = \sup_{r > 0} \frac{1}{|B(p, r)|} \int_{B(p, r)} |f(z)| \, dz.
\]

**Definition 2** (Strong maximal function). Let $f$ be a measurable function defined on $\mathbb{R}^2$. The strong maximal function of $f$ is defined on $\mathbb{R}^2$ by

\[
M_{S}f(x, y) = \sup_{x_1 < x < x_2, y_1 < y < y_2} \frac{1}{(x_2 - x_1)(y_2 - y_1)} \int_{x_1}^{x_2} \int_{y_1}^{y_2} |f(u, v)| \, dv \, du.
\]

**Definition 3** (Horizontal maximal function). Let $f$ be a measurable function defined on $\mathbb{R}^2$. The horizontal maximal function of $f$ is defined on $\mathbb{R}^2$ by
by
\[
M_x f(u, v) = \sup_{u_1 < u < u_2} \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} |f(w, v)| \, dw.
\]

**Definition 4** (Vertical maximal function). Let \( f \) be a measurable function defined on \( \mathbb{R}^2 \). The vertical maximal function of \( f \) is defined on \( \mathbb{R}^2 \) by
\[
M_y f(u, v) = \sup_{v_1 < v < v_2} \frac{1}{v_2 - v_1} \int_{v_1}^{v_2} |f(u, w)| \, dw.
\]

The first rearrangement result we consider is a simple consequence of the following result due to E. M. Stein [6]:

**Theorem 5.** Let \( f \) be a measurable function supported on \( I^n \), the unit \( n \)-cube in \( \mathbb{R}^n \). There exist positive, finite constants \( c, C \) (depending on \( n \)) such that
\[
c\|f\|_{L^\log L(I^n)} \leq \int_{I^n} M_{HL} f \leq C\|f\|_{L^\log L(I^n)}.
\]

Inequalities such as (5) will often be denoted by
\[
\|f\|_{L^\log L(I^n)} \sim \int_{I^n} M_{HL} f
\]
for the remainder of this paper. Also, the unit square \( I^2 \) in \( \mathbb{R}^2 \) will be denoted by \( Q \).

**Corollary 6.** Let \( f \) and \( \widetilde{f} \) be equidistributed functions supported on \( I^n \). Then
\[
\int_{I^n} M_{HL} f \sim \int_{I^n} M_{HL} \widetilde{f}.
\]

*Proof.* As \( \|f\|_{L^\log L(I^n)} = \|\widetilde{f}\|_{L^\log L(I^n)} \), this follows directly from Theorem 5. \( \square \)

**Corollary 7.** Suppose \( f \) and \( \widetilde{f} \) are functions supported on \( Q \) and also suppose that \( f(\cdot, y) \) and \( \widetilde{f}(\cdot, y) \) are equidistributed for each \( y \in [0, 1] \). Then
\[
\int_{Q} M_x \widetilde{f} \sim \int_{Q} M_x f.
\]

*Proof.* This is an application of Corollary 6 and the Fubini theorem. \( \square \)

Far more interesting is the relationship between \( \int_{Q} M_y f \) and \( \int_{Q} M_y \widetilde{f} \) when \( f \) and \( \widetilde{f} \) are horizontal rearrangements of each other, i.e., \( \widetilde{f}(\cdot, y) \) and \( f(\cdot, y) \) are equidistributed on \( [0, 1] \) for any value of \( y \). In general these integrals are not comparable. For example, define \( g \) by
\[
g(x, y) = \frac{1}{\|x - y\|(\log |x - y|)^2} \chi_Q.
\]
Let \( \tilde{g} \) be the function supported on \( Q \) which is a horizontal rearrangement of \( g \) and such that \( \tilde{g}(x, y) \) is nonincreasing in \( x \), i.e., \( \tilde{g}(x_1, y) \geq \tilde{g}(x_2, y) \) for any \( 0 \leq x_1 \leq x_2 \leq 1 \) and any value of \( y \). One can readily compute that \( \int_Q M_y \tilde{g} < \infty \) but \( \int_Q M_y g = \infty \). We do have the following result, however:

**Theorem 8.** Let \( f \) be a nonnegative measurable function supported on \( Q \). Let \( \tilde{f} \) be the function supported on \( Q \) which is nonincreasing in \( x \) and such that, for each \( y \in [0, 1] \), \( f(\cdot, y) \) and \( f(\cdot, y) \) are equidistributed.

Then

\[
\int_Q M_y \tilde{f} \leq c \int_Q M_y f,
\]

where \( c \) is a universal constant.

**Proof.** Let \( \alpha > 0 \). Let \( \lambda(\alpha) = \{|(u, v) \in Q : M_y f(u, v) > \alpha\}|. \) Define \( \tilde{\lambda}(\alpha) \) similarly. It suffices to show that \( \tilde{\lambda}(\alpha) \leq 400 \lambda(\alpha/64) \).

Without loss of generality, assume \( f \) is smooth on \( Q \). Take the Calderón–Zygmund decomposition of \( f \) with respect to \( \alpha \) on each vertical segment in \( \{s \times [0, 1], s \in [0, 1]\} \) of \( Q \), yielding for each \( x \in [0, 1] \) disjoint sets \( Q_{x,j,\alpha} \subseteq [0, 1] \) such that \( \alpha < |Q_{x,j,\alpha}|^{-1} \int_{Q_{x,j,\alpha}} f(x, z) \, dz \leq 2\alpha \). (In the case that \( \int_0^1 f(x, z) \, dz > 2\alpha \), set \( Q_{x,1,\alpha} = [0, 1] \).) Note that \( f(p) \leq \alpha \) for almost every \( p \) in the complement of \( \bigcup_{x \in [0, 1], j \in \mathbb{Z}_+} (x \times Q_{x,j,\alpha}) \). For \( \tilde{f} \) one may produce the associated sets \( \tilde{Q}_{x,j,\alpha} \) in a similar fashion.

Let \( E_\alpha = \{(x, y) \in Q : y \in \bigcup_{x \in [0, 1], j \in \mathbb{Z}_+} Q_{x,j,\alpha}\} \). Define \( \tilde{E}_\alpha \) similarly. It suffices to show that \( |\tilde{E}_{4\alpha}| \leq 2|E_\alpha| \). To see this, recall from the theory of the Hardy–Littlewood maximal operator \( M_{HL} \) that if \( g \) is a measurable function supported on the unit interval \( [0, 1] \), \( g \geq 0 \), \( \int_0^1 g \leq 2\alpha \),

\[
\lambda_{HL}(\alpha) = |\{x \in [0, 1] : M_{HL} g(x) > \alpha\}|,
\]

and \( E_{HL,\alpha} = \bigcup Q_{j,\alpha} \), where the \( Q_{j,\alpha} \) are the intervals obtained by taking the Calderón–Zygmund decomposition of \( g \) with respect to \( \alpha \), then \( |E_{HL,\alpha}| \leq \lambda_{HL}(\alpha/2) \leq 200 |E_{HL,\alpha}|/8 \). This readily yields that, if \( |\tilde{E}_{4\alpha}| \leq 2|E_\alpha| \), then \( \tilde{\lambda}(\alpha) \leq 200 \tilde{\lambda}(\alpha)/8 \leq 400 |E_{\alpha}/32| \leq 400 \lambda(\alpha/64) \). Hence \( \lambda(\alpha) \leq 400 \lambda(\alpha/64) \), as desired.

To show that \( |\tilde{E}_{4\alpha}| \leq 2|E_\alpha| \) we proceed as follows:

First we consider the special case in which \( \int_0^1 f(x, y) \, dy \leq \alpha \) for any \( x \in [0, 1] \). Having taken the Calderón–Zygmund decomposition of \( f \) with respect to \( \alpha \) described above, we obtain the disjoint sets \( Q_{x,j,\alpha} \subseteq [0, 1] \) for each \( x \in [0, 1] \) and the associated set \( E_\alpha \). Now, \( f(p) \leq \alpha \) for almost every \( p \) in the complement of \( E_\alpha \). So if \( S \) is a measurable subset of \( Q \) and \( |S| > 2|E_\alpha| \), then \( \int_S f \leq 2\alpha |S| \). Now let \( \phi : Q \to Q \) be a measure-preserving bijection such that \( f(\phi(p)) = f(p) \) for any \( p \in Q \). Using \( \phi \) we see that \( |\tilde{E}_{4\alpha}| \leq 2|E_\alpha| \). Otherwise, if \( |\tilde{E}_{4\alpha}| > 2|E_\alpha| \) we would have \( |\tilde{E}_{4\alpha}|^{-1} \int_{\tilde{E}_{4\alpha}} f = \int_Q f \leq \alpha \).
$|\tilde{E}_{4\alpha}|^{-1} \int_{\phi^{-1}(\tilde{E}_{4\alpha})} f \leq 2\alpha$ by the above. But $|\tilde{E}_{4\alpha}|^{-1} \int_{\tilde{E}_{4\alpha}} \tilde{f} > 4\alpha$ by the construction of $\tilde{E}_{4\alpha}$. So $|\tilde{E}_{4\alpha}| \leq 2|E_{\alpha}|$ if $\int_{0}^{1} f(x, y) \, dy \leq \alpha$ for all $x \in [0, 1]$.

Now we let $f$ be an arbitrary nonnegative smooth function on $Q$. Without loss of generality assume there exists $c \in (0, 1)$ such that $\int_{0}^{1} f(x, y) \, dy > \alpha$ if $x < c$, and $\int_{0}^{1} f(x, y) \, dy \leq \alpha$ if $x \geq c$. Form the Calderón–Zygmund decomposition of $f$ with respect to $\alpha$ as before, obtaining the $Q_{x,j,\alpha}$ and $E_{\alpha}$. Note that $Q_{x,1,\alpha} = [0, 1]$ if $x < c$.

For each $y \in [0, 1]$ we define the functions $f_{y}(x)$ on $[0, 1]$ by $f_{y}(x) = f(x, y)$. We construct a function $f'_{y}(x)$ on $[0, 1]$ equidistributed to $f_{y}(x)$ such that $f'_{y}(x) \leq f_{y}(x)$ if $x_{2} \geq c \geq x_{1}$ and $f'_{y}(x) \leq f_{y}(x)$ if $x \geq c$ as follows:

Let $B_{y} = \{ x \in [0, c) : f_{y}(x) < \tilde{f}_{y}(c) \}$. Let $A_{y} \subset \{ x \in [c, 1) : f_{y}(x) \geq \tilde{f}_{y}(c) \}$ be such that the measure of its interior is equal to $|B_{y}|$. Let $A_{y}^{o}$ and $B_{y}^{o}$ respectively denote the interiors of $A_{y}$ and $B_{y}$. Let $\phi_{y} : A_{y}^{o} \to B_{y}^{o}$ be a measure-preserving bijection such that if $x \in A_{y}^{o}$, $\{ b \in B_{y}^{o} : b < \phi_{y}(x) \} = |\{ a \in A_{y}^{o} : a < x \}|$. Define $f'_{y}(x)$ by

$$f'_{y}(x) = \begin{cases} f_{y}(x) & \text{if } x \notin A_{y}^{o} \cup B_{y}^{o}, \\ f_{y}(\phi_{y}^{-1}(x)) & \text{if } x \in B_{y}^{o}, \\ f_{y}(\phi_{y}(x)) & \text{if } x \in A_{y}^{o}. \end{cases}$$

Note that $f'_{y}(x) \leq f_{y}(x)$ if $x > c$. Define the function $f'$ on $Q$ by $f'(x, y) = f'_{y}(x)$. For the Calderón–Zygmund decomposition of $f'$ with respect to $\alpha$ as above, obtaining the associated sets $Q'_{x,j,\alpha}$, $E'_{\alpha}$. Note that $E_{\alpha} \supseteq E'_{\alpha}$; so without loss of generality we may assume $f = f'$. Hence, without loss of generality, $f(x_{1}, y) \geq f(x_{2}, y)$ if $0 \leq x_{1} < c \leq x_{2} \leq 1$, and $\int_{0}^{1} f(x, y) \, dy > \alpha$ if and only if $x < c$.

Let $f_{1} = f \chi_{[0 \leq x < c]}$, $f_{2} = f \chi_{[c \leq x \leq 1]}$. So $f = f_{1} + f_{2}$. Let $\tilde{f}_{1}$ be a rearrangement of $f_{1}$ such that, for each $y \in [0, 1]$, $f_{1}(\cdot, y)$ and $f_{1}(\cdot, y)$ are equidistributed and $\tilde{f}_{1}(x, y)$ is nonincreasing in $x$. Define $\tilde{f}_{2}$ to be the rearrangement of $f_{2}$ within $\{ Q \cap \{ (x, y) : c \leq x \leq 1 \} \}$ such that, for each $y \in [0, 1]$, the functions $f_{2}(\cdot, y)$ and $f_{2}(\cdot, y)$ are equidistributed and such that $f_{2}(x, y)$ is nonincreasing in $x$ in $\{ x : c \leq x \leq 1 \}$. Now $\tilde{f} = \tilde{f}_{1} + \tilde{f}_{2}$. Let

$$E_{1,\alpha} = \bigcup_{x \in [0,c)} \{ x \times Q_{x,j,\alpha} \}, \quad E_{2,\alpha} = \bigcup_{x \in [c,1]} \{ x \times Q_{x,j,\alpha} \}.$$}

Define $\tilde{E}_{1,\alpha}$ and $\tilde{E}_{2,\alpha}$ similarly. Note that $|\tilde{E}_{1,4\alpha}| \leq 2|E_{1,\alpha}|$ trivially (as $Q_{x,1,\alpha} = [0, 1]$ if $x < c$) and $|\tilde{E}_{2,4\alpha}| \leq 2|E_{2,\alpha}|$ by the special case argument, since $\int_{0}^{1} f(x, y) \, dy \leq \alpha$ for $x \geq c$. As $|E_{\alpha}| = |E_{1,\alpha}| + |E_{2,\alpha}|$ and $|\tilde{E}_{\alpha}| = |\tilde{E}_{1,\alpha}| + |\tilde{E}_{2,\alpha}|$, we see that $|\tilde{E}_{4\alpha}| \leq 2|E_{\alpha}|$, as desired. $\square$
Theorem 8 yields the following two easy but useful corollaries:

**Corollary 9.** Let $f$ be a nonnegative measurable function supported on $Q$. Let $\tilde{f}(x,y)$ be the function supported on $Q$ which is nonincreasing in $x$ and such that, for each $y \in [0,1]$, $\tilde{f}(\cdot,y)$ and $f(\cdot,y)$ are equidistributed. Then

$$\int_Q M_{HL} M_y \tilde{f} \leq c \int_Q M_{HL} M_y f,$$

where $c$ is a universal constant.

**Proof.** Define $\lambda(\alpha), \tilde{\lambda}(\alpha)$ as in the proof of Theorem 8. As the proof of Theorem 8 indicated that $\tilde{\lambda}(\alpha) \leq 400 \lambda(\alpha/64)$, we see that $\|M_y \tilde{f}\|_{L \log L} \leq \|M_y f\|_{L \log L}$. Hence Theorem 5 yields $\int_Q M_{HL} M_y \tilde{f} \lesssim \int_Q M_{HL} M_y f$. \hfill $\square$

**Corollary 10.** Let $f$ be a nonnegative measurable function supported on $Q$. Let $\tilde{f}$ be a function on $Q$ which is nonincreasing in $x$ and such that, for each $y \in [0,1]$, $\tilde{f}(\cdot,y)$ and $f(\cdot,y)$ are equidistributed. Then

$$\int_Q M_y M_x \tilde{f} \leq c \int_Q M_y M_x f,$$

where $c$ is a universal constant.

**Proof.** By Theorem 8, $\int_Q M_y(M_x f) \leq c \int_Q M_y M_x f$. So it suffices to show that $\int_Q M_y M_x \tilde{f} \leq c \int_Q M_y (M_x \tilde{f})$. To do this, it suffices to show that, given a nonnegative function $g$ supported on $[0,1]$ and $p \in (0,1)$, we have $\tilde{M}_{HL} g(p) \geq c M_{HL} \tilde{g}(p)$, where $c$ is independent of $g$ and $p$, and $\tilde{g}$ is the function supported on $[0,1]$ which is equidistributed to $g$ and such that $\tilde{g}(x_1) \geq \tilde{g}(x_2)$ whenever $0 < x_1 \leq x_2 < 1$. Well, suppose $M_{HL} \tilde{g}(p) = \alpha$. Then on a set $A \subseteq [0,1]$ of measure $|A| = |p|$, we have $|A|^{-1} \int_A |g| = \alpha$.

Assuming without loss of generality that $\int_0^1 g \leq \alpha/2$, we take a Calderón–Zygmund decomposition of $g$ with respect to $\alpha/4$, yielding intervals $Q_{j,\alpha/4}$.

Now $|\bigcup Q_{j,\alpha/4}| \geq |A|$. Otherwise we would have $|\bigcup Q_{j,\alpha/4}| < |A|$, implying that if $E$ is a set contained in $[0,1]$ such that $|E| \geq |A|$, then

$$\frac{1}{|E|} \int_E |g| \leq \frac{1}{|E|} \left(|E| \frac{\alpha}{2} + \left| \bigcup Q_{j,\alpha/4} \right| \frac{\alpha}{2} \right) = \frac{\alpha}{2} \left(1 + \frac{\left| \bigcup Q_{j,\alpha/4} \right|}{|E|} \right),$$

contradicting the fact that $|A|^{-1} \int_A |g| = \alpha$.

So $M_{HL} g > \alpha/8$ on a set contained in $[0,1]$ of measure greater than or equal to $|A|$. So $M_{HL} g(p) > \alpha/8$. Hence $\tilde{M}_{HL} \tilde{g}(p) \geq \frac{1}{8} M_{HL} \tilde{g}(p)$. \hfill $\square$

The statement of Corollary 10 is false if the operator $M_y M_x$ is replaced by either $M_x M_y$ or $M_S$. To exhibit a counterexample, we first define the
functions $h_{2^n}$ as follows:

\begin{equation}
7. \quad h_{2^n}(x, y) = \sum_{m=0}^{2^n-1} 2^{2^n-m-1} \chi_{[0,2^{-2^n+m+1}]}(y) \chi_{[m-2^{-n},(m+1)-2^{-n}]}(x).
\end{equation}

Let now the function $h$ be defined by

\begin{equation}
8. \quad h = \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} \frac{1}{2^{2k-1}} h_{2^{4k-1}}.
\end{equation}

One can show that, if $\tilde{h}(x, y)$ is a horizontal rearrangement of $h$ which is nonincreasing in $x$, then $\int_Q M_x M_y h = \int_Q M_y h < \infty$, but $\int_Q M_x M_y \tilde{h}$ and $\int_Q M_y \tilde{h}$ are infinite. More details in this regard are found in [4].

We now consider rearrangement results involving sums of maximal operators. A good example of such a result is the following:

**Theorem 11.** Let $f$ and $\tilde{f}$ be equidistributed functions supported on $Q$. Then

\begin{equation}
9. \quad \int_Q (M_x f + M_y f) \sim \int_Q (M_x \tilde{f} + M_y \tilde{f}).
\end{equation}

**Proof.** In [3] it is shown that if $f$ is supported on $Q$, then

\begin{equation}
10. \quad \int_Q M_{HL} f \sim \int_Q M_x f + M_y f.
\end{equation}

The result then follows from Theorem 5. \hfill \Box

We now strengthen this result. Define the maximal operators $M, \overline{M}$ as follows:

**Definition 12.** Let $f$ be a measurable function supported on $Q$. The associated maximal function $Mf$ is defined on $Q$ by

\[
Mf(p_1, p_2) = \sup_{x_1 < p_1 < x_2} \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} \int_0^1 |f(x, y)| \, dy \, dx.
\]

The associated maximal function $\overline{M}f$ is defined on $Q$ by

\[
\overline{M}f(p_1, p_2) = \sup_{y_1 < p_2 < y_2} \frac{1}{y_2 - y_1} \int_{y_1}^{y_2} \int_0^1 |f(x, y)| \, dx \, dy.
\]

**Theorem 13.** Let $f$ and $f^*$ be equidistributed functions supported on $Q$. Then

\begin{equation}
11. \quad \int_Q (Mf + M_y f) \sim \int_Q (Mf^* + M_y f^*).
\end{equation}
Moreover,

\[ \int_Q M_{HL}f \sim \int_Q (Mf + M_y f). \tag{12} \]

**Proof.** By Theorem 5, we need only to show (12). Now, by a theorem of Jessen, Marcinkiewicz, and Zygmund \cite{5} we already know that

\[
\int_Q (Mf + M_y f) \leq \int_Q (M_x f + M_y f) \lesssim \|f\|_{L^\infty(Q)} \sim \int_Q M_{HL}f.
\]

So it suffices to show that \( \int_Q M_{HL}f \lesssim \int_Q (Mf + M_y f) \).

We may assume without loss of generality that \( f \) is smooth and nonnegative. Let \( \tilde{f}(x, y) \) be the function supported on \( Q \) which is nonincreasing in \( x \) and such that, for each \( y \in [0, 1], \ f(\cdot, y) \) and \( \tilde{f}(\cdot, y) \) are equidistributed. It is enough to show that \( \int_Q M\tilde{f} \lesssim \int_Q (Mf + M_y f) \). For then

\[
\|f\|_{L^\infty(Q)} = \|\tilde{f}\|_{L^\infty(Q)}
\]

\[
\sim \int_Q M_{HL}\tilde{f} \quad \text{(by Theorem 5)}
\]

\[
\sim \int_Q (M_x \tilde{f} + M_y \tilde{f}) \quad \text{(by (10))}
\]

\[
= \int_Q M\tilde{f} + \int_Q M_y \tilde{f}
\]

\[
\lesssim \int_Q (Mf + M_y f) + \int_Q M_y \tilde{f}
\]

\[
\lesssim \int_Q (Mf + M_y f) \quad \text{(by Theorem 8)}.
\]

We now show \( \int_Q M\tilde{f} \lesssim \int_Q (Mf + M_y f) \). To do this, it suffices to show that

\[
|\{p \in Q : M\tilde{f}(p) > 2\alpha\}| \leq 2 |\{p \in Q : Mf(p) + M_y f(p) > \alpha\}|
\]

for each \( \alpha > 0 \).

Now, let \( A = \{p \in Q : Mf(p) > \alpha\} \).

Let \( B = \{p \in Q : p \notin A, M_y f(p) > \alpha\} \). If \( A = Q \), we’re done. Otherwise, \( \frac{1}{|A|} \int_A |f| \leq 2\alpha \). Also, if \( p \notin A \cup B \), then \( f(p) \leq \alpha \). Note also that \( \frac{1}{|B|} \int_B \tilde{f} \leq 2\alpha \).

Now, let \( f' \) be a function supported on \( Q \) which is equidistributed to \( f \) and such that \( f'(x_1, y_1) \geq f'(x_2, y_2) \) whenever \( 0 \leq x_1 \leq x_2 \leq 1, \ 0 \leq y_1 \leq 1, \) and \( 0 \leq y_2 \leq 1 \). Note if \( p \in Q, \ Mf'(p) \geq Mf(p) \). So it suffices to show

\[
|\{p \in Q : Mf'(p) > 2\alpha\}| \leq 2 |\{p \in Q : Mf(p) + M_y f(p) > \alpha\}|
\]
Let $D = [0, |A \cup B|] \times [0, 1]$ be a subset of $Q$. Note that $|D|^{-1} \int_D f' \leq 2\alpha$. If $p \notin D$, then $f'(p) \leq \alpha$. So if $p \notin D$, $Mf'(p) \leq 2\alpha$. So

$$\left| \{p \in Q : Mf'(p) > 2\alpha \} \right| \leq |A| + |B|$$

$$\leq \left| \{p \in Q : Mf(p) > \alpha \} \right| + \left| \{p \in Q : M_y f(p) > \alpha \} \right|$$

$$\leq 2 \left| \{p \in Q : Mf(p) + M_y f(p) > \alpha \} \right|.$$  \(\square\)

We now turn to a substantially more sophisticated rearrangement result involving sums of iterated maximal operators. The following result, besides being of intrinsic interest, is used in [4] to show that for any measurable function $f$ supported on $Q$,

$$\|f\|_{L^2(\log L)^2(Q)} \sim \int_Q (Mf + Mf + M_y M_x f).$$

This is in turn used to show that if $f$ is a function supported on $Q$ such that $\int_Q M_y M_x f < \infty$ but $\int_Q M_x M_y f = \infty$ (such functions do exist: the function $f(x, y) = h(y, x)$, where $h$ is defined as in (7), (8) provides an example), then there exists a set $A$ of finite measure in $\mathbb{R}^2$ such that $\int_A M_S f = \infty$. This result is particularly striking considering that there exists a function $g$ constructed by M. E. Gomez [2] such that $g$ is supported in $Q$, $\int_Q M_x M_y g = \infty$, and $\int_Q M_y M_x g = \infty$, but $M_S g$ is integrable over every set of finite measure in $\mathbb{R}^2$.

**Theorem 14.** Let $f$ be a measurable function supported on $Q$. Let $f'$ be a function supported on $Q$ such that $f'(-, y)$ and $f(-, y)$ are equidistributed for each $y \in [0, 1]$. Then

$$\int_Q (Mf + M_y M_x f') \sim \int_Q (Mf + M_y M_x f).$$

**Proof.** Without loss of generality, assume $f$ is nonnegative. Let $\tilde{f}$ be a function supported on $Q$ which is nonincreasing in $x$ and such that $f(-, y)$ and $f(-, y)$ are equidistributed for each $y \in [0, 1]$. It suffices to show that

$$\int_Q (Mf + M_y M_x \tilde{f}) \sim \int_Q (Mf + M_y M_x f).$$

Well,

$$\int_Q (Mf + M_y M_x f) \lesssim \int_Q (M_x M_y f + M_y M_x f) \sim \int_Q M_{HL} M_x f \quad (\text{by (10)})$$

$$\sim \int_Q M_{HL} M_x \tilde{f}$$
(the last step by Theorem 5, since $$\|M_x f\|_{L \log L} \sim \|M_x \tilde{f}\|_{L \log L}$$). Further,
$$\int_Q M_{HL} M_x \tilde{f} \sim \int_Q (M_x M_x \tilde{f} + M_y M_x \tilde{f})$$ (by (10))
$$= \int_Q (M M \tilde{f} + M_y M_x \tilde{f}) .$$

So
$$\int_Q (M M f + M_y M_x f) \lesssim \int_Q (M M \tilde{f} + M_y M_x \tilde{f}) .$$

It suffices then to prove the reverse of this last inequality. This step is somewhat involved and will be the focus of the remainder of this paper.

It will be technically convenient to work with the dyadic analogues of the maximal operators $$M_{HL}, M, M^\Delta, M_x, \text{ and } M_y$$. Recall that a **dyadic interval** in $$[0, 1]$$ is an interval of the form $$[k \cdot 2^j, (k + 1) \cdot 2^j]$$, where $$j$$ is a nonpositive integer and $$k$$ is a nonnegative integer such that $$(k + 1) \cdot 2^j \leq 1$$. We denote the set of dyadic subintervals of $$[0, 1]$$ by $$I^\Delta$$. A **dyadic square** in $$Q$$ is a set of the form $$I \times J$$, where $$I$$ and $$J$$ are dyadic intervals in $$[0, 1]$$ of the same length. We denote the set of dyadic squares in $$Q$$ by $$S^\Delta$$. We formally define the dyadic maximal operators $$M_{HL}^\Delta, M^\Delta, M^\Delta_x, M^\Delta_y$$ as follows:

**Definition 15.** Let $$f$$ be a measurable function supported on $$Q$$. The dyadic Hardy–Littlewood maximal function $$M_{HL}^\Delta f$$ is defined on $$Q$$ by
$$M_{HL}^\Delta f (p) = \sup_{p \in I \in I^\Delta} \frac{1}{|I|} \int_I |f| .$$

The maximal function $$M^\Delta f$$ is defined by
$$M^\Delta f (p_1, p_2) = \sup_{p_1 \in I \in I^\Delta} \frac{1}{|I|} \int_I \int_0^1 |f(x, y)| dy dx .$$

Similarly, we define $$M^\Delta_x f, M^\Delta_y f$$ by
$$M^\Delta_x f (p_1, p_2) = \sup_{p_2 \in I \in I^\Delta} \frac{1}{|I|} \int_I \int_0^1 |f(x, y)| dy dx ,$$
$$M^\Delta_y f (p_1, p_2) = \sup_{p_1 \in I \in I^\Delta} \frac{1}{|I|} \int_I |f(x, p_2)| dx ,$$
$$M^\Delta_y f (p_1, p_2) = \sup_{p_2 \in I \in I^\Delta} \frac{1}{|I|} \int_I |f(p_1, y)| dy .$$

The dyadic Hardy–Littlewood maximal function $$M_{HL}^\Delta f$$ of a measurable function $$f$$ supported on $$[0, 1]$$ is defined on $$[0, 1]$$ by
$$M_{HL}^\Delta f (p) = \sup_{p \in I \in I^\Delta} \frac{1}{|I|} \int_I |f| .$$
We will also require the following definition and theorem introduced in [3]:

**Definition 16.** Let \( \beta \) be a countable collection of Lebesgue measurable subsets of the unit \( n \)-cube \( I^n \) in \( \mathbb{R}^n \) of positive measure. A (possibly finite) subset \( \{R_i\} \) of \( \beta \) is said to be a Córdoba–Fefferman collection with respect to \( \beta \) if there exists an enumeration \( \tilde{R}_1, \tilde{R}_2, \tilde{R}_3, \ldots \) of the elements of \( \{R_i\} \) such that \( |\tilde{R}_i \cap \bigcup_{j<i} \tilde{R}_j| \leq \frac{1}{2}|\tilde{R}_i| \) for each \( i \). In this case we write \( \{R_i\} \in \text{CFC}(\beta) \).

**Theorem 17.** Let \( \beta \) be a countable collection of Lebesgue measurable subsets of the unit \( n \)-cube \( I^n \) in \( \mathbb{R}^n \) of positive measure. Let \( \beta \) be such that for any point \( x \) in \( I^n \), \( x \in R \) for some \( R \in \beta \). Define the maximal operator \( M_{\beta} \) on \( L^1(I^n) \) by

\[
M_{\beta} f(x) = \sup_{x \in R \in \beta} \frac{1}{|R|} \int_R |f(y)| dy.
\]

Suppose \( M_{\beta} \) satisfies the (Tauberian) condition

\[
\left| \{x \in I^n : M_{\beta} \chi_E(x) \geq \frac{1}{2} \} \right| \leq C_\beta |E|
\]

for all measurable sets \( E \subset I^n \). Then if \( f \in L^1(I^n) \),

\[
\int_{I^n} M_{\beta} f \sim \sup_{\{R_i\} \in \text{CFC}(\beta)} \int_{I^n} |f| \sum_i \chi_{R_i}.
\]

In particular,

\[
\frac{1}{2} \sup_{\{R_i\} \in \text{CFC}(\beta)} \int_{I^n} |f| \sum_i \chi_{R_i} \leq \int_{I^n} M_{\beta} f \leq 4 C_\beta \sup_{\{R_i\} \in \text{CFC}(\beta)} \int_{I^n} |f| \sum_i \chi_{R_i}.
\]

The maximal operators \( M, M^+, M_{HL}, M_S \), and their corresponding dyadic analogues satisfy the desired Tauberian condition, since \( M_{HL} \) is of weak type \((1,1)\). Also, as a matter of notation, if a given maximal operator \( M_{\beta} \) is naturally associated to a collection \( \beta \), as, say, \( M_S \) is associated to the set of rectangles with sides parallel to the axes, we will frequently denote the Córdoba–Fefferman collection \( \text{CFC}(\beta) \) by \( \text{CFC}(M_{\beta}) \).

We will need the following lemma:

**Lemma 18.** Suppose \( f \) is a measurable function supported on \( Q \). Then

\[
\int_Q M^{\Delta} M^{\Delta} f \lesssim \int_Q M^{\Delta} f + \int_Q M^{\Delta} M^{\Delta} f.
\]

**Proof.** By Equations (10) and (12) we realize

\[
\int_Q M_x f \lesssim \int_Q M f + \int_Q M_y f.
\]

Since \( \int_Q M_x f \sim \int_Q M_x^\Delta f \), and in view of the inequality

\[
\left| \{p \in Q : M_x f(p) > \alpha \} \right| < 100 \left| \{p \in Q : M_x^\Delta f(p) > \frac{1}{100} \alpha \} \right|,
\]

we have

\[
\int_Q M_x f \lesssim \int_Q M f + \int_Q M_y f.
\]

Therefore,

\[
\int_Q M^{\Delta} M^{\Delta} f \lesssim \int_Q M^{\Delta} f + \int_Q M^{\Delta} M^{\Delta} f.
\]
valid for all $\alpha > 0$, we see by symmetry and the Fubini theorem that
\begin{equation}
\int_Q M^\Delta_x f \lesssim \int_Q M^\Delta f + \int_Q M^\Delta_y f.
\end{equation}
Hence
\begin{equation}
\int_Q M^\Delta_x M^\Delta_x f \lesssim \int_Q M^\Delta_x M^\Delta f + \int_Q M^\Delta_y M^\Delta f.
\end{equation}
If $\int_Q M^\Delta_x M^\Delta_x f \lesssim \int_Q M^\Delta_y M^\Delta f$ we’re done, since $\int_Q M^\Delta_x M^\Delta f \lesssim \int_Q M^\Delta_x M^\Delta f$. So we may assume without loss of generality that
\[\int_Q M^\Delta_x M^\Delta f \lesssim \int_Q M^\Delta f.
\]
Hence it is enough to prove the following:

**Lemma 19.** Suppose $f$ is a measurable function supported on $Q$ such that $\int_Q M^\Delta_x M^\Delta f \lesssim \int_Q M^\Delta y M^\Delta f$. Then $\int_Q M^\Delta_x M^\Delta f \lesssim \int_Q M^\Delta M^\Delta f$.

**Proof.** We first prove the following claim:

**Claim 20.** Suppose $f \in L^\infty([0,1])$. Let $\epsilon > 0$. Then there exists $\{R_i\} \subset \text{CFC}(M^\Delta_{HL})$ such that:

(i) $M^\Delta_{HL} f \sim \sum_i \frac{1}{|R_i|} (\int_{R_i} |f|) \chi_{R_i}$ in $[0,1]$ except on a set of measure less than $\epsilon$; and

(ii) $\int_0^1 M^\Delta_{HL} M^\Delta_{HL} f \sim \int_0^1 |f| \left( \sum_i \chi_{R_i} \right)^2$.\]

**Proof.** We may assume without loss of generality that $f$ is a smooth non-negative function and $\|f\|_{L(\log L)^2([0,1])} = 1$. Hence $\|f\|_1 \leq 1$. Let $\ell$ be the largest integer such that $2^\ell < \|f\|_1$. Take the Calderón–Zygmund decomposition of $f$ with respect to $2^\ell$, yielding intervals $\{Q_{1,i}\}_{i \in \mathbb{Z}^+}$. Let $\{Q_{1,i}\}_{i=1,\ldots,j_1} \subset \{Q'_{1,i}\}_{i \in \mathbb{Z}^+}$ such that
\[\left| \bigcup_{i=1}^\infty Q'_{1,i} \setminus \bigcup_{i=1}^{j_1} Q_{1,i} \right| < \epsilon/2.\]
(Note here that $Q_{1,1} = [0,1]$, and $j_1 = 1$.)

We continue by induction. Suppose $\{Q_{k,i}\}_{i=1,\ldots,j_k}$ has been selected. Take the Calderón–Zygmund decomposition of $f$ with respect to $2^{\ell+2k}$, yielding intervals $\{Q'_{k+1,i}\}_{i \in \mathbb{Z}^+}$. Let $\{Q_{k+1,i}\}_{i=1,\ldots,j_{k+1}} \subset \{Q'_{k+1,i}\}_{i \in \mathbb{Z}^+}$ be such that $\bigcup_{i=1}^\infty Q_{k+1,i} \subset \bigcup_{i=1}^{j_{k+1}} Q_{k+1,i}$ and $\left| \bigcup_{i=1}^\infty Q'_{k+1,i} \setminus \bigcup_{i=1}^{j_{k+1}} Q_{k+1,i} \right| < \epsilon/2^{k+1}$.

In this manner, we obtain intervals
\[(*) \quad Q_{1,1}, \ldots, Q_{1,j_1}, Q_{2,1}, \ldots, Q_{2,j_2}, \ldots, Q_{m,1}, \ldots, Q_{m,j_m},\]
where $m$ is the largest integer such that $2^{\ell+2(m-1)} < \|f\|_\infty$. Let $\mathcal{N}$ be the union of all the $Q'_{i,j}$ that are not on the list $(*)$. Note $|\mathcal{N}| < \epsilon$. 


Let \( p \in [0, 1] \setminus \mathcal{N} \). Let \( Q_p \) be the smallest interval on the list \((\ast)\) that contains \( p \). Note that

\[
M_{hl} f(p) \sim \frac{1}{|Q_p|} \int_{Q_p} f \sim \sum_{i,j} \frac{1}{|Q_{i,j}|} \left( \int_{Q_{i,j}} f \right) \chi_{Q_{i,j}}(p).
\]

Also, for almost every \( p \in [0, 1] \setminus \mathcal{N} \), if \( f(p) > 4^j \geq 1 \) then \( p \in \cup_i Q_{j,i} \). Hence \( p \) lies in at least \( j \) intervals listed in \((\ast)\).

Let \( R_1 = Q_{m,1} \), \( R_2 = Q_{m,2} \), \( \ldots \), \( R_{j_m} = Q_{m,j_m} \), \( R_{j_m+1} = Q_{m-1,1} \), \( \ldots \), \( R_{j_m+j_m−1} = Q_{m−1,j_m−1} \), \( \ldots \), \( R_{j_1+\cdots+j_m} = Q_{1,j_1} \).

If \( j < k \), we have \(|R_k \cap \bigcup_{j=1}^{k−1} R_j| \leq \frac{1}{2}|R_k|\) by the Calderón–Zygmund construction of the \( Q_{i,j} \). To see this consider, say, \( Q_{1,1} = [0, 1] \). Now

\[
\frac{1}{|Q_{1,1}|} \int_{Q_{1,1}} f \leq 2^{\ell+1}.
\]

Suppose \(|Q_{1,1} \cap \bigcup_i Q_{2,i}| > \frac{1}{2}|Q_{1,1}|\). Then

\[
\frac{1}{|Q_{1,1}|} \int_{Q_{1,1}} f \geq \frac{1}{|Q_{1,1}|} \sum_{i : |Q_{2,i} \cap Q_{1,1}| > 0} \int_{Q_{2,i}} f > \frac{1}{|Q_{1,1}|} \sum_{i : |Q_{2,i} \cap Q_{1,1}| > 0} 2^{\ell+2} = \frac{1}{|Q_{1,1}|} 2^{\ell+2} |Q_{1,1} \cap \bigcup_i Q_{2,i}| > 2^{\ell+2} \cdot \frac{1}{2} = 2^{\ell+1},
\]

contradicting \( \frac{1}{|Q_{1,1}|} \int_{Q_{1,1}} f \leq 2^{\ell+1} \). Likewise, if \( Q_{j,k} \) is on the list \((\ast)\),

\[
|Q_{j,k} \cap \text{union of the } Q_{r,s} \text{ following } Q_{j,k} \text{ on the list } (\ast)| \leq \frac{1}{2} |Q_{j,k}|.
\]

Hence \( \{R_i\} \in \text{CFC}(M_{hl}^\Delta) \).

So there exists \( \{R_i\} \in \text{CFC}(M_{hl}^\Delta) \) such that

\[
M_{hl} f(p) \sim \sum_i \frac{1}{|R_i|} \left( \int_{R_i} f \right) \chi_{R_i}(p)
\]

except on a set of measure less than \( \epsilon \). Condition \((i)\) is thus satisfied by \( \{R_i\} \).

We now show that condition \((ii)\) is satisfied by the same Córdoba–Fefferman collection \( \{R_i\} \) as well.

We have already demonstrated that, if \( p \in [0, 1] \setminus \mathcal{N} \) and \( 4^{j+1} \geq f(p) \geq 4^j > 1 \), then \( p \) lies in at least \( j \) of the \( R_i \). Because in this case

\[
f(p)(\log(3 + f(p)))^2 \sim f(p) (\log(3 + 4^j))^2 \sim f(p) j^2,
\]
we obtain
\[ f(p)(\log(3 + f(p)))^2 \lesssim f(p) \left( \sum_i \chi_{R_i}(p) \right)^2. \]

If \( f(p) < 4 \), then
\[ f(p)(\log(3 + f(p)))^2 \lesssim f(p) \lesssim f(p) \left( \sum_i \chi_{R_i}(p) \right)^2, \]
since \([0, 1]\) is one of the \( R_i \).

Let \( \epsilon \) be small enough that
\[ \int_0^1 f(\log(3 + f))^2 \sim \int_{[0, 1] \setminus \mathcal{N}} f(\log(3 + f))^2. \]

Since
\[ \int_{[0, 1] \setminus \mathcal{N}} f(\log(3 + f))^2 \lesssim \int_{[0, 1] \setminus \mathcal{N}} f \left( \sum_i \chi_{R_i} \right)^2 \lesssim \|f\|_{L(\log L)^2([0, 1])}^2 \]
(the latter inequality holding by duality), \( \|f\|_{L(\log L)^2([0, 1])} = 1 \), and
\[ \int_0^1 M_{HL}^\Delta M_{HL}^\Delta f \sim \|f\|_{L(\log L)^2([0, 1])}, \]
we see that \( \int_0^1 M_{HL}^\Delta M_{HL}^\Delta f \sim \int_0^1 f \left( \sum_i \chi_{R_i} \right)^2 \), as desired. \( \square \)

**Corollary 21.** Let \( \epsilon > 0 \) and \( f \in L^\infty(Q) \). There exists \( \{R_i\} \in \text{CFC}(M^\Delta) \) such that:

i) \( M^\Delta f \sim \sum_i \frac{1}{|R_i|} (\int_{R_i} f) \chi_{R_i} \) on \( Q \) except on a set of measure less than \( \epsilon \); and

ii) \( \int_Q M^\Delta M^\Delta f \sim \int_Q |f| \left( \sum_i \chi_{R_i} \right)^2. \)

**Proof.** Apply Claim 20 and the Fubini theorem. \( \square \)

**Lemma 22.** Let \( \{R_i\}, \{S_i\} \) be in \( \text{CFC}(M^\Delta) \). Then
\[ \sum_j \frac{1}{|R_j|} \left( \int_{R_j} \sum_i \chi_{S_i} \right) \chi_{R_j} \lesssim \left( \sum_i \chi_{S_i} + 1 \right) \left( \sum_i \chi_{R_i} + 1 \right). \]

**Proof.** Let \( j \in \mathbb{Z}_+ \). It is enough to show that if \( p \in R_j \), then
\[ \frac{1}{|R_j|} \int_{R_j} \sum_i \chi_{S_i} \lesssim \left( \sum_i \chi_{S_i}(p) + 1 \right). \]

Let \( \{S_{i, \text{int}}\} \) denote the set of the \( S_i \) contained in \( R_j \). Let \( \{S_{i, \text{ext}}\} \) denote the set of the \( S_i \) strictly containing \( R_j \). Note that, for some positive finite constant \( C \),
\[ \frac{1}{|R_j|} \int_{R_j} \sum_i \chi_{S_{i, \text{int}}} \leq C. \]
since \( \int_0^1 |\log x| \, dx = 1 \). Now, \( \frac{1}{|R_j|} \int_{R_j} \sum \chi_{S_i,ext} \) equals the number of \( S_i,ext \) strictly containing \( R_j \), which equals \( \sum \chi_{S_i,ext}(p) \). So

\[
\frac{1}{|R_j|} \int_{R_j} \sum_i \chi_{S_i,ext} \lesssim \sum_i \chi_{S_i,ext}(p) \lesssim \sum_i \chi_{S_i}(p).
\]

Hence, as desired,

\[
\frac{1}{|R_j|} \int_{R_j} \sum_i \chi_{S_i} = \frac{1}{|R_j|} \int_{R_j} \left( \sum_i \chi_{S_i,int} + \sum_i \chi_{S_i,ext} \right) \lesssim \sum_i \chi_{S_i}(p) + 1. \quad \square
\]

**Lemma 23.** Suppose \( f \), \( g \), and \( h \) are nonnegative measurable functions on \([0, 1]\). Also suppose \( \int_0^1 fg^2 \leq \int_0^1 fgh \). Then \( \int_0^1 fg^2 \leq \int_0^1 fh^2 \).

**Proof.** \( \int_0^1 fg^2 \leq \int_0^1 fgh \) implies

\[
\|f^{1/2}g\|_2^2 \leq \int_0^1 fgh \leq \|f^{1/2}g\|_2 \|f^{1/2}h\|_2.
\]

Hence

\[
\|f^{1/2}g\|_2 \leq \|f^{1/2}h\|_2.
\]

So \( \int_0^1 fg^2 \leq \int_0^1 fh^2 \). \quad \square

We now finish the proof of Lemma 19, and hence the proof of Lemma 18 as well. Without loss of generality, we assume \( f \) is a nonnegative smooth function supported on \( Q \).

Let \( 0 < \epsilon = 2^{-k} < 1 \), \( k \in \mathbb{Z}_+ \). Let \( R_{x,\epsilon} \) denote the set of dyadic rectangles in \( Q \) of height \( \epsilon \). Let the maximal operator \( M_{x,\epsilon}^{\Delta} \) be given by

\[
M_{x,\epsilon}^{\Delta}f(p) = \sup_{p \in R_{x,\epsilon}} \frac{1}{|R|} \int_R |f|.
\]

For sufficiently small \( \epsilon \), \( M_{x,\epsilon}^{\Delta}f \sim M_{x}^{\Delta}f \). Assume \( \epsilon \) is indeed sufficiently small.

Now, since \( \int_Q M_{x}^{\Delta}M_{x}^{\Delta}f \leq \int_Q M_{x}^{\Delta}M_{x}^{\Delta}f \) by hypothesis, we realize by Theorem 17 and Claim 20 that for some \( \{R_i\} \in \text{CFC}(M_{x,\epsilon}^{\Delta}) \) and some
\( \{A_i\} \in \text{CFC}(\Delta M) \),

\[
\int_Q M_x^\Delta M_x^\Delta f \sim \int_Q f \left( \sum_i \chi_{R_i} \right)^2 \\
\lesssim \int_Q \left( \sum_i \chi_{A_i} \right) \left( \sum_j \frac{1}{|R_j|} \left( \int_{R_j} f \right) \chi_{R_j} \right) \\
\sim \int_Q f \left( \sum_j \frac{1}{|R_j|} \int_{R_j} \sum_i \chi_{A_i} \right) \chi_{R_j} \\
\lesssim \int_Q f \left( \sum_j \chi_{R_j} \right) \left( \sum_j \chi_{A_j} \right),
\]

the latter inequality being justified by Lemma 22 and the fact that, letting \( B_{j,k} = [0, 1] \times [j \cdot 2^{-k}, (j + 1)2^{-k}] \), we have

\[
\{A_i \cap B_{j,k}\}_{i \in \mathbb{Z}^+, 0 \leq j \leq 2^{k-1}} \in \text{CFC}(\Delta M_{x,\epsilon}).
\]

(Recall here that \( \epsilon = 2^{-k}. \))

By Lemma 23, we see that the above result

\[
\int_Q f \left( \sum_i \chi_{R_i} \right)^2 \lesssim \int_Q f \left( \sum_j \chi_{R_j} \right) \left( \sum_j \chi_{A_j} \right)
\]

implies

\[
\int_Q f \left( \sum_i \chi_{R_i} \right)^2 \lesssim \int_Q f \left( \sum_i \chi_{A_i} \right) \lesssim \int_Q M_x^\Delta M_x^\Delta f.
\]

Hence \( \int_Q M_x^\Delta M_x^\Delta f \lesssim \int_Q M_x^\Delta M_x^\Delta f \), as desired. \( \square \)

**Corollary 24.** Suppose \( f \) is a measurable function supported on \( Q \). Then

\[
\int_Q M_x M_x f \lesssim \int_Q MMf + \int_Q M_y M_x f.
\]

**Proof.** By (19) we have

\[
\int_Q M_x^\Delta M_x^\Delta \lesssim \int_Q M_x^\Delta M_x^\Delta f + \int_Q M_y^\Delta M_x^\Delta f.
\]

Applying Lemma 19, we see that

\[
\int_Q M_x^\Delta M_x^\Delta \lesssim \int_Q M_x^\Delta M_x^\Delta f + \int_Q M_y^\Delta M_x^\Delta f.
\]

Hence, by (17) we have

\[
\int_Q M_x M_x f \lesssim \int_Q MMf + \int_Q M_y M_x f. \quad \square
\]

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We are now in position to complete the proof of Theorem 14 by showing that

\[ \int_Q (MM\tilde{f} + M_y M_x \tilde{f}) \lesssim \int_Q (MMf + M_y M_x f) . \]

We have \( \int_Q M_y M_x \tilde{f} \lesssim \int_Q M_y M_x f \) by Corollary 10, and

\[ \int_Q MM\tilde{f} = \int_Q M_y M_x \tilde{f} \lesssim \int_Q M_x M_y f \lesssim \int_Q (MMf + M_y M_x f) \]

by Corollary 24. So

\[ \int_Q (MM\tilde{f} + M_y M_x \tilde{f}) \lesssim \int_Q (MMf + M_y M_x f) , \]

as desired. \( \square \)

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DIFFERENTIABILITY OF QUANTUM MOMENT MAPS
AND G-INARIANT STAR PRODUCTS

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We study quantum moment maps of $G$-invariant star products, a quantum analogue of the moment map for classical Hamiltonian systems. Introducing an integral representation, we show that any quantum moment map for a $G$-invariant star product is differentiable. This property gives us a new method for the classification of $G$-invariant star products on regular coadjoint orbits of compact semisimple Lie groups.

1. Introduction

Deformation quantization was introduced in the 1970s by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer [2]. It is one of the important methods for quantizing classical systems. This quantization scheme provides an autonomous theory based on deformations of the ring of classical observables on a phase space (Poisson algebra), and does not involve a radical change in the nature of the observables.

Star products invariant under the action of a Lie group $G$ have been studied with increasing generality from the beginning of the deformation quantization. They appear naturally in the quantization of classical systems with group symmetries, or in the star representation theory of Lie groups.

Quantum moment maps have been introduced in [21], and are the natural quantum analogue of moment maps on Hamiltonian $G$-spaces [16]; see Definition 3.1. A quantum moment map plays an important role for the study of $G$-invariant star products, similar to the one played by a (classical) moment map for classical systems. One of the interesting applications of quantum moment maps is to provide an example of quantum dual pair [21, 20]. Another remarkable result is the quantum reduction theorem, which says that a quantization commutes with reduction [9]. We also give an application of quantum moment map by providing an invariant, called $c_*$ in [12], for a $G$-invariant star product $\ast$ on a $G$-transitive symplectic manifold [12]. This $c_*$ is computed with the help of a quantum moment map and depends only on the class of $G$-equivalent star products. In [12, 13], we give a few examples of $c_*$ for a SO(3)-invariant star product on the coadjoint orbit $S^2$.

But there are serious problems with quantum moment maps. First, there is no obvious way to compute an explicit expression for a quantum moment
map for a given $G$-invariant star product. We provide a partial answer to this problem in [13]. Another important problem is the differentiability of quantum moment maps. Originally, a quantum moment map is defined only on the universal enveloping algebra $\mathfrak{U}(\mathfrak{g}_\lambda)$, that is, the set of polynomials on $\mathfrak{g}^*$. But this definition of quantum moment maps does not directly imply its differentiability. A priori, a quantum moment map has only an algebraic meaning, and cannot be studied in the category of differentiable deformations, which can be inconvenient.

In this article, we give another expression for quantum moment maps which is differentiable. This expression is an analogue of Weyl correspondence that can be formally written as

$$\Phi_*(u) = \int \mathfrak{F} u(\xi) \exp_*(i\xi \Phi_*(X)) d\xi,$$

where $\mathfrak{F} u$ and $\Phi_*$ denote the Fourier transform of $u$ and the quantum moment map of $*$ on $\mathfrak{g}$ respectively. To make sense of this formula, it is necessary to address two questions: defining the function $\exp_*(i\xi \Phi_*(X))$ and giving a meaning to the integral.

For the first, we simply define $\exp_*(i\xi \Phi_*(X))$ by power series with respect to the star product. We show that this naive definition of $\exp_*(i\xi \Phi_*(X))$ is well-defined and it is a product of $e^{i\xi \Phi_0(X)}$ and a polynomial in $\xi$. This is an ingredient to make the quantum moment map differentiable. For the second question, since the domain of a quantum moment map contains any polynomial, $u$ in the formula above should be considered as a tempered distribution. In fact, for any slowly increasing infinitely differentiable function $u$, one can provide the integration as

$$\mathfrak{F}^{-1} \left[ \mathfrak{F} [u](\xi) \exp_*(i\xi \Phi_*(X)) e^{-i\xi \Phi_0(X)} \right] |_{x = \Phi_0(X)} \quad [18].$$

We prefer to use oscillatory integrals rather than tempered distributions in order to make computations easier. We give a brief review on oscillatory integrals in Appendix A; see also [15].

As an application of the differentiability of a quantum moment map we give a structure theorem for $G$-invariant star products on a coadjoint orbit of compact semisimple Lie groups. The class of $G$-invariant star products is parametrized by $G$-invariant Weyl curvature, that is, the second $G$-invariant de Rham cohomology [3]. However, this classification does not give enough information on the structure of these star products.

Regarding the structure of star products, there is an interesting study in [10]. It provides a family of algebraic star products on a coadjoint orbit of semisimple Lie group by a quotient algebra of the Gutt star product. This work has the advantage of giving an explicit representation for this kind of star products.
We provide here a similar structure theory of $G$-invariant star products on such orbits in the differentiable category as an application of the differentiability of quantum moment maps. So we have another classification of such star products by using quantum moment maps. Moreover, as a corollary of the structure theorem, we answer the problem we introduced in [13]: Does $c_*$ parametrize the class of $G$-invariant star products? The answer is yes for regular coadjoint orbits of compact semisimple Lie groups.

The paper is organized as follows: in Section 2, we recall basic concepts and results in deformation quantization, $\lambda$-formal analytic functions and the Gutt star product on $\mathfrak{g}$. The main results are in Section 3: the exponential $\exp_*(i\xi\Phi_*(X))$, an integral expression for $\Phi_*$, and a proof of the differentiability of $\Phi_*$. In Section 4, we show the structure theorem of $G$-invariant star products on a coadjoint orbit.

2. Preliminaries

2.1. Star products. Let $(M,\omega)$ be a symplectic manifold and $C^\infty(M)$ the set of smooth functions on $M$. The Poisson bracket on $C^\infty(M)$ associated to $\omega$ is denoted by $\{,\}$. Let $C^\infty(M)[\lambda]$ be the space of power series in a formal parameter $\lambda$ with coefficients in $C^\infty(M)$.

A (differentiable) star product is an associative multiplication $*$ on the space $C^\infty(M)[\lambda]$, having the form

$$u*v = uv + \sum_{n=1}^{\infty} \left(\frac{\lambda}{2}\right)^n C_n(u,v) \quad \text{for any } u,v \in C^\infty(M),$$

where each $C_k$ is a bidifferential operator annihilating constants and $C_1(u,v) - C_1(v,u) = 2\{u,v\}$.

In the situation where a Lie group $G$ acts on $M$, a star product $*$ is said to be $G$-invariant if $g(u*v) = gu*gv$ for any $u,v \in C^\infty(M)[\lambda]$ and $g \in G$, where $gu(x) = u(g^{-1}x)$ for $x \in M$. There exists a star product on any symplectic manifold $[5, 17, 7]$, and the existence of $G$-invariant star products is equivalent to the existence of a $G$-invariant connection on $M$ $[21, 8]$. When $G$ is compact, $G$-invariant connections always exist and consequently there always exist $G$-invariant star products on $M$.

Two star products $*_1$ and $*_2$ on $C^\infty(M)[\lambda]$ are said to be formally equivalent if there is a formal series

$$T = \text{Id} + \sum_{n=1}^{\infty} \lambda^n T_n$$

of differential operators on $C^\infty(M)$ annihilating constants such that $u*_2 v = T(T^{-1}u*_1 T^{-1}v)$. In this case, $T$ is called an equivalence between $*_1$ and $*_2$, and $*_2$ is denoted by $*_1$. If $*_1$ and $*_2$ are equivalent $G$-invariant star
products and if the equivalence $T$ is $G$-invariant, then $*_{1}$ and $*_{2}$ are said to be formally $G$-equivalent and $T$ is called $G$-equivalence; see also [4, 3].

2.2. Formal analytic functions. We now make some simple but useful remarks on the convergence of the power series valued in $\mathbb{C}[\lambda]$, that will be needed for calculus of functions in $C^\omega(\mathbb{R}^n)[\lambda]$.

**Definition 2.1.** A function $u = u_0 + \lambda u_1 + \cdots \in C^\infty(\mathbb{R}^n)[\lambda]$ is called formal analytic if each $u_i$ is analytic on $\mathbb{R}^n$. We denote the set of formal analytic functions by $C^\omega(\mathbb{R}^n)[\lambda]$.

Let $u$ and $v$ be formal analytic functions. We define the composition $u(v)$. If $u$ is a polynomial, there is no difficulty: just substitute $v$ in $u$. For the general case, we define the composition by using power series. We begin with the following definition:

**Definition 2.2.** Let $a_J = \sum_{k=0}^{\infty} a_{J,k} \lambda^k \in \mathbb{C}[\lambda]$ be a multi-indexed sequence with respect to $J = (j_1, \ldots, j_n)$. The series $\sum_J a_J$ is said to converge formally absolutely if, for any $k$, the series $\sum_J a_{J,k}$ converges absolutely.

If a power series $\sum_J a_J y^J$ converges formally absolutely for some radius $\rho > 0$, it defines a formal analytic function on $|y| < \rho$.

Let $p^J(x) = \sum_{k=1}^{\infty} p_{J,k}^y(x) \lambda^k : \mathbb{R}^n \to \mathbb{R}^n[\lambda]$, $j = 1, \ldots, n$, be a formal analytic map. A formal differential operator $p\partial$ is defined by $(p\partial)u(y) = \sum p^J(x) (\partial_J u)(y)$ for any smooth function $u : \mathbb{R}^n \to \mathbb{R}$. We define a formal operator $e^{p\partial}$ for $u$ by

$$ (e^{p\partial} u)(y) = u(y) + \sum_{0 < |J|} \frac{1}{|J|!} p^J(x) (\partial_J u)(y). $$

Here $|J| = j_1 + \cdots + j_n$, $p^J = (p^1)^{j_1} \cdots (p^n)^{j_n}$ and $\partial_J = (\partial/\partial y_1)^{j_1} \cdots (\partial/\partial y_n)^{j_n}$. Note that the right-hand side converges with respect to the filtration of $\lambda$ since $\deg p > 0$. It is easy to show that $e^{p\partial} u$ is an automorphism, that is, $e^{p\partial}(u_1 u_2) = (e^{p\partial} u_1)(e^{p\partial} u_2)$.

If $u$ is a polynomial on $\mathbb{R}^n$ then $u(y+p(x))$ is a function of $(x, y)$ that can be defined by substituting $y+p(x)$ in $u$, and we have $u(y+p(x)) = (e^{p\partial} u)(y)$. If $u$ is given by a power series, $u(y) = \sum_J a_J y^J$, one can see that the series $\sum_J a_J(y+p(x))^J$ is equal to $(e^{p\partial} u)(y)$ as formal power series in $y$. Since $(e^{p\partial} u)(y)$ converges formally absolutely on the same domain of $y \in \mathbb{R}^n$ where $\sum_J a_J y^J$ converges, we can define $u(y+p(x)) = \sum_J a_J(y+p(x))^J$ as a formal analytic function. Therefore we can define $u(v)$ for any formal analytic map $v : \mathbb{R}^m \to \mathbb{R}^n[\lambda]$, and $u(v) = (e^{v-v_0 \partial} u)(v_0)$ holds, where $v = v_0 + v_1 \lambda + \cdots$.

**Definition 2.3.** Let $u : \mathbb{R}^n \to \mathbb{R}$ be an analytic map and $v : \mathbb{R}^m \to \mathbb{R}^n[\lambda]$ a formal analytic map. Then we define a formal analytic map $u(v) : \mathbb{R}^m \to \mathbb{R}[\lambda]$ by the power series

$$ u(v(x)) = \sum_J a_J(v(x))^J, $$
where $u = \sum_j a_j y^j$.

**Remark.** The equation

$$u(v(x)) = (e^{(v-v_0)(x)}u)(v_0(x))$$

holds for any formal analytic map, and it gives the Taylor theorem for formal analytic functions.

### 2.3. The Gutt Star Product

Let $\mathfrak{g}$ be a real Lie algebra and $\mathfrak{g}^*$ its dual. The universal enveloping algebra of $\mathfrak{g}$ is denoted by $\mathcal{U}(\mathfrak{g})$, and the universal symmetric algebra of $\mathfrak{g}$ by $\mathcal{S}(\mathfrak{g})$. We also denote the space of polynomials on $\mathfrak{g}^*$ by $\text{Pol}(\mathfrak{g}^*)$. Let $\mathfrak{g}[\lambda]$ be the set of formal power series in $\lambda$ with coefficients in $\mathfrak{g}$. We define a Lie algebra structure $[\ , \ ]_\lambda$ on $\mathfrak{g}[\lambda]$ by $[\xi,\eta]_\lambda = \lambda[\xi,\eta]$ for any $\xi,\eta \in \mathfrak{g}$ and extend it by $\lambda$-linearity, where $[\ , \ ]$ is the Lie bracket of $\mathfrak{g}$. We denote this Lie algebra by $\mathfrak{g}_\lambda$. One can introduce a grading on $\mathfrak{g}_\lambda$ by assigning degree 2 to $\xi \in \mathfrak{g}$ and to $\lambda$, and $[\ , \ ]_\lambda$ has degree 0. This grading induces a grading on the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_\lambda)$ of $\mathfrak{g}_\lambda$.

It is well-known that the space of smooth functions on $\mathfrak{g}^*$ admits a natural Poisson structure defined by the Kirillov–Poisson bracket $\Pi$. For any smooth functions $u$ and $v$ on $\mathfrak{g}^*$, $\Pi$ is given by $\Pi(u,v)(\mu) = \langle [du, dv], \mu \rangle$, where $du(\mu)$ is an element of $\mathfrak{g}$ considered as 1-form on $\mathfrak{g}^*$.

S. Gutt has defined a star product on $\mathfrak{g}^*$ [11]. We shall call this product the Gutt star product, denoted by $\ast^G$. The Gutt star product can be directly obtained by transposing the algebraic structure of $\mathcal{U}(\mathfrak{g}_\lambda)$ to $C^\infty(\mathfrak{g}^*)[\lambda]$. This is achieved through the natural isomorphism between $\text{Pol}(\mathfrak{g}^*)[\lambda]$ and $\mathcal{S}(\mathfrak{g}_\lambda)$ and with the help of the symmetrization map $s : \mathcal{S}(\mathfrak{g}_\lambda) \rightarrow \mathcal{U}(\mathfrak{g}_\lambda)$. For polynomials $u$ and $v$, the Gutt star product is given by

$$u \ast^G v = s^{-1}(s(u) \cdot s(v)), \quad (3)$$

where $\cdot$ is the product of $\mathcal{U}(\mathfrak{g}_\lambda)$. Formula (3) defines an associative differentiable deformation of the usual product on $\text{Pol}(\mathfrak{g}^*)$ which admits a unique extension to $C^\infty(\mathfrak{g}^*)[\lambda]$.

As a direct consequence of Equation (3) $\ast^G$ is a Weyl star product, that is, for any linear function $\xi$ on $\mathfrak{g}^*$, we have $\xi^{\ast^G k} = \xi^k$, where $\xi^{\ast^G k} = \xi^{\ast^G} \cdots \ast^G \xi$ ($k$ factors). Moreover, $\ast^G$ is $\mathfrak{g}$-covariant,

$$\xi^{\ast^G \eta} = \eta^{\ast^G \xi} = 2\lambda \Pi(\xi, \eta) \quad \text{for} \ \xi, \eta \in \text{Lin}(\mathfrak{g}^*),$$

and $\text{Ad}^*(G)$-invariant,

$$g(u \ast^G v) = (gu) \ast^G (gv) \quad \text{for} \ u, v \in C^\infty(\mathfrak{g}^*)[\lambda], \ g \in G.$$

There is a characterization of the Gutt star product:

**Proposition 2.1 ([6]).** The Gutt star product is the unique $\mathfrak{g}$-covariant Weyl star product on $(\mathfrak{g}^*, \Pi)$. Any $\mathfrak{g}$-covariant star product on $(\mathfrak{g}^*, \Pi)$ is equivalent to the Gutt star product.
Let \( \xi = \sum_{k=0}^{\infty} \xi_k \lambda^k \in \text{Lin}(\mathfrak{g}^*)[\lambda] \cong \mathfrak{g}_\lambda \). Then a power series
\[
 e^\xi = \sum_{k=0}^{\infty} \frac{1}{k!} \xi^k
\]
can be defined in the sense of formal absolute convergence, and satisfies Equation (2). A simple computation implies that there are polynomials \( p_k(\xi_1, \xi_2, \ldots, \xi_k) \) such that
\[
 e^\xi = e^{\xi_0} \sum_{k=0}^{\infty} p_k(\xi_1, \xi_2, \ldots, \xi_k) \lambda^k.
\]
(5)

Since \(*^G\) is a Weyl star product, we also have
\[
 e^\xi = \exp_{*^G}(\xi) \equiv \sum_{k=0}^{\infty} \frac{1}{k!} \xi^{*^G_k}.
\]

For any \( \xi, \eta \in \mathfrak{g}_\lambda \), we denote by \( \text{CH}_\lambda(\xi, \eta) \) the Campbell–Hausdorff series of a Lie algebra \( \mathfrak{g}_\lambda \). We note that \( \text{CH}_\lambda(\xi, \eta) \) is an element of \( \mathfrak{g}_\lambda \) since \([ , ]_\lambda\) has degree 0 and \( \text{CH}_\lambda \) converges with respect to the filtration of \( \lambda \).

Since \(*^G\) is \( \mathfrak{g} \)-covariant, we have \( \exp_{*^G}(\xi) *^G \exp_{*^G}(\eta) = \exp_{*^G}(\text{CH}_\lambda(\xi, \eta)) \), that is,
\[
 e^\xi *^G e^\eta = e^{\text{CH}_\lambda(\xi, \eta)} \quad \text{for } \xi, \eta \in \mathfrak{g}_\lambda.
\]
(6)

Therefore, the set \( G_\lambda \equiv \{ e^\xi : \xi \in \mathfrak{g}_\lambda \} \subset C^\infty(\mathfrak{g}^*)[\lambda] \) is closed under multiplication by \(*^G\). It is also easy to show that \( G_\lambda \) is a group.

2.4. Oscillatory integral formula for star products. For later use, we provide an oscillatory integral expression of the Gutt star product. We shall use the notations and the definitions given in Appendix A for oscillatory integrals; see also [15].

**Definition 2.4.** A function \( u \in C^\infty(\mathbb{R}^n) \) has polynomial growth of degree \( \tau > 0 \) if for any multi-index \( I = (i_1, i_2, \ldots, i_n) \), there is a constant \( C_I \) such that
\[
 |\partial_{\zeta}^I u(\zeta)| \leq C_I \langle \zeta \rangle^\tau,
\]
where \( \langle \zeta \rangle = \sqrt{1 + |\zeta|^2} \). We denote the set of such functions by \( A_\tau^0 \).

Let \( \mathcal{A}^0 = \bigcup_{\tau \geq 0} \mathcal{A}_\tau \). If we identify \( \mathfrak{g}^* \) with \( \mathbb{R}^n \), \( \mathcal{A}^0[\lambda] \) is a subalgebra of \( (C^\infty(\mathfrak{g}^*)[\lambda], *^G) \), which contains all polynomials.

**Definition 2.5.** Let \( u \in \mathcal{A}^0 \). The oscillatory integral expression of \( u \) is given by
\[
 u(\zeta) = \text{Os-} \int e^{i\alpha(\zeta-\beta)} u(\alpha) \, d\alpha d\beta,
\]
where the right-hand side means oscillatory integral.
Since $*^G$ is differentiable, the $*^G$ operation commutes with integration. Therefore we have the oscillatory integral expression of the Gutt star product as follows: for any $u, v \in A^0$,

$$u *^G v(x) = \text{Os-} \int e^{-i(\alpha \beta + \alpha' \beta')} f(\alpha) g(\alpha') e^{i\beta x} *^G e^{i\beta' x} \, d\alpha \, d\alpha' \, d\beta \, d\beta'.$$

We remark that $e^{CH_\lambda(i\beta x, i\beta' x)} \in A^0[\lambda] \times A^0[\lambda]$ because of Equation (5). Hence the computation above makes sense and $A^0[\lambda]$ is a subalgebra of $(C^\infty(\mathfrak{g}^*)[\lambda], *^G)$.

3. Differentiability of quantum moment maps

This section is devoted to the study of quantum moment maps, a main subject of this paper. The definition of quantum moment maps adopted here is given in [21]; see Definition 3.1. This definition is a natural analogue of the definition of classical moment maps in Hamiltonian systems.

However, quantum moment maps differ from their classical counterparts in a significant feature, locality. In the classical case, giving a ring morphism of $C^\infty(\mathfrak{g}^*)$ into $C^\infty(M)$ is equivalent to giving a differential map of $M$ into $\mathfrak{g}^*$; this is a consequence from the locality of the ring of functions and its ring morphisms. So this implies that any ring morphism of $\text{Pol}(\mathfrak{g}^*)$ into $C^\infty(M)$ has a natural extension to $C^\infty(\mathfrak{g}^*)$. But the problem is not clear for the quantum case. There is no guarantee that a homomorphism of star algebras is local or differentiable.

We show here that any quantum moment map is differentiable.

3.1. Definition of quantum moment maps. Let $(M, \omega)$ be a symplectic $G$-space and $*$ a $G$-invariant star product. We denote the star commutator by $[a, b]* = a * b - b * a$.

Definition 3.1 ([21]). A quantum moment map is a homomorphism of associative algebras

$$\Phi_* : \mathfrak{u}(\mathfrak{g}_\lambda) \to C^\infty(M)[\lambda]$$

that satisfies

$$[\Phi_*(\xi), u]* = \lambda \xi u,$$

where the right-hand side of (8) is the infinitesimal action of $\xi \in \mathfrak{g}$ on $C^\infty(M)[\lambda]$.

It is easy to see that (7) is equivalent to

$$\Phi_*([\xi, \eta]_\lambda) = [\Phi_*(\xi), \Phi_*(\eta)]_* \quad \text{for any } \xi, \eta \in \mathfrak{g}.$$
On the existence and the uniqueness of quantum moment maps, some simple criteria are known.

**Theorem 3.1** ([21]). Let $H^*_\text{dR}(M)$ be the de Rham cohomology group and $H^*(\mathfrak{g}, \mathbb{R})$ the Lie algebra cohomology group with coefficients in $\mathbb{R}$. There exists a quantum moment map if $H^1_{\text{dR}}(M) = 0$ and $H^2(\mathfrak{g}, \mathbb{R}) = 0$.

**Theorem 3.2** ([21]). The set of quantum moment maps of a $G$-invariant star product is parametrized by $H^1(\mathfrak{g}, \mathbb{R})$.

The following proposition says that quantum moment maps are natural analogues of the classical ones.

**Proposition 3.1** ([21]). Let $\Phi_* : \text{Pol}(\mathfrak{g}^*[[\lambda]]) \to C^\infty(M)[[\lambda]]$ be a quantum moment map. Then $M$ is a Hamiltonian $G$-space. Moreover $\Phi_*$ satisfies

\[ \Phi_*(u) = \Phi_0(u) + O(\lambda) \text{ for any } u \in \text{Pol}(\mathfrak{g}^*), \]

where $\Phi_0 : \text{Pol}(\mathfrak{g}^*) \to C^\infty(M)$ is the corresponding classical moment map.

An important property of $\Phi_*$ is its covariance under $G$-equivalence.

**Proposition 3.2.** Let $*$ be a $G$-invariant star product and $\Phi_*$ a quantum moment map of $*$. If $*'$ is a $G$-invariant star product which is $G$-equivalent to $*$, then $T\Phi_*$ is a quantum moment map of $*'$, where $T$ is a $G$-equivalence between $*$ and $*'$.

**Proof.** It is enough to show that $[T\Phi_*(X), f]_{*'} = \lambda Xf$, since $T\Phi_*$ is an algebra homomorphism from $(\text{Pol}(\mathfrak{g}^*)[[\lambda]], *^G)$ to $(C^\infty(M)[[\lambda]], *')$.

\[ [T\Phi_*(\xi), f]_{*'} = T[\Phi_*(\xi), T^{-1}f]_{*} = T(\lambda \xi T^{-1}f) = \lambda \xi f. \]

Since quantum moment maps are parametrized by $H^1(\mathfrak{g}, \mathbb{R})$ we have:

**Corollary 3.1.** Assume $H^1(\mathfrak{g}, \mathbb{R}) = \{0\}$. Let $*, *'$ be $G$-invariant star product and $\Phi_*, \Phi_{*'}$ quantum moment maps of $*, *'$ respectively. If $*'$ is $G$-equivalent to $*$ then $T\Phi_* = \Phi_{*'}$.

3.2. Exponential function of a quantum moment map. We define a function $\exp_*(\Phi_*(X))$ in $C^\infty(M)[[\lambda]]$ for $X \in \mathfrak{g}_\lambda$. This function “generates” $\Phi_*(\text{Pol}(\mathfrak{g}^*))$, and we will use it to obtain another expression for $\Phi_*$. An important property of $\exp_*(\Phi_*(X))$ is that it is a product of $e^X$ and a polynomial of $X$. This is essential for the differentiability of $\Phi_*$. Assume that $*$ is a $G$-invariant star product of Fedosov type. Recall that $Q$, $\sigma$ and $\circ$ denote the Fedosov quantization procedure corresponding to $*$, the projection of $W_D$ onto $C^\infty(M)[[\lambda]]$ and the Weyl product on $\Gamma W$ respectively. See also Appendix B, where we give a brief summary of these terms.
Lemma 3.1. Let \( \xi = \xi_0 + \xi_1 \lambda + \cdots \in \mathfrak{g}_\lambda \). The series
\[
\sum_{k=0}^{\infty} \frac{1}{k!} Q(\Phi_*(\xi))^\circ k
\]
converges \( \lambda \)-formally absolutely and uniformly on any compact subset of \( M \), and defines an element of \( W_D \). Moreover, (10) has the expression
\[
\sum_{k=0}^{\infty} \frac{1}{k!} Q(\Phi_*(\xi))^\circ k = e^{\Phi_0(\xi_0)} \sum_{I,j} p_{I,j}(\Phi_*(\xi), \partial \Phi_*(\xi), \ldots) y^I \lambda^j,
\]
where the \( p_{I,j} \) are polynomials in \( \{ \Phi_*(\xi), \partial \Phi_*(\xi), \ldots \} \).

Proof. Decompose \( Q(\Phi_*(\xi)) = \Phi_0(\xi_0) + R(\xi) \), where \( \Phi_0 \) is the classical moment map, and \( \deg R(\xi) \geq 1 \). Since \( \Phi_0(\xi_0) \) is a central element of \( \Gamma W \), we have \( [\Phi_0(\xi_0), R(\xi)]_0 = 0 \), where \( [\cdot, \cdot]_0 \) denotes the Weyl product commutator. Therefore we have the (formal) equation
\[
\sum_{k=0}^{\infty} \frac{1}{k!} Q(\Phi_*(\xi))^\circ k = e^{\Phi_0(\xi_0)} \sum_{k=0}^{\infty} \frac{1}{k!} R(\xi)^{\circ k}.
\]
The last sum on the right converges with respect to the filtration of \( \Gamma W \) since \( \deg R(\xi) \geq 1 \). So, it is easy to see that the right-hand side of (12) converges absolutely and uniformly on any compact subset of \( M \). Applying the Weyl derivation \( D \) on (10) term by term, we see that (10) is a flat section.

To show the last statement, we express \( R(\xi) \) as
\[
R(\xi) = \sum_{|\ell|+j \geq 1} r_{I,j}(\Phi_*(\xi), \partial \Phi_*(\xi), \ldots) y^I \lambda^j.
\]
Each \( r_{I,j} \) is a polynomial in \( \{ \Phi_*(\xi), \partial \Phi_*(\xi), \ldots \} \), since it is obtained by the Fedosov quantization procedure. Thus each coefficient of \( y^I \lambda^j \) in the series
\[
\sum_{k=0}^{\infty} \frac{1}{k!} R(\xi)^{\circ k}
\]
is also a polynomial. \( \square \)

Definition 3.2. For any \( \xi \in \mathfrak{g}_\lambda \), the function \( \exp_*(\Phi_*(\xi)) \) in \( C^\infty(M)[\lambda] \) is defined by
\[
\exp_*(\Phi_*(\xi)) = \sigma \left( \sum_{k=0}^{\infty} \frac{1}{k!} Q(\Phi_*(\xi))^\circ k \right).
\]

In the proof of Lemma 3.1, setting \( \xi = \alpha^l X_l \), where \( \{ \alpha^l \} \in \mathbb{C}^n[\lambda] \) and \( \{ X_l \} \) is a basis of \( \mathfrak{g}_0 \), we have:

Corollary 3.2. \( \exp_*(\Phi_*(\alpha^l X_l)) \) is a product of \( e^{\alpha^l_0 \Phi_0(X_l)} \) and a polynomial in \( \alpha^l \) taking values in \( C^\infty(M)[\lambda] \).
Proof. Since a quantum moment map is linear with respect to $\xi = \alpha^l X_l \in g[\lambda]$, each $r_{ij}$ in (13) is also linear with respect to $\xi$ and (14) is a polynomial in $\alpha^l$.

Lemma 3.2. Assume $\xi, \eta \in g[\lambda]$. Then

\begin{equation}
\exp_*(\Phi_*(\xi)) \ast \exp_*(\Phi_*(\eta)) = \exp_*(\Phi_*(CH_\lambda(\xi, \eta))).
\end{equation}

Proof. By the definition of $\exp_*(\Phi_*(\xi))$,

\begin{equation}
Q(\exp_*(\Phi_*(CH_\lambda(\xi, \eta)))) = \sum \frac{1}{k!} Q(\Phi_*(CH_\lambda(\xi, \eta)))^{\circ k} = \sum \frac{1}{k!} (CH_\circ(Q(\Phi_*(\xi)), Q(\Phi_*(\eta))))^{\circ k},
\end{equation}

where $CH_\circ$ denotes the Campbell–Hausdorff series with respect to the Weyl product $\circ$ of the Weyl bundle $\Gamma_W$. Since

\begin{equation}
\sum \frac{1}{k!} (CH_\circ(Q(\Phi_*(\xi)), Q(\Phi_*(\eta))))^{\circ k} \circ \sum \frac{1}{k!} (Q(\Phi_*(\eta)))^{\circ k}
\end{equation}
in $\Gamma_W$, we have the lemma.

For each multi-index $J = (j_1, j_2, \ldots, j_n)$, define a differential operator

\begin{equation}
D^J_\alpha = \left(-i \frac{\partial}{\partial \alpha^1}\right)^{j_1} \cdots \left(-i \frac{\partial}{\partial \alpha^n}\right)^{j_n}.
\end{equation}

Lemma 3.3. Assume $\{\alpha^l\} \in \mathbb{R}^n$. Then

\begin{equation}
(D^J \exp_*(\Phi_*(i\alpha^l X_l)))|_{\alpha=0} = \Phi_*(X^J).
\end{equation}

Proof. Let $\tilde{X}_l = Q(\Phi_*(X_l))$.

\begin{equation}
D^J(Q(\exp_*(\Phi_*(i\alpha^l X_l))))|_{\alpha=0} = D^J\left(\sum_{k=0}^{\infty} \frac{1}{k!} (i\alpha^l \tilde{X}_l)^{\circ k}\right)|_{\alpha=0}
= D^J\left(\frac{1}{|J|!} (i\alpha^l \tilde{X}_l)^{\circ |J|}\right)|_{\alpha=0}
= Q(\Phi_*(X^J)).
\end{equation}

3.3. Oscillatory integral expression for $\Phi_*$. We shall provide another expression for a quantum moment map $\Phi_*$ by using $\exp_*(\Phi_*(X))$ and an oscillatory integral. This expression gives us a clear understanding of quantum moment maps and enables us to show the differentiability of $\Phi_*$.

Definition 3.3. Let $\{X_l\}$ be a basis of $g$ and $\{X^l\}$ its dual basis. We define the map $\overline{\Phi}_*$ from $A^0$ into $C^\infty(M)[\lambda]$ as follows:

\begin{equation}
\overline{\Phi}_*(u) = \text{Os} \int u(\mu X)e^{-i\mu \nu} \exp_*(\Phi_*(i\nu X)) \, d\mu \, d\nu, \quad u \in A^0,
\end{equation}
where $\mu X = \mu_l X_l$ and $\nu X = \nu_l X_l$.

This definition makes sense since $u(\mu X) \exp_s(\Phi_s(i\nu X)) \in \mathcal{A}$. It is easy to see that the definition does not depend on a choice of a basis $\{X_l\}$.

**Lemma 3.4.** $\Phi_s$ coincides with $\Phi_*$ on polynomials.

**Proof.** Let $X^J$ be a monomial on $g^*$. Then

$$\Phi_s(X^J) = \text{Os-}\int \beta^J e^{-i\alpha\beta} \exp_s(\Phi_s(i\alpha X)) \, d\alpha \, d\beta = \text{Os-}\int e^{-i\alpha\beta} D^J_\alpha \exp_s(\Phi_s(i\alpha X)) \mid_{\alpha = 0} = \Phi_s(X^J),$$

where we have applied Equation (16) in the last line. □

So we shall also use the notation $\Phi_*$ for $\Phi_s$.

The following proposition says that $\exp_s(\Phi_s(X))$ can be considered as the image of $e^X$ under a quantum moment map.

**Proposition 3.3.** Assume that $p^k = p^k_j \lambda_j \in \mathbb{C}[\lambda]$ satisfies $p^k_0 \in i\mathbb{R}$. Then $e^{pX} = e^{p^k X_k} \in \mathcal{A}$ and $\Phi_*(e^{pX}) = \exp_s(\Phi_*(pX))$.

**Proof.** Let $pX = iaX + rX$, where $a \in \mathbb{R}$ and $r \in \lambda \mathbb{C}[\lambda]$. Then $e^{pX} \in \mathcal{A}$ by Equation (5). By the definition of $\Phi_*$,

$$\Phi_*(e^{pX}) = \text{Os-}\int e^{p\mu} e^{-i\mu \nu} \exp_s(\Phi_s(i\nu X)) \, d\mu \, d\nu = \text{Os-}\int e^{i(a - \nu)\mu} e^{rD_\nu} \exp_s(\Phi_s(i\nu X)) \, d\mu \, d\nu = \text{Os-}\int e^{i(a - \nu)\mu} \exp_s(\Phi_s(i(\nu - ir)X)) \, d\mu \, d\nu = \exp_s(\Phi_*(i(a - ir)X)) = \exp_s(\Phi_*(pX)),$$

where we have applied Equation (2). □

As a corollary of Proposition 3.3 and Lemma 3.2, we have

$$\Phi_*(e^{i\xi}) \ast \Phi_*(e^{i\eta}) = \Phi_*(e^{i\xi + G e^{i\eta}}).$$

**Theorem 3.3.** The quantum moment map $\Phi_*$ is differentiable. Moreover, if $*$ is of Fedosov type, there are functions $S_{I,j} \in C^\infty(M)$, $I = (i_1, \ldots, i_n)$, $j = 0, 1, \ldots$ such that

$$\Phi_*(u) = \sum_{j=0}^\infty \sum_{0 \leq |I| \leq 2j} S_{I,j} \Phi_0(D^I_\mu u) \text{ for any } u(\mu) \in C^\infty(g^*).$$
Proof. First we assume that $\ast$ is of Fedosov type. By Corollary 3.2, we have an expression for $\exp_s(\Phi_s(i\alpha^l X_l))$:

$$e^{i\alpha^l \Phi_0(X_l)} \sum_{j=0}^{\infty} \lambda^j \sum_{0 < |l| \leq 2j} S_{l,j} \alpha^l,$$

where $S_{l,j} \in C^\infty(M)$ depends on $\Phi_s(X_l)$ and on the Weyl connection $D$ of $\ast$. By the definition of $\Phi_s$, we have

$$\Phi_s(u) = \text{Os-} \int u(\mu)e^{-i\mu^\ast \nu^\ast} \exp_s(\Phi_s(i\nu^\ast X_l)) \, d\mu \, d\nu$$

$$= \sum_{j=0}^{\infty} \lambda^j \sum_{0 < |l| \leq 2j} \text{Os-} \int u(\mu)e^{-i\mu^\ast \nu^\ast} e^{i\nu^\ast \Phi_0(X_l)} S_{l,j} \nu^l \, d\mu \, d\nu$$

$$= \sum_{j=0}^{\infty} \lambda^j \sum_{0 < |l| \leq 2j} \text{Os-} \int S_{l,j} (D^l_\mu u)(\mu)e^{-i\nu^\ast (\mu^\ast - \Phi_0(X_l))} \, d\mu \, d\nu$$

$$= \sum_{j=0}^{\infty} \lambda^j \sum_{0 < |l| \leq 2j} S_{l,j} \Phi_0(D^l_\mu u).$$

For a general $G$-invariant star product $\ast'$, we have a $G$-invariant star product $\ast$ of Fedosov type which is $G$-equivalent to $\ast'$ by Theorem B.3. If $T$ denotes a $G$-equivalence between $\ast$ and $\ast'$, then any quantum moment map $\Phi_{\ast'}$ of $\ast'$ has the form $T \Phi_s$. Therefore $\Phi_{\ast'}$ is differentiable. \qed

By Theorem 3.3, $\Phi_s$ admits a unique extension to $C^\infty(\mathfrak{g}^*)[\lambda]$. Since $\Phi_s$ is an algebra homomorphism on polynomials on $\mathfrak{g}^*$, the differentiability of $\Phi_s$ implies that $\Phi_s$ is an algebra homomorphism on $C^\infty(\mathfrak{g}^*)[\lambda]$.

It is not difficult to compute $S_{l,j}$ for lower degrees in $l, j$. For instance, $S_{0,0} = 1$, $S_{1,1} = \Phi_1(X_1)$, $S_{m,1} = \{ \Phi_0(X_l), \Phi_0(X_m) \}$, and so on. Therefore $\Phi_s(u) = \Phi_0(u) + o(\lambda)$ for any $u \in C^\infty(\mathfrak{g}^*)$.

### 3.4. Properties of $\Phi_s$

**Proposition 3.4.** A quantum moment map is a $\mathfrak{g}$-equivariant map from $C^\infty(\mathfrak{g}^*)[\lambda]$ to $C^\infty(M)[\lambda]$. Therefore, $\Phi_s$ is $G$-equivariant if $G$ is connected.

**Proof.** This is a direct consequence of the definition of quantum moment maps and the $G$-invariance of $\ast^G$ and $\ast$: for any $u \in C^\infty(\mathfrak{g}^*)$ and $\xi \in \mathfrak{g}$,

$$\Phi_s(\lambda \xi u) = \Phi_s([\xi, u]_\ast^G) = [\Phi_s(\xi), \Phi_s(u)]_\ast = \lambda \xi \Phi_s(u).$$ \qed

**Proposition 3.5.** If $f \in C^\infty(M)[\lambda]$ commutes with any $\Phi_s(u)$, $u \in C^\infty(\mathfrak{g}^*)$, then $f$ is a $G$-invariant function.
Proof. It is easy. \qed

**Proposition 3.6.** A quantum moment map $\Phi_*$ is surjective if and only if its classical part $\Phi_0$ is surjective.

**Proof.** Assume that $\Phi_0$ is surjective. For $u = \sum u_i\lambda^i \in C^\infty(\mathfrak{g}^*)[\lambda]$ and $\varphi \in C^\infty(M)$ the equation $\Phi_*(u) = \varphi$ is equivalent to

$$\Phi_0(u_0) = \varphi,$$

$$\Phi_0(u_k) = -\sum_{j=1}^k \Phi_j(u_{k-j}) \quad \text{for any } k > 0. \quad (20)$$

One can solve this system of equations by induction, since $\Phi_0$ is surjective. The converse is trivial. \qed

**Lemma 3.5.** Let $\varphi$ be a smooth function on $M$. Assume there exists a solution $u \in C^\infty(\mathfrak{g}^*)[\lambda]$ of the equation $\Phi_*(u) = \varphi$. Then $u$ depends locally on $\varphi$. More precisely, the dependence of $u$ at $J(q)$ on $\varphi$ is described by differentials of $\varphi$ at $q \in M$, where $J : M \to \mathfrak{g}^*$ is the dual form of $\Phi_*$, that is, $(\Phi_0(u))(q) = u(J(q))$.

**Proof.** Equation (19) says that $u_0(J(q))$ depends on $\varphi(q)$. Since $\Phi_*$ is differentiable, the right-hand side of Equation (20) also depends on differentials of $\varphi$ if $0, \ldots, u_{k-1}$ depend on differentials of $\varphi$. \qed

**3.5. Invariants for $G$-invariant star products on transitive spaces.**

In this subsection we review the results of [12, 13], where we define invariants for $G$-invariant star products on $G$-transitive symplectic manifolds.

Let $M$ be a $G$-transitive symplectic manifold and $*$ a $G$-invariant star product on $M$. We assume that there is a unique quantum moment map $\Phi_*$ for $*$. Let $\mathfrak{z}$ be the center of $C^\infty(\mathfrak{g}^*)[\lambda]$, that is, the set of functions that commute with any smooth function on $\mathfrak{g}^*$ with respect to the Gutt star product. One can show that $\mathfrak{z}$ is equal to the set of $G$-invariant functions on $\mathfrak{g}^*$. For any $l \in \mathfrak{z}$ we have $[\Phi_*(l), \Phi_*(C^\infty(\mathfrak{g}^*))]_* = \Phi_*(l, C^\infty(\mathfrak{g}^*))_{*,*} = 0$, so that Proposition 3.5 implies $\Phi_*(l)$ is a $G$-invariant function on $M$. Since $M$ is transitive, $\Phi_*(l)$ is constant. Consequently, we make the following definition:

**Definition 3.4.** Let $M$ be a $G$-transitive symplectic manifold and $*$ a $G$-invariant star product admitting a unique quantum moment map $\Phi_*$. Define a map $c_* : \mathfrak{z} \to \mathbb{C}[\lambda]$ by $c_*(l) = \Phi_*(l)$ for any $l \in \mathfrak{z}$.

The following simple proposition is important:

**Proposition 3.7.** If $\ker \Phi_* = \ker \Phi_*$ then $c_* = c_*'$.

**Proof.** Any function on $C^\infty(\mathfrak{g}^*)[\lambda]$ of the form $l - c_*(l)$ for $l \in \mathfrak{z}$ is an element of $\ker \Phi_*$. Therefore if $\ker \Phi_* = \ker \Phi_*$, we have $\Phi_*(l - c_*(l)) = 0$, that is, $c_*(l) = c_*(l)$. \qed
Corollary 3.3. If $\ast'$ is $G$-equivalent to $\ast$ then $c_\ast = c_{\ast'}$.

This means that $c_\ast$ depends only on the class of $G$-equivalent star products. We have computed $c_\ast$ for a few examples in [13].

There is a natural question to ask: Does $c_\ast$ parametrize the class of $G$-invariant star products on a $G$-transitive space? In the next section, we give a complete answer of this question when $M$ is a coadjoint orbit of a compact semisimple Lie group.

4. Star products on regular coadjoint orbits of compact semisimple Lie groups

Let $G$ be a real compact semisimple Lie group and $O \subset g^*$ a regular coadjoint orbit of $G$. $O$ has a natural symplectic structure that is induced from the Kirillov–Poisson structure $\Pi$.

Recall that there is a $G$-invariant star product on $O$ since $G$ is compact. Since $G$ is semisimple, Theorems 3.1 and 3.2 imply that for each $G$-invariant star product $\ast$ on $O$ there is a unique quantum moment map of $\ast$.

We study here $G$-invariant star products on $O$, and our goal is to present a structure theory for them.

4.1. Structure theory. Let $\ast$ be a $G$-invariant star product on $O$ and $\Phi_\ast = \Phi_0 + \Phi_1 \lambda_1 + \cdots$ a quantum moment map of $\ast$. The classical moment map $\Phi_0$ is given simply by the pullback of the embedding map of $O$ in $g^*$; that is, $\Phi_0$ is surjective. Therefore, Proposition 3.6 implies that the quantum moment map $\Phi_\ast$ is also surjective. As a direct consequence:

Proposition 4.1. We have a $G$-equivariant isomorphism

\[ C^\infty(g^*)[\lambda]/\ker \Phi_\ast \cong C^\infty(O)[\lambda]. \]

(21)

Let $\ast$ and $\ast'$ be $G$-invariant star products. If we assume $\ker \Phi_\ast = \ker \Phi_{\ast'}$, Proposition 4.1 defines a morphism $S : (C^\infty(O)[\lambda], \ast) \to (C^\infty(O)[\lambda], \ast')$. Let $\varphi$ be a smooth function on $O$. There exists $u \in C^\infty(g^*)[\lambda]$ such that $\Phi_\ast(u) = \varphi$, and we define $S(\varphi) \equiv \Phi_{\ast'}(u)$. Lemma 3.5 and the differentiability of $\Phi_{\ast'}$ imply that the morphism $S$ is differentiable. Therefore:

Lemma 4.1. $\ker \Phi_\ast = \ker \Phi_{\ast'}$ if and only if $\ast$ is $G$-equivalent to $\ast'$.

As we have seen, $\ker \Phi_\ast = \ker \Phi_{\ast'}$ implies $c_\ast = c_{\ast'}$. The next proposition shows that if there are “good coordinates” on $g^*$, the converse also holds:

Proposition 4.2. Assume there are functionally independent $G$-invariant functions $p_i : g^* \to \mathbb{R}$, $1 \leq i \leq r$, such that $O$ is given by the level set $\{ \xi \in g^* : p_i(\xi) = c_i \}$ for some regular value $\{c_i\}$ of $\{p_i\}$. Then $\ker \Phi_\ast$ is equal to the ideal of $(C^\infty(g^*)[\lambda], \ast^G)$ generated by $\{p_i - c_\ast(p_i) : 1 \leq i \leq r \}$. 
**Proof.** Let \( f = f_0 + f_1 \lambda + \cdots \in \ker \Phi_* \). It is easy to see that \( f_0 \in \ker \Phi_0 \), that is, \( f_0 \) is null on \( \mathcal{O} \). So there are functions \( g_i \in C^\infty(\mathfrak{g}^*) \) such that

\[
f_0 = \sum_{i=1}^r g_i (p_i - c_i).
\]

(22)

Setting

\[
f(0) = \sum_{i=1}^r g_i (p_i - c_i(p_i)),
\]

(23)

we have \( f(0) \in \ker \Phi_* \) and \( f(0) = f_0 \). Applying the same argument to \( (f - f(0))/\lambda \) inductively we find a sequence of functions \( f^{(k)} \) satisfying

\[
f = \sum_{k=0}^\infty f^{(k)} \lambda^k.
\]

(24)

Since each \( f^{(k)} \) has the form (23), this completes the proof. \( \square \)

Let \( I \subset \mathrm{Pol}(\mathfrak{g}^*) \) be the set of polynomials on \( \mathfrak{g}^* \) invariant under \( G \). By Chevalley’s theorem [19] one has \( I = \mathbb{C}[p_1, \ldots, p_r] \), where \( p_1, \ldots, p_r \) are algebraically independent homogeneous polynomials and \( r \) is the rank of \( \mathfrak{g} \). One can also see that any regular coadjoint orbit \( \mathcal{O} \) is given by the level set \( \{ \xi \in \mathfrak{g}^* : p_1(\xi) = c_1, \ldots, p_r(\xi) = c_r \} \) for some regular value \( \{c_j\} \) [14]. Therefore, \( \{p_j\} \) satisfies the condition of Proposition 4.2. So we have the inverse of Proposition 3.7.

**Proposition 4.3.** Let \( \ast, \ast' \) be \( G \)-invariant star products and let \( \Phi_*, \Phi_{*'} \) be quantum moment maps of \( \ast, \ast' \) respectively. Then \( c_* = c_{*'} \) implies \( \ker \Phi_* = \ker \Phi_{*'} \). Moreover, if we take \( \{p_j\} \) as the algebraically independent homogeneous polynomials obtained from Chevalley’s theorem, we have

\[
\ker \Phi_* = \langle p_j - \Phi_*(p_j) \rangle,
\]

where \( \langle p_j - \Phi_*(p_j) \rangle \) denotes the ideal of \( C^\infty(\mathfrak{g}^*)[\lambda] \) generated by \( p_j - \Phi_*(p_j) \).

And we have also the following structure theorem:

**Theorem 4.1.** For any \( G \)-invariant star product \( \ast \) on \( \mathcal{O} \), there are constants \( c_{*,j} \in \mathbb{C}[\lambda], j = 1, 2, \ldots, r \) such that

\[
(C^\infty(\mathcal{O})[\lambda], \ast) \cong C^\infty(\mathfrak{g}^*)[\lambda]/\langle p_i - c_{*,j} \rangle.
\]

Moreover, this isomorphism is \( G \)-equivariant.

**Proof.** Let \( \Phi_* \) be a quantum moment map of \( \ast \). If we set \( c_{*,j} = \Phi_*(p_j) \), the theorem is a direct consequence of Propositions 4.1 and 4.3. \( \square \)

**Corollary 4.1.** There is a one-to-one correspondence between the classes of \( G \)-invariant star products on \( \mathcal{O} \) and \( c_* \).
4.2. Example. Coadjoint orbits of SO(3). Let $\mathcal{O}$ be a regular coadjoint orbit of SO(3). It is well-known that $\mathcal{O}$ is a two-dimensional sphere in $\mathfrak{so}(3)^*$ given by the Casimir polynomial $p(x, y, z) = x^2 + y^2 + z^2$ on $\mathfrak{so}(3)^* = \mathbb{R}^3$; that is, there is a real number $r > 0$ such that

$$\mathcal{O} = \{(x, y, z) \in \mathfrak{g}^* : p(x, y, z) = r^2\}.$$ 

The class of SO(3)-invariant star products is parametrized by the second equivariant de Rham cohomology $H^2_{dR}(\mathcal{O}, \mathbb{R})^{SO(3)}$. Let $\star$ be a SO(3)-invariant star product on $\mathcal{O}$. Then $p$ satisfies the conditions of Proposition 4.2, so that $\ker \Phi_\star$ is described by $c_\star(p)$ and we obtain

$$(25) \quad (C^\infty(\mathcal{O})[\lambda], \star) \cong (C^\infty(\mathfrak{g}^*)[\lambda], \star^{\mathcal{O}})/\langle p - c_\star(p) \rangle.$$ 

Hence, $G$-invariant star products on $\mathcal{O}$ have the form of the right-hand side of (25) and are parametrized by $c_\star(p)$. This gives another classification of $G$-invariant star products on $\mathcal{O}$.

Appendices

A. Oscillatory integrals

We provide here a brief review on oscillatory integrals in order to fix some definitions and notations. The following is based on [15], with little modifications adapted to our problem:

**Definition A.1.** A function $a(\xi, x) \in C^\infty(\mathbb{R}_\xi^n \times \mathbb{R}_x^n)$ is said to be of $\mathcal{A}_\tau^m$-class, where $-\infty < m < \infty$ and $0 \leq \tau$, if for any multi-indices $I$ and $J$, there exists a constant $C_{I, J}$ such that

$$|\partial^I_\xi \partial^J_x a(\xi, x)| \leq C_{I, J} \langle \xi \rangle^m \langle x \rangle^{-\tau},$$

where $\langle \xi \rangle = \sqrt{1 + |\xi|^2}$. Set

$$\mathcal{A} = \bigcup_{-\infty < m < \infty} \bigcup_{0 \leq \tau} \mathcal{A}_\tau^m. $$

For $a(\xi, x) \in \mathcal{A}_\tau^m$, we define a family of seminorms $|a|_l, \ l = 0, 1, \ldots$, by

$$|a|_l = \max_{|I| + |J| \leq l} \sup_{(\xi, x)} \{|\partial^I_\xi \partial^J_x a(\xi, x)| \langle \xi \rangle^{-m} \langle x \rangle^{-\tau}\}.$$ 

Then $\mathcal{A}_\tau^m$ becomes a Fréchet space. A subset $B$ of $\mathcal{A}$ is called bounded if there is a $\mathcal{A}_\tau^m$ such that $B \subset \mathcal{A}_\tau^m$ and $\sup_{a \in B} \{|a|_l\} < \infty$ for any $l = 0, 1, \ldots$. 
Definition A.2. For any \(a(\xi, x) \in \mathcal{A}\), we define the oscillatory integral \(\text{Os}[e^{-i\xi x} a]\) by

\[
\text{Os}[e^{-i\xi x} a] \equiv \text{Os} \int e^{-i\xi x} a(\xi, x) \, d\xi \, dx = \frac{1}{(2\pi)^n} \lim_{\varepsilon \to 0} \int \chi(\varepsilon \xi, \varepsilon x) e^{-i\xi x} a(\xi, x) \, d\xi \, dx,
\]

where \(\chi(\xi, x)\) is any function of \(S(\mathbb{R}^{2n})\) satisfying \(\chi(0, 0) = 1\).

Lemma A.1. If \(\chi(x) \in S(\mathbb{R}^n)\) satisfies \(\chi(0) = 1\) then

\[
\chi(\varepsilon x) \to 1 \quad (\text{uniformly on compact sets}),
\]

\[
\partial_I \chi(\varepsilon x) \to 0 \quad (\text{uniformly on } \mathbb{R}^n, |I| > 0),
\]

and for any multi-index \(I\) there is a constant \(C_I\) independent of \(0 < \varepsilon < 1\) such that for any \(\sigma\) such that \(0 \leq \sigma \leq |I|\),

\[
|\partial_I \chi(\varepsilon x)| \leq C_I \varepsilon^\sigma \langle x \rangle^{-(|I| - \sigma)}.
\]

Proof. \((27)\) is clear since \(\partial_I \chi(\varepsilon x) = \varepsilon^{|I|} \partial_I \chi(y)|_{y = \varepsilon x}\). If \(|x| \leq 1\) then \((28)\) is obtained from the equality \(|\partial_I \chi(\varepsilon x)| = \varepsilon^\sigma (\varepsilon^{(|I| - \sigma)}|\partial_I \chi(y)|_{y = \varepsilon x})\). If \(|x| > 1\) and \(0 \leq \sigma \leq |I|\), we have

\[
\varepsilon^{(|I| - \sigma)}|\partial_I \chi(y)|_{y = \varepsilon x} = \left(\frac{|y|^{(|I| - \sigma)}|\partial_I \chi(y)|}{|y = \varepsilon x|} \right)|x|^{-(|I| - \sigma)} \leq C_I \langle x \rangle^{-(|I| - \sigma)} \]

\(\square\)

Theorem A.1. For any \(a \in \mathcal{A}\), \(\text{Os}[e^{-i\xi x} a]\) is independent of the choice of \(\chi \in S\) satisfying \(\chi(0, 0) = 1\). For \(a \in \mathcal{A}_\tau^m\), if we take integers \(l, l'\) satisfying

\[
-2l + m < -n, \quad -2l' + \tau < -n,
\]

then

\[
|\langle x \rangle^{-2l'} \langle D_\xi \rangle^{2l'} \{ \langle \xi \rangle^{-2l} \langle D_x \rangle^{2l} a(\xi, x) \}| \in L_1(\mathbb{R}^{2n})
\]

and

\[
\text{Os}[e^{-i\xi x} a] = \text{Os} \int e^{-i\xi x} \langle x \rangle^{-2l'} \langle D_\xi \rangle^{2l'} \{ \langle \xi \rangle^{-2l} \langle D_x \rangle^{2l} a(\xi, x) \} \, d\xi \, dx.
\]

Moreover, for \(a \in \mathcal{A}_\tau^m\) there is a constant \(C\) such that

\[
|\text{Os}[e^{-i\xi x} a]| \leq C|a|_{2(l + l')}.
\]
Proof. Fix $0 < \varepsilon < 1$. Integrating by parts, we have

$$I_\varepsilon \equiv \int e^{-i\xi x} \chi(\varepsilon \xi, \varepsilon x) a(\xi, x) \, d\xi \, dx$$

$$= \int e^{-i\xi x} (\xi)^{-2l} \langle \chi(\varepsilon \xi, \varepsilon x) a(\xi, x) \rangle \, d\xi \, dx$$

$$= \int e^{-i\xi x} (\xi)^{-2l'} \langle D_\xi \rangle^{2l'} (\xi)^{-2l} \langle D_x \rangle^{2l} \langle \chi(\varepsilon \xi, \varepsilon x) a(\xi, x) \rangle \, d\xi \, dx.$$ 

Lemma A.1 implies the set $\{\chi(\varepsilon \xi, \varepsilon x)\}_{0 < \varepsilon < 1}$ is a bounded subset of $A_0^0$, so that for any $I$ and $J$ there is a constant $C_{I, J}$ independent of $\varepsilon, a \in A_\tau^m$ such that

$$\left| \partial^I_I \partial^J_J (\chi(\varepsilon \xi, \varepsilon x) a(\xi, x)) \right| \leq C_{I, J} |a| (|I|+|J|) \langle \xi \rangle^m \langle x \rangle^\tau.$$ 

On the other hand, for any $s$, there is a constant $C_{s, I}$ such that

$$\left| \partial^I_I \langle \xi \rangle^s \right| \leq C_{s, I} \langle \xi \rangle^{-|I|},$$

obtained by induction from $\partial_{\xi_j} \langle \xi \rangle^s = s \xi_j \langle \xi \rangle^{s-1}$. From these facts we deduce that there is for any $I$ a constant $C_{l, I}$ independent of $\varepsilon, a \in A_\tau^m$ and such that

$$\left| \partial^I_I \langle \xi \rangle^{-2l} \langle D_x \rangle^{2l} (\chi(\varepsilon \xi, \varepsilon x) a(\xi, x)) \right| \leq C_{l, I} |a| (2l+|I|) \langle \xi \rangle^{m-2l} \langle x \rangle^\tau.$$ 

Hence there is a constant $C_{l, l'}$ independent of $\varepsilon, a \in A_\tau^m$ such that

$$(\xi)^{-2l'} \langle D_\xi \rangle^{2l'} \left\{ \langle \xi \rangle^{-2l} \langle D_x \rangle^{2l} (\chi(\varepsilon \xi, \varepsilon x) a(\xi, x)) \right\} \leq C_{l, l'} |a| (l'+|I|) \langle \xi \rangle^{m-2l} \langle x \rangle^{\tau-2l'}.$$ 

The right-hand side of (31) is in $L_1(\mathbb{R}^{2n})$ because of (29). Hence Lebesgue’s convergence theorem gives

$$\text{Os}[e^{i\xi x} a] = \lim_{\varepsilon \to 0} \frac{I_\varepsilon}{(2\pi)^2}$$

$$= \frac{1}{(2\pi)^2} \int e^{-i\xi x} (\xi)^{-2l'} \langle D_\xi \rangle^{2l'} \left\{ \langle \xi \rangle^{-2l} \langle D_x \rangle^{2l} a(\xi, x) \right\} \, d\xi \, dx,$$

and proves (30). \qed

Theorem A.2. Assume $\{a_j\}_{j=1}^\infty$ is a bounded set of $A$ and there is $a \in A$ such that

$$a_j(\xi, x) \to a(\xi, x) \text{ uniformly on compact sets of } \mathbb{R}^{2n}.$$ 

Then

$$\lim_{j \to \infty} \text{Os}[e^{-i\xi x} a_j] = \text{Os}[e^{-i\xi x} a].$$
**Theorem A.3.** The oscillatory integral satisfies the following formulas:
\[ \text{Os}[e^{-i\xi x}a(\xi, x)] = \text{Os}[e^{-i(\xi - \xi_0)(x - x_0)}a(\xi - \xi_0, x - x_0)], \quad (\xi_0, x_0) \in \mathbb{R}^{2n}, \]
\[ \text{Os}[e^{-i\xi x}x^I a] = \text{Os}[(\xi - D_x)^I e^{-i\xi x}a] = \text{Os}[e^{-i\xi x} D^I_x a], \]
\[ \text{Os}[e^{-i\xi x}\xi^I a] = \text{Os}[(\xi - D_x)^I e^{-i\xi x}a] = \text{Os}[e^{-i\xi x} D^I_x a]. \]

**Theorem A.4.** Let \( a = a(x) \in A \) be a function depending only on \( x \). Then
\[ \text{Os} \int e^{-i\xi(x-y)}a(x)\,d\xi\,dx = a(y). \]

**B. The Fedosov construction of star products**

We provide here a brief summary of the Fedosov construction, which is one of the most useful method of constructing a star product on a symplectic manifold \((M, \omega)\). For details see [7, 8].

A formal Weyl algebra \( W_x \) associated with \( T_x M \) for \( x \in M \) is an associative algebra with unit over \( \mathbb{C} \) defined as follows: each element of \( W_x \) is a formal power series in \( \lambda \) with coefficients being formal polynomials in \( T_x M \), that is, each element has the form
\[ a(y, \lambda) = \sum_{k, J} \lambda^k a_{k, J} y^J, \]
where \( y = (y^1, \ldots, y^{2n}) \) are linear coordinates on \( T_x M \), \( J = (j_1, \ldots, j_{2n}) \) is a multi-index and \( y^J = (y^1)^{j_1} \cdots (y^{2n})^{j_{2n}} \). The product \( \circ \) is defined by the Moyal–Weyl rule,
\[ a \circ b = \sum_{k=0}^{\infty} \frac{(\lambda^2)^k}{k!} y^{i_1j_1} \cdots \omega^{i_kj_k} \frac{\partial^k a}{\partial y^{i_1} \cdots \partial y^{i_k}} \frac{\partial^k b}{\partial y^{j_1} \cdots \partial y^{j_k}}, \]
where the \( \omega^{lm} \) are the coefficients of \( \omega \) with respect to \( y^J \). If we assign \( \deg y^j = 1 \) and \( \deg \lambda = 2 \), the algebra \( W_x \) becomes a filtered algebra.

Let \( W = \bigcup_{x \in M} W_x \). Then \( W \) is a bundle of algebras over \( M \), called the Weyl bundle over \( M \). Each section of \( W \) has the form
\[ a(x, y, \lambda) = \sum_{k, \alpha} \lambda^k a_{k, \alpha}(x) y^\alpha, \]
where \( x \in M \). We call \( a(x, y, \lambda) \) smooth if each coefficient \( a_{k, \alpha}(x) \) is smooth in \( x \). We denote the set of smooth sections by \( \Gamma W \). It constitutes an associative algebra with unit under the fibrewise multiplication.

Let \( \nabla \) be a torsion-free symplectic connection on \( M \), which always exists and \( \partial : \Gamma W \to \Gamma W \otimes \Lambda^1 \) be its induced covariant derivative. Consider a connection on \( W \) of the form
\[ Da = -\delta a + \partial a - \frac{1}{\lambda} [\gamma, a] \quad \text{for} \ a \in \Gamma W \]
with \( \gamma \in \Gamma W \otimes \Lambda^1 \), where
\[
\delta a = dx^k \wedge \frac{\partial a}{\partial y^k}.
\]
Clearly, \( D \) is a derivation for the Moyal–Weyl product. A simple computation shows that
\[
D^2 a = \frac{1}{\lambda} [\Omega, a] \quad \text{for any } a \in \Gamma W,
\]
where
\[
\Omega = \omega - R + \delta \gamma - \partial \gamma + \frac{1}{\lambda} \gamma^2.
\]
Here \( R = \frac{1}{4}i R_{ijkl} y^i \, dx^k \wedge dx^l \) and \( R_{ijkl} = \omega_{im} R^{m}_{jkl} \) is the curvature tensor of the symplectic connection.

A connection of the form (33) is called Abelian if \( \Omega \) is a scalar 2-form, that is, \( \Omega \in \Lambda^2 \mathbb{R} \). We call \( D \) a Fedosov connection if it is Abelian and \( \deg \gamma \geq 3 \). For an Abelian connection, the Bianchi identity implies that \( d \Omega = D \Omega = 0 \), that is, \( \Omega \) is closed. In this case, we call \( \Omega \) a Weyl curvature.

**Theorem B.1** ([7]). Let \( \nabla \) be any torsion-free symplectic connection, and \( \Omega = \omega + \lambda \omega_1 + \cdots \in Z^2(M)[\lambda] \) a perturbation of the symplectic form \( \omega \). There exists a unique \( \gamma \in \Gamma W \otimes \Lambda^1 \) such that \( D \) given by Equation (33) is a Fedosov connection which has Weyl curvature \( \Omega \) and satisfies \( \delta^{-1} \gamma = 0 \).

We denote \( W_D \) the set of smooth and flat sections, that is, sections \( a \) in \( \Gamma W \) satisfying \( Da = 0 \). The space \( W_D \) becomes a subalgebra of \( \Gamma W \). Let \( \sigma \) denote the projection of \( W_D \) onto \( C^\infty(M)[\lambda] \) defined by \( \sigma(a) = a|_{y=0} \).

**Theorem B.2** ([7]). Let \( D \) be an Abelian connection. For any \( a_0(x, \lambda) \in C^\infty(M)[\lambda] \) there exists a unique section \( a \in W_D \) such that \( \sigma(a) = a_0 \). Thus \( \sigma \) establishes an isomorphism between \( W_D \) and \( C^\infty(M)[\lambda] \) as \( \mathbb{C}[\lambda] \)-vector spaces.

We denote the inverse map of \( \sigma \) by \( Q \) and call it a quantization procedure. The Weyl product \( \circ \) on \( W_D \) is translated to \( C^\infty(M)[\lambda] \), yielding a star product \( * \). Namely, we set for \( a, b \in C^\infty(M)[\lambda] \)
\[
a * b = \sigma(Q(a) \circ Q(b)).
\]

For \( G \)-invariant star products, there is a simple criterion.

**Proposition B.1** ([8]). Let \( \nabla \) be a \( G \)-invariant connection, \( \Omega \) a \( G \)-invariant Weyl curvature and \( D \) the Fedosov connection corresponding to \( (\nabla, \Omega) \). The star product corresponding to \( D \) is \( G \)-invariant.

We study mainly star products of Fedosov type because of the following theorem:
Theorem B.3 ([3]). Every $G$-invariant star product is $G$-equivalent to a Fedosov star product.

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FLAT FRONTS IN HYPERBOLIC 3-SPACE

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We investigate flat surfaces in hyperbolic 3-space with admissible singularities, called flat fronts. An Osserman-type inequality for complete flat fronts is shown. When equality holds in this inequality, we show that all the ends are embedded, and give new examples for which equality holds.

Introduction

It is a classical fact that any complete flat surface in the hyperbolic 3-space $H^3$ must be a horosphere or a hyperbolic cylinder. However, this does not imply the lack of an interesting global theory for flat surfaces. Recently, Gálvez, Martínez and Milán [4] established a Weierstrass-type representation formula for such surfaces. More recently, the authors [7] proved another representation formula constructing a flat surface from a given pair of hyperbolic Gauss maps, and also gave new examples.

In this paper, we investigate global properties of flat surfaces with admissible singularities, accounting for all the previous examples in [4] and [7]. (A singular (i.e., degenerate) point is called admissible if the corresponding points on nearby parallel surfaces are regularly immersed. See Section 2.) Such surfaces are characterized as the projections of Legendrian immersions in the unit cotangent bundle $T^*_1H^3$ of $H^3$, called flat fronts. The 5-manifold $T^*_1H^3$ has the canonical contact form $\eta$. If we identify $H^3$ with the Poincaré ball $(D^3; x^1, x^2, x^3)$, any element $\alpha$ of the cotangent bundle $T^*H^3$ can be written as

$$\alpha = p_1(\alpha) \, dx^1 + p_2(\alpha) \, dx^2 + p_3(\alpha) \, dx^3 \quad (\in T^*H^3).$$

Then $(p_1, p_2, p_3, x^1, x^2, x^3)$ gives a canonical coordinate system of $T^*H^3$ and the canonical form on $T^*H^3$,

$$\eta = p_1 \, dx^1 + p_2 \, dx^2 + p_3 \, dx^3,$$

which induces a canonical contact form on $T^*_1H^3$. An immersion $L: M^2 \rightarrow T^*_1H^3$ is called Legendrian if the pullback $L^*\eta$ vanishes identically. For a given immersion $f: M^2 \rightarrow H^3$, there exists a unique Legendrian immersion $L_f: M^2 \rightarrow T^*_1H^3$. 

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such that $\pi \circ L_f = f$, where $\pi : T^*_1H^3 \to H^3$ is the projection. That is, any immersion can be lifted to a Legendrian immersion. However, the converse is not true. A projection

$$\pi \circ L : M^2 \longrightarrow H^3$$

of a Legendrian immersion $L$ is called a \textit{(wave) front}, which may have singular points (points where the Jacobi matrix degenerates.) A point which is not singular is called \textit{regular}, where the first fundamental form is positive definite. The Gaussian curvature is well-defined at regular points. A front is called \textit{flat} if the Gaussian curvature vanishes at each regular point.

A front $f$ is called \textit{complete} if there is a symmetric tensor $T$ on $M^2$ which has compact support such that $T + ds^2$ is a complete Riemannian metric on $M^2$, where $ds^2$ is the first fundamental form of $f$. If $M^2$ is orientable, $M^2$ can be regarded as a Riemann surface whose complex structure is compatible with respect to the pullback of the Sasakian metric on $T^*_1H^3$ by $L_f$. Moreover, the second fundamental form is Hermitian with respect to this structure, and there is a closed Riemann surface $\bar{M}^2$ such that $\bar{M}^2$ is biholomorphic to $\bar{M}^2 \setminus \{p_1, \ldots, p_n\}$. The points $p_1, \ldots, p_n$ are called the \textit{ends} of $f$.

For each point $p \in M^2$, there exists a pair $(G(p), G_*(p)) \in S^2 \times S^2$ of distinct points on the ideal boundary $S^2 = \partial H^3$ such that the geodesic in $H^3$ starting from $G_*(p)$ towards $G(p)$ coincides with the oriented normal geodesic at $p$ (see Figure 1). The maps

$$G, \ G_* : \bar{M}^2 \setminus \{p_1, \ldots, p_n\} \longrightarrow S^2$$

are called the \textit{positive} and \textit{negative hyperbolic Gauss maps} of $f$, respectively. They are holomorphic if we regard $S^2 = \partial H^3$ as the Riemann sphere. An end $p_j$ is called \textit{regular} if both $G$ and $G_*$ extend holomorphically across it. As we shall show later, there are many flat fronts with regular ends. Moreover, such surfaces satisfy the following global property:
Theorem. An orientable complete flat front $f: \mathcal{M}^2 \setminus \{p_1, \ldots, p_n\} \to H^3$ with regular ends satisfies the inequality

$$\deg G + \deg G_* \geq n,$$

where $\deg G$ is the degree of the holomorphic map $G: \mathcal{M}^2 \to \mathbb{CP}^1 = S^2$. Equality holds if and only if all ends are embedded.

This inequality is an analogue of the Osserman inequality

$$2 \deg G + \chi(\mathcal{M}^2 \setminus \{p_1, \ldots, p_n\}) \geq n,$$

which holds for the Gauss map $G$ of either a complete minimal surface $f: \mathcal{M}^2 \setminus \{p_1, \ldots, p_n\} \to \mathbb{R}^3$ with finite total curvature, or a surface $f: \mathcal{M}^2 \setminus \{p_1, \ldots, p_n\} \to H^3$ of mean curvature 1. In these two cases, as in ours, equality implies the embeddedness of ends. (See [8, 5] for the minimal surface case and [12] for the hyperbolic case.)

To prove that equality implies the ends are embedded, a criterion for embeddedness of ends given in [4] will be applied. Furthermore, we shall classify flat 3-noids and exhibit a genus-1 flat front with regular ends (Section 4).

On the other hand, since the pullback of the Sasakian metric $d\sigma^2$ by the Legendrian lift of a complete flat front $f$ is complete, it satisfies the Cohn-Vossen inequality

$$\frac{1}{2\pi} \int_{M^2} (-K_{d\sigma^2}) dA_{d\sigma^2} \geq -\chi(M^2),$$

(0.1)

where $dA_{d\sigma^2}$ is the area element of $d\sigma^2$ and $\chi(M^2)$ is the Euler number of $M^2$. In Section 3, we shall prove that equality holds if and only if all ends are asymptotic to a hyperbolic cylinder.

Note that flat hypersurfaces in $H^n$ ($n \geq 4$) are totally umbilic. So $n = 3$ is the interesting case.

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1. Local properties of flat surfaces

In this section, we review local properties of flat immersions. We denote by $H^3$ the hyperbolic 3-space of constant curvature $-1$. Let $M^2$ be a 2-manifold and

$$f: M^2 \to H^3$$

a flat immersion, meaning that the Gaussian curvature of the induced metric vanishes. It follows from the Gauss equation that the second fundamental form is positive or negative definite and thus $M^2$ is orientable. We fix an orientation of $M^2$. Then $M^2$ can be regarded as a Riemann surface such
that the second fundamental form $dh^2$ is Hermitian. A holomorphic map or immersion

$$E = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : M^2 \to \text{SL}(2, \mathbb{C})$$

is called a Legendrian curve or immersion if

$$(1.1) \quad D \, dA - B \, dC = 0.$$  

Indeed, (1.1) implies the vanishing of the pullback of a holomorphic contact form on $\text{SL}(2, \mathbb{C})$. As is shown in [4], there exists a holomorphic Legendrian immersion

$$E_f : \tilde{M}^2 \to \text{SL}(2, \mathbb{C})$$

defined on the universal cover $\tilde{M}^2$ of $M^2$ such that $f$ is the projection of $E_f$ onto $H^3 = \text{SL}(2, \mathbb{C})/\text{SU}(2)$. $E_f$ is called a holomorphic Legendrian lift of $f$. Since (1.1) implies $E_f^{-1} dE_f$ is off-diagonal, we can set

$$E_f^{-1} dE_f = \begin{pmatrix} 0 & \theta \\ \omega & 0 \end{pmatrix}.$$  

The holomorphic 1-forms $\omega$ and $\theta$ are called the first canonical form and the second canonical form, respectively. We have

$$(1.2) \quad \omega = \begin{cases} \frac{dA}{B} & (\text{if } dA \neq 0 \text{ or } B \neq 0), \\ \frac{dC}{D} & (\text{if } dC \neq 0 \text{ or } D \neq 0), \end{cases}$$

$$(1.3) \quad \theta = \begin{cases} \frac{dB}{A} & (\text{if } dB \neq 0 \text{ or } A \neq 0), \\ \frac{dD}{C} & (\text{if } dD \neq 0 \text{ or } C \neq 0). \end{cases}$$

Here $\neq 0$ means that the 1-form or function in question are not identically zero.

In particular,

$$(1.4) \quad \omega = \frac{dA}{B} = \frac{dC}{D} \quad \text{and} \quad \theta = \frac{dB}{A} = \frac{dD}{C}$$

if all cases in (1.2) and (1.3) are well-defined. Then the first and second fundamental forms $ds^2$ and $dh^2$ have the expressions

$$(1.5) \quad ds^2 = (\omega + \bar{\theta})(\bar{\omega} + \theta) = \omega \theta + \bar{\omega} \theta + |\omega|^2 + |\theta|^2,$$

$$(1.6) \quad dh^2 = |\theta|^2 - |\omega|^2.$$
Though $\omega$ and $\theta$ are defined only on the universal cover $\tilde{M}^2$, the first fundamental form $ds^2$ is well-defined on $M^2$, and then so is the $(1,1)$-part of $ds^2$:

$$ds_{1,1}^2 := |\omega|^2 + |\theta|^2. \quad (1.7)$$

Since (1.6) is well-defined on $M^2$, so are $|\omega|^2$ and $|\theta|^2$. Moreover, we can deduce that

$$|\omega|^2 \text{ and } |\theta|^2 \text{ define flat pseudometrics on } M^2 \text{ compatible} \quad (1.8)$$

with the complex structure of $M^2$.

The $(2,0)$-part of $ds^2$ is called the Hopf differential and is denoted by $Q$:

$$Q := \omega \theta. \quad (1.9)$$

The positive hyperbolic Gauss map $G$ and the negative hyperbolic Gauss map $G_*$ of the flat surface are defined by

$$G = \frac{A}{C}, \quad G_* = \frac{B}{D}. \quad (1.10)$$

They are single-valued on $M^2$. The geometric meaning of $G$ and $G_*$ is described in the Introduction. (See also [4].) By definition,

$$dG = d\left(\frac{A}{C}\right) = \frac{dAC - A dC}{C^2} = \frac{BC - DA}{C^2} \omega = -\frac{\omega}{C^2}. \quad (1.11)$$

Similarly,

$$dG_* = d\left(\frac{B}{D}\right) = \frac{dB \, D - B \, dD}{D^2} = \frac{AD - BC}{D^2} \theta = \frac{\theta}{D^2}. \quad (1.12)$$

On the other hand,

$$G - G_* = \frac{A}{C} - \frac{B}{D} = \frac{AD - BC}{CD} = \frac{1}{CD}. \quad (1.13)$$

We have the identity

$$Q = \omega \theta = -(CD)^2 dGdG_* = -\frac{dGdG_*}{(G - G_*)^2}. \quad (1.14)$$

Now we set

$$g(q) := \int_{p_0}^q \omega, \quad g_*(q) := \int_{p_0}^q \theta \quad (q \in M^2), \quad (1.15)$$

where $p_0$ is a base point. Then $g$ and $g_*$ are holomorphic functions defined on $\tilde{M}^2$. We remark that $(g, G)$ and $(g_*, G_*)$ satisfy the important relation

$$S(g) - S(G) = 2Q, \quad S(g_*) - S(G_*) = 2Q, \quad (1.16)$$

(see [4]), where $S(G)$ is the Schwarzian derivative

$$S(G) = \left( \frac{G''}{G'} \right)' - \frac{1}{2} \left( \frac{G''}{G'} \right)^2 \, dz^2 \quad \left( ' = \frac{d}{dz} \right).$$
with respect to a local complex coordinate \( z \) on \( M^2 \). Though the meromorphic 2-differentials \( S(g) \) and \( S(G) \) depend on complex coordinates, the difference \( S(g) - S(G) \) does not.

**Remark 1.1.** Hyperbolic 3-space \( H^3 \) can be realized as a hyperboloid in Minkowski 4-space \( (L^4, (x^0, x^1, x^2, x^3)) \):

\[
H^3 = \left\{ (x^0, x^1, x^2, x^3) \in L^4 ; x^0 > 0, -(x^0)^2 + \sum_{j=1}^{3} (x^j)^2 = -1 \right\}.
\]

Let \( f : M^2 \to H^3 \) be a flat immersion and assume \( M^2 \) is connected. Then the universal cover \( \tilde{M}^2 \) of \( M^2 \) is diffeomorphic to \( \mathbb{R}^2 \) and has a coordinate system \((x, y)\) defined on \( \tilde{M}^2 \) such that the first fundamental form \( ds^2 \) can be written as

\[
ds^2 = dx^2 + dy^2.
\]

Then we have an orthonormal frame field

\[
e : \tilde{M}^2 \ni p \mapsto (f(p), f_x(p), f_y(p), \nu(p)) \in \text{SO}(3,1),
\]

where \( \nu(p) \in T_p H^3(\subset L^4) \) is the unit normal vector of the immersion \( f \) at \( p \).

Now, we can identify \( L^4 \) with the set \( \text{Herm}(2) \) of 2 by 2 Hermitian matrices:

\[
L^4 \ni (x^0, x^1, x^2, x^3) \longleftrightarrow \left( \begin{array}{cc} x^0 + x^3 & x^1 + ix^2 \\ x^1 - ix^2 & x^0 - x^3 \end{array} \right) \in \text{Herm}(2).
\]

Then the hyperbolic 3-space \( H^3 \) can be rewritten as

\[
H^3 = \{ X \in \text{Herm}(2) ; \det(X) = 1, \text{trace} X > 0 \}
\]

\[
= \{ a a^* ; a \in \text{SL}(2, \mathbb{C}) \},
\]

where \( a^* = t^* a \). Setting

\[
v_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad v_1 := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad v_2 := \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad v_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

there is a lift \( E : \tilde{M}^2 \to \text{SL}(2, \mathbb{C}) \) of the orthonormal frame \( e \) such that \( e = \pi \circ E \), where \( \pi : \text{SL}(2, \mathbb{C}) \to \text{SO}(3,1) \) is the 2-fold covering homomorphism, that is,

\[
(1.17) \quad f = EE^*, \quad f_x = Ev_1 E^*, \quad f_y = Ev_2 E^*, \quad \nu = Ev_3 E^*.
\]

Thus \( E \) coincides with \( E_f \). This implies that \( E \) itself is holomorphic with respect to the complex structure induced from the second fundamental form. Multiplication \( E \mapsto aE \) by a matrix \( a = (a_{ij}) \in \text{SL}(2, \mathbb{C}) \) corresponds to an isometric change of the surface, \( f \mapsto af a^* \). This induces the change of hyperbolic Gauss maps, as follows:

\[
G \mapsto a \ast G := \frac{a_{11}G + a_{12}}{a_{21}G + a_{22}}, \quad G_s \mapsto a \ast G_s := \frac{a_{11}G_s + a_{12}}{a_{21}G_s + a_{22}}.
\]

(1.19)
It is interesting to compare this with the case of surfaces of constant mean curvature 1 in $H^3$. In that case, there is a holomorphic immersion $F: \tilde{M}^2 \to \text{SL}(2, \mathbb{C})$ such that $f = FF^*$, but it does not coincide with the lift $E: \tilde{M}^2 \to \text{SL}(2, \mathbb{C})$ of an orthonormal frame. We must adjust $E$ by multiplying by a local SU(2)-section $s: \tilde{M}^2 \to \text{SU}(2)$ so that $F := Es$ becomes holomorphic. (See Bryant [1].)

2. Flat surfaces as (wave) fronts

In this section, we define flat fronts as projections of Legendrian immersions into the unit cotangent bundle $T^*_1H^3$. Since $T^*_1H^3$ is isomorphic to the unit tangent bundle $T_1H^3$, we can make the identification

$$T^*_1H^3 \cong \mathcal{F} := \{(x, v) \in L^4 \times L^4; -\langle x, x \rangle = \langle v, v \rangle = 1, \langle x, v \rangle = 0\},$$

where $\langle \cdot , \cdot \rangle$ is the inner product of $L^4$. The metric

$$d\sigma^2_0 := \sum_{j=0}^3 (dx^j)^2 + \sum_{j=0}^3 (dv^j)^2 \quad (x = (x^0, x^1, x^2, x^3), \ v = (v^0, v^1, v^2, v^3))$$

on $\mathcal{F}$ induced from the product of Lorentzian metrics of $L^4 \times L^4$ is positive definite, and is called the Sasakian metric. In fact, if we identify $\mathcal{F}$ with $T_1H^3$, it coincides with the metric on the unit tangent bundle defined by Sasaki [9, 10]. The contact form of $\mathcal{F}$ is given by

$$\eta := \sum_{j=0}^3 v^j dx^j.$$

Now, a Legendrian immersion $L$ of a 2-manifold $M^2$ into the unit cotangent bundle can be identified with an immersion

$$L: M^2 \longrightarrow \mathcal{F}$$

such that $L^*\eta$ vanishes. We denote the two canonical projections by

$$\pi_F: \mathcal{F} \ni (x, v) \longmapsto x \in H^3, \quad \pi'_F: \mathcal{F} \ni (x, v) \longmapsto v \in L^4.$$ 

A map $f: M^2 \to H^3$ is called a front if there exists a Legendrian immersion $L_f: M^2 \to \mathcal{F}$ such that

$$\pi_F \circ L_f = f.$$ 

$L_f$ is called the Legendrian lift of $f$.

By definition, any immersion $f: M^2 \to H^3$ is a front if $M^2$ is orientable. In fact, $L_f$ is given by the pair $(f, \nu_f)$ consisting of $f$ and the unit normal vector $\nu_f$ of $f$.

For a given front $f: M^2 \to H^3$, we can define a parallel front $f_t: M^2 \to H^3$ of distance $t$ by

$$f_t := (\cosh t)f + (\sinh t)\nu_f = \pi_F \circ L_t,$$
where

\[ L_t := (f_t, \nu_f) \quad (\nu_f := (\sinh t)f + (\cosh t)\nu_f) \]

is a Legendrian immersion and

\[ \nu_f := \pi'_F \circ L_f : \mathbb{M}^2 \to L^4. \]

When \( f \) is an immersion, this is nothing but the definition of a parallel surface. So we call \( \nu_f \) the unit normal vector (field) of the front \( f \).

For a given front \( f : \mathbb{M}^2 \to H^3 \),

\[ ds^2 := \langle df, df \rangle \quad \text{and} \quad dh^2 := -\langle df, d\nu_f \rangle \]

are called the first and the second fundamental forms, respectively.

**Definition 2.1.** A front \( f : \mathbb{M}^2 \to H^3 \) is called flat if, for each \( p \in \mathbb{M}^2 \), there exists a real number \( t \in \mathbb{R} \) such that the parallel front \( f_t \) gives a flat immersion at \( p \).

**Remark 2.2.** An equivalent definition of a flat front is that the Gaussian curvature of \( f \) vanishes at all regular points. However, this definition is not suitable when all points of \( f \) are degenerate, and such a case really occurs, since hyperbolic cylinders can collapse to a geodesic.

As shown in the following proposition, all parallel fronts \( f_t \) \((t \in \mathbb{R})\) of a flat front \( f \) are also flat fronts.

**Proposition 2.3.** Let \( f : \mathbb{M}^2 \to H^3 \) be a flat front. Then the second fundamental form \( dh^2 \) is proportional to the pullback of the Sasakian metric \( d\sigma^2 = L_f^* d\sigma^2_0 \). The parallel front \( f_t \) of \( f \) is also a flat front for all \( t \). In particular, the Gaussian curvature of \( f_t \) at the regular point vanishes.

**Remark 2.4.** As in [4], the lift \( E_{f_t} \) of \( f_t \) is given by

\[ E_{f_t} = E_f \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}. \]

**Proof of Proposition 2.3.** We fix a point \( p \in \mathbb{M}^2 \). By definition, there is a parallel front \( f_{t_0} : \mathbb{M}^2 \to H^3 \) such that \( f_{t_0} \) is regular at \( p \) and the Gaussian curvature of \( f_{t_0} \) vanishes around \( p \). Without loss of generality, we may assume that \( t_0 = 0 \), that is, \( f = f_{t_0} \).

First, we consider the case that the first and second fundamental forms are proportional. Then \( f \) must be a horosphere and the statement of the theorem is obvious.

So we may assume that the second fundamental form is not proportional to the first. We can write the Legendrian lift \( L_f \) as a pair \( L_f = (f, \nu_f) \), where \( \nu_f \) is the unit normal vector field of \( f \). Then

\[ d\sigma^2 = \langle df, df \rangle + \langle d\nu, d\nu \rangle. \]
Now fix a local coordinate neighborhood \((U; u, v)\) of \(M^2\) and define three 2 by 2 matrices:

\[
M_1 := \begin{pmatrix}
\langle f_u, f_u \rangle & \langle f_u, f_v \rangle \\
\langle f_v, f_u \rangle & \langle f_v, f_v \rangle
\end{pmatrix},
\]

\[
M_2 := -\begin{pmatrix}
\langle f_u, \nu_u \rangle & \langle f_u, \nu_v \rangle \\
\langle f_v, \nu_u \rangle & \langle f_v, \nu_v \rangle
\end{pmatrix},
\]

\[
M_3 := \begin{pmatrix}
\langle \nu_u, \nu_u \rangle & \langle \nu_u, \nu_v \rangle \\
\langle \nu_v, \nu_u \rangle & \langle \nu_v, \nu_v \rangle
\end{pmatrix}.
\]

We set \(A := M_1^{-1}M_2\), which is the shape operator of \(f\). The Gauss equation implies that

\[
det A = 1 + K_{ds^2},
\]

where \(K_{ds^2}\) is the Gaussian curvature of \(ds^2\). On the other hand, by the definition of \(M_3\), we have

\[
M_3 = M_2A = M_1A^2.
\]

By the Cayley–Hamilton theorem, we have

\[
M_3 = M_1(2(\text{trace } A)A - (1 + K_{ds^2})I) = 2(\text{trace } A)M_2 - K_{ds^2}M_1,
\]

where \(I\) is the identity matrix. Thus

\[
M_1 + M_3 - 2(\text{trace } A)M_2 = -K_{ds^2}M_1.
\]

Since \(M_1\) is not proportional to \(M_2\), this implies that \(M_1 + M_3\) is proportional to \(M_2\) if and only if \(K_{ds^2}\) vanishes. So the second fundamental form \(dh^2\) is proportional to \(d\sigma^2\) when \(f\) is flat. Now we shall show that \(f_t\) is also flat. Indeed, \(f_t\) and its unit normal vector \(\nu_t\) have the expressions

\[
f_t = (\cosh t)f + (\sinh t)\nu, \quad \nu_t = (\sinh t)f + (\cosh t)\nu.
\]

The fundamental forms are

\[
\begin{align*}
\text{ds}_t^2 &= (\cosh^2 t)\text{ds}^2 + 2(\cosh t \sinh t)\text{dh}^2 + (\sinh^2 t)\langle \text{dv}, \text{dv} \rangle, \\
\text{dh}_t^2 &= (\cosh t \sinh t)\text{ds}^2 + 2(\cosh^2 t + \sinh^2 t)\text{dh}^2 + (\cosh t \sinh t)\langle \text{dv}, \text{dv} \rangle, \\
\langle \text{dv}_t, \text{dv}_t \rangle &= (\sinh^2 t)\text{ds}^2 + 2(\cosh t \sinh t)\text{dh}^2 + (\cosh^2 t)\langle \text{dv}, \text{dv} \rangle,
\end{align*}
\]

where \(\langle dv, dv \rangle\) is the third fundamental form of \(f\). Since \(d\sigma^2 = \text{ds}^2 + \langle dv, dv \rangle\), we have

\[
\text{dh}_t^2 = (\cosh t \sinh t)d\sigma^2 + 2(\cosh^2 t + \sinh^2 t)\text{dh}^2,
\]

and

\[
d\sigma_t^2 := \text{ds}_t^2 + \langle \text{dv}_t, \text{dv}_t \rangle = (\cosh^2 t + \sinh^2 t)d\sigma^2 + 4 \cosh t \sinh t\text{dh}^2.
\]

Since \(dh^2\) is proportional to \(d\sigma^2\), \(dh_t^2\) and \(d\sigma_t^2\) are also proportional. Since \(f\) is not a horosphere, \(\text{ds}^2\) is not proportional to \(\text{dh}^2\) and thus \(f_t\) is flat for all \(t \in \mathbb{R}\). \(\square\)
From now on, we assume that $M^2$ is oriented. (If $M^2$ is not orientable, we can take the double cover.) Then there is a complex structure on $M^2$ such that $d\sigma^2$ is Hermitian. Since the second fundamental form is proportional to $d\sigma^2$, this complex structure of $M^2$ coincides with the one treated in Section 1, as long as $f$ is an immersion. So, we shall call this complex structure the canonical complex structure, and $M^2$ is always considered as a Riemann surface.

**Proposition 2.5.** Let $M^2$ be a Riemann surface and $E: \tilde{M}^2 \to \text{SL}(2, \mathbb{C})$ a holomorphic Legendrian immersion defined on the universal cover $\tilde{M}^2$ and such that $f = EE^*$ is single-valued on $M^2$. Then $f$ is a flat front. If

\[
E^{-1}dE = \begin{pmatrix} 0 & \theta \\ \omega & 0 \end{pmatrix},
\]

the first and the second fundamental forms are represented as

\[
ds^2 = \omega \theta + \bar{\omega} \bar{\theta} + \left( |\omega|^2 + |\theta|^2 \right),
\]

\[dh^2 = |\theta|^2 - |\omega|^2.
\]

Conversely, any flat front is given as a projection of a holomorphic Legendrian immersion.

**Proof.** Let $E: \tilde{M}^2 \to \text{SL}(2, \mathbb{C})$ be a holomorphic Legendrian curve and let $(\omega, \theta)$ be as in (2.3). Then $E$ is an immersion if $|\omega|^2 + |\theta|^2$ is positive definite. On the other hand, we have

\[
df = dE E^* + E dE^* = E(E^{-1}dE + (E^{-1}dE)^*)E^*
\]

\[
= E \begin{pmatrix} 0 & \theta + \bar{\omega} \\ \omega + \bar{\theta} & 0 \end{pmatrix} E^*,
\]

\[
d\nu = dEv_3 E^* + Ev_3 dE^* = E(E^{-1}dEv_3 + v_3(E^{-1}dE)^*)E^*
\]

\[
= E \begin{pmatrix} 0 & -\theta + \bar{\omega} \\ \omega - \bar{\theta} & 0 \end{pmatrix} E^*.
\]

In the identification as in (1.17), the canonical Lorentzian inner product is given by

\[
\langle X, Y \rangle := -\frac{1}{2} \text{trace}(XY) \quad (X, Y \in \text{Herm}(2)),
\]

where $\widetilde{Y}$ is the cofactor matrix of $Y$. If $Y \in \text{SL}(2, \mathbb{C})$, we have $\widetilde{Y} = Y^{-1}$. Then

\[
ds^2 = \langle df, df \rangle = -\frac{1}{2} \text{trace} \left\{ \begin{pmatrix} 0 & \theta + \bar{\omega} \\ \omega + \bar{\theta} & 0 \end{pmatrix} \begin{pmatrix} 0 & -\theta + \bar{\omega} \\ -\omega - \bar{\theta} & 0 \end{pmatrix} \right\}
\]

\[
= (\omega + \bar{\theta})(\bar{\omega} + \theta).
\]
Similarly, since $dh^2 = -\langle df, d\nu \rangle$, we have (2.4). Thus, the pullback of the Sasakian metric by $(f, \nu)$ as in (2.1) is represented as

$$d\sigma^2 = \langle df, df \rangle + \langle d\nu, d\nu \rangle = 2(|\omega|^2 + |\theta|^2).$$

Hence $L_f$ is an immersion if and only if $|\omega|^2 + |\theta|^2$ is positive definite. This proves the assertion. \qed

**Remark 2.6.** As seen in the proof of Proposition 2.5, the $(1, 1)$-part of the first fundamental form

$$ds^2 = |\omega|^2 + |\theta|^2$$

is equal to one-half of $d\sigma^2$, the pullback of the Sasakian metric by $(f, \nu)$. Also, $ds^2$ is the pullback of the bi-invariant Hermitian metric of $\text{SL}(2, \mathbb{C})$ by $E$.

A flat front $f: M^2 \to H^3$ can be interpreted from two points of view. The first is the projection of a (real) Legendrian immersion $L_f: M^2 \to T^*H^3 \cong \mathcal{F}$, and the second is the projection of a (holomorphic) Legendrian immersion $E_f: M^2 \to \text{SL}(2, \mathbb{C})$. One can naturally expect that these two Legendrian immersions are related. In fact, $\text{SL}(2, \mathbb{C})$ acts transitively and we can write

$$T^*H^3 \cong T_1H^3 \cong \mathcal{F} \cong \text{SL}(2, \mathbb{C})/U(1).$$

We denote the canonical projection by

$$p_{\text{SL}}: \text{SL}(2, \mathbb{C}) \longrightarrow \mathcal{F}.$$ 

**Proposition 2.7.** The pullback of the contact form $\eta$ by $p_{\text{SL}}$ is equal to the real part of the holomorphic contact form on $\text{SL}(2, \mathbb{C})$, that is,

$$p_{\text{SL}}^*(\eta) = 2 \text{Re}(s_{22}ds_{11} - s_{12}ds_{21})$$

holds, where $(s_{ij}) \in \text{SL}(2, \mathbb{C})$. In particular, the real Legendrian immersion $L_f$ can be interpreted as the projection of a holomorphic Legendrian immersion $E_f$.

**Proof.** Since

$$x^{-1} = (ss^*)^{-1} = (s^*)^{-1}s^{-1}, \quad dv = s(s^{-1}dv_3 + v_3(s^{-1}ds^*)s^*)$$

we have

$$\eta = \langle x, dv \rangle = \frac{1}{2} \text{trace}(x^{-1}dv) = \frac{1}{2} \text{trace}(s^{-1}ds_2v_3 + v_3(s^{-1}ds^*))$$

$$= -\text{Re}(\text{trace}(s^{-1}ds_2v_3)) = 2 \text{Re}(s_{22}ds_{11} - s_{21}ds_{12}),$$

where we set $s = (s_{ij}) \in \text{SL}(2, \mathbb{C})$. \qed
Remark 2.8. Since the holomorphic Legendrian lift $E_f$ of a flat front $f$ is not single-valued on $M^2$ in general, $E_f$ has the monodromy representation $\rho_f: \pi_1(M^2) \to \text{SU}(2)$ such that $E_f \circ \tau = E_f \rho_f(\tau)$ for any deck transformation $\tau \in \pi_1(M)$. On the other hand, since $E_f \circ \tau$ is also Legendrian, the representation $\rho_f$ is reducible, that is, it reduces to the isotropy group $U(1)$ of the action of $\text{SL}(2, \mathbb{C})$ to $\mathcal{F}$.

We can define the hyperbolic Gauss maps of the flat front $f$ in the same way as for an immersion:

$$G = \frac{A}{C}, \quad G_* = \frac{B}{D},$$

where

$$E_f = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

These are single-valued on $M^2$. Since $AD - BC = 1$, $G(p)$ never coincides with $G_*(p)$, and we get the holomorphic map

$$\mathcal{G} = (G, G_*): M^2 \to S^2 \times S^2 \setminus \{\text{the diagonal set}\} =: \text{Geod}(H^3),$$

where $\text{Geod}(H^3)$ is the set of oriented geodesics in $H^3$.

**Theorem 2.9.** Let $\tilde{M}^2$ be the universal cover of a Riemann surface $M^2$ and let $E: \tilde{M}^2 \to \text{SL}(2, \mathbb{C})$ be a holomorphic Legendrian curve such that $f = EE^*$ is single-valued on $M^2$. The following assertions are equivalent:

1. $E$ is an immersion.
2. $L_f$ is an immersion.
3. The $(1,1)$-part of the first fundamental form

$$ds^2_{1,1} = |\omega|^2 + |\theta|^2$$

is positive definite, where $\omega$ and $\theta$ are the off-diagonal components of $E^{-1}dE$.
4. $\mathcal{G} = (G, G_*): M^2 \to \text{Geod}(H^3)$ is an immersion.

**Remark 2.10.** A point where $E$ degenerates is called a branch point of $E$ (or of $f = EE^*$), and the projection of a holomorphic Legendrian curve is called a brachided flat front. The conditions above imply that $f$ is free of branch points.

**Proof of Theorem 2.9.** The equivalence of the first three assertions follows from the proof of Proposition 2.5. So it is sufficient to prove that (3) and (4) are equivalent. By (1.11) and (1.12),

$$ds^2_{1,1} = |\omega|^2 + |\theta|^2 = |C^2 dG|^2 + |D^2 dG_*|^2.$$

If both $C$ and $D$ are nonvanishing, the equivalence of (3) and (4) is obvious. If instead $C = 0$, then $D \neq 0$, $\omega = 0$,

$$ds^2_{1,1} = |\theta|^2 = |D^2 dG_*|^2,$$
and $\text{d}s^2$ is positive definite if and only if $dG_* \neq 0$. Similarly, if $D = 0$, we get $C \neq 0$, $\theta = 0$,

$$ds^2_{1,1} = |\omega|^2 = |C^2 dG|^2,$$

and $\text{d}s^2$ is positive definite if and only if $dG \neq 0$. □

In [7], the authors gave a representation formula for Legendrian curves in $\text{SL}(2, \mathbb{C})$ via the data $(G, G_*)$. We now reformulate it for the construction of flat fronts in $H^3$:

**Theorem 2.11.** Let $G$ and $G_*$ be nonconstant meromorphic functions on a Riemann surface $M^2$ such that $G(p) \neq G_*(p)$ for all $p \in M^2$. Assume that:

1. All poles of the 1-form $dG / (G - G_*)$ are of order 1.
2. $\int_{\gamma} dG / (G - G_*)$ is purely imaginary for each loop $\gamma$ on $M^2$.

Set

$$\xi(z) := c \exp \int_{z_0}^{z} \frac{dG}{G - G_*},$$

where $z_0 \in M^2$ is a base point and $c \in \mathbb{C} \setminus \{0\}$ is an arbitrary constant. Then

$$E := \left( \frac{G}{\xi}, \frac{G_*/(G - G_*)}{\xi} \right)$$

is a nonconstant meromorphic Legendrian curve defined on $\tilde{M}^2$ in $\text{SL}(2, \mathbb{C})$ whose hyperbolic Gauss maps are $G$ and $G_*$, and such that the projection $f = EE^*$ is single-valued on $M^2$. Moreover, $f$ is a front if and only if $G$ and $G_*$ have no common branch points. Conversely, any non-totally-umbilical flat fronts can be constructed in this manner.

**Proof.** Given a pair $(G, G_*)$ of nonconstant meromorphic functions, on a Riemann surface $M^2$ satisfying (1), the meromorphic map $E$ defined by (2.8) is a holomorphic Legendrian curve in $\text{SL}(2, \mathbb{C})$, as a consequence of Theorem 3 of [7]. Then condition (2) implies that $f = EE^*$ is single-valued on $M^2$. Now, by Theorem 2.9, the branched flat front $f$ is free of branch points if and only if the pair $(G, G_*)$ gives an immersion of $M^2$ into $S^2 \times S^2$.

Since any flat front can be lifted to a holomorphic Legendrian curve defined on $\tilde{M}^2$, Theorem 3 of [7] also implies that any non-totally-umbilical flat front can be constructed in this manner. (If one of the hyperbolic Gauss maps is constant, it is totally umbilic, i.e., locally a horosphere.) □

### 3. Flat fronts with complete ends

We define completeness of fronts as follows:
Definition 3.1. Let $M^2$ be a 2-manifold. A front $f : M^2 \to H^3$ is called complete if there is a symmetric 2-tensor $T$ with compact support such that the sum $T + ds^2$ is a complete Riemannian metric of $M^2$, where $ds^2$ is the first fundamental form of $f$.

Remark 3.2. Note that the parallel family of a complete front $f$ may contain an incomplete flat front. For example, the hyperbolic cylinder, that is, the surface equidistant from a geodesic (Example 4.1 in Section 4), contains a geodesic in its parallel family.

The following assertion is a simple consequence of Lemma 2 of [4]:

Lemma 3.3. Let $M^2$ be an oriented 2-manifold and $f : M^2 \to H^3$ a complete flat front. There exists a compact Riemann surface $\overline{M}^2$ and finitely many points $p_1, \ldots, p_n \in \overline{M}^2$ such that $M^2$ (as a Riemann surface) is biholomorphic to $\overline{M}^2 \setminus \{p_1, \ldots, p_n\}$. The Hopf differential $Q$ of $f$ can be extended meromorphically on $\overline{M}^2$.

These points $p_1, \ldots, p_n$ are called ends of the front $f$.

Proof of Lemma 3.3. Since $f$ is complete, there exists a symmetric tensor $T$ supported in a compact subset of $M^2$ and such that $ds^2 := T + ds^2$ is complete, where $ds^2$ is the first fundamental form. Since the Gaussian curvature of $ds^2$ vanishes, the total absolute curvature of $ds^2$ is finite. Then by Huber’s theorem, there is a compact 2-manifold $\overline{M}^2$ and finite points $p_1, \ldots, p_n \in \overline{M}^2$ such that $M^2$ (as a Riemann surface) is diffeomorphic to $\overline{M}^2 \setminus \{p_1, \ldots, p_n\}$. Now we take a sufficiently small neighborhood $U_j$ of an end $p_j$ such that $ds^2 = ds^2$ holds on $U_j$. If $|\omega| = |\theta|$ at a point $q \in U_j$, the first fundamental form $ds^2$ is degenerate at $q$ because of (1.5). Hence $|\omega| \neq |\theta|$ on $U_j$. If $|\omega| > |\theta|$, by (25) of [4], we have

$$ds^2 = \omega \theta + \overline{\omega} \overline{\theta} + |\omega|^2 + |\theta|^2 \leq 2|\omega||\theta| + |\omega|^2 + |\theta|^2 = (|\omega| + |\theta|)^2 \leq 4|\omega|^2.$$  

Since $ds^2$ is complete at $p_j$, so is a metric $|\omega|^2$. Moreover, by the holomorphicity of $\omega$ with respect to the complex structure induced from the second fundamental form, the metric $|\omega|^2$ is a flat metric conformal to the complex structure of $M^2$ and complete at $p_j$. This proves the first assertion. (See also Lemma 1 of [4].) In the case of $|\omega| < |\theta|$, we reach the conclusion using the conformal metric $|\theta|^2$.

The meromorphicity of $Q$ is proved in Lemma 2 of [4].

As seen in the proof of Lemma 3.3, we have:
Corollary 3.4. If a flat front $f$ is complete, so is the $(1,1)$-part
$$ds^2_{1,1} = |\omega|^2 + |\theta|^2$$
of the first fundamental form.

Remark 3.5. If a flat front is complete and is also a proper mapping, then its image is a closed subset of $H^3$. However, a proper flat front $f$ whose image is closed in $H^3$ may not be complete. (When $f$ has no singularity, it is complete by the Hopf–Rinow theorem.) In fact, we consider a flat front $f = EE^*: \mathbb{C} \to H^3$, where
$$E := \begin{pmatrix} z e^{-z} & (z - 1) e^z \\ e^{-z} & e^z \end{pmatrix}.$$  

It can easily be checked that $f(z)$ tends to the north pole of the ideal boundary in the Poincaré ball as $z \to \infty$, which implies that $f$ is a proper mapping. But the first fundamental form vanishes on the imaginary axis, which appears as cuspidal edges. (See Figure 2. The criterion for singularities of flat fronts will appear in the forthcoming paper [6].)

It is a classical fact that there are no compact flat surfaces in $H^3$. We can also prove the nonexistence of compact flat fronts.

Proposition 3.6. There are no compact flat fronts without boundary.

Proof. Let $f: M^2 \to H^3$ be a compact flat front, and take a holomorphic Legendrian lift $E$ of $f$. Recall that $f$ and $E$ are matrix-valued. The trace of $f$ satisfies
$$(\operatorname{trace} f) z = \operatorname{trace}(f z) = \operatorname{trace} \{E_z (E_z)^*\} \geq 0,$$
where $z$ is a complex coordinate of $M^2$. Hence the function $\operatorname{trace} f: M^2 \to \mathbb{R}$ is subharmonic, and must be constant, since $M^2$ is compact. By an isometry in $H^3$, we may assume that $f(z_0) = I$, where $I$ is the $2 \times 2$ identity matrix. Then $\operatorname{trace} f$ is identically 2. At the same time, $\det f$ is identically 1 so the eigenvalues $\lambda_1, \lambda_2$ of $f$ satisfy
$$\lambda_1 + \lambda_2 = 2, \quad \lambda_1 \lambda_2 = 1.$$
Hence \( \lambda_1 = \lambda_2 = 1 \). Since \( f \) is Hermitian, this implies that \( f(z) \) is equal to the identity matrix, a contradiction. \( \square \)

Gálvez, Martínez and Milán investigated complete ends of flat surfaces deeply. The following fact is proved in [4]:

**Lemma 3.7** (Theorem 4 of [4]). Let \( p \) be an end of a complete flat front. The following three conditions are equivalent:

1. The Hopf differential \( Q \) has at most a pole of order 2 at \( p \).
2. The positive hyperbolic Gauss map \( G \) has at most a pole at \( p \).
3. The negative hyperbolic Gauss map \( G_* \) has at most a pole at \( p \).

**Remark 3.8.** The hyperbolic Gauss maps and the Hopf differential of the flat front \( f = EE^* \) as in (3.1) are

\[
G = z, \quad G_* = z - 1, \quad Q = dz^2.
\]

This means that meromorphicity of \( G \) and \( G_* \) does not imply that \( Q \) has at most poles of order 2 without assuming the completeness of ends. In fact, \( Q \) has pole of order 4 at \( z = \infty \).

If an end of a flat front satisfies one of the three conditions above, it is called a **regular** end. An end that is not regular is called an **irregular** end. An end \( p \) is said to be **embedded** if there is a neighborhood \( U \) of \( p \in \overline{M}^2 \) such that the restriction of the front to \( U \setminus \{p\} \) is an embedding.

**Lemma 3.9** (Theorem 5 of [4]). Let \( p \) be a regular end of a complete flat front. Suppose that \( |\theta| < |\omega| \) at \( p \). Then \( p \) is embedded if and only if it is not a branch point of the positive hyperbolic Gauss map \( G \).

**Lemma 3.10.** The two hyperbolic Gauss maps take the same value at a regular end of a complete flat front. That is, \( G(p) = G_*(p) \) if \( p \) is a regular end.

**Proof.** Assume that \( G(p) \neq G_*(p) \) for a regular end \( p \). By Lemma 3.7, \( G(z) \) and \( G_*(z) \) are both meromorphic at \( p \). In particular, the function \( \xi(z) \) defined in Theorem 2.11 is holomorphic. Then so is \( E \), contradicting the completeness of the first fundamental form of the front at \( p \). Thus \( G(p) = G_*(p) \). \( \square \)

Let \( f: \overline{M}^2 \setminus \{p_1, \ldots, p_n\} \to H^3 \) be a complete flat front and \( \omega, \theta \) its first and second canonical forms. Suppose all ends \( p_1, \ldots, p_n \) are regular. By Lemma 2 of [4], there exist real numbers \( \mu_j \) and \( \mu_j^* \) \( (j = 1, \ldots, n) \) such that

\[
\omega(z) = (z - p_j)^{\mu_j} \omega_0(z) \quad (\omega_0(p_j) \neq 0),
\]

\[
\theta(z) = (z - p_j)^{\mu_j^*} \theta_0(z) \quad (\theta_0(p_j) \neq 0),
\]

where \( U; z \) is a complex coordinate around \( p_j \) and \( \omega_0(z) \) and \( \theta_0(z) \) are holomorphic 1-forms defined on \( U \). The real numbers \( \mu_j \) and \( \mu_j^* \) do not depend
on the choice of the coordinate $z$ and equal the order of the pseudometrics $|\omega|^2$ and $|\theta|^2$, respectively:

\begin{align}
|\omega|^2 &= |z - p_j|^{2\mu_j} (a_j^* + o(1)) |dz|^2, \\
|\theta|^2 &= |z - p_j|^{2\mu_j^*} (a_j^{**} + o(1)) |dz|^2,
\end{align}

where $a_j$ and $a_j^*$ are positive real numbers and $o(1)$ denotes higher order terms. By (1.14), we have

\begin{align}
\mu_j + \mu_j^* &= \text{ord}_{p_j} Q,
\end{align}

where $\text{ord}_{p_j} Q$ is the order of the Hopf differential $Q$ at $p_j$. Suppose that the Laurent expansion of $Q$ at $z = p_j$ is

\[ Q = \frac{1}{(z - p_j)^2(q - 2(p_j) + o(1))} dz^2, \]

where $o(1)$ is a function satisfying $\lim_{z \to p_j} o(1) = 0$. The following lemma is a direct consequence of the formula (1.16):

**Lemma 3.11.** The identity

\[ 4q - 2(p_j) = m_j(m_j + 2) - \mu_j(\mu_j + 2) = m_j^*(m_j^* + 2) - \mu_j^*(\mu_j^* + 2) \]

holds, where $\mu_j$ and $\mu_j^*$ are the orders of the pseudometrics $|\omega|^2$ and $|\theta|^2$, respectively, and $m_j$ and $m_j^*$ are the branching orders of $G$ and $G_*$ at $p_j$. (For instance, $m_j = 1$ implies $p_j$ is a double point of $G$.)

**Proposition 3.12.** A regular end $p_j$ of a complete flat front is embedded if and only if either $G$ or $G_*$ does not branch at $p_j$.

**Proof.** First, we assume the end $p_j$ is embedded. As pointed out in a previous paper [7], the holomorphic Legendrian lift $\hat{E}_f$ enjoys the following duality. We set

\[ \hat{E}_f := E_f \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \]

Then $\hat{E}_f$ is also a holomorphic Legendrian immersion such that $f = \hat{E}_f \hat{E}_f^*$, and the roles of $(G, \omega)$ and $(G_*, \theta)$ are interchanged. Then, replacing $E_f$ by $\hat{E}_f$ if necessary, we may assume $|\theta| < |\omega|$ near $p_j$. By Lemma 3.9, $G$ does not branch at $p_j$.

Conversely, we assume either $G$ or $G_*$ does not branch at $p_j$. Replacing $E_f$ by $\hat{E}_f$ if necessary, we may assume $G$ does not branch at $p_j$, that is, $m_j = 0$. If $|\theta| < |\omega|$ near $p_j$, the assertion follows from Lemma 3.9. So we may assume $|\theta| > |\omega|$ near $p_j$, and then we have $\mu_j \geq \mu_j^*$. By Lemma 3.11,

\[ (m_j - m_j^*)(m_j + m_j^* + 2) = (\mu_j - \mu_j^*)(\mu_j + \mu_j^* + 2). \]

By (3.3),

\[ \mu_j + \mu_j^* + 2 = \text{ord}_{p_j} Q + 2. \]
Since \( p_j \) is regular, \( \text{ord}_{p_j} Q \geq -2 \). Thus \( 0 = m_j \geq m^*_j (\geq 0) \), and so neither \( G \) nor \( G_* \) branch at \( p \). Replacing \( E_f \) by \( \hat{E}_f \) if necessary, we may assume \( |\theta| < |\omega| \) and get the embeddedness of \( p \) directly from Lemma 3.11. \( \square \)

Now we shall prove an assertion stated in the Introduction:

**Theorem 3.13.** Let \( f : \overline{M^2 \setminus \{p_1, \ldots, p_n\}} \rightarrow H^3 \) be a complete flat front whose ends are all regular. Then
\[
\deg G + \deg G_* \geq n,
\]
and equality holds if and only if all ends are embedded.

To prove the theorem, we shall prepare two lemmas.

**Lemma 3.14.** Let \( g \) and \( h \) be meromorphic functions on a compact Riemann surface \( \overline{M^2} \). Suppose that \( g \) and \( h \) have no common poles. Then
\[
\deg (ag + bh) = \deg g + \deg h,
\]
where \( a, b \in \mathbb{C} \) are nonzero constants.

**Proof.** Since \( g, h \) are meromorphic, their degrees equal the number of their poles, counting multiplicities. If \( P(g) \) is the divisor of poles of \( g \) (so that \( P(g) = s_1q_1 + \cdots + s_nq_n \), with \( q_1, \ldots, q_n \) the poles of \( g \) and \( s_1, \ldots, s_n \) their multiplicities), the degree of \( g \) is the sum of the coefficients of \( P(g) \), and likewise for \( h \). But \( P(ag + bh) = P(g) + P(h) \) unless \( ab = 0 \). Thus
\[
\deg g + \deg h = \deg (ag + bh). \quad \square
\]

**Lemma 3.15.** Let \( f : \overline{M^2 \setminus \{p_1, \ldots, p_n\}} \rightarrow H^3 \) be a complete flat front. Suppose that \( p = p_j \) is a regular end. Then \( p \) is an embedded end if and only if the difference
\[
h := G - G_*
\]
of the two hyperbolic Gauss maps does not branch at \( p \).

**Proof.** If \( p \) is not a branch point of \( h \), then either \( G \) or \( G_* \) does not branch at \( p \). Then embeddedness of the end \( p = p_j \) follows from Proposition 3.12. Conversely, suppose now that an end \( p \) is embedded. We take a complex coordinate \( z \) around \( p \) such that \( z(p) = 0 \). Then, by an isometry of \( H^3 \), we may assume \( G(0) = G_*(0) = 0 \) because of Lemma 3.10 and (1.19). It follows from Proposition 3.12 that \( G \) and \( G_* \) are expanded as
\[
G(z) = az + o(z) \quad \text{and} \quad G_*(z) = a_*z + o(z),
\]
where \( a \) and \( a_* \) are complex numbers such that \( a \neq 0 \) or \( a_* \neq 0 \), and \( o(z) \) denotes a higher order term. Thus by (1.14), the Hopf differential \( Q \) has the expansion
\[
Q = -\frac{aa_* + o(1)}{(a - a_*)z + o(z)} dz^2.
\]
By Lemma 3.7, we have $aa_*=0$ or $a-a_*\neq 0$. If $a-a_*\neq 0$, it follows that $h$ does not branch at $0$. If $aa_*=0$, one of $G$ and $G_*$ branches at $0$ and the other does not. Then $h=G-G_*$ does not branch at $0$. □

**Proof of Theorem 3.13.** Taking an isometry if necessary, we may assume that all ends $p_1, \ldots, p_n$ are not poles of both hyperbolic Gauss maps $G$ and $G_*$. The ends of the front are equal to the zeros of $h := G - G_*$, so $G$ and $G_*$ have no common poles. The zero divisor $Z(h)$ of the meromorphic function $h$ is of the form

$$Z(h) = \sum_{j=1}^{n} m_j p_j,$$

where $m_1, \ldots, m_n$ are positive integers. Then by Lemma 3.14

$$\deg G + \deg G_* = \deg h = \sum_{j=1}^{n} m_j \geq n,$$

which proves the inequality. Moreover, the equality holds if and only if

$$m_1 = \cdots = m_n = 1.$$  

By Lemma 3.15, this is the case if and only if all ends are embedded. □

**Remark 3.16.** As seen in the proof, the inequality in Theorem 3.13 holds even if $f$ has branch points (see Remark 2.10), that is, common branch points of $G$ and $G_*$ on $M^2$. However, the category of branched flat front seems too wide for the study of flat surfaces. In fact, branched covers of flat fronts are all branched flat fronts whose images are the same as the original fronts.

**Remark 3.17.** Let $\overline{M}^2$ be a compact Riemann surface with positive genus. Since there are no meromorphic functions on $\overline{M}^2$ of degree 1, a complete flat front defined on $\overline{M}^2$ minus a finite number of points must have at least 4 ends. *Is there a flat front with positive genus and exactly 4 embedded ends?* There does exist a genus-1 flat front with 5 ends (Example 4.6), but it is still unknown whether there one with 4 ends. (One can construct a genus-1 branched flat front with 4 embedded ends, but the image of such a front is a double cover of a genus-0 flat front.)

As seen in Remark 3.16, the inequality of Theorem 3.13 is valid for branched flat fronts. On the other hand, we show an inequality which reflects properties of fronts. First some terminology:

**Definition 3.18.** Let $p$ be a regular end of a flat front and let $(\omega, \theta)$ be as in (2.3). The end $p$ is called cylindrical if

$$\ord_p |\omega|^2 = \ord_p |\theta|^2 = -1,$$
where $\text{ord}_p |\omega|^2$ and $\text{ord}_p |\theta|^2$ are the orders of the pseudometrics $|\omega|^2$ and $|\theta|^2$, as in (3.2), respectively.

The ends of a hyperbolic cylinder, a surface equidistant from a geodesic, are cylindrical. (See Example 4.1, and also [4, p. 427] or [7, Example 4.1].)

Lemma 3.19 (Theorem 6 of [4]). A cylindrical end is asymptotic to a finite cover of a hyperbolic cylinder. □

Lemma 3.20. A regular end $p$ of a complete flat front is cylindrical if and only if $\text{ord}_p |\omega|^2 = -1$ or $\text{ord}_p |\theta|^2 = -1$.

Proof. Assume $\text{ord}_p |\omega| = -1$. Then $\omega$ is written as $\omega = (z - p)^{-1}\omega_0(z)$, where $\omega_0$ is a holomorphic 1-form such that $\omega_0(p) \neq 0$ and $z$ is a complex coordinate around $p$. After an isometry of $H^3$ if necessary, we may assume $G(p) = 0$ because of (1.19). Then $G$ is written as $(z - p)^m G_0(z)$, where $m \geq 1$ is an integer and $G_0(z)$ is a holomorphic function such that $G_0(p) \neq 0$. From (1.16) we conclude that the order of the Hopf differential $Q$ at $z = p$ is $-2$. Hence by (1.9), $\text{ord}_p |\theta|^2 = -1$. Similarly, if $\text{ord}_p |\theta|^2 = -1$, we have $\text{ord}_p |\omega|^2 = -1$. □

Let $f : M^2 \setminus \{p_1, \ldots, p_n\} \to H^3$ be a complete flat front and let $(\omega, \theta)$ be as in (2.3). By (2.5), the pullback of the Sasakian metric of $T^*_1 H^3$ by the Legendrian lift of $f$ is

$$d\sigma^2 = 2(|\omega|^2 + |\theta|^2).$$

(3.4)

Since $\omega$ and $\theta$ are holomorphic 1-forms, the Gaussian curvature $K_{d\sigma^2}$ of $d\sigma^2$ is nonpositive. Moreover, $d\sigma^2$ is complete because of Corollary 3.4. Thus it satisfies the Cohn-Vossen inequality (0.1) in the Introduction.

Proposition 3.21. Equality holds in (0.1) if and only if all ends are regular and cylindrical.

Proof. In fact,

$$\frac{1}{2\pi} \int_{M^2} (-K_{d\sigma^2}) dA_{d\sigma^2} = -\chi(M^2) + \sum_{j=1}^n \text{ord}_{p_j} d\sigma^2$$

$$= -\chi(M^2) + \sum_{j=1}^n (\text{ord}_{p_j} d\sigma^2 + 1).$$

(See [11] or Corollary 1 of [3].) In our case $\text{ord}_{p_j} d\sigma^2 \leq -1$, since $d\sigma^2$ is complete. Hence equality holds if and only if $\text{ord}_{p_j} d\sigma^2 = -1$ for all $j = 1, \ldots, n$. But by (3.4),

$$\text{ord}_{p_j} d\sigma^2 = \min \{\text{ord}_{p_j} |\omega|^2, \text{ord}_{p_j} |\theta|^2\}.$$

Hence if $\text{ord}_{p_j} d\sigma^2 = -1$, the end $p_j$ is regular because of (1.9) and Lemma 3.7, and then $p_j$ is cylindrical because of Lemma 3.20. □
Note that the left-hand side of (0.1) may not be an integer.

4. Examples and a classification

In this section, we investigate complete flat fronts all of whose ends are regular and embedded. We shall classify them when the number of ends is less than or equal to 3. We begin by reviewing known examples and their hyperbolic Gauss maps.

Example 4.1 (flat fronts of revolution). Let \( \mathcal{M} \) denote the Riemann sphere \( S^2 = \mathbb{C} \cup \{ \infty \} \) and consider a pair \((\mathcal{G}, \mathcal{G}_*)\) of meromorphic functions on \( \mathcal{M} \) defined by \( \mathcal{G}(z) = z \) and \( \mathcal{G}_*(z) = \alpha z \), for some constant \( \alpha \in \mathbb{R} \setminus \{1\} \).

Define \( \mathcal{M} \) by

\[
\mathcal{M} := \begin{cases} 
\mathcal{M} \setminus \{0\} & \text{if } \alpha = 0 \\
\mathcal{M} \setminus \{0, \infty\} & \text{otherwise.}
\end{cases}
\]

(4.1)

One can easily check that \( \mathcal{M} \) and \((\mathcal{G}, \mathcal{G}_*)\) satisfy conditions (1) and (2) of Theorem 2.11. Indeed, these data give a Legendrian immersion

\[
\mathcal{E} = \begin{pmatrix} 
\frac{z^{-\alpha/(1-\alpha)}}{c} & \frac{\alpha z^{1/(1-\alpha)}}{1-\alpha} \\
\frac{z^{-1/(1-\alpha)}}{c} & \frac{cz^{\alpha/(1-\alpha)}}{1-\alpha}
\end{pmatrix}
\]

for some constant \( c \)

(4.2)

and a resulting flat front \( f := \mathcal{E} \mathcal{E}^*: \mathcal{M} \to \mathbb{H}^3 \).

This flat front \( f \) is a horosphere if \( \alpha = 0 \) or a hyperbolic cylinder if \( \alpha = -1 \). We shall call \( f \) an hourglass if \( \alpha < 0 \), or a snowman if \( \alpha > 0 \). The first and second canonical forms and the Hopf differential are

\[
\omega = -\frac{1}{c^2} z^{-2/(1-\alpha)} \, dz, \quad \theta = \frac{c^2 \alpha}{(1-\alpha)^2} z^{2\alpha/(1-\alpha)} \, dz, \quad Q = -\frac{\alpha}{(1-\alpha)^2} z^2 \, dz^2.
\]

The total curvature of the pullback of the Sasakian metric is calculated as

\[
\frac{1}{2\pi} \int_{\mathcal{M}} (-K_{d\sigma^2}) \, dA_{d\sigma^2} = 2 \left| \frac{1 + \alpha}{1 - \alpha} \right|.
\]

Horospheres can be characterized by the hyperbolic Gauss maps.

Proposition 4.2. Let \( f: \mathcal{M} \to \mathbb{H}^3 \) be a complete flat front. Assume that one of the hyperbolic Gauss maps \( \mathcal{G}, \mathcal{G}_* \) is constant. Then \( f \) is a horosphere.

Proof. It suffices to prove the case of \( \mathcal{G}_* \) constant. In this case, \( \mathcal{G} \) must be nonbranched, because \( \mathcal{G} = (\mathcal{G}, \mathcal{G}_*) \) is an immersion. On the other hand, \( Q \) is identically zero. It follows from Lemma 3.7 that \( \mathcal{G} \) has at most poles, that is, \( \mathcal{G} \) is a meromorphic function on a compact Riemann surface \( \overline{\mathcal{M}} \). This implies that \( \mathcal{G} \) gives a biholomorphism \( \overline{\mathcal{M}} \cong S^2 \). Therefore we may assume
\[ \alpha = -1 \quad \alpha < 0 \quad \alpha > 0 \]

Figure 3. Flat fronts of revolution.

\[ G(z) = z \text{ on } \widetilde{M}^2(\cong S^2 \cong \mathbb{C} \cup \{\infty\}). \] Then it follows from Example 4.1 that \( f \) is a horosphere. \( \square \)

**Lemma 4.3.** Let \( f: M^2 = \widetilde{M}^2 \setminus \{p_1, \ldots, p_n\} \to H^3 \) be a complete flat front with embedded regular ends \( p_1, \ldots, p_n \). If \( n \leq 3 \), then \( \widetilde{M}^2 \) is biholomorphic to the Riemann sphere.

**Proof.** By Proposition 4.2, it suffices to prove this when both \( G \) and \( G^* \) are nonconstant, i.e., \( \deg G \geq 1 \) and \( \deg G^* \geq 1 \). Since all ends are regular and embedded, \( \deg G + \deg G^* = n \leq 3 \). Therefore \( \deg G = 1 \) or \( \deg G^* = 1 \). Thus \( G \) or \( G^* \) is a biholomorphism to the Riemann sphere. \( \square \)

Let us investigate complete flat fronts \( f: \widetilde{M}^2 \setminus \{p_1, p_2\} \to H^3 \) with 2 embedded regular ends \( p_1, p_2 \). As stated in Lemma 4.3, \( \widetilde{M}^2 = S^2 \cong \mathbb{C} \cup \{\infty\} \). Without loss of generality, we may assume that the images of the 2 ends are 0, \( \infty \in S^2 \) (= \( \partial H^3 \)):

\[ G(p_1) = G^*(p_1) = 0, \quad G(p_2) = G^*(p_2) = \infty. \] (4.3)

Since the ends are embedded, \( G \) and \( G^* \) have degree 1. We identify \( \widetilde{M}^2 \) with \( S^2 \) via \( G \), that is, \( G(z) = z \). Then the coordinates of \( p_1, p_2 \) are \( z = 0, \infty \), respectively. We can also set \( G^*(z) = (az + b)/(cz + d) \). It follows from (4.3) that \( b = c = 0 \). Therefore \( G^*(z) = \alpha z \) for some nonzero constant \( \alpha \). Moreover, conditions (1) and (2) of Theorem 2.11 imply \( \alpha \in \mathbb{R} \setminus \{0, 1\} \).

To summarize, \( f \) is congruent to a flat front of \((G, G^*) = (z, \alpha z)\) for some \( \alpha \in \mathbb{R} \setminus \{0, 1\} \). Hence, it is a flat front of revolution (see Example 4.1).
Next we investigate complete flat fronts $f : \mathbb{M}^2 \setminus \{p_1, p_2, p_3\} \to H^3$, called \textit{trinoids}, with 3 embedded regular ends $p_1, p_2, p_3$. We may assume $\mathbb{M}^2 = S^2 \cong \mathbb{C} \cup \{\infty\}$ by Lemma 4.3 and $\deg G = 1$, $\deg G_* = 2$. As in the case of 2-end fronts above, we may assume that $G(z) = z$ and that
\begin{equation}
G_*(0) = 0, \quad G_*(1) = 1, \quad G_*(\infty) = \infty
\end{equation}
are the images of the ends. Since $G_*$ is a meromorphic function on $\mathbb{M}^2 = S^2$ of degree 2, it is a quotient of polynomials of degree $\leq 2$. Indeed, one can check from (4.4) that $G_*$ has the form
\begin{equation}
G_*(z) = \frac{z(\alpha z + \beta)}{\gamma z + 1},
\end{equation}
where $\alpha, \beta, \gamma \in \mathbb{C}$ satisfy
\begin{equation}
\alpha + \beta = \gamma + 1, \quad \alpha \neq 0, \quad \alpha - \beta \gamma \neq 0.
\end{equation}
The conditions (4.6) can be rewritten as
\begin{equation}
\beta = -\alpha + \gamma + 1, \quad \alpha(\alpha - \gamma)(\gamma + 1) \neq 0.
\end{equation}
By straightforward computation, we see that
\begin{equation}
\frac{dG}{G - G_*} = \left(\frac{1}{\gamma - \alpha}\right) \frac{\gamma z + 1}{z(z - 1)} dz,
\end{equation}
which has poles only at $z = 0, 1, \infty$. All of them are simple poles, with residues $-1/(\gamma - \alpha)$, $(\gamma + 1)/(\gamma - \alpha)$, $-\gamma/(\gamma - \alpha)$, respectively. These residues must be real, because of condition (2) of Theorem 2.11. Hence $\alpha, \gamma \in \mathbb{R}$ ($\beta \in \mathbb{R}$). It follows from Theorem 2.11 and (1.11) that
\begin{equation}
\xi = c \exp \int \frac{dG}{G - G_*} = cz^{1/(\alpha - \gamma)}(z - 1)/(\gamma + 1)/(\gamma - \alpha),
\end{equation}
\begin{equation}
\omega = -\xi^{-2}dG = -c^{-2}z^{2/(\gamma - \alpha)}(z - 1)^{(2\gamma + 2)/(\gamma - \alpha)} dz.
\end{equation}
The Hopf differential $Q$ is computed as
\begin{equation}
Q = -\frac{dGdG_*}{(G - G_*)^2} = -\frac{1}{(\gamma - \alpha)^3} \frac{\alpha \gamma z^2 + 2 \alpha z + \beta}{z^2(z - 1)^2} dz^2.
\end{equation}
Thus $Q$ has poles only at $z = 0, 1, \infty$ with orders at most 2. Indeed,
\begin{align*}
(\text{ord}_0 Q, \text{ord}_1 Q, \text{ord}_\infty Q) &= \begin{cases}
(-1, -2, -2) & \text{if } \alpha = \gamma + 1 \ (\iff \beta = 0), \\
(-2, -1, -2) & \text{if } \alpha = -1 \ (\iff \alpha \gamma + 2 \alpha + \beta = 0), \\
(-2, -2, -1) & \text{if } \gamma = 0 \ (\iff \alpha = 0), \\
(-2, -2, -2) & \text{otherwise}.
\end{cases}
\end{align*}
To summarize, we have obtained the following classification theorem:

\textbf{Theorem 4.4.} Let $f : M^2 \to H^3$ be a complete flat front of which all ends are regular and embedded. If $f$ has at most 3 ends, it is congruent to one of the following:
(i) a horosphere if it has a single end,
(ii) a hyperbolic cylinder, an hourglass, or a snowman if it has 2 ends,
(iii) a trinoid with 
\[ (G, G_*) = \left( z, \frac{z(a \alpha + \beta)}{\gamma z + 1} \right), \]
where \( \alpha, \beta, \gamma \) are real constants satisfying (4.7), if it has 3 ends.

For arbitrary distinct points \( p_1, \ldots, p_n \in \partial H^3 = C \cup \{ \infty \}, \) we can construct a flat front of genus zero with embedded regular ends \( p_1, \ldots, p_n \) as follows:

**Example 4.5.** Let \( p_1, \ldots, p_n \) be arbitrary distinct points in \( \partial H^3 = C \cup \{ \infty \}. \) Without loss of generality, we may assume that \( p_n = \infty. \) Choose nonzero real numbers \( a_1, \ldots, a_{n-1} \) such that \( a_1 + \cdots + a_{n-1} \neq 0, 1. \) Set

\[ M^2 = C \setminus \{ p_1, \ldots, p_{n-1} \}, \]
\[ G = z, \]
\[ G_* = \left( z \sum_{k=1}^{n-1} \left( a_k \prod_{j \neq k} (z - p_j) \right) - \prod_{j=1}^{n-1} (z - p_j) \right) / \sum_{k=1}^{n-1} \left( a_k \prod_{j \neq k} (z - p_j) \right). \]

Then

\[ (4.8) \quad \frac{dG}{G - G_*} = \left( \frac{a_1}{z - p_1} + \frac{a_1}{z - p_1} + \cdots + \frac{a_{n-1}}{z - p_{n-1}} \right) dz. \]

It follows that \( M^2 \) and \( (G, G_*) \) satisfy the conditions of Theorem 2.11. Therefore these data yield a flat front. A straightforward computation shows that

\[ \xi = c \prod_{j=1}^{n-1} (z - p_j)^{a_j}, \]
\[ \omega(= -dG/\xi^2) = -c^{-2} \left( \prod_{j=1}^{n-1} (z - p_j)^{-2a_j} \right) dz, \]
\[ Q = \left( \sum_{j=1}^{n-1} \frac{a_j}{(z - p_j)^2} - \left( \sum_{j=1}^{n-1} \frac{a_j}{z - p_j} \right)^2 \right) dz^2. \]

It follows that \( p_1, \ldots, p_{n-1}, \infty \) are complete regular ends. Moreover, they are embedded ends since \( \deg G + \deg G_* = 1 + (n - 1) = n \) is equal to the number of ends (see Theorem 3.13).

Finally, we give examples of a complete flat front of genus 1.
Example 4.6 (flat front of genus 1 with 5 embedded ends). Let \( \wp \) denote the Weierstrass \( \wp \) function on the square torus \( T^2 = \mathbb{C}/\{\mathbb{Z} \oplus i\mathbb{Z}\} \). We note that \( \wp \) satisfies the differential equation
\[
(\wp')^2 = 4\wp^2 - e_1^2,
\]
\( e_1 = \wp(1/2) \).

Take the meromorphic functions
\[
G = \wp', \quad G_* = -\frac{8e_1^2}{3} \frac{\wp}{\wp'}
\]
on \( T^2 \). Let \( M^2 \) be a Riemann surface \( T^2 \) punctured at 5 points where \( G \) and \( G_* \) take the same value:
\[
M^2 := T^2 \setminus \{z \mid \wp(z)(3\wp(z)^2 - e_1^2) = 0\}.
\]

We remark that \( \wp \) has a double zero at \( z = (1+i)/2 \), and \( 3\wp^2 - e_1^2 \) has 4 simple zeros.

For these data, a computation gives
\[
\frac{dG}{G - G_*} = \frac{3}{2} \frac{\wp'}{\wp} dz.
\]
This implies that conditions (1) and (2) of Theorem 2.11 are satisfied. Therefore, the Riemann surface (4.10) and the meromorphic functions (4.9) define a flat front.
The first canonical form $\omega$ and the Hopf differential $Q$ are computed to be

$$\omega = -\frac{2}{c^2} \frac{3\varphi^2 - e_1^2}{\varphi^3} dz, \quad Q = -\frac{6e_1^2(\varphi^2 + e_1^2)}{\varphi(3\varphi^2 - e_1^2)} dz^2,$$

from which the completeness of the ends $\{ z : \varphi(3\varphi^2 - e_1^2) = 0 \}$ follows. One can also verify the consistency of the data $G$, $\omega$ and $Q$ by formula (1.10) in [7]. Obviously, all ends are regular.

Since $G_*$ has only simple zeros, at $z = 0$ and $\frac{1}{2}(1 + i)$, its degree is 2. Clearly $\deg G = 3$. Hence, the equality in Theorem 3.13 is attained. Therefore all 5 ends are embedded.

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INDUCTIVE ALGEBRAS FOR TREES

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Let $G$ be a locally compact group and $\pi: G \to U(\mathcal{H})$ a unitary representation of $G$. A commutative subalgebra of $\mathcal{B}(\mathcal{H})$ is called $\pi$-inductive when it is stable through conjugation by every operator in the range of $\pi$. This concept generalizes Mackey’s definition of a system of imprimitivity for $\pi$; it is expected that studying inductive algebras will lead to progress in the classification of realizations of representations on function spaces. In this paper we take as $G$ the automorphism group of a locally finite homogeneous tree; we consider the principal spherical representations of $G$, which act on a Hilbert space of functions on the boundary of the tree, and classify the maximal inductive algebras of such representations. We prove that, in most cases, there exist exactly two such algebras.

1. Introduction. Inductive algebras

Let $G$ be a separable locally compact group and $\pi: G \to \mathcal{B}(\mathcal{H})$ a unitary representation of $G$ on a Hilbert space $\mathcal{H}$. A classical tool for studying the realizations of $\pi$ on function spaces is the concept of a system of imprimitivity for $\pi$. In Mackey’s formulation a self-adjoint, weakly closed commutative subalgebra $A$ of $\mathcal{B}(\mathcal{H})$ is called a system of imprimitivity for $\pi$ when $\pi(g^{-1})A\pi(g) = A$ for every $g \in G$. Given such an algebra, by Mackey’s imprimitivity theorem [8, Theorem 3.10] there exist a $G$-space $X$, a measure $\mu$ on $X$, a cocycle $A$ and a Hilbert space $\tilde{\mathcal{H}}$ such that:

- $\pi$ is equivalent to the representation $\pi_X$ on $L^2(X, d\mu, \tilde{\mathcal{H}})$ defined in the following way:

$$ (\pi_X(g)f)(x) = A(g, x) \left( \frac{d\mu(g^{-1}x)}{d\mu(x)} \right)^{1/2} f(g^{-1}x). $$

- Through the equivalence, $A$ corresponds to $L^\infty(X, d\mu)$, regarded as an algebra of multiplication operators on $L^2(X, d\mu, \tilde{\mathcal{H}})$.

More generally, given $\pi: G \to \mathcal{B}(\mathcal{H})$ as above, define a (not necessarily self-adjoint) subalgebra of $\mathcal{B}(\mathcal{H})$ to be $\pi$-inductive algebras when it is
commutative and stable through conjugation by $\pi$. An operator $T \in \mathcal{B}(\mathcal{H})$ belongs to some $\pi$-inductive algebra if and only if

$$[T, \pi(g)T\pi(g^{-1})] = 0 \quad \text{for all} \; g \in G.$$  

(1.2)

If $\pi$ can be realized as acting on a space of functions on $X$ (not necessarily an $L^2$-space) it often happens that there is an associated $\pi$-inductive algebra of multipliers (not necessarily an $L^\infty$ one). Vice versa, a systematic search for $\pi$-inductive algebras is one way to identify possible realizations of $\pi$.

Through Mackey’s theorem and the replacement a system of imprimitivity by a strictly smaller one we get a quotient of the first realization. So it is usually enough to study maximal systems of imprimitivity (those contained in no strictly bigger one relative to the same representation). Accordingly, we restrict ourselves here to the identification of maximal inductive algebras.

This approach to the identification of realizations was proposed by Tim Steger. Although no result is known making the correspondence between realizations and inductive algebras precise, or ensuring that restriction to maximal algebras causes no loss of generality, this method precisely recovers the classical realizations in examples about matrix groups studied by Vemuri and Steger, suggesting that the failure of the approach must be connected with the existence of somewhat unnatural realizations. For other representations unexpected maximal inductive algebras might exist, associated with hitherto unknown realizations.

We review Steger and Vemuri’s work more closely, highlighting the distinction between systems of imprimitivity and inductive algebras found in those cases. In [13], Vemuri dealt with the standard representation of the Heisenberg group, where he found that all maximal inductive algebras are self-adjoint, hence imprimitivity systems. For the three-dimensional Heisenberg group he confirms that no realization exists besides the standard one on $L^2(\mathbb{R}^2)$, those obtained from it through Fourier transform and (more generally) through the metaplectic group, and finally those on $L^2((\mathbb{R}/\mathbb{Z})^2)$.

In [12], Steger and Vemuri dealt with representations of $\text{SL}(2, \mathbb{R})$; see [7, Chapter II, 5-6]. The principal series can be realized in two different ways on $L^2(\partial D)$, where $D$ is the unit disc. Correspondingly, there exist exactly two maximal inductive algebras, each identifiable in the corresponding realization with the self-adjoint multiplier space $L^\infty(\partial D)$.

For the discrete series the natural realization is an $L^2$-space of holomorphic functions on $D$ and the multiplier algebra is $H^\infty(D)$, which is the only maximal inductive algebra. For the complementary series exactly two maximal inductive algebras exist, and they are adjoint to each other. They correspond to the two known realizations: a Sobolev space on the circle $\partial D$ and its dual.
In this paper we take as $G$ the full automorphism group of a homogeneous tree and classify maximal inductive algebras for its spherical principal series representations, which are realized on the boundary of the tree. There is a close analogy here (see [1], [2]) with the principal series of $\text{SL}(2, \mathbb{R})$, the role of $L^2(\partial D)$ being played now by $L^2(\Omega)$ (where $\Omega$ is the boundary of the tree) on which $L^\infty(\Omega)$ acts by multiplication.

As in Steger and Vemuri’s work our results show, by ruling out the existence of unexpected maximal inductive algebras, that the list of realizations seems to be complete. A forthcoming paper will deal with further results about the principal series representation. For example, there exists exactly one more maximal inductive algebra for the midpoint representation. The new algebra is still self-adjoint but, unlike the previous examples, it is not maximal abelian. For general representations it will be shown in the same paper that restricting to the subgroup of even automorphisms does not increase the number of maximal inductive algebras.

A natural extension of this work would deal with complementary series representations ([10]) of the automorphism group, which are obtained by analytic continuation of the principal series. Since these representations are realized on Sobolev-like completions of the space of locally constant functions on the boundary it is easy to forecast, as in the case of $\text{SL}(2, \mathbb{R})$, that non-self-adjoint maximal inductive algebras (corresponding to spaces of Sobolev space multipliers) do exist.

For cuspidal series representations ([10]), instead, one knows of realizations on discrete spaces. Indeed such a representation is constructed by inducing from a compact open subgroup $H$ of $G$. It remains to discover how many inductive algebras exist for a given cuspidal series representation; also whether inductive algebras other than $\ell^\infty(G/H)$ exist, and what sorts of realizations are associated to them.

Finally, we remark that these results ought to be extended to the principal series of $\text{PGL}(2, F)$ (for $F$ a nonarchimedean local field such as the $p$-adics), regarded as a group of automorphisms of its tree ([1], [2]).

2. Homogeneous trees and their automorphisms

The results quoted in this section are taken from [3, Chapter I], where detailed proofs can be found.

2.1. Chains and geodesics on trees. A tree is a connected graph with no circuits. This means that, given two vertices $y, z$, there exists exactly one finite sequence $\{x_0 = y, \ldots, x_n = z\}$ ($n \geq 1$) such that $\{x_j, x_{j+1}\}$ is an edge for $j < n$ and $x_j \neq x_{j+2}$ for $j < n - 1$. Such a sequence is said to be a chain of length $n$ and is denoted by $(x_0, x_1, \ldots, x_n)$ or $[y, z]$. Setting $d(y, z) = n$ if the chain connecting $y$ with $z$ has length $n$ one defines an integer-valued distance on the tree. The set $\{x \in \mathcal{X} : d(x, y) \leq k\}$ (the
sphere of radius $n$ centered in $y$) will be denoted by $B(y, k)$. If a reference vertex $O$ is fixed we will put $|x| := d(x, O)$.

We shall suppose that, for some positive integer $q$, each vertex of the tree has exactly $q + 1$ neighbours. Such a tree is said to be homogeneous and locally finite with degree of homogeneity $q + 1$. Both the tree and its set of vertices will be denoted by $\mathcal{X}$; note that $\mathcal{X}$ is infinite if $q \geq 1$. We shall always assume that $q \geq 2$.

The concept of a chain can be generalized in a natural way by letting the index $j$ range over all of $\mathbb{N}$ or $\mathbb{Z}$. One obtains, respectively, infinite and doubly infinite chains. A doubly infinite chain is also called a geodesic.

Two infinite chains $(x_i)_{i=0}^\infty$ and $(y_i)_{i=0}^\infty$ are declared to be equivalent when, for some $n$, we have $x_i = y_{n+i}$ infinitely often (or, which is the same, from some $i$ on). This is indeed an equivalence relation on the set of infinite chains; the quotient space $\Omega$ is called the boundary of the tree. Given a vertex $z$, every class $\omega \in \Omega$ contains exactly one infinite chain through $z$; one can think of $\omega$ as its limit point.

Given two vertices $x, y$, we denote by $\mathcal{X}(x, y)$ the subset of $\mathcal{X}$ that can be reached by chains starting in $x$ and having $[x, y]$ as a subchain. Call $\Omega(x, y)$ the set of limit points of infinite chains contained in $\mathcal{X}(x, y)$. Varying $x, y$, and considering intersections with $\Omega$, one obtains a basis for a compact topology on $\Omega$. The family of all sets $\mathcal{X}(x, y) \cup \Omega(x, y)$ ($x, y \in \mathcal{X}$), together with the singletons of $\mathcal{X}$, is a basis for a compact topology on $\mathcal{X} \cup \Omega$. With respect to this topology $\Omega$ is closed. Functions defined on $\Omega$ will be represented as in Figure 1. That particular example shows the function equal to $\eta$ on $\Omega(O, x)$, to $\theta$ on $\Omega(O, y)$ and to $\varepsilon$ on $\Omega(x, O) \sim \Omega(O, y)$. When no value is specified for some subset of $\Omega$, it will be understood that the function is zero there.

![Figure 1](image)

2.2. The automorphism group $G$. A distance-preserving bijection of the set of vertices of $\mathcal{X}$ is called an automorphism of $\mathcal{X}$. The group of such maps will be denoted by $G$. Notice that $G$ also acts naturally on $\Omega$.

Now, given $O \in \mathcal{X}$, let $O'$ be a neighbour of $O$. For $x \in \mathcal{X}$, let $d(x, \{O, O'\}) = \min(d(x, O), d(x, O'))$ and

$$
\begin{align*}
\mu(\Omega(O, x)) &= \frac{1}{2}q^{-d(x, \{O, O'\})} \\
\mu(\Omega(O, O')) &= \mu(\Omega(O', O)) = \frac{1}{2}.
\end{align*}
$$

(2.1)
It is easy to see that $\mu$ can be extended to a Borel probability measure on $\Omega$ which is invariant through the subgroup $K'$ that stabilizes $\{O, O'\}$. In other words, given $k \in K'$, the measure $\mu_k$ such that $\mu_k(A) = \mu(k^{-1}A)$ for every Borel set $A$ is equal to $\mu$.

**Remark 1.** From now on we depart slightly from the line of [3], where a different measure $\nu$ is considered. That measure is invariant through the subgroup that fixes $O$. However $\nu \cong \mu$ and every result about $\nu$ applies, with minor modifications, to the present case.

For general $g$, $\mu_g$ need not be equal to $\mu$; the argument in [3, Section II.1] nonetheless shows that $\mu_g \cong \mu$, and the Radon–Nikodym derivative $d\mu_g/d\mu$ takes only finitely many values on $\Omega$.

### 2.3. Spherical representations of $G$.

Fix $s \in \mathbb{R}$; for $g \in G$ let $P(g, \cdot) := (d\mu_g/d\mu)$ and put, for $f \in \mathcal{H} := L^2(\Omega, \mu)$,

$$
\pi_s(g)f(\omega) = P(g, \omega)^{1/2+is}f(g^{-1}\omega) \quad \text{for almost every } \omega \in \Omega.
$$

The cocycle identity $P(gh, \omega) = P(g, \omega)P(h, g^{-1}\omega)$ implies that $\pi_s$ is a unitary representation of $G$ on $\mathcal{H}$. The family $\{\pi_s : s \in \mathbb{R}\}$ is called the principal spherical series of $G$.

**Proposition 2.1.** Let $\pi_s$ be a spherical representation of the principal series and $\mathcal{M}$ the algebra of multiplication operators on $\mathcal{H}$. Then $\mathcal{M}$ is a maximal $\pi_s$-inductive algebra.

*Proof.* For $\phi \in L^\infty(\Omega)$, $f \in \mathcal{H}$ let $M_{\phi}f := \phi \cdot f$. Let $(\lambda(g)\phi)(\omega) = \phi(g^{-1}\omega)$ for every $g \in G$. A computation shows that $\pi_s(g)M_{\phi}\pi_s(g^{-1}) = M_{\lambda(g)\phi}$, so that $\mathcal{M}$ is $\pi_s$-inductive. It is well-known that $\mathcal{M}$ is maximal commutative (see [11, Proposition 4.7.6]), hence maximal $\pi_s$-inductive. \qed

We recall now (see [3, Section II.3]) that $\pi_s$ is equivalent to $\pi_{-s}$ for every value of $s$. Let $I_s$ be a unitary intertwining operator: then $\pi_{-s} \mid_{K'}$ fixes $I_s \mathbf{1}$ since $\pi_s \mid_{K'}$ fixes $\mathbf{1}$. So $I_s \mathbf{1}$ is constant, and by unitarity its modulus is one. Since we are mainly interested in conjugation by $I_s$, we may well assume that $I_s \mathbf{1} = \mathbf{1}$ for every $s$. With this normalization $I_{-s} = I_s^{-1}$ for every $s$. Since $\pi_s(g)\mathbf{1} = P^{1/2+is}(g, \cdot)$, the intertwining property of $I_s$ also gives

$$
I_s(P^{1/2+is}(g, \omega)) = P^{1/2-is}(g, \omega) \quad \text{for almost every } \omega \in \Omega.
$$

It follows from Proposition 2.1 that, for every $s$, the algebra $I_{-s}\mathcal{M}I_s$ is maximal $\pi_s$-inductive. We will see in Corollary 3.4 that for most values of $s$ the new $\pi_s$-inductive algebra is different from $\mathcal{M}$, though clearly isomorphic to it.
3. Classification of maximal inductive algebras

3.1. Operators that transform according to a character. As will soon be clear, the classification of maximal $\pi_s$-inductive algebras can be accomplished by studying their elements that transform in a simple way under conjugation by $\pi_s$ (when the latter is restricted to certain compact subgroups of $G$).

**Definition 1.** Fix two adjacent vertices $O, O'$; let $K := K_O \cap K_{O'}$ (the compact subgroup of automorphisms that fix both $O$ and $O'$). Furthermore, fix $j \in G$ that exchanges $O$ with $O'$, and let $K'$ be, as in Section 2.2, the compact subgroup of automorphisms that stabilize $\{O, O'\}$. Then $K' = K \cup jK$.

For $k \in K'$, we put

$$\chi(k) := \begin{cases} 1 & \text{if } kO = O, \\ -1 & \text{if } kO = O'. \end{cases}$$

(3.1)

Finally, we say that an operator $T \in B(\mathcal{H})$ transforms according to $\chi$ when

$$\pi_s(k)T\pi_s(k^{-1}) = \chi(k)T \quad \text{for every } k \in K'.$$

(3.2)

If $\pi_s(k)T\pi_s(k^{-1}) = T$ for every $k \in K'$, we simply call $T$ a $K'$-invariant operator. Denote by $\mathcal{F}$ the subalgebra of $K'$-invariant operators and by $\mathcal{E}$ the subspace of operators that transform according to $\chi$.

It is easy to see that

$$\mathcal{M} \cap \mathcal{F} = CM_1 = CI,$$

(3.3)

$$\mathcal{M} \cap \mathcal{E} = CM_W,$$

(3.4)

where $W := 1_{\Omega(O,O')} - 1_{\Omega(O',O)}$. Indeed, for $\phi \in L^\infty$, the condition $M_\phi \in \mathcal{F}$ means that, for any $k \in K'$, $\phi(k\omega) = \phi(\omega)$ almost everywhere. Let $dk$ be the normalized Haar measure on $K'$. Then, by Fubini’s Theorem,

$$0 = \int_{K'} \int_{\Omega} |\phi(k\omega) - \phi(\omega)| \, d\mu dk = \int_{\Omega} \int_{K'} |\phi(k\omega) - \phi(\omega)| \, dk d\mu \geq \int_{\Omega} \left| \int_{K'} \phi(\omega) - \phi(k\omega) dk \right| d\mu.$$

The last integral over $K'$ is independent of $\omega$ by transitivity, so $\phi$ must be constant. The second statement is proved by applying the first one to the function $W \cdot \phi$.

**Lemma 3.1.** Let $\mathcal{A}$ be a maximal $\pi_s$-inductive algebra. If $M_W \in \mathcal{A}$, then $\mathcal{A} = \mathcal{M}$.

**Proof.** Let $T \in \mathcal{A}$ and $g \in G$; since $1_{\Omega(O',O)} = \frac{1}{2}(W + I)$ we have

$$[M_{\lambda(g)}1_{\Omega(O',O)}, T] = [M_{\lambda(g)}1_{\Omega(O',O)}, \pi(g^{-1})T\pi(g)] = 0.$$
Translates of $1_{\Omega(O', O)}$ span a dense subspace of $C(\Omega)$, so $T$ commutes with the subalgebra

$$\mathcal{N} = \{ M_\phi : \phi \in C(\Omega) \}$$

of $\mathcal{M}$. But $\mathcal{N}' = \mathcal{M}$ (see again [11, Section 4.7]), hence $\mathcal{A} = \mathcal{M}$ by maximality.

The next result describes, for any maximal inductive algebra $\mathcal{A}$, the subspaces $\mathcal{A} \cap \mathcal{E}$ and $\mathcal{A} \cap \mathcal{F}$. This is the crucial step towards the classification of maximal inductive algebras, which will be summarized in Section 3.3.

**Theorem 3.2.** Let $\pi_s$ be a representation of the principal spherical series, with $\Re q^s \neq 0$, and let $\mathcal{A}$ be a maximal $\pi_s$-inductive algebra. Then:

(i) $\mathcal{A} \cap \mathcal{F} = C1$.

(ii) Either $\mathcal{A} \cap \mathcal{E} \subseteq CM_W$ or $\mathcal{A} \cap \mathcal{E} \subseteq C(I_s M_W I_s)$.

**Remark 2.** One could be tempted to prove part (ii) only and then deduce part (i) on the following grounds: if $T_1 \in \mathcal{E}$ and $T_2 \in \mathcal{F}$, then $T_1 T_2 \in \mathcal{E}$. Unfortunately $T_2$ and $T_1$ need not belong to the same maximal inductive algebra, so in general part (ii) does not apply to $T_1 T_2$. The reasoning can be justified because any maximal inductive algebra containing $T_2$ must also contain some nontrivial element of $\mathcal{E}$, but our proof of this fact (Theorem 4.1) relies on Theorem 3.2.

**Remark 3.** It will be clear that part (i) holds regardless of the value of $\Re q^s$.

In the proof of Theorem 3.2 we will use the following orthogonal decomposition of the Hilbert space $\mathcal{H}$, which provides an easy description of the relevant operators as infinite matrices:

**Definition 2.** For $n \geq 1$, let $\mathcal{M}_n$ be the subspace of $\mathcal{H}$ generated by the set

$$\{ 1_{\Omega(O,x)} : d(x, \{O, O'\}) = n \}.$$

In other words, $\mathcal{M}_n$ is the subspace of locally constant functions on $\Omega$ that depend only of the first $n$ steps from $\{O, O'\}$.

Moreover, let

$$\begin{align*}
\mathcal{H}_{n+1} &= \mathcal{M}_{n+1} \oplus \mathcal{M}_n \quad (n \geq 1), \\
\mathcal{H}_0 &= C1, \\
\mathcal{K} &= CW := C(1_{\Omega(O,O')} - 1_{\Omega(O',O)}), \\
\mathcal{M}_0 &= \mathcal{H}_0 \oplus \mathcal{K}, \\
\mathcal{H}_1 &= \mathcal{M}_1 \oplus (\mathcal{H}_0 \oplus \mathcal{K}).
\end{align*}$$

See Figure 2 for examples. Notice that the mean of elements of $\mathcal{H}_n$ $(n \geq 1)$ on each half of the boundary is zero while the function $W$, which generates $\mathcal{K}$, is orthogonal to 1 but not to $1_{\Omega(O,O')}$ and $1_{\Omega(O',O')}$. 
Figure 2. General elements of $\mathcal{M}_1$ and of $\mathcal{H}_1$ $(q = 2)$.

We let $\mathcal{H}_n^+$ be the subspace of functions of $\mathcal{H}_n$ whose support lies in $\Omega(O', O)$. Now $\dim \mathcal{H}_0 = \dim K = 1$, while $\dim \mathcal{M}_n = 2q^n$ and $\dim \mathcal{H}_n = 2(q - 1)q^{n-1}$ ($n \geq 1$). It follows that, for $n \geq 1$, $\mathcal{M}_n = \mathcal{H}_0 \oplus K \oplus \bigoplus_{j=1}^{n} \mathcal{H}_j$. Since locally constant functions are dense in $\mathcal{H}$ we conclude that

$$\mathcal{H} = \mathcal{H}_0 \oplus K \oplus \bigoplus_{j=1}^{\infty} \mathcal{H}_j.$$  

We compute, for later use, the action of the intertwining operator on $K$:

**Lemma 3.3.** Let $W$ be as in (3.4) and let $I_s$ be, as in Section 2.3, the unitary intertwiner of $\pi_s$ and $\pi_{-s}$ such that $I_s 1 = 1$. Let

$$\psi(s) := \frac{q - 1 + 2 \text{Im} q^{1/2+is}}{q - 1 - 2 \text{Im} q^{1/2+is}}.$$ 

Then $I_s W = \psi(s) W$.

**Proof.** We have

$$I_s \left( \frac{d\mu_g}{d\mu} \right)^{1/2+is} = I_s (\pi_s(g) 1) = \pi_{-s}(g) I_s^{-1} 1 = \pi_{-s}(g) 1 = \left( \frac{d\nu_g}{d\nu} \right)^{1/2-is}.$$  

Figure 3. $d\mu_g/d\mu$.

Take a geodesic $(x_i)_{i \in \mathbb{Z}}$ containing $O$ and $O'$, and suppose that $gx_i = x_{i+1}$ for all $i$. In this case $g$ is called a one step translation along $[O, O']$. The Radón–Nikodým derivative $d\mu_g/d\mu$ can be evaluated as in [3, Chapter II]; see Figure 3 for an example with $gO' = O$. The sum of all the functions $(d\mu_g/d\mu)^{1/2+is}$, with $g$ ranging over the set of translations just described, equals $(q - 1 - 2 \text{Im} q^{1/2+is}) W$, and the thesis follows from the two previous equalities. 

**Corollary 3.4.** With the same notation as in Lemma 3.3, $I_{-s} M I_s \neq M$ if and only if $q^{2is} \neq 1$. 

Proof. Let $T = I_{-s}M_{W}I_{s}$, and suppose that $T = M_{f}$. Since $I_{s}1 = 1$ by definition, Lemma 3.3 gives $T1 = \psi(-s)W$, hence $f = \psi(-s)W$. On the other hand $TW = I_{-s}M_{W}\psi(s)W = \psi(s)I_{-s}1 = \psi(s)1$, because $W^{2} = 1$. So we also get $f = \psi(s)W$. But $\psi(s) \neq \psi(-s)$ when $q^{2is} \neq 1$. This contradiction shows that $I_{s}M_{W}I_{-s}$ is not a multiplication operator.

Conversely, if $q^{2is} = 1$, the same reasoning shows that $I_{s}M_{W}I_{-s} \in \mathcal{M}$. Subtracting $I$ we get $I_{s}M_{1_{\Omega(O',O)}}I_{-s} \in \mathcal{M}$. Conjugating by $\pi_{s}$ we find that $I_{s}M_{\lambda(g)1_{\Omega(O',O)}}I_{-s} \in \mathcal{M}$ for every $g$. We conclude that $I_{s}M_{I_{-s}} = \mathcal{M}$ as in the proof of Lemma 3.1. □

Definition 3. For two representations $\pi_{1}, \pi_{2}$ of a group $H$, let $c_{H}(\pi_{1}, \pi_{2})$ denote the dimension of the space of unitary intertwining operators. 

Let 

\begin{equation}
(3.8) \quad \pi_{K'} := \text{Res}(\pi_{s}, K'|_{K}) \quad \text{and} \quad \pi_{n} := \text{Res}(\pi_{s}, K'|_{\mathcal{M}_{n}}) \quad (n \geq 0).
\end{equation}

Lemma 3.5. The representations $\pi_{K'}, \pi_{n} \ (n \geq 0)$ defined as in (3.8) are irreducible and inequivalent.

Proof. Let $\tau$ be a finite chain of length $n$, such that $\tau \cap \{O, O'\} = \{O'\}$, and let $K_{\tau}$ be the subgroup of $K'$ which fixes every vertex of $\tau$. Then $\pi_{n}$ is isomorphic to the representation $\text{Ind}(K_{\tau}, K', 1)$ induced on $K'$ from the trivial representation $1$ of $K_{\tau}$. By Frobenius’ reciprocity theorem ([6, VI.11, Theorem 7]),

\begin{equation}
(3.9) \quad c_{K'}(\text{Ind}(K_{\tau}, K', 1), \text{Ind}(K_{\tau}, K', 1)) = c_{K'}(1, \pi_{n})
\end{equation}

and the last number is the dimension of the subspace of $\pi_{n}|_{K'}$-invariant elements of $\mathcal{M}_{n}$. This subspace is easily seen to be $n + 2$-dimensional. So we have proved that $n + 2 = c(\pi_{n}, \pi_{n})$. Since $\pi_{n} = \pi|_{K} \oplus \bigoplus_{j=0}^{\infty} \pi|_{H_{j}}$, it follows that the summands are irreducible and inequivalent. □

Definition 4. For $n \in \mathbb{N}$ define $\iota_{n}$ to be the projection on $\mathcal{H}_{n}$, and $\iota_{K}$ to be the projection on $K$. Given an operator $T \in \mathcal{B}(\mathcal{H})$, let $T_{m,n} := \iota_{n}T\iota'_{n}$. If one thinks of $T$ as an infinite matrix $A$, then $T_{n,n}$ corresponds to a block $A_{m,n}$. Define similarly, for every $n$, the operators $T_{K,n}, T_{n,K}$ and the blocks $A_{K,n}, A_{K,n}$ of $A$.

When $T$ belongs to an inductive algebra, some information about its matrix $A$ can be obtained from Lemma 3.5.

Proposition 3.6. Let $A \subset \mathcal{B}(\mathcal{H})$ be a maximal inductive algebra and $T \in A \cap \mathcal{E}$. Consider the matrix $A$ of $T$ and its finite blocks $A_{i,j}$ as in Definition 4. Then:

(i) For $i, j \geq 1$, $A_{i,j}$ is a multiple of $M_{W}|_{\mathcal{H}_{i}}$ when $i = j$, and zero otherwise.

(ii) $A_{i,K}, A_{K,i}$ are zero unless $i = 0$. 

Proof. Since $T$ transforms according to $\chi$, for every $n, m \geq 1$ $\iota_n T\iota^*_m$ intertwines $\Res(\pi_s, K')|_{\mathcal{H}_n}$ with $\chi \otimes \Res(\pi_s, K')|_{\mathcal{H}_m}$, while $\iota_j T\iota^*_j$ intertwines $\Res(\pi_s, K')|_{\mathcal{H}_j}$ with $\chi \otimes \Res(\pi_s, K')|_{K}$. A similar result holds for $\iota K T\iota^*_j$, and the thesis follows from Lemma 3.5 and Schur’s Lemma. □

We are now ready to classify the operators of $\mathcal{A}$ that are $K$-invariant or transform according to the character $\chi$.

3.2. Proof of Theorem 3.2.

Part (i). Let $T \in \mathcal{A} \cap \mathcal{F}$. By Proposition 3.6 there exists a sequence \{\lambda_0, \lambda_K, \lambda_1, \lambda_2, \ldots\} such that

$$T = \lambda_0 I_{\mathcal{H}_0} \oplus \lambda_K I_{K} \oplus \bigoplus_{j \geq 1} \lambda_j I_{\mathcal{H}_j}. \quad (3.10)$$

Refer to Definition 2 for details about the subspaces of $\mathcal{H}$ involved. To prove that the relevant sequence is constant, we remark first that $[T, T^*] = 0$, and so eigenspaces of different eigenvalues of $T$ are orthogonal. Put $T^g := \pi(g^{-1}) T \pi(g)$ and suppose that there exist $v_l \in \mathcal{H}_l$, $v_m \in \mathcal{H}_m$ and $g \in G$ such that

$$\langle T \pi(g) v_l, \pi(g) v_m \rangle = \langle T^g v_l, v_m \rangle \neq 0.$$ 

Then $v_m$ is an eigenvector of $\lambda_m$; since $[T, T^g] = 0$, $T^g v_l$ is an eigenvector of $\lambda_l$. Hence $\lambda_l = \lambda_m$.

Keeping this in mind, we prove that $\lambda_0 = \lambda_K$. By maximality $T - \lambda_0 I \in \mathcal{A}$, so we may assume that $\lambda_0 = 0$. As in the proof of Lemma 3.3, let $g$ be a translation by $-l$ along a geodesic $(x_j)_{j=-\infty}^{\infty}$, with $x_{-1} = O$ and $x_0 = O'$, and choose $v_l$ as in Figure 4. Using $v_1 \in \mathcal{H}_1$ as in Figure 5, write

$$\pi(g) v_l = \frac{(q-1)(q-q^z)}{2q} 1 + \frac{(q-1)(q+q^z)}{2q} W + \frac{q^z}{q} v_1. \quad (3.12)$$

Since $\lambda_0 = 0$, we have

$$T \pi(g) v_l = \lambda_K \frac{(q-1)(q+q^z)}{2q} W + \lambda_1 \frac{q^z}{q} v_1. \quad (3.13)$$

Figure 4. $v_l \in \mathcal{H}_l$.  

• • • • •... 

\begin{center}
\begin{tikzpicture}
\clip (-3,-2) rectangle (3,2);
\draw (-3,0) -- (3,0) node[midway, above] {$x_{l-1}$} -- (-1,0) node[midway, above] {$x_l$} -- (0,1) -- (0,-1) -- (1,0) -- (3,0);
\draw (-3,0) -- (0,0) node[midway, above] {$q^lz$} -- (0,1) -- (0,-1) -- (1,0) -- (3,0);
\draw (-3,0) -- (0,0) node[midway, above] {$O$} -- (0,1) -- (0,-1) -- (1,0) -- (3,0);
\draw (-3,0) -- (0,0) node[midway, above] {$O'$} -- (0,1) -- (0,-1) -- (1,0) -- (3,0);
\end{tikzpicture}
\end{center}
Then

\[
\zeta : = (T \pi (g) v_l) \mid \Omega (O, x_{-1}) = -\lambda_\kappa \frac{(q - 1)(q + q^z)}{2q} + \lambda_1 \frac{(q - 1)q^z}{q} = \frac{q - 1}{2q} (-\lambda_\kappa (q + q^z) + 2\lambda_1 q^z).
\]

Refer to Figure 6 for pictures of \(\pi (g) 1\) and \(\pi (g) W\).

If \(\zeta \neq 0\) we can choose \(v_l\) in such a way that \(\langle T \pi (g) v_l, \pi (g) 1 \rangle \neq 0\) and \(\langle T \pi (g) v_l, \pi (g) W \rangle \neq 0\) and then conclude that \(\lambda_\kappa = \lambda_l = \lambda_0 = 0\) using the reasoning explained above.

Now suppose that \(\zeta = 0\). Then, unless \(\lambda_\kappa = 0\), we will have

\[
(3.14) \quad \frac{\lambda_1}{\lambda_\kappa} = \frac{q + q^z}{2q^z} = \frac{q^{1-z} + 1}{2}.
\]

If this is the case then \((\lambda_1/\lambda_\kappa)^2 \neq \lambda_1/\lambda_\kappa\), since \(q > 1\), so applying these same arguments to \(T^2\) instead of \(T\) we find \(\lambda_\kappa^2 = 0\), hence \(\lambda_\kappa = 0\).
Now we prove that $\lambda_j = \lambda_0$ for $j \geq 1$, given that $\lambda_0 = \lambda_K$. By maximality, again, we can subtract $\lambda_0 I$. So we reduce to prove that $\lambda_1 = \lambda_2 = \cdots = 0$ when $\lambda_0 = \lambda_K = 0$. Suppose, by induction, that $\lambda_0 = \lambda_K = \lambda_1 = \cdots = \lambda_{l-1} = 0$. We wish to show that $\lambda_l = 0$. If we can find $g \in G$, $v \in M_{l-1}$, and $v_1 \in H_l$ so that

$$\langle T^g v, v_1 \rangle = \langle \pi(g^{-1}) T \pi(g) v, v_1 \rangle = \langle T \pi(g) v, \pi(g) v_1 \rangle \neq 0$$

then $T^g v$, like $v$, is in the 0-eigenspace of $T$, while $v_1$ is in the $\lambda_l$-eigenspace, so $\lambda_l = 0$.

Consider first the case $l = 1$. Let $g$ be a translation one step to the left along the usual geodesic, $(x_j)_{j=-\infty}^\infty$, and choose $v$ and $v_1$ as in Figure 7.

![Figure 7. The case $l = 1$.](image)

Under the hypothesis $\lambda_0 = \lambda_K = 0$, one easily calculates $\langle T \pi(g) v, \pi(g) v_1 \rangle = \lambda_1 (q - 1)/2q^2$. Therefore, if $\lambda_1$ were nonzero, the reasoning following (3.15) would show that it was zero after all.

Now consider the case of general $l$. Let $g$ be a translation by $2l - 1$ steps to the left along the usual geodesic, and choose $v$ and $v_1$ as in Figure 8. Figure 8 also shows the exact form of $\pi(g) v$. The form of $\pi(g) v_1$ is similar and all one needs to know of it is that $\pi(g) v_1 \mid \Omega(O,x_{-l-1}) = 0$ and $\pi(g) v_1 \mid \Omega(O,x_{-l}) \sim \Omega(O,x_{-l-1}) = 1$, exactly as for $\pi(g) v$. Under the hypothesis that $T \mid M_{l-1} = 0$ one calculates that $\langle T \pi(g) v, \pi(g) v_1 \rangle = \lambda_l (q - 1)/2q^{l+1}$ and then applies the reasoning following (3.15).

**Part (ii).** By Proposition 3.6, again, every operator $T \in \mathcal{A} \cap \mathcal{E}$ can be decomposed as

$$T = \lambda_{K0} M_W \mid \mathcal{H}_0 \oplus \lambda_0 \mathcal{K} M_W \mid \mathcal{K} \oplus \bigoplus_{j \geq 1} \lambda_j M_W \mid \mathcal{H}_j$$
for some sequence \( \{ \lambda_{K_0}, \lambda_{0K}, \lambda_1, \lambda_2, \ldots \} \). Obviously \( T \in \mathbf{CM}_W \) if and only if the relevant sequence is constant. We will show that this is indeed the case, up to replacing possibly \( T \) with \( I_s T I_s^{-1} \). By specializing Equations (1.2), (3.2) to matrix coefficients we get, for every \( v \in \mathcal{H}_1^+ \),

\[
\begin{align*}
(3.17) \quad & \lambda_{0K} \langle T^g \mathbf{1}, W \rangle = \langle T^g T W, W \rangle = \langle T^g W, T^* W \rangle = \lambda_{K0} \langle T^g W, \mathbf{1} \rangle \\
& \lambda_{0K} \langle T^g W, W \rangle = \langle T^g W, T^* \mathbf{1} \rangle = \langle T^g T W, \mathbf{1} \rangle = \lambda_{K0} \langle T^g 1, \mathbf{1} \rangle \\
& \lambda_1 \langle T^g \mathbf{1}, v \rangle = \langle T^g \mathbf{1}, T^* v \rangle = \langle T^g T \mathbf{1}, v \rangle = \lambda_{K0} \langle T^g W, v \rangle \\
& \lambda_1 \langle T^g W, v \rangle = \langle T^g W, T^* v \rangle = \langle T^g T W, v \rangle = \lambda_{K0} \langle T^g \mathbf{1}, v \rangle.
\end{align*}
\]

If

\[
(3.18) \quad \langle T^g W, v \rangle \langle T^g \mathbf{1}, v \rangle \neq 0 \quad \text{for some } g \in G, v \in \mathcal{H}_1^+,
\]

we multiply the last two equations in (3.17) and get

\[
(3.19) \quad \lambda_{K0} \lambda_{0K} = \lambda_1^2.
\]

Suppose now that condition (3.18) fails. Observe that, if exactly one of the factors in (3.18) (the first one, say) vanishes, Equations (3.17) give \( \lambda_1 = \lambda_{K0} = 0 \), so that (3.19) still holds. So we reduce to the case in which

\[
(3.20) \quad \langle T^g W, v \rangle = \langle T^g \mathbf{1}, v \rangle = 0
\]

for all \( v \in \mathcal{H}_1^+ \) and \( g \in G \). In particular, let \( g \) be a translation one step on the left along the usual geodesic. Take \( v \in \mathcal{H}_1 \) such that \( \langle v, \pi(g) \mathbf{1} \rangle \neq 0 \) and renormalize \( v \) so that \( \|v\|^2 = (q - 1)/2 \). Refer again to Figure 6 for a representation of \( \pi(g) \mathbf{1}, \pi(g)W \).
Put $z = 1/2 + is$ and

\[(3.21)\] \[\alpha := \frac{q - 1}{2q}; \quad \beta := \text{Re} \frac{q^z}{q}; \quad \gamma := \text{Im} \frac{q^z}{q}.\]

Then, for a suitable $v' \in \langle \{1, W, v\}\rangle^\perp$,

\[(3.22)\]
\[
\pi(g)1 = (\alpha + \beta)1 + (\alpha + i\gamma)W + q^{-1}(q^{-1} - 1)v \\
\pi(g)W = (\alpha - i\gamma)1 + (\alpha - \beta)W - q^{-1}(q^{-1} + 1)v \\
\pi(g)v = (2q)^{-1}\{(1 - q^2)1 + (1 + q^2)W - 2(q^{-1} - 1)v\} + v'.
\]

Looking at the action of $T$, and assuming (3.20), we get the system

\[(3.23)\]
\[
\lambda_0\kappa(\alpha + i\gamma)(q^z - 1) + \lambda\kappa_0(\alpha + \beta)(-q^{-1} - 1) + \lambda_1q^{-1}(q^{-1} - 1) = 0, \\
\lambda_0\kappa(\beta - \alpha)(q^z - 1) + \lambda\kappa_0(-\alpha + i\gamma)(-q^{-1} - 1) + \lambda_1q^{-1}(q^{-1} + 1) = 0.
\]

Since $q \geq 2$, we readily obtain

\[(3.24)\]
\[
\lambda_0\kappa = \lambda_1 \frac{q^z(-2\alpha - q^2q^{-1} - q^{-1})}{q^z - 1} \\
\lambda\kappa_0 = \lambda_1 \frac{q^z(q^2q^{-1} - 2\alpha - q^{-1})}{q^z + 1}
\]

(recall that $|q^z| \geq \sqrt{2}$) and finally, by a simple computation,

\[(3.25)\]
\[
\lambda_0\kappa\lambda\kappa_0 = \lambda_1^2.
\]

We will now use for the first time the condition $\beta \neq 0$, that is, $\text{Re} q^{1/2 + is} \neq 0$; under this condition we will find a further, linear relation between the first coefficients of $T$. Indeed, taking $g \in G, v \in \mathcal{H}_1^+\ )$ as in (3.22), the second equation in (3.17) reads, with the same notation as in (3.21),

\[(3.26)\]
\[
\lambda_0\kappa(\lambda\kappa_0(\alpha - i\gamma)(\alpha - \beta) + \lambda_0\kappa(\alpha + i\gamma)(\alpha + \beta) + \frac{\lambda_1}{2}q^{-2}(q - 1)|q^z + 1|^2) \\
= \lambda_0\kappa(\lambda\kappa_0(\alpha - i\gamma)(\alpha + \beta) + \lambda_0\kappa(\alpha + i\gamma)(\alpha + \beta) + \frac{\lambda_1}{2}q^{-2}(q - 1)|q^z - 1|^2).
\]
If \( \lambda_0K = 0 \) then \( \lambda_1 = 0 \) by (3.19). Moreover, by (3.22) and the first equation in (3.17),

\[
0 = \lambda_0K \langle T^g1, W \rangle = \lambda_{K0} \langle T^gW, 1 \rangle = \lambda_{K0}^2 |\alpha - i\gamma|^2;
\]

that is, \( \lambda_{K0} = 0 \), since \( \gamma \in \mathbb{R} \) and \( \alpha = (q-1)/(2q) > 0 \) by hypothesis. Therefore, excluding the case in which \( \lambda_{K0} = \lambda_0K = \lambda_1 = 0 \) we can divide both sides in (3.26) by \( \lambda_0K \). Since \( \beta \neq 0 \) (equivalently, \( |q^* + 1| \neq |q^* - 1| \)), the resulting relation

\[
2\beta(\lambda_{K0}(\alpha - i\gamma) + \lambda_0K(\alpha + i\gamma)) = \frac{\lambda_1}{2} q^{-2}(q - 1)(|q^* + 1|^2 - |q^* - 1|^2)
\]

is nontrivial. Taking \( \lambda_1 \) as a parameter one sees that the solutions to the system of equations (3.28) and (3.25) are

\[
\lambda_{K0} = \lambda_0K = \lambda_1
\]

and

\[
\lambda_{K0} = \psi(s)\lambda_1, \quad \lambda_0K = (\psi(s))^{-1}\lambda_1,
\]

with \( \psi(s) \) as in (3.6). Under condition (3.29), \( T = \lambda_1I \) on \( C_1 \oplus K \). By Lemma 3.3, \( T \) satisfies (3.30) if and only if \( I_s^{-1}TI_s \) satisfies (3.29). Since \( \psi(s) \neq 1 \) the same solution must be chosen for every \( T \in \mathcal{A} \); else the product of two elements of \( \mathcal{A} \cap E \) would fail to be scalar, contradicting part (i).

Finally, when (3.29) holds, the same reasoning as in part (i) shows that \( \lambda_j = \lambda_1 \) for all \( j \geq 1 \). Indeed, the proof of the analogous fact for \( K' \)-invariant operators only involves vectors in \( \mathcal{H}_0 \oplus K \oplus \bigoplus_{j \geq 1} \mathcal{H}_j^+ \). So the method also applies to operators in \( \mathcal{A} \cap E \), which act scalarly on any \( \mathcal{H}_j^+ \).

We conclude that either \( T \in CM_W \) or \( I_sTI_s^{-1} \in CM_W \).

\[\square\]

3.3. Proof of the main classification result. Suppose that \( \text{Im} q^{is} \neq 0 \) and \( \text{Re} q^{is} \neq 0 \). Take a maximal \( \pi_s \)-inductive algebra \( \mathcal{A} \) and assume, for a moment, that \( \mathcal{A} \cap E \neq \{0\} \). By Theorem 3.2, either \( \mathcal{A} \) or \( I_{-s}AI_s \) contains a nonzero scalar multiple of \( M_W \). Hence, by Lemma 3.1, either \( \mathcal{A} = \mathcal{M} \) or \( I_{-s}AI_s = \mathcal{M} \). So, for these values of \( s \), the classification of maximal inductive algebras is achieved, up to verifying that any such algebra must contain operators that transform according to \( \chi \). The latter turns out to be true, but the proof — to be found in the next section — is not trivial.

4. Nontrivial operators that transform according to \( \chi \)

4.1. Nonscalar \( K \)-invariant operators.

Theorem 4.1. Let \( \pi_s \) be a spherical representation of the principal series, with \( q^{2is} \neq -1 \), and let \( \mathcal{A} \) be a maximal \( \pi_s \)-inductive algebra. Then:

(i) \( \mathcal{A} \cap \mathcal{F} \) contains a nonscalar operator.
(ii) $\mathcal{A} \cap \mathcal{E}$ contains a nonzero operator.

We prove first that (i) implies (ii). Indeed, given a nonscalar $K$-invariant element of $\mathcal{A}$, say $T$, take $j \in K'$ such that $\chi(j) = -1$. Then, obviously:

- $T_1 := T + \pi(j)T\pi(j^{-1})$ is $K'$-invariant.
- $T_2 := T - \pi(j)T\pi(j^{-1})$ transforms according to $\chi$.

Now $T_1, T_2 \in \mathcal{A}$, and by Theorem 3.2, $T_1$ must be scalar. Since $T$ is nonscalar, $T_2 = 2T - T_1 \neq 0$, so part (ii) of Theorem 4.1 holds.

The proof of part (i) is harder and will be accomplished in several steps. We recall that $K$ is the stabilizer of the subtree $\{O, O'\}$, and set about proving first a weaker statement, replacing $\{O, O'\}$ with a bigger subtree. More precisely, we fix $O \in X$ and let $K_n (n \in \mathbb{N})$ be the compact subgroup of $G$ that stabilizes the subtree $B(O, n) = \{x \in X : d(x, O) \leq n\}$ (a sphere of radius $n$). The normalized Haar measure on $K_n$ will be denoted by $dk$.

Given $S \in \mathcal{A}$, we let

$$S_n := \int_{K_n} \pi(k)S\pi(k^{-1}) \, dk.$$  \hspace{1cm} (4.1)

Then $S_n$ is $K_n$-invariant and $S_n \in \mathcal{A}^w = \mathcal{A}$. We know from Proposition 2.1 that $CI$ is not maximal, so we can take as $S$ a nonscalar element of $\mathcal{A}$. In this case $S_n$ is also nonscalar for some $n$. To see this, consider two locally constant functions $\xi, \eta$; then, for $n$ large enough, $\pi|_{K_n}$ fixes both $\xi$ and $\eta$, so that $\langle S_n\xi, \eta \rangle = \langle S\xi, \eta \rangle$. Since locally constant functions are dense in $\mathcal{H}$ and $\{S_n\}$ is uniformly bounded, $S_n$ weakly approaches $S$ and so $S_n$ is nonscalar for some $n$. We conclude that part (i) of Theorem 4.1 holds when $K$ is replaced by the stabilizer of a suitable sphere.

4.2. Complete subtrees. Note that every vertex of a sphere $B$ has either one or $q + 1$ neighbours in $B$. With the notation of [3, Chapter III], we can say that every vertex is either a boundary vertex or an interior vertex. This is also trivially true for the one-edge subtree. Subtrees verifying this condition are said to be complete (see Figure 10 for an example). We will now prove that part (i) of Theorem 4.1 holds for the stabilizer of a finite complete subtree $F$, of more than one edge, only if it also holds for the stabilizer of some smaller complete subtree. This, by induction, will conclude the proof.

Given a complete subtree $J$, we denote by $\partial J$ the set of its boundary vertices. We say that a vertex $x \in J$ is almost terminal if it is interior and exactly $q$ of its neighbours are boundary vertices. See again Figure 10.

Remark 4. From now on, given a terminal vertex $P$ of a complete subtree $J$, we denote by $P'$ its unique neighbour in $J$. The subtree $J'$ obtained as in Figure 11 by erasing from $J$ the neighbours of $P'$ that lie in $\partial J$ is also complete, and $P' \in \partial J'$. 
Figure 10. A complete subtree, with almost terminal vertices $A$, $B$, and a subtree which is not complete.

Consider the subgroup
\begin{equation}
H_P := \{ g \in G : g \text{ fixes } \mathcal{X}(P, P') \}
\end{equation}
of the automorphisms that only affect the vertices lying in the subtree of $\mathcal{X} \sim \mathcal{J}$ sprouting from $P$. It is shown in [3, Section III.3] that
\begin{equation}
K_{\mathcal{J}} = \prod_{R \in \partial \mathcal{J}} H_R.
\end{equation}
In particular, if $R \in \partial \mathcal{J}$ and $R \neq P$, then $H_R$ lies in the centralizer of $H_P$.

When $\mathcal{J}$ is replaced by $\mathcal{J}'$ as in Remark 4, formula (4.3) becomes
\begin{equation}
K_{\mathcal{J}'} = \left( \prod_{R \in \partial \mathcal{J}} H_R \right) \times H_{P'};
\end{equation}
of course, all the relevant subgroups are still defined by (4.2), but with respect to the new subtree $\mathcal{J}'$.

It follows from (4.4) that, given a $K_{\mathcal{J}}$-invariant operator $T$,
\begin{equation}
\mathcal{T} := \int_{H_P} k(T) \, dk,
\end{equation}
defined as in (4.1) is $K_{\mathcal{J}'}$-invariant. Moreover, if $T$ lies in some maximal inductive algebra $\mathcal{A}$, then also $\mathcal{T} \in \mathcal{A}$. The trouble is that starting from a nonscalar operator $T$ does not guarantee that $\mathcal{T}$ will be nonscalar. We will show, however, that this is true for a suitable choice of $P$ in $\partial \mathcal{J}$; to this purpose we will describe more precisely the nonscalar $K_{\mathcal{J}}$-invariant elements of maximal inductive algebras.
4.3. The Hilbert spaces $\mathcal{H}^P$. Recall from (2.1) the definition of the subsets $\mathcal{X}(P_1, P_2) \subset \mathcal{X}$ for $P_1, P_2 \in \mathcal{X}$. Consider a complete subtree $\mathcal{I}$, and associate to every $P \in \partial \mathcal{I}$ a sequence of subspaces of $\mathcal{H}$ in the following way:

\[(4.6) \quad \mathcal{H}^P := \{f \in \mathcal{H} : f = f \cdot 1_{\Omega(P', P)}\},
\]
\[
\mathcal{H}_0^P := \mathcal{M}_0^P = C1_{\Omega(P', P)},
\]
\[
\mathcal{M}_n^P := \langle \{1_{\Omega(P', Q)} : Q \in \mathcal{X}(P', P) \text{ and } d(Q, P) = n\} \rangle \quad (n \geq 1),
\]
\[
\mathcal{H}_{n+1}^P = \mathcal{M}_{n+1}^P \ominus \mathcal{M}_n^P \quad (n \geq 1).
\]

This construction should be compared with (3.5); one can say that $\mathcal{M}_n^P$ is the space of locally constant functions, with support in $\Omega(P', P)$, that only depend on the first $n$ steps along some finite chain in $\mathcal{X}(P', P)$. Note that definition (4.6) does depend on the choice of the subtree.

As in the reasoning following (3.5), we see that
\[
\mathcal{H}^P = \bigoplus_{j=0}^{\infty} \mathcal{H}_j^P.
\]

From this decomposition we will get more information about the action of $K_\mathcal{I}$-invariant operators on the subspaces $\{\mathcal{H}^P : P \in \partial \mathcal{I}\}$.

**Lemma 4.2.** Take a finite complete subtree $\mathcal{I}$ and two vertices $P, Q \in \partial \mathcal{I}$. If $T \in B(\mathcal{H})$ is $K_\mathcal{I}$-invariant, then $T(\mathcal{H}^P \ominus \mathcal{H}_0^P) \subseteq (\mathcal{H}^Q)^\perp$ and $T(\mathcal{H}^P) \subseteq (\mathcal{H}_0^Q)^\perp$.

**Proof.** Consider $H_P$ as in (4.2). Then $\mathcal{H}^P \ominus \mathcal{H}_0^P$ is generated by
\[
\{v \in \mathcal{H}^P : \pi(h)v = -v \quad \text{for some } h \in H_P\}.
\]

At the same time, $\pi(h)w = w$ for all $w \in \mathcal{H}^Q$ and $h \in H_P$. Thus any $v$ as in (4.8) is orthogonal to $\mathcal{H}^Q$. The second inclusion follows by applying the first to $T^*$. \(\square\)

4.4. The inductive step.

**Theorem 4.3.** Let $A$ be a maximal inductive algebra, and $\mathcal{I} \subset \mathcal{X}$ a finite complete subtree with at least two edges, such that $A$ contains a nonscalar $K_\mathcal{I}$-invariant operator. Then, for some complete subtree $\mathcal{J} \subset \mathcal{I}$, there exists in $A$ a nonscalar $K_\mathcal{J}$-invariant operator. In other words, $\mathcal{I}$ is not minimal among the complete subtrees verifying Theorem 4.1.

The proof of Theorem 4.3 will take up the remainder of the paper, and will be divided into several steps. To begin with, we deduce from (4.7) and Lemma 4.2 that, for any nonscalar $K_\mathcal{I}$-invariant operator $T$, exactly one of the following holds:

(a) For every $P \in \partial \mathcal{J}$ there exists $\lambda_P \in \mathbb{C}$ such that $T |_{\mathcal{H}^P} = \lambda_P I_{\mathcal{H}^P}$, but $P \mapsto \lambda_P$ is not constant.
(b) Every subspace $H^{P'}$ is $T$-invariant, but the restriction of $T$ to some $H^{P'}$ is nonscalar.

(c) $T H^{P'} \not\subseteq H^{Q'}$ for some pair of different terminal vertices $P, Q$. In this case $(T 1_{\Omega(P', P)}, T 1_{\Omega(Q', Q)}) \neq 0$.

Case (b) is easily dealt with. Indeed, suppose $P \in \partial I$ is such that $T |_{H^P}$ is nonscalar. Let $H$ be the subgroup of $G$ that fixes all vertices in $\mathcal{X}(P', P)$. Then

\[ T^\sharp := \int_H \pi(h)T \pi(h^{-1}) \, dh \tag{4.9} \]

agrees with $T$ on $H^P$. Consequently, $T^\sharp$ is nonscalar. Moreover, since $T^\sharp$ is fixed by $H$ and by $H_P$, it is fixed by $K_{\{P, P\}}$. So $T^\sharp$ and the nontrivial complete subtree $I \sim B(P', 1) \cap \partial I$ (as in Figure 11) satisfy the thesis.

For case (a), suppose first that $I$ is not a sphere of radius one. Take $Q \in \partial I$ and its almost terminal neighbour $Q'$; put $J := I \sim B(Q', 1) \cap \partial I$. Then $J$ is a nontrivial complete subtree. Take $H_Q, H_{Q'}$ and $\tilde{T}$ as in (4.2) and (4.5). Since $T$ acts as a scalar on each subspace $H^P$ ($P \in \partial I$), we have $\tilde{T} H^{Q'} \subseteq H^{Q'}$ and

\[ \tilde{T}|_{H^{Q'}} = \frac{1}{q} \left( \sum_{R \in \partial I \cap B(Q', 1)} \lambda_R \right) I_{H^{Q'}}, \tag{4.10} \]

so that $\tilde{T}$ acts as a scalar on $H^R$ for every $R$ in the boundary of the smaller complete subtree $J$. As remarked above, $\tilde{T}$ may be scalar, but in that case

\[ \sum_{R \in \partial I \cap B(Q', 1)} \lambda_R = q\beta \quad \text{for every } \beta \in \{\lambda_P : P \in \partial I\}. \tag{4.11} \]

If $I$ is a sphere of radius one, with interior vertex $O$, consider $P \in \partial I$ and the subgroup $K_{O,P}$ that fixes $O$ and $P$. Construct $\tilde{T}$ as in (4.5). If $\tilde{T}$ is scalar then $\lambda_P = (1/q) \sum_{R \in \partial I, R \neq P} \lambda_R$. This must fail for some $P \in \partial I$, because $P \mapsto \lambda_P$ is not constant on $\partial I$. So only case (c) remains to be considered.

4.5. Proof of Theorem 4.3 in case (c). If $I$ satisfies the additional hypothesis that there exist at least three almost terminal vertices $P', Q', R'$, the proof still runs easily. Indeed, let $P, Q, R$ be boundary neighbours of $P', Q', R'$, respectively. Then the operator

\[ \hat{T} := \int_{H_{R'}} \pi(h)T \pi(h^{-1}) \, dh, \tag{4.12} \]

defined as in (4.2), but relatively to the subtree $I \sim (B(R', 1) \cap \partial I)$, is invariant with respect to the stabilizer of some proper complete subtree; moreover $\hat{T} = T$ on $H^P$ and $H^Q$, so that $\hat{T} 1_{\Omega(P', P)} \not\subseteq 1_{\Omega(Q', Q)}$. Hence $\hat{T}$ is nonscalar.
In order to achieve the proof of Theorem 4.3 we must finally consider the finite complete subtrees which fail to satisfy the additional hypothesis stated just before (4.12). It is clear that any such subtree belongs to one of the following classes:

(i) unit spheres,
(ii) unions of pairs of unit spheres, with centers at distance 1, or
(iii) unions of families of unit spheres, with centers on a given chain (other than an edge).

We first deal with case (ii). Let \( O \) be an interior vertex of the subtree. Call \( P_1, P_2, \ldots, P_{2q} \) the boundary vertices of \( \mathcal{T} \), with \( d(P_i, O) > d(P_i, O') \) if and only if \( i \leq q \); let \( f_i := 1_{\Omega(O, P_i)} \). We know from Lemma 4.2 that \( T \) acts on the subspace of \( \mathcal{H} \) generated by \( \{f_1, f_2, \ldots, f_{2q}\} \). With respect to this basis \( T \) is represented by a nonscalar matrix

\[
A := \begin{pmatrix}
B & C \\
D & E
\end{pmatrix}.
\]

(4.13)

If, for example, \( \langle Tf_i, f_j \rangle \neq 0 \) for some \( i, j \leq q \ (i \neq j) \), then, just as in the first remark in this subsection, we see that the operator

\[
\tilde{T} := \int_{H_{O'}} \pi(h)T\pi(h^{-1}) \, dh
\]

(4.14)

is still nonscalar and is invariant through the smaller subgroup that fixes the sphere of radius one with center in \( O' \). The same reasoning applies if \( Tf_h \not\perp f_k \) with \( h, k \geq q + 1 \ (h \neq k) \). So we restrict to the case in which the blocks \( B \) and \( D \) are both diagonal. We consider \( \tilde{T} \) as in (4.14) and compute its matrix \( \tilde{A} \) with respect to \( \{f_1, f_2, \ldots, f_{2q}\} \). Since \( H_{O'} \) acts on \( \{f_1, f_2, \ldots, f_{2q}\} \) as a permutation group, it turns out, with the notation of (4.13), that

\[
\tilde{A} = \begin{pmatrix}
\tilde{B} & \tilde{C} \\
\tilde{D} & \tilde{E}
\end{pmatrix},
\]

(4.15)

with

\[
\tilde{B} = q^{-1}(B_{11} + \cdots + B_{qq})I_q,
\]

\[
\tilde{E} = E,
\]

\[
\tilde{C}_{ij} = q^{-1} \sum_{h} C_{hj} \quad \text{for every } i, j,
\]

\[
\tilde{D}_{ij} = q^{-1} \sum_{h} D_{ih} \quad \text{for every } i, j.
\]

(4.16)

By replacing \( H_{O'} \) with \( H_O \) in (4.14) we obtain yet another element \( \tilde{T}' \) of \( \mathcal{A} \). It is easy to see that its matrix \( \tilde{A}' \) satisfies an analogue of (4.16) (where \( B \)
and $C$ interchange their roles, respectively, with $E$ and $D$). If $\bar{T}$ and $\bar{T}'$ are both scalar, we obtain from (4.16) the following conditions about $A$:

(i) $B = E = \lambda I_q$ for some $\lambda \in \mathbb{C}$.
(ii) Each row and column in $C$ and $D$ adds up to zero (hence $\det C = \det D = 0$).

We digress for a moment to prove the following simple result:

**Lemma 4.4.** Let $C$ be a $q \times q$ complex matrix such that every row and every column adds up to zero. For $\sigma, \tau \in S_q$, the group of permutations on $q$ objects, let $C^{\sigma, \tau}$ be the matrix such that $(C^{\sigma, \tau})_{ij} = C_{\sigma(i)\tau(j)}$. Suppose, finally, that $CC^{\sigma, \tau} = 0$ for every $\sigma, \tau$. Then $C = 0$.

**Proof.** By hypothesis

$$\sum_{i=1}^{q} C_{hi} C_{\sigma(i)\tau(j)} = 0 \quad \text{for every } h, j \in \{1, \ldots, q\}, \sigma, \tau \in S_q. \quad (4.17)$$

Since $\tau$ is bijective we can replace it by the identity. Setting

$$\rho(\sigma)((v_1, \ldots, v_q)) = (v_{\sigma^{-1}(1)}, \ldots, v_{\sigma^{-1}(q)}), \quad \sigma \in S_q,$$

defines an irreducible representation of $S_q$ on the space

$$\{v \in \mathbb{C}^q : v_1 + v_2 + \cdots + v_q = 0\}$$

(see [5, §1.3]). Therefore, if $C_{hi} \neq 0$ for some pair of indexes, a combination of vectors of the form $(C_{\sigma(1)j}, \ldots, C_{\sigma(q)j})$ gives $(\overline{C_{h1}}, \ldots, \overline{C_{hq}})$. This contradicts (4.17). \qed

Now, resuming the proof of Theorem 4.3, consider the set of automorphisms

$$G_S := \{ g \in G : g(\{f_1, f_2, \ldots, f_q\}) = \{f_{q+1}, f_2, \ldots, f_{2q}\}\}; \quad (4.19)$$

that is, with the notation of Definition 1 in Section 3, the coset $jK'$. When $g \in G_S$, $K_Tg$ is contained in $gK_T$, so that $T^g$ is $K_T$-invariant (since $T$ is). For $g \in G_S$ there exist $\sigma, \tau \in S_q$ such that

$$\pi(g)f_i = \begin{cases} f_{q+\sigma(i)} & \text{if } i \leq q, \\ f_{\tau(i-q)} & \text{if } i \geq q + 1. \end{cases} \quad (4.20)$$

From the hypotheses on $T$ we see that $T^g$, as an operator on the span of $\{f_1, \ldots, f_{2q}\}$, is represented by the block matrix

$$A^g := \begin{pmatrix} \lambda I_q & D^{\tau, \sigma} \\ C^{\sigma, \tau} & \lambda I_q \end{pmatrix}, \quad (4.21)$$

with $C^{\sigma, \tau}, D^{\tau, \sigma}$ defined as in Lemma 4.4. Hence

$$A^g A = \begin{pmatrix} \lambda^2 I_q + DD^{\tau, \sigma} & \lambda D^{\tau, \sigma} + \lambda C \\ \lambda D + \lambda C^{\sigma, \tau} & \lambda^2 I_q + CC^{\sigma, \tau} \end{pmatrix}. \quad (4.22)$$
Repeating the reasoning above for $T^g$ and $T^gT$ we find that $\tilde{T}^gT$ is nonscalar for some $g \in G_S$, unless the blocks along the diagonal of $A^gA$ are scalar. Then $CC^\sigma,\tau$ and $DD^\sigma,\tau$ are also scalar, and since $\det C = \det D = 0$ we get $CC^\sigma,\tau = DD^\sigma,\tau = 0$ for every $\sigma, \tau \in S_q$. By Lemma 4.4, $C = D = 0$, which is impossible because $T$ is nonscalar. This concludes the proof in case (ii).

Case (iii) is dealt with much like case (ii). Indeed, let $(x_1, x_2, \ldots, x_n)$ ($n \geq 3$) be the central chain of the subtree:

![Figure 12. A subtree of the kind considered in case (iii).](image)

and recall the notations set in Sections 4.2 and 4.3. If $P'$ or $Q'$ can be chosen in $\{x_2, \ldots, x_{n-1}\}$ then a boundary vertex $R$ with $P' \neq R' \neq Q'$ is easily found (for example, if $Q' = x_1$, it suffices to choose $R \in B(x_n, 1) \sim \{x_{n-1}\}$).

So we restrict to the case in which $\{P', Q'\} = \{x_1, x_n\}$ and $T(\mathcal{H}^R) \subseteq \mathcal{H}^R$ for $P \neq R \neq Q$. Now the subspace

$$\bigoplus_{R \neq P, Q} \mathcal{H}^R$$

is invariant under conjugation by $H_{x_1}$, $H_{x_n}$ (defined as in (4.2)) and also by the subgroup of inversions that switch $x_1$ with $x_n$. So $T$ can be represented by a diagonal block matrix, the two diagonal blocks corresponding to the orthogonal subspaces

$$\bigoplus_{R \neq P, Q} \mathcal{H}^R \quad \text{and} \quad \mathcal{H}^{P'} \oplus \mathcal{H}^{Q'}.$$  

Next, we consider the operators

$$\bar{T} := \int_{H_P} \pi(h)T\pi(h^{-1}) \, dh \quad \text{and} \quad \bar{T}' := \int_{H_Q} \pi(h)T\pi(h^{-1}) \, dh$$

and apply the same reasoning as in point (ii) to the block along the diagonal which corresponds to $\mathcal{H}^{P'} \oplus \mathcal{H}^{Q'}$.

We deal finally with case (i). Let $O$ be the unique interior vertex of $I$, and let $A$ be the $(q + 1) \times (q + 1)$ matrix of $T$ with respect to the basis $\{1_{\Omega(O, P)} : P \in \partial I\}$. Suppose that, for every $P \in \partial I$,

$$\bar{T} := \int_{H_{OP}} \pi(h)T\pi(h^{-1}) \, dh,$$

where $H_{OP} := \{h \in G : gO = O \text{ and } hP = P\}$ is scalar. Then, as in case (ii), the following conditions on $A$ are found:

- In every row and column, nondiagonal entries add up to zero.
Every diagonal entry is the average of the \( q \) remaining ones. Hence, all the diagonal entries are equal.

We can write \( A \) as \( \lambda I_{q+1} + A_0 \), where \( A_0 \) satisfies the last two conditions and every diagonal element is zero. Now let \( g \) be an automorphism that fixes \( O \), and hence acts on \( \partial I \) as a permutation \( \sigma \) on \( q + 1 \) elements. The matrix of \( T^g \) is \( \lambda I_{q+1} + A_0^{\sigma, \sigma} \). Apply construction (4.26) to \( TT^g \). If a scalar operator results for all \( P \in \partial I \), then \( TT^g \) satisfies the same conditions as \( T \).

Consideration of the action of \( TT^g \) on the vector \((1, 1, \ldots, 1)\) shows that the common value of the diagonal elements is \( \lambda^2 \). It follows that \( A_0 A_0^{\sigma, \sigma} = 0 \) for every permutation \( \sigma \). As in case (iii), Lemma 4.4 gives \( A_0 = 0 \), hence \( A \in CI_{q+1} \), a contradiction.

This completes the proof of Theorem 4.3 and hence of Theorem 4.1. \( \square \)

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References


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