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WAVE EQUATIONS FOR GRAPHS AND THE EDGE-BASED LAPLACIAN

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In this paper we develop a wave equation for graphs that has many of the properties of the classical Laplacian wave equation. This wave equation is based on a type of graph Laplacian we call the "edge-based" Laplacian. We give some applications of this wave equation to eigenvalue/geometry inequalities on graphs.

1. Introduction

The main goal of this paper is to develop a "wave equation" for graphs that is very similar to the wave equation $u_{tt} = \Delta u$ in analysis. Whenever this type of wave equation is involved in a result in analysis, our graph theoretic wave equation seems likely to provide the tool to link the result in analysis to an analogous result in graph theory.

Traditional graph theory defines a Laplacian, Δ, as an operator on functions on the vertices. This gives rise to a wave equation $u_{tt} = -\Delta u$ (since graph theory Laplacians are positive semidefinite). However, this wave equation fails to have a "finite speed of wave propagation". In other words, if $u = u(x,t)$ is a solution, we may have $u(x,0) = 0$ for all vertices x within a distance $d > 0$ to a fixed vertex, x_0, without having $u(x_0, \epsilon)$ vanishing for any $\epsilon > 0$. As such, this graph theoretic wave equation cannot link most results in analysis involving the wave equation to a graph theoretic analogue.

In this paper we study what appears to be a new type of wave equation on graphs. This wave equation (1) involves a reasonable analogue of $u_{tt} = \Delta u$ in analysis, (2) has "finite speed of wave propagation" and many other basic properties shared by its analysis counterpart, and (3) seems to be a good vehicle for translating results in analysis to those in graph theory, and vice versa. This wave equation cannot be expressed in the language of traditional graph theory; it requires some of the notions of "calculus on graphs" in [**FT99**]. It does, however, have a simple physical interpretation — namely, the edges are taut strings, fused together at the vertices. And in fact, the type of Laplacian we use has appeared in the physics literature as the "limiting case" of a "quantum wire" (see [**Hur00**, **RS01**, **KZ01**] for example); but our type of development of the wave equation and its applications to graph theory seem to have escaped the interest of physicists.

A second goal of this paper is to point out that whereas in analysis there is really *one* fundamental Rayleigh quotient, heat equation, and wave equation, in graph theory there are always *two*. These two fundamental types for an equation or concept result from the fact that graph theory involves two different volume measures. So while in analysis there is usually one fundamental, top (space) dimension volume, in graph theory one has a "vertex-based" measure, $\mathcal{V}$ (a type of vertex counting measure), and an "edge-based" measure, $\mathcal{E}$ (essentially Lebesgue measure on edges viewed as real intervals) (see below and Section 2 for details); both $\mathcal{V}, \mathcal{E}$ seem to play a volume measure type of role. So we get a "vertex-based" Rayleigh quotient, heat equation, wave equation, etc. and their "edge-based" counterpart. (We can also form "mixed" equations and concepts from these two pure types, as well as vary coefficients, add lower order terms, etc.) However, sometimes it turns out that one of the two types of equation or concept is less interesting. This definitely seems to be the case in the wave equation, where the vertex-based equation does not have finite propagation speed.

A third goal of this paper is to give some examples of how to translate results in analysis to graphs and vice versa using the edge-based Laplacian (including the wave equation). As an example, we give a simple proof of a relation between distances of sets, their sizes, and the first nonzero edge-based eigenvalue; our result can be better (or worse) than that of Chung–Faber–Manteuffel; our proof also works in analysis, and rederives the results in [**FT**] with a simpler proof. As another example, we show that the optimal generalized Alon–Milman type bound is essentially achieved by a generalized Chung–Faber–Manteuffel bound derived by Chung–Grigor'yan–Yau. We briefly describe how to convert graph theoretic results into analysis results. The results in analysis turn out to be independent of any discussion of edge-based Laplacians, and hence appear as a short, separate article [**FT**] (especially for analysts who wish to learn as little graph theory as possible).

In this paper we at times restrict ourselves to finite graphs; at other times we insist that the graphs be locally finite, i.e., that each vertex meets only finitely many edges; finally, some discussion is valid for arbitrary graphs. We will indicate at the beginning of each section and/or subsection when assumptions are made on the graphs therein.

The rest of this introduction, aside from closing remarks, is devoted to giving an informal overview of the simplest form of our wave equation. If this overview seems cryptic, the reader may wish to consult [**FT99**] or Section 2 of this paper.

Let $G = (V, E)$ be a graph. Let $\mathcal{G}$ be its *geometric realization*, i.e., the metric space consisting of V with a real interval of length 1 joining u and v "glued in" for edge $\{u, v\}$. Let $\mathcal{V}$ be the vertex counting measure, and $\mathcal{E}$ be Lebesgue measure on the edges. Then the (positive semidefinite) Laplacian, as in [**FT99**], takes a function, f, and returns an "integrating factor" (essentially a measure, see Section 2),

$$\Delta f = (\Delta_V f)\, d\mathcal{V} + (\Delta_E f)\, d\mathcal{E}$$

where Δ_E is minus the usual real Laplacian (i.e., second derivative), and Δ_V is essentially a sum of normal derivatives along edges incident with each vertex. It therefore makes no sense to write a wave equation[1]

$$u_{tt} = -\Delta u,$$

for the left-hand side should be a function, and the right-hand side an integrating factor.

The vertex-based wave equation is

$$u_{tt}\, d\mathcal{V} = -\Delta u.$$

This means that $\Delta_E u = 0$, and so u is "edgewise linear" (i.e., a linear function when restricted to any edge). For such a u, $\Delta_V u$ coincides with the traditional graph theoretic Laplacian, and we recover the wave equation based on traditional graph theory.

The edge-based wave equation is the equation

$$u_{tt}\, d\mathcal{E} = -\Delta u,$$

or $\Delta_V u = 0$ and $u_{tt} = -\Delta_E u$. This equation has wave propagation speed 1, and has many other properties befitting a wave equation.

When using an edge-based concept, one may speak of Laplacian eigenvalues. In this case one is referring to the set Λ_E of λ with $\Delta_E f = \lambda f$ and $\Delta_V f = 0$. However, traditional graph theory deals with Λ_V, defined analogously. Fortunately, it is easy to relate the two notions of eigenvalues, assuming we "normalize" the Laplacian Δ_V (see Section 3). Namely, assuming Δ_V is normalized and our graph has all edge lengths one, we have

$$\lambda \in \widetilde{\Lambda_E} \iff 1 - \cos\sqrt{\lambda} \in \Lambda_V$$

where $\widetilde{\Lambda_E}$ is Λ_E with some "less interesting" eigenvalues (certain squares of multiples of π) discarded. Λ_V is a finite set of values between 0 and 2, and Λ_E is an infinite set of nonnegative values (whose square roots are periodic, and whose values satisfy a one-dimensional Weyl's asymptotic law).

In this paper we will mildly generalize this setup, allowing for edges of variable "length" and "weight," and vertices of various "weight."

The rest of this paper is organized as follows: in Section 2 we review some notions from the calculus on graphs of [**FT99**]. In Section 3 we discuss the edge-based eigenfunction theory; it closely resembles standard eigenfunction theory. In Section 4 we discuss the wave equation and its basic properties. In Section 5 we give some applications of the edge-based Laplacian and the wave equation.

[1]Recall that the minus sign appears in the wave equation since the Laplacian is positive semidefinite.

2. Calculus on graphs

2.1. The setup. We use a similar setting as in [**Fri93**], and we recall this setting here. Let $G = (V, E)$ be a graph (undirected), such that with each edge, $e \in E$, we have associated a length, $\ell_e > 0$. We form the *geometric realization*, $\mathcal{G}$, of G, which is the metric space consisting of V and a closed interval of length ℓ_e from u to v for each edge $e = \{u, v\}$. When there is no confusion, we identify a $v \in V$ with its corresponding point in $\mathcal{G}$ and identify an $e \in E$ with its corresponding closed interval in $\mathcal{G}$. By an *edge interior* we mean the interior of an edge in $\mathcal{G}$.

Definition 2.1. The *boundary*, $\partial\mathcal{G}$, of a graph, $\mathcal{G}$, is simply a specified subset of its vertices. By the *interior of* $\mathcal{G}$, denoted $\mathring{\mathcal{G}}$, we mean $\mathcal{G} \setminus \partial\mathcal{G}$; similarly the *interior vertices*, denoted $\mathring{V}$, we mean $V \setminus \partial\mathcal{G}$. We say that $\partial\mathcal{G}$ is separated if each boundary vertex is incident upon exactly one edge.

Boundary separation is a property whose analogue for manifolds is always true. In most practical situations one can assume the boundary is separated.

Convention 2.2. Unless specified, in this article we assume all graphs have a separated boundary.

In this article we will give “boundary condition” for functions to satisfy at the boundary. Neumann or mixed boundary conditions (see the next section) behave bizarrely unless the boundary is separated.[2]

Convention 2.3. By a *traditional graph* we mean an undirected graph $G = (V, E)$. In this article we assume our graphs are always given with:

(1) lengths associated to each edge,
(2) a specified boundary (i.e., a specified subset of vertices).

Whenever an edge length is not specified, it is taken to be one. Whenever a boundary is not specified, it is taken to be empty. We refer to the geometric realization, $\mathcal{G}$, of the graph as the graph, when no confusion may arise.

An edge $e = \{u, v\}$ of length, ℓ, is a real interval of length ℓ, and as such has two *standard coordinates*, one that sets u to 0 and v to ℓ, and the other vice versa. Whenever we speak of a property such as differentiability, we always mean with respect to these standard coordinates.

Definition 2.4. By $C^k(\mathcal{G})$, the set of *k-times continuously differentiable functions on* $\mathcal{G}$, we mean the set of continuous functions on $\mathcal{G}$ whose restriction to each edge interior is k-times uniformly continuously differentiable

[2] It is not hard to see that the Neumann condition for a function, f, on a boundary vertex, v, is equivalent to $\Delta_V f = 0$ at v (see this section and the next). This is only equivalent to the normal derivative at f vanishing along all boundary edges if v is incident upon only one edge.

(as a function on that real interval). The same definition applies with $\mathcal{G}$ replaced by $C^k(\mathcal{G} \setminus V)$.

We cannot differentiate functions on $\mathcal{G}$ without orienting the edges; however, we can always take the gradient of a differentiable function as long as we know what is meant by a vector field. Recall that a vector field on an interval is a section of its tangent bundle or, what is the same, a function on the interval with an orientation of the interval, where we identify f plus an orientation with $-f$ with the opposite orientation.

Definition 2.5. By $C^k(\mathrm{T}\mathcal{G})$, the set of k-times continuously differentiable vector fields on $\mathcal{G}$, we mean those data consisting of a k-times uniformly continuously differentiable vector field on each open interval corresponding to each edge interior.

Notice that a vector field is not defined at a vertex, only on edge interiors.

Definition 2.6. For $f \in C^k(\mathcal{G})$ we may form, by differentiation, its *gradient*, $\nabla f \in C^{k-1}(\mathrm{T}\mathcal{G})$. For $X \in C^k(\mathrm{T}\mathcal{G})$ we can form, by differentiation, its *calculus divergence*, $\nabla_{\mathrm{calc}} \cdot X \in C^{k-1}(\mathrm{T}\mathcal{G} \setminus V)$.

Definition 2.7. A subset $\Omega \subset \mathcal{G}$ is of *finite type* if it lies in the union of finitely many vertices and edges. A function on $\mathcal{G}$ is of *finite type* if its support (i.e., the closure of the set where it does not vanish) is of finite type. We set $C^k_{\mathrm{fn}}(\mathcal{G})$ to be those elements of $C^k(\mathcal{G})$ of finite type; we similarly define $C^k_{\mathrm{fn}}(\mathcal{G} \setminus V)$ and $C^k_{\mathrm{fn}}(\mathrm{T}\mathcal{G})$.

Definition 2.8. An $f \in C^k_{\mathrm{fn}}(\mathcal{G})$ is said to satisfy the *Dirichlet condition* if f vanishes on $\partial\mathcal{G}$. We let $C^k_{\mathrm{Dir}}(\mathcal{G})$ denote the set of such functions.

Definition 2.9. $\mathrm{Lip}(\mathcal{G})$ denotes the class of *Lipschitz continuous* functions on $\mathcal{G}$, i.e., those $f \in C^0(\mathcal{G})$ whose restriction to each edge interior is uniformly Lipschitz continuous. We similarly define $\mathrm{Lip}_{\mathrm{fn}}(\mathcal{G})$ and $\mathrm{Lip}_{\mathrm{Dir}}(\mathcal{G})$.

2.2. Two volume measures. In analysis concepts such as Laplacians, Rayleigh quotients, and isoperimetric constants are defined using one volume measure; in calculus on graphs we use two "volume" measures.

Definition 2.10. A *vertex measure*, $\mathcal{V}$, is a measure supported on V with $\mathcal{V}(v) > 0$ for all $v \in V$. An *edge measure*, $\mathcal{E}$, is a measure with $\mathcal{E}(v) = 0$ for all $v \in V$ and whose restriction to any edge interior, $e \in E$, is Lebesgue measure (viewing the interior as an open interval) times a constant $a_e > 0$.

Traditional graph theory usually works with the traditional vertex and edge measures, $\mathcal{V}_{\mathrm{T}}$ and $\mathcal{E}_{\mathrm{T}}$, given by $\mathcal{V}_{\mathrm{T}}(v) = 1$ for all $v \in V$ and $a_e = 1$ for all $e \in E$, i.e., $\mathcal{E}_{\mathrm{T}}$ is just Lebesgue measure at each edge.

Convention 2.11. Henceforth we assume that any graph has an associated vertex measure, $\mathcal{V}$, and an edge measure, $\mathcal{E}$. When $\mathcal{V}$ is not specified we take it to be $\mathcal{V}_{\mathrm{T}}$; similarly, when unspecified we take $\mathcal{E}$ to be $\mathcal{E}_{\mathrm{T}}$.

In this article we write $\int f\,d\mathcal{E}$ and $\int f\,d\mathcal{V}$ for $\int_{\mathcal{G}} f\,d\mathcal{E}$ and $\int_{\mathcal{G}} f\,d\mathcal{V}$.

2.3. Integrating factors. In this paper a somewhat formal notion will become extremely important.

Definition 2.12. By an *integrating factor* on $\mathcal{G}$ we mean a formal expression of the form $\mu = \alpha\,d\mathcal{V} + \beta\,d\mathcal{E}$ where α is a function defined (at least) on the vertices of $\mathcal{G}$, and $\beta \in C^0(\mathcal{G} \setminus V)$.

The continuity assumption on β is not essential, but it makes things nicer for the following reason: an integrating factor as above determines a linear functional $\mathcal{L}_\mu$, on $C^0_{\mathrm{Dir}}(\mathcal{G})$ via

$$\mathcal{L}_\mu = \int f\mu = \int f\alpha\,d\mathcal{V} + \int f\beta\,d\mathcal{E}.$$

We say that two integrating factors $\mu_i = \alpha_i(x)\,d\mathcal{V} + \beta_i(x)\,d\mathcal{E}$ for $i = 1, 2$ are equal if the $\mathcal{L}_{\mu_i}$ are equal; clearly this amounts to $\alpha_1 = \alpha_2$ at the interior vertices and $\beta_1 = \beta_2$ everywhere on $\mathcal{G}\setminus V$ (since the β_i are continuous there).

We will sometimes wish to insist that $\alpha_1 = \alpha_2$ on boundary vertices as well; this corresponds to viewing integrating factors as linear functionals on $C^0(\mathcal{G})$. In this case we will speak of *boundary inclusive equality.*

In the calculus on graphs we have two measures, and thus a need for integrating factors, i.e., the need to mark functions with a $d\mathcal{V}$ or $d\mathcal{E}$ to remind us how to integrate the function against other functions. For example, we shall soon see that the divergence of a vector field or the Laplacian of a function is an integrating factor. Consequently, any wave or heat equation is most correctly regarded as an equation between integrating factors. (In traditional graph theory, all integrating factors have a vanishing $d\mathcal{E}$ component, i.e., $\beta = 0$ in the above, and they may be considered as functions on the vertices, i.e., they may be identified with α's values on the vertices.)

2.4. Regular graphs. In this article, r-regularity has a slightly more general meaning than in traditional graph theory where all vertices and edges have weight 1 (in other words $\mathcal{E} = \mathcal{E}_{\mathrm{T}}$ and $\mathcal{V} = \mathcal{V}_{\mathrm{T}}$).

Here we mean the following:

Definition 2.13. We say that a graph $\mathcal{G}$ is *r-regular* if for any $v \in \mathring{V}$ we have $\rho(v) = r$, where

$$\rho(v) = \mathcal{V}(v)^{-1} \sum_{e \ni v} \mathcal{E}(e).$$

Clearly graphs that are r-regular in the traditional sense are regular with our definition. The quantity $\rho(v)$ arises quite naturally when we consider *edgewise linear* functions, i.e., continuous functions whose restriction to each edge is a linear function. For these functions we clearly have

$$\int_e f\,d\mathcal{E} = \tfrac{1}{2}\big(f(u) + f(v)\big)\mathcal{E}(e)$$

for each edge $e = \{u, v\}$. Hence

$$\int f\, d\mathcal{E} = \int \tfrac{1}{2} f\rho\, d\mathcal{V}. \tag{2.1}$$

2.5. The divergence. The divergence of a vector field and the Laplacian of a function can be defined in terms of concepts that are already fixed, namely a graph (encompassing measures $\mathcal{E}$ and $\mathcal{V}$) and the gradient. Interestingly enough, the divergence turns out to be different from the "calculus divergence" described earlier.

Before defining the divergence we record a "divergence theorem" for the calculus divergence.

Let $X \in C^1(\mathrm{T}\mathcal{G})$. For any edge $e = \{u, v\}$ let $X|_e$ denote X restricted to the interior of e and then extended to u and v by continuity. We clearly have

$$\int_e \nabla_{\mathrm{calc}} \cdot X\, d\mathcal{E} = a_e\big(\mathbf{n}_{e,u} \cdot X|_e(u) + \mathbf{n}_{e,v} \cdot X|_e(v)\big),$$

where $\mathbf{n}_{e,u}, \mathbf{n}_{e,v}$ denote outward pointing unit (normal) vectors. Hence we obtain:

Proposition 2.14. *For all $X \in C^1_{\mathrm{fn}}(\mathcal{G})$ we have*

$$\int \nabla_{\mathrm{calc}} \cdot X\, d\mathcal{E} = \int \widetilde{\mathbf{n}} \cdot X\, d\mathcal{V}, \tag{2.2}$$

where

$$(\widetilde{\mathbf{n}} \cdot X)(v) = \mathcal{V}(v)^{-1} \sum_{e \ni v} a_e \mathbf{n}_{e,v} \cdot X|_e(v).$$

Let $C^k_{\mathrm{Dir}}(\mathcal{G})$ denote those functions in $C^k_{\mathrm{fn}}(\mathcal{G})$ that vanish on the boundary of $\mathcal{G}$.

Definition 2.15. For a vector field, X, its *divergence functional* is the linear functional $\mathcal{L}_X : C^\infty_{\mathrm{Dir}}(\mathcal{G}) \to \mathbb{R}$ given by

$$\mathcal{L}_X(g) = -\int X \cdot \nabla g\, d\mathcal{E}.$$

Proposition 2.16. *For any $X \in C^1(\mathrm{T}\mathcal{G})$ and $g \in C^\infty_{\mathrm{Dir}}(\mathcal{G})$ we have*

$$\mathcal{L}_X(g) = \int (\nabla_{\mathrm{calc}} \cdot X) g\, d\mathcal{E} - \int (\widetilde{\mathbf{n}} \cdot X) g\, d\mathcal{V},$$

i.e., the divergence functional of X is represented by $(\nabla_{\mathrm{calc}} \cdot X)\, d\mathcal{E} - (\widetilde{\mathbf{n}} \cdot X)\, d\mathcal{V}$ (viewed as a linear functional via integration).

Proof. We substitute Xg for X in Equation (2.2), and note that

$$\nabla_{\mathrm{calc}} \cdot (Xg) = g\nabla_{\mathrm{calc}} \cdot X + X \cdot \nabla g.$$

Definition 2.17. For $X \in C^1(\mathrm{T}\mathcal{G})$ we define its *divergence*, $\nabla \cdot X$, to be the integrating factor

$$(\nabla_{\mathrm{calc}} \cdot X)\, d\mathcal{E} - (\widetilde{\mathbf{n}} \cdot X)\, d\mathcal{V}.$$

If X is edgewise constant, so that $\nabla_{\mathrm{calc}} \cdot X = 0$, we will also refer to

$$-\widetilde{\mathbf{n}} \cdot X$$

(a function defined only on vertices) as its divergence, and write it $\nabla \cdot X$.

We conclude:

Proposition 2.18. *For any $g \in C^1_{\mathrm{fn}}(\mathcal{G})$ and $X \in C^1_{\mathrm{fn}}(\mathrm{T}\mathcal{G})$ we have*

$$\int_{\mathcal{G}} (\nabla \cdot X) g + \int X \cdot \nabla g \; d\mathcal{E} = 0.$$

To make this look more like analysis we can write this as:

$$\int_{\mathcal{G} \setminus \partial\mathcal{G}} (\nabla \cdot X) g + \int X \cdot \nabla g \, d\mathcal{E} = \int_{\partial\mathcal{G}} (\widetilde{\mathbf{n}} \cdot X) g \, d\mathcal{V}.$$

2.6. The Laplacian. In graph theory we usually define positive semidefinite Laplacians. So we define

$$\Delta f = -\nabla \cdot (\nabla f).$$

As integrating factors we have

$$\Delta f = (\Delta_E f)\, d\mathcal{E} + (\Delta_V f)\, d\mathcal{V},$$

with $\Delta_E f = -\nabla_{\mathrm{calc}} \cdot \nabla f$ and $\Delta_V f = \widetilde{\mathbf{n}} \cdot \nabla f$. It is easy to see that:

Proposition 2.19. *For all $f \in C^2_{\mathrm{fn}}(\mathcal{G})$ and $g \in C^1_{\mathrm{fn}}(\mathcal{G})$ we have*

$$\int (\Delta f) g = \int \nabla f \cdot \nabla g \, d\mathcal{E}. \tag{2.3}$$

If also $g \in C^2_{\mathrm{fn}}(\mathcal{G})$ we have

$$\int (\Delta f) g = \int (\Delta g) f. \tag{2.4}$$

The link with the traditional graph theoretic Laplacian is as follows:

Proposition 2.20. *For $f \in C^2_{\mathrm{fn}}(\mathcal{G})$ edgewise linear we have*

$$\nabla_{\mathrm{calc}} \cdot \nabla f = 0, \quad \textit{hence} \quad \Delta f = \widetilde{\mathbf{n}} \cdot \nabla f \, d\mathcal{V}.$$

Viewing Δf as a function on vertices we therefore have:

$$(\Delta f)(v) = \mathcal{V}(v)^{-1} \sum_{e \sim \{u,v\}} a_e \frac{f(v) - f(u)}{\ell_e}. \tag{2.5}$$

When restricting to edgewise linear functions, it is common (in graph theory) to write Δ as $D - A$, where D is the diagonal matrix or operator (classically the "degree" matrix) whose v, v entry is

$$L(v) = \mathcal{V}(v)^{-1} \sum_{e\sim\{u,v\}} a_e/\ell_e,$$

where we omit e's that are self-loops from the summation, and where A is the "adjacency" matrix or operator given by

$$(Af)(v) = \mathcal{V}(v)^{-1} \sum_{e\sim\{u,v\}} (a_e/\ell_e) f(u),$$

again omitting self-loops, e.

2.7. Variable integrals. We shall wish to consider the derivative at $t = t_0$ of the function

$$\mathcal{I}(t) = \int_{S(t)} f(x,t)\, d\mathcal{E}(x),$$

where t is a real parameter, $S(t)$ is a decreasing family of open subsets of $\mathcal{G}$, and $f(x,t)$ is continuous in x (taking values in $\mathcal{G}$) and differentiable in t (taking values near t_0).

For this paper we only need consider $S(t)$ given by the set of points within a distance $\tau - ct$ from a fixed set $A \subset \mathcal{G}$, with τ fixed and $0 \le t_0 < \tau$. We will assume $S(t)$ is of finite type for t near t_0, and that $\partial S(t_0)$ is finite and contains no vertices.

With the above assumptions, the formula for $\mathcal{I}'(t_0)$ is very easy. We will discuss more general variants of these formulas later. These more general variants are not needed in this paper, but are interesting to consider and compare with the co-area formula and its problems at the vertices as described in [**FT99**].

Calculus says that if a, b, c are constants with $c > 0$, and $f = f(x,t) : \mathbb{R}^2 \to \mathbb{R}$ is continuous in x and differentiable in t, then

$$\frac{d}{dt}\int_{a+ct}^{b-ct} f(x,t)\, dx = -c\big(f(a+ct,t) + f(b-ct,t)\big) + \int_{a+ct}^{b-ct} f_t(x,t)\, dx,$$

where $f_t = \partial f/\partial t$. If we replace $a + ct$ by a in the above integral, the $f(a + ct, t)$ drops out, and similarly for $b - ct$ replaced by b.

Summing the above calculus equality over all edges yields the following easy proposition:

Proposition 2.21. *Let $S(t), f(x,t), t_0, \mathcal{I}(t)$ be as above. Then*

$$\mathcal{I}'(t_0) = -\sum_{\substack{(x,e)\ \text{s.t. } e\ni x,\\ x\in\partial S(t_0)}} f(x,t_0) c a_e + \int_{S(t_0)} f_t(x,t_0)\, d\mathcal{E}(x). \tag{2.6}$$

In other words, the above sum is over all boundary points, x, of $S(t_0)$, and involves the edge, e, on which x lies.

Finally, we remark that if $S(t)$ does not decrease "linearly" with speed c everywhere, then we simply replace c by the speed at with $\partial S(t)$ moves at x in the summation of any of the above formulas.

We finish this subsection by describing what happens when $\partial S(t_0)$ contains vertices. If so, then the left and right derivatives of $\mathcal{I}(t)$ at t_0 will exist but won't usually be equal. Equation 2.6 will essentially hold, but different (x, e) pairs appear in the sum. So consider pairs (x, e) with $x \in e \cup \partial S(t_0)$ (remember we identify an edge with its *closed* interval in $\mathcal{G}$, so x may be a vertex); we call a pair *future active* if e's interior intersected with $S(t_0)$ contains an open interval ending at x. In other words, the picture of $S(t)$ near x and along e is changing for $t > t_0$ near t_0 (since $S(t)$ is decreasing in t). We say that a pair (x, e) is *past active* if either x is not a vertex, or x is a vertex and (x, e) is not future active, see Figure 1. (Geometrically, since $S(t)$ is decreasing in t, we are saying that for t near t_0 and at a boundary point, x, the picture of $S(t)$ always changes when x is not a vertex, and when x is a vertex it changes along some edges in the future ($t > t_0$) and other edges in the past ($t < t_0$).)

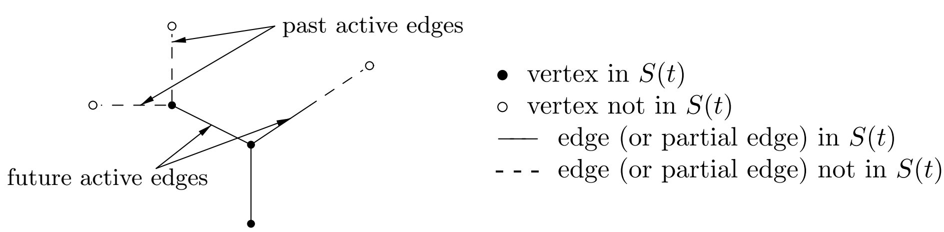

Figure 1.

Summing the calculus formula shows that the right derivative of $\mathcal{I}$ at t_0 exists and equals

$$\mathcal{I}'(t+0) = -\sum_{(x,e) \text{ future active}} f(x, t_0) c a_e + \int_{S(t_0)} f_t(x, t_0)\, d\mathcal{E}(x). \tag{2.7}$$

Similarly the left derivative is the same, with future active replaced by past active pairs (x, e).

Notice that the notion of "future active" essentially arose in the definition of the area of the boundary of a subset, in [**FT99**], Section 2, in connection with the co-area formula.

3. The edge-based eigenvalues and eigenfunctions

In this section we discuss some facts about the "edge-based" Laplacian and its eigenpairs, i.e., pairs (f, λ) with $\Delta_E f = \lambda f$ and $\Delta_V f = 0$. Such pairs are crucial to understanding the solutions to the wave equation and invariants associated with it. We concern ourselves with the basic cases at first, later illustrating fancier boundary conditions and mixed edge and vertex Laplacians.

It is worth mentioning that the equations $\Delta_E f = \lambda f$ and $\Delta_V f = 0$ describe the modes associated with a physical object with a metal string for each edge, with strings being fused together at the vertices. For example, if we "pluck" such an object, it would produce tones with the frequencies of $\sqrt{\lambda}$ with λ ranging over the edge-based eigenvalues (this is seen from considering the wave equation of the next section).

3.1. Basic existence theory.

Definition 3.1. We say that (f, λ) is an *eigenpair for the "edge-based" Laplacian* if $f \in C^\infty(\mathcal{G})$ and satisfies $\Delta_E f = \lambda f$ and $\Delta_V f = 0$. We say that f satisfies the *Dirichlet condition* if f vanishes at all boundary points; the similarly for the *Neumann condition*, where f's normal derivatives along its edges evaluated at any boundary point vanish.

Notice that the condition $\Delta_E f = \lambda f$ implies that f's restriction to each edge is given by $A\cos(\omega x + B)$, where A, B are constants, $\omega = \lambda^{1/2}$, and x represents one of the two standard coordinates on the edge.

The existence of a complete set of eigenpairs for the Laplacian is well understood in analysis for compact domains, and the same techniques carry over to our setting, for finite graphs, with almost no modifications. We only summarize the theory, and refer the reader to [**GT83**] or [**Fri69**].

Proposition 3.2. *Let $\mathcal{G}$ be a finite graph. There exists eigenpairs (f_i, λ_i) for the edge based Laplacian, such that:*

(1) $0 \le \lambda_1 \le \lambda_2 \le \cdots$,
(2) *the f_i satisfy the Dirichlet condition,*
(3) *the f_i form a complete orthonormal basis for $L^2_{\mathrm{Dir}}(\mathcal{G}, \mathcal{E})$, and*
(4) $\lambda_i \to \infty$.

The same statement holds with Dirichlet replaced by Neumann and $L^2_{\mathrm{Dir}}(\mathcal{G}, \mathcal{E})$ replaced by $L^2(\mathcal{G}, \mathcal{E})$.

Proof. Consider the Rayleigh quotient

$$\mathcal{R}(f) = \frac{\int |\nabla f|^2 \, d\mathcal{E}}{\int |f|^2 \, d\mathcal{E}},$$

which is certainly defined for $f \in C^1(\mathcal{G})$. Let $u_1, u_2, \ldots$ be a minimizing sequence for $\mathcal{R}$ in $C^1_{\mathrm{Dir}}(\mathcal{G})$, i.e., $u_i \in C^1_{\mathrm{Dir}}(\mathcal{G})$ with

$$\mathcal{R}(u_i) \to \inf_{f \in C^1_{\mathrm{Dir}}(\mathcal{G})} \mathcal{R}(f).$$

We may assume $\int |u_i|^2 \, d\mathcal{E} = 1$, and thus that $\int |\nabla u_i|^2 \, d\mathcal{E}$ are bounded. Let $H^1_{\mathrm{Dir}}(\mathcal{G})$ be the closure of $C^1_{\mathrm{Dir}}(\mathcal{G})$ under the norm

$$\|f\|^2_{H^1} = \int [|\nabla f|^2 + |f|^2] \, d\mathcal{E};$$

$H^1_{\mathrm{Dir}}(\mathcal{G})$ can also be described with Fourier transforms, or as the set of L^2 functions with a weak derivative in L^2; it is well-known to be a separable Hilbert space, therefore having a weakly compact unit ball. Hence by passing to a subsequence we may assume that the u_i converge weakly in H^1 to a $u \in H^1_{\mathrm{Dir}}(\mathcal{G})$.

The u_i are uniformly Hölder continuous of exponent $1/2$. To see this let $x, y \in \mathcal{G}$ be of distance ρ, and fix a path γ of length ρ from x to y. We have

$$\big|u_i(x) - u_i(y)\big| \le \int_\gamma |\nabla u_i| \, d\mathcal{E} \le \left(\int_\gamma d\mathcal{E}\right)^{\frac{1}{2}} \left(\int_\gamma |\nabla u_i|^2 \, d\mathcal{E}\right)^{\frac{1}{2}} \le C\rho^{1/2} \|u_i\|_{H^1},$$

where C is the maximum over edges e of $a_e^{-1/2}$. Hence our claim holds, and by Ascoli's lemma we can pass to a further subsequence and assume that the u_i converge uniformly to a u that is Hölder continuous of exponent $1/2$; since the u_i vanish on the boundary, so does u.

We have (by the uniform convergence and the weak H^1 convergence)

$$\mathcal{R}(u) \le \liminf \mathcal{R}(u_i),$$

and so equality must hold and u minimizes $\mathcal{R}$ over all of $H^1_{\mathrm{Dir}}(\mathcal{G})$.

Now we claim that u is our desired eigenfunction, and $\lambda = \mathcal{R}(u)$ its eigenvalue. This is seen by setting $\lambda = \mathcal{R}(u)$ and considering $\mathcal{R}(u + \epsilon w)$ for various $w \in H^1_{\mathrm{Dir}}(\mathcal{G})$ and taking $\epsilon \to 0$. We conclude that

$$\int \nabla u \nabla w \, d\mathcal{E} = \lambda \int uw \, d\mathcal{E} \tag{3.1}$$

for all $w \in H^1_{\mathrm{Dir}}(\mathcal{G})$. Now standard estimates for elliptic equations (e.g., Lemma 15.4 of [**Fri69**]) show that in fact u is C^∞ at all edge interiors, and satisfies

$$\Delta_E u = \lambda u. \tag{3.2}$$

Hence u's restriction to any edge is given as $A\cos(\omega x + B)$ for $\omega = \lambda^{1/2}$, and A, B are constants depending on e (and which of the two standard coordinates we place on e). Since u is Hölder continuous on $\mathcal{G}$, it is certainly continuous everywhere, including all its vertices. Hence $u \in C^\infty_{\mathrm{Dir}}(\mathcal{G})$. Finally, given a vertex, v, let us show that $(\Delta_V u)(v) = 0$. From Proposition 2.19 we know that

$$\int \nabla u \cdot \nabla w \, d\mathcal{E} = \int (\Delta_E u) w \, d\mathcal{E} + \int (\Delta_V u) w \, d\mathcal{V} = \lambda \int uw \, d\mathcal{E} + \int (\Delta_V u) w \, d\mathcal{V}.$$

From (3.1) and (3.2) we obtain

$$\int (\Delta_V u) w \, d\mathcal{V} = 0. \tag{3.3}$$

We choose $w \in C^1_{\mathrm{fn}}(\mathcal{G})$ in Equation (3.1), such that $w(v) = 1$ and $w(v') = 0$ on other vertices and conclude

$$(\Delta_V u)(v) = 0.$$

Letting $h \to 0$ we conclude that u is an "edge-based" Laplacian eigenfunction satisfying the Dirichlet condition, with eigenvalue λ.

Set $f_1 = u$ and $\lambda_1 = \lambda$. Now repeat the same argument, except minimizing $\mathcal{R}$ over those functions that are orthogonal to f_1. With the same argument, we find an eigenpair (f_2, λ_2) with $\lambda_1 \le \lambda_2$ and f_1 orthogonal to f_2. Now repeat again, minimizing over functions orthogonal to f_1 and f_2. In this way we get a sequence of orthogonal eigenpairs (f_i, λ_i) with λ_i nondecreasing in i.

Next we show that $\lambda_i \to \infty$. Otherwise the H^1 norms of the f_i are uniformly bounded, and so the f_i are uniformly Hölder continuous, and so a subsequence of the f_i converges uniformly to a g. But since the f_i are orthogonal, g would be orthogonal to all f_i and therefore to itself, so $g = 0$. This contradicts the uniform convergence of the subsequence of f_i to g.

It remains to show that the f_i are complete. If the f_i are not complete, there is a nonzero $g \in L^2_{\mathrm{Dir}}(\mathcal{G})$ orthogonal to all the f_i. By convolving g with smooth approximations to Dirac's delta function, and modifying it at the vertices, we can find a $g_\epsilon \in C^\infty_{\mathrm{Dir}}(\mathcal{G})$ for any $\epsilon > 0$ with $\|g - g_\epsilon\|_{L^2} \le \epsilon$. Hence for small ϵ the function, h, which is the projection of g_ϵ onto the complement of the f_i's, is: (1) nonzero, (2) orthogonal to all f_i, and (3) lies in $H^1_{\mathrm{Dir}}(\mathcal{G})$. Now $\mathcal{R}(h)$ must upper bound the λ_i, by their minimizing property. Hence λ_i are bounded, which we know is impossible.

We finish by remarking that the same proof holds for the Neumann condition, except that we begin by working with $C^1(\mathcal{G})$ instead of $C^1_{\mathrm{Dir}}(\mathcal{G})$. $\square$

The same theorem holds for more general "mixed" boundary conditions. Namely, we consider the condition that a function, f, satisfy the mixed condition

$$f = 0 \quad \text{on } K_1, \tag{3.4}$$

$$\widetilde{\mathbf{n}} \cdot \nabla f + \sigma f = 0 \quad \text{on } K_2, \tag{3.5}$$

where K_1, K_2 are a partition of $\partial\mathcal{G}$ and where σ is nonnegative. Then the same theorem and proof hold, provided that we replace $C^1_{\mathrm{Dir}}(\mathcal{G})$ by

$$\{ f \in C^1(\mathcal{G}) \mid f = 0 \text{ on } K_1 \},$$

and replace the Rayleigh quotient, $\mathcal{R}$, by

$$\widetilde{\mathcal{R}}(f) = \frac{\int |\nabla f|^2 \, d\mathcal{E} + \int_{K_2} \sigma f^2 \, d\mathcal{V}}{\int |f|^2 \, d\mathcal{E}}. \tag{3.6}$$

We mention that $\mathcal{V}$ does not enter in any essential way into the edge-based eigenvalues. Only $\mathcal{E}$ should affect those eigenvalues.

We can get mixed edge-vertex Laplacians. So consider the Rayleigh quotient:

$$\mathcal{R}(f) = \frac{\int \gamma |\nabla f|^2 \, d\mathcal{E}}{\int \alpha f^2 \, d\mathcal{V} + \int \beta f^2 \, d\mathcal{E}},$$

with α, β, γ continuous, nonnegative and γ differentiable and never vanishing. Its successive minimizers satisfy

$$-\nabla \cdot (\gamma \nabla f) = \lambda f (\alpha \, d\mathcal{V} + \beta \, d\mathcal{E}). \tag{3.7}$$

The theorem above gives us a complete eigenbasis for $L^2(\mathcal{G}, \mu)$ with $\mu = \alpha\mathcal{V} + \beta\mathcal{E}$. This basis is infinite provided that β is not identically zero.

Finally, we mention that it is easy to modify the above to work for mixed boundary conditions with mixed edge-vertex Laplacians.

3.2. Weyl's law. One fundamental result about edge-based eigenvalues that is true for any finite graph is that their growth rate is determined, to first-order, by the sum of the lengths of their edges. In analysis the analogous quantity is the volume of the subdomain or manifold, and Weyl's proof of this fact (see [**Wey12**]) in analysis immediately carries over here.

For a finite graph, $\mathcal{G}$, let $N_{\mathrm{Dir}}(\lambda, \mathcal{G})$ be the number of Dirichlet edge-based eigenvalues $\leq \lambda$ for $\mathcal{G}$, and similarly for $N_{\mathrm{Neu}}(\lambda, \mathcal{G})$.

Proposition 3.3 (Weyl's Law)**.** *Fix a finite graph, $\mathcal{G}$. Let $N(\lambda)$ be either $N_{\mathrm{Dir}}(\lambda, \mathcal{G})$ or $N_{\mathrm{Neu}}(\lambda, \mathcal{G})$. There is a constant, C, such that*

$$|N(\lambda) - L\lambda^{1/2}/\pi| \leq C,$$

where L is the sum of all the lengths of the edges in the graph.

Proof. Consider the graph, $\mathcal{G}_1$, where every vertex is a boundary point. Then the edge-based eigenvalues of $\mathcal{G}_1$ are found by minimizing the same Rayleigh quotient over a more restrictive class of functions. Hence, by the min-max principle,

$$N_{\mathrm{Dir}}(\lambda, \mathcal{G}_1) \leq N_{\mathrm{Dir}}(\lambda, \mathcal{G}).$$

Similarly,

$$N_{\mathrm{Neu}}(\lambda, \mathcal{G}) \leq N_{\mathrm{Neu}}(\lambda, \mathcal{G}_1),$$

for $N_{\mathrm{Neu}}(\lambda, \mathcal{G}_1)$ corresponds to a Rayleigh quotient over the space of functions that needn't be continuous at any vertex. For similar reasons we have

$$N_{\mathrm{Dir}}(\lambda, \mathcal{G}) \leq N_{\mathrm{Neu}}(\lambda, \mathcal{G}),$$

the former N corresponding to the same class of functions as the latter except the latter need not vanish at boundary vertices. To summarize, we have shown

$$N_{\mathrm{Dir}}(\lambda, \mathcal{G}_1) \le N_{\mathrm{Dir}}(\lambda, \mathcal{G}) \le N_{\mathrm{Neu}}(\lambda, \mathcal{G}) \le N_{\mathrm{Neu}}(\lambda, \mathcal{G}_1).$$

Hence it suffices to prove the proposition for $\mathcal{G}_1$.

But the values of the $\mathcal{G}_1$ eigenfunctions don't interact across vertices, so

$$N_{\mathrm{Dir}}(\lambda, \mathcal{G}_1) = \sum_{e \in E} N_{\mathrm{Dir}}(\lambda, e),$$

where edges, e, are also viewed as graphs. If e is an edge of length ℓ, its Dirichlet eigenfunctions are $f_n(x) = \sin(nx\pi/\ell)$ for $n = 1, 2, \ldots$, with eigenvalues $\lambda_n = (n\pi/\ell)^2$. Hence we have

$$N_{\mathrm{Dir}}(\lambda, e) = \lfloor \lambda^{1/2}\ell/\pi \rfloor.$$

It follows that for some constant $C > 0$,

$$N_{\mathrm{Dir}}(\lambda, \mathcal{G}_1) \ge -C + (\lambda^{1/2}/\pi) \sum_{e \in E} \ell_e = -C + L\lambda^{1/2}/\pi.$$

For similar reasons the result also holds for $N_{\mathrm{Neu}}(\lambda, \mathcal{G}_1)$, where the eigenfunctions on an edge are $f_n(x) = \cos\big((n-1)x\pi/\ell\big)$ for $n = 1, 2, \ldots$, with eigenvalues $\lambda_n = \big((n-1)\pi/\ell\big)^2$. □

Again, a similar result holds for mixed boundary conditions. To see this, we shall show that for any fixed mixed boundary condition (as in the end of the previous subsection)

$$N_{\mathrm{Dir}}(\lambda, \mathcal{G}) \le N_{\mathrm{mixed}}(\lambda, \mathcal{G}) \le N_{\mathrm{Neu}}(\lambda, \mathcal{G}),$$

where N_{mixed} counts eigenvalues with a mixed condition. Indeed, N_{Dir} can be viewed as having the same Rayleigh quotient, as in Equation (3.6), as N_{mixed}, except over a smaller space (i.e., the space of functions vanishing over all of $\partial\mathcal{G}$, not just K_1). Furthermore, the Rayleigh quotient for N_{mixed} is no less than that for N_{Neu}, and the space for the former is more restrictive. Hence the claim that $N_{\mathrm{Dir}} \le N_{\mathrm{mixed}} \le N_{\mathrm{Neu}}$, and hence the asymptotic law.

To get an asymptotic law for mixed edge-vertex Laplacians (as in Equation (3.7)), the above arguments show it suffices to consider Dirichlet and Neumann eigenvalues for an edge, e. Partition e into k intervals, $I_1, \ldots I_k$, and on a fixed interval, I_j, set $\gamma_{\max}$ to be the maximum value of γ there, and $\beta_{\min}$ similarly. The Rayleigh quotient with $\gamma_{\max}$ replacing γ and $\beta_{\min}$ replacing β is never smaller, and so the Dirichlet eigenvalue counting function on I_j for the Rayleigh quotient with β and γ is at least

$$\left\lfloor \frac{\lambda^{1/2}|I_j|\,\beta_{\min}}{\gamma_{\max}\pi} \right\rfloor.$$

We conclude that for any $\epsilon > 0$ we have that the Dirichlet eigenvalue counting function on e is at least

$$\lambda^{1/2}(1/\pi)\left(-\epsilon + \int_e \beta(x)/\gamma(x)\,dx\right),$$

where x is a standard coordinate on e. We conclude a similar upper bound for the Neumann eigenvalues, and conclude that $N(\lambda) \sim \lambda^{1/2}C/\pi$, where

$$C = \sum_e \int_e \frac{\beta(x)}{\gamma(x)}\,dx.$$

3.3. A condition on edge-based eigenfunctions.

Proposition 3.4. *Let $\mathcal{G}$ be a locally finite graph. Let (f, ω) be an edge-based eigenpair for the Laplacian, and let $\omega = \lambda^{1/2}$. Let v be an interior vertex such that $\omega\ell_e$ is not a multiple of π for any e incident upon v. Then*

$$\sum_{e \sim \{v,u\}} a_e \frac{f(u) - \cos(\omega\ell_e) f(v)}{\sin(\omega\ell_e)} = 0.$$

If the degree of v is infinite, the theorem still holds provided the above sum converges absolutely (and Δ_V is understood in the natural way).

Proof. This is a simple consequence of the fact that if f is an eigenfunction then $\Delta_V f(v) = 0$ at any interior vertex. Fix an edge $e \sim \{v, u\}$, and let x be the standard coordinate on e with $x(v) = 0$ and $x(u) = \ell_e$. We have f's restriction to e with coordinate x, $f_e = f_e(x) = A\cos(\omega x + B)$ for some A, B. Hence

$$f(v) = f_e(0) = A\cos B,$$

and

$$f(u) = f_e(\ell_e) = A\cos B\cos(\omega\ell_e) - A\sin B\sin(\omega\ell_e),$$

and the outward normal derivative of f at v is

$$f'(0) = -A\omega\sin B = -\omega\frac{f(u) - \cos(\omega\ell_e) f(v)}{\sin(\omega\ell_e)}.$$

Since $\Delta_V f = \widetilde{\mathbf{n}} \cdot \nabla f$ at v is $\mathcal{V}(v)^{-1}$ times the sum of the above times a_e, we conclude the proposition. □

Using this proposition one can rather easily determine all the edge-based eigenpairs in terms of the eigenpairs of the "normalized" adjacency matrix of the graph, provided that all edge lengths are equal; the ω's will turn out to be periodic of period 2π. However, we do not know of such a determination when edge lengths vary; we shall show that a graph with two vertices joined by three edges of varying edge lengths will not have periodic ω's.

3.4. The equilength case. In this subsection we consider a finite graph, $\mathcal{G}$, all of whose edges have length 1 (however the a_e can vary). We can similarly deal with any graph whose edge lengths are equal.

For any $v \in \mathring{V}$, let a_v be the sum of the a_e over all edges, e, incident with v (these edges, e, may be incident with boundary vertices). We will assume that $\mathcal{G}$ is connected with at least one edge, so that $a_v > 0$ for all v. Let $\widetilde{A}$ be the "normalized" adjacency matrix, which is just the adjacency matrix of Section 2, normalized by dividing each row by its corresponding a_v. $\widetilde{A}$ represents a Markov chain if and only if $\mathcal{G}$ has no boundary vertices (and it always represents a Markov chain if we add in the boundary vertices and make them "absorbing" states).

Our main result describes the edge-based eigenvalues in terms of $\widetilde{A}$ and the number of edges and interior vertices. We state this first, and then prove it in a series of propositions. First we set

$$\mathbb{Z}_{\geq 0} = \{0, 1, 2, 3, \dots\},$$

so that for any $\tau \in \mathbb{R}$ we can write

$$\tau + 2\pi\mathbb{Z}_{\geq 0} = \{\tau, \tau + 2\pi, \tau + 4\pi, \dots\}.$$

Theorem 3.5. *Let $\mathcal{G}$ be a connected graph with at least one edge, and let $\widetilde{A}$ be its normalized adjacency matrix, as above. The edge-based eigenvalues is the multiset sum*[3] *of the following sets: for each eigenvalue, λ, of $\widetilde{A}$, there is a unique $\cos^{-1}(\lambda) \in [0, \pi]$; corresponding to this λ we have eigenvalues*

$$\cos^{-1}(\lambda) + 2\pi\mathbb{Z}_{\geq 0}, \qquad \text{and} \qquad 2\pi - \cos^{-1}(\lambda) + 2\pi\mathbb{Z}_{\geq 0}. \tag{3.8}$$

Additionally, the sets

$$\pi + 2\pi\mathbb{Z}_{\geq 0}, \qquad \text{and} \qquad 2\pi + 2\pi\mathbb{Z}_{\geq 0} \tag{3.9}$$

occur with multiplicity $|E| - |\mathring{V}|$. This means that if $|E| - |\mathring{V}| = -1$, i.e., $\mathcal{G}$ is a tree without boundary, then we subtract the list in Equation (3.9) once from the union over Equation (3.8); i.e., $n\pi$ for nonnegative integer n occurs with multiplicity one.

Let us mention that if $\mathcal{G}$ has separated boundary, then the Neumann condition at a boundary vertex is equivalent to considering that vertex to be an interior vertex. Hence our theorem really also handles the case where we impose a Dirichlet condition on some vertices, and a Neumann condition on the rest, assuming the rest are separated.

Proof. The proof of this theorem occupies the rest of this section. We prove it in a sequence of propositions. Notice that our proof provides a method for finding a basis for the eigenspaces.

[3] i.e., if λ occurs five times in the lists below, then its multiplicity is five.

Proposition 3.6. *Let $\omega \in \mathbb{R} \setminus (\pi\mathbb{Z})$. Then ω^2 is an edge-based Dirichlet eigenvalue if and only if $\cos\omega$ is an eigenvalue of $\widetilde{A}$. If so, ω^2's multiplicity is that of $\cos\omega$ in $\widetilde{A}$, and for each corresponding eigenfunction, f, of $\widetilde{A}$ (therefore defined on the vertices), we may extend f along each edge (as $A\cos(\omega x + B)$ for some A, B and standard edge coordinate x) to an edge-based eigenfunction.*

Proof. If $\omega \in \mathbb{R} \setminus (\pi\mathbb{Z})$ has ω^2 an edge-based eigenvalue, then by Proposition 3.4 we have for each v

$$\sum_{e\sim\{v,u\}} a_e f(u) = \cos(\omega) f(v) \sum_{e\sim\{v,u\}} a_e.$$

In other words, since f is Dirichlet, $\widetilde{A}f = \cos(\omega)f$; that is, $\cos\omega$ is an eigenvalue of $\widetilde{A}$.

Conversely, let $\cos\omega \neq \pm 1$ be an eigenvalue of $\widetilde{A}$ with eigenfunction g. We claim that for any fixed $e = \{u, v\}$, there is a unique way to extend g to a function along e of the form $A\cos(B + \omega x)$, where x is the standard coordinate on e with $x(v) = 0$. Indeed, consider the equations in A, B for fixed $g(u), g(v)$:

$$g(v) = A\cos B, \qquad g(u) = A\cos(B + \omega).$$

Since

$$A\sin B = \frac{g(u) - g(v)\cos\omega}{-\sin\omega},$$

$A\cos B$ and $A\sin B$ are uniquely determined by $g(u), g(v)$. From what we know of polar coordinates, this means either $A = 0$ and B is arbitrary, in which case g's extension along e is by 0 (and $g(u) = g(v) = 0$), or there is a unique positive $A = A_0$ and unique $B = B_0$ modulo 2π satisfying the equations, with the only other solution being $A = -A_0$ and, modulo 2π, $B = B_0 + \pi$. Since the function $A\cos(B + \omega x)$ is the same for these solutions, and does have the right value at $x = x(v) = 0$ and $x = x(u) = 1$, g can be extended to satisfy $\Delta_E g = \omega^2 g$. Also clearly g satisfies the condition in Proposition 3.4, which is equivalent to $\Delta_V g = 0$.

To sum up, we know that ω^2 is an edge-based eigenvalue for $\omega \in \mathbb{R} \setminus (\mathbb{Z}\pi)$ if and only if $\cos\omega$ is an eigenvalue of $\widetilde{A}$. We know that the restriction of any Δ_E eigenfunction gives an $\widetilde{A}$ eigenfunction, and we know that (given ω) a $\widetilde{A}$ eigenfunction g has a unique extension to a Δ_E eigenfunction. Hence the multiplicities of ω^2 in Δ_E and $\cos\omega$ in $\widetilde{A}$ are equal. $\square$

The story when $\omega \in \pi\mathbb{Z}$ is less elegant. Indeed, there are many edge-based eigenfunctions, f, whose restriction to the vertices vanishes.

Let Y_ω be the eigenspace corresponding to the Dirichlet edge-based eigenfunctions with eigenvalue ω^2,

$$Y_\omega = Y_\omega(\mathcal{G}) = \{f \in C^\infty_{\mathrm{Dir}}(\mathcal{G}) \mid \Delta_E f = \omega^2 f, \Delta_V f = 0\},$$

and let Z_ω be those elements of Y_ω vanishing on all vertices,

$$Z_\omega = Z_\omega(\mathcal{G}) = \bigl\{ f \in Y_\omega \mid f|_V = 0 \bigr\}.$$

We reduce the study of Y_ω for $\omega \in \pi\mathbb{Z}$ to that of Z_ω by the following proposition:

Proposition 3.7. *Y_ω/Z_ω for $\omega \in 2\pi\mathbb{Z}$ is one-dimensional if $\partial\mathcal{G} = \emptyset$, and otherwise zero. Similarly for $\omega \in \pi + 2\pi\mathbb{Z}$, except that we require $\partial\mathcal{G} = \emptyset$ and that $\mathcal{G}$ is bipartite.*

Proof. If $f \in Y_{2\pi n}$ with $n \in \mathbb{Z}$, and $e = \{u, v\}$ is an edge, then $f(u) = f(v)$, since f's restriction to e is of the form $A\cos(B + 2\pi nx)$. Hence f must be constant on V. This implies that Y_ω/Z_ω is at most one-dimensional, and must be zero if $\partial\mathcal{G} \neq \emptyset$. On the other hand, if $\partial\mathcal{G} = \emptyset$ then the function whose restriction to each edge is $\cos(\omega x)$ gives a nonzero element of Y_ω/Z_ω. The case $\omega = (2n + 1)\pi$ is handled similarly. □

The next two propositions essentially finish our work in this section.

Proposition 3.8. *For $\omega > 0$ there is a natural isomorphism of Y_ω with $Y_{\omega+2\pi}$ that restricts to an isomorphism of Z_ω with $Z_{\omega+2\pi}$. The same is true if $\omega + 2\pi$ is replaced by $\omega + \pi$, provided that $\mathcal{G}$ is bipartite.*

Proof. For an arbitrary $f \in Y_\omega$, let ιf be the function whose restriction to e is $A\cos\bigl(B + (2\pi + \omega)x\bigr)$, where A, B are given by f's restriction to e being $A\cos(B + \omega x)$. Then $\Delta_V f = \Delta_V(\iota f)\omega/(\omega + 2\pi)$ so $\Delta_V(\iota f) = 0$. From here it is clear that ι is the desired isomorphism. □

Proposition 3.9. *Let $\mathcal{G}' = \mathcal{G} \cup \{e\}$ be the graph formed by adding an edge, e, to $\mathcal{G}$. Then $Z_\omega(\mathcal{G})$ naturally injects into $Z_\omega(\mathcal{G}')$, and the quotient is of dimension 1 or zero.*

Proof. If $f \in Z_\omega(\mathcal{G})$, we simply extend it by zero on e to get a member of $Z_\omega(\mathcal{G}')$. Member of $Z_\omega(\mathcal{G}')$ restricts to $A\sin(\omega x)$ along e for some A, and hence any two of them are scalar multiples of each other modulo $Z_\omega(\mathcal{G})$. □

First a corollary of these two propositions:

Corollary 3.10. *Let b be the number of ± 1's that appear among $\widetilde{A}$'s eigenvalues, i.e.,*

$$b = b(\mathcal{G}) = \begin{cases} 0 & \textit{if } \partial\mathcal{G} \neq \emptyset, \\ 1 & \textit{if } \partial\mathcal{G} = \emptyset \textit{ and } \mathcal{G} \textit{ is not bipartite, and} \\ 2 & \textit{if } \partial\mathcal{G} = \emptyset \textit{ and } \mathcal{G} \textit{ is bipartite.} \end{cases}$$

Then

$$\dim Y_\pi + \dim Y_{2\pi} + 2(|\mathring{V}| - b) = 2|E|.$$

Hence, if $\mathcal{G}$ is bipartite, we further have

$$\dim Y_\pi = \dim Y_{2\pi} = |E| - |\mathring{V}| + b.$$

Proof. The first part follows from the above and Weyl's law. The second part follows from Proposition 3.8. □

To prove Theorem 3.5, first note that since we are working with the Dirichlet condition we can assume the boundary is separated — if not, we just give each boundary edge its own boundary vertex.

The corollary above shows that the theorem is true for a tree. Any connected graph is the union of a tree and a number of edges. Hence it suffices to show that if the theorem is true for $\mathcal{G}$ then it is true for $\mathcal{G}$ with an edge thrown in, $\mathcal{G}' = \mathcal{G} \cup \{e\}$.

So assume the theorem is true for $\mathcal{G}$. If $b(\mathcal{G}) = b(\mathcal{G}')$ then $\dim Y_\pi + \dim Y_{2\pi}$ increases by 2. But each dimension can increase by at most 1, so they increase precisely by that much, and the theorem holds for $\mathcal{G}'$.

The only change in b that can happen by adding an edge is that $b(\mathcal{G}) = 2$ but $b(\mathcal{G}') = 1$, i.e., $\mathcal{G}$ is bipartite but $\mathcal{G}'$ isn't. In this case $\dim Y_\pi + \dim Y_{2\pi}$ remains the same. While Z_π cannot decrease in going from $\mathcal{G}$ to $\mathcal{G}'$, Y_π/Z_π goes from one to zero-dimensional from $\mathcal{G}$ to $\mathcal{G}'$. Since $Y_{2\pi}/Z_{2\pi}$ remains one-dimensional, $Z_{2\pi}$ increases by one. Once again we see how the Z_ω's and Y_ω/Z_ω's change for $\mathcal{G}'$, and it is easy to see the theorem holds there. □

4. The wave equation

In this section we usually only assume that the graphs are locally finite. This is because the wave equation has finite propagation speed, and "cannot tell" whether or not a graph is finite (in any fixed interval of time).

Definition 4.1. Given a graph, fix nonnegative $\alpha, \beta \in C^0(\mathcal{G})$, nonnegative $\gamma \in C^1(\mathcal{G})$, and an interval $I \subset \mathbb{R}$. A function $u = u(x,t) : \mathcal{G} \times I \to \mathbb{R}$ is said to satisfy the wave equation with coefficients α, β, γ if:

(1) u is continuous on $\mathcal{G} \times I$ and $u(\,\cdot\,, t) \in C^2(\mathcal{G})$ for all $t \in I$;
(2) for all $x \in \mathring{\mathcal{G}}$ and $t \in \mathring{I}$ the derivative u_{tt} exists at (x,t) and is continuous in x; and
(3) for fixed t we have
$$(\alpha\, d\mathcal{V} + \beta\, d\mathcal{E})u_{tt} = \nabla \cdot (\gamma \nabla u)$$
as integrating factors.

If u vanishes on $\partial\mathcal{G} \times I$ we say that u satisfies the *Dirichlet condition*; similarly for the *Neumann condition* if ∇u vanishes along all edges at all boundary vertices.

Having the equality above as integrating factors means that
$$\alpha u_{tt} = -\gamma \Delta_V u$$
at all (x,t) with $x \in \mathring{V}$ and $t \in \mathring{I}$, and that
$$\beta u_{tt} = -\gamma \Delta_E u + \nabla\gamma \cdot \nabla u$$

at all (x,t) with $x \in \mathring{\mathcal{G}} \setminus V$ and $t \in \mathring{I}$.

In the above definition we may also allow mixed boundary conditions as in Equations (3.4) and (3.5).

The the vertex-based wave equation, we mean the wave equation with coefficients $\alpha = \gamma = 1$ and $\beta = 0$. In this case $\Delta_E u = 0$ and so u must be edgewise linear. As remarked in Section 2, Δ_V on a finite graph can be viewed as a bounded operator on $L^2_{\mathrm{Dir}}(\mathcal{G}, \mathcal{V})$. As such, for any edgewise linear $f \in L^2_{\mathrm{Dir}}(\mathcal{G}, \mathcal{V})$ we can form

$$u(x,t) = \cos\bigl(t\sqrt{\Delta_V}\,\bigr)f = f - t^2\Delta_V f/2 + t^4\Delta_V^2 f/4! - \cdots$$

and we easily see it satisfies the wave equation with $u(x,0) = f(x)$. The boundedness of Δ_V implies that if $f = \chi_v$ is the edgewise linear characteristic function[4] at v, then for fixed x,

$$u(x,t) = \cos\bigl(t\sqrt{\Delta_V}\,\bigr)\chi_v = (-1)^d t^{2d} N_{x,v}/(2d)! + O(t^{2d+2})$$

where d is x's distance to v and $N_{x,v}$ is the number of paths from x to v of length d. It follows that $u(x,t) > 0$ for small t, and so the vertex-based wave equation does not have a finite wave propagation speed.

Since traditional graph theory is based on Δ_V restricted to edgewise linear functions, the above explains why approaching the wave equation with traditional graph theory leads to unsatisfactory results.

A much better model of the wave equation appearing in analysis is the edge-based wave equation, where $\beta = \gamma = 1$ and $\alpha = 0$. We will show in the next subsection that these waves propagate "along the edges" and have finite wave speed equal to 1.

4.1. The energy inequality. Many basic properties of the wave equation follow from well-known energy inequality, which we state and apply in this subsection.

If $A \subset \mathcal{G}$, then the energy of $u = u(x,t)$ over A at time t is defined to be

$$\mathrm{Energy}(A;t) = \int_A \bigl(\gamma(\nabla u)^2\, d\mathcal{E} + u_t^2(\alpha\, d\mathcal{V} + \beta\, d\mathcal{E})\bigr).$$

For real $h > 0$ let

$$A^h = \bigl\{x \in \mathcal{G} \mid \mathrm{dist}(x,A) < h\bigr\}.$$

Now fix coefficients α, β, γ for a wave equation and let c be the smallest constant such that $\gamma \le c^2\beta$ throughout $\mathcal{G}$ (we assume this c exists).

Theorem 4.2. *Let A be an open set with A^{ct_0} of finite type for some $t_0 > 0$. Then if u is a solution to the Dirichlet or Neumann wave equation we have*

$$\mathrm{Energy}(A^{ct_0};0) \ge \mathrm{Energy}(A;t_0).$$

[4] i.e., this function is 1 on v, 0 on other vertices, and edgewise linear.

The same holds of the mixed boundary condition, as in Equations (3.4) *and* (3.5)*, provided we add*

$$\int_{K_2\cap A} \gamma\sigma u^2\, d\mathcal{V}$$

to the energy.

This theorem will be proven in the next subsection; its proof is virtually identical to its well-known proof in analysis.

We remark that the wave equation has a time symmetry, in that if u satisfies the wave equation then so does $w(x,t) = u(x,-t)$. We may therefore conclude the symmetric fact that

$$\mathrm{Energy}(A;0) \le \mathrm{Energy}(A^{ct_0};t_0).$$

From the energy inequality we easily conclude the following proposition:

Proposition 4.3. *Assume $\mathcal{G}$ is locally finite, that β is strictly positive on $\mathcal{G}$, and that c exists as before (i.e., $\gamma \le c^2\beta$ everywhere). Let u be a solution to the wave equation on $\mathcal{G}\times[0,T]$ with $u(x,0)=0$ and $u_t(x,0)=0$ for all x within a distance ct to a fixed $y\in\mathcal{G}$. Then $u(y,t)=0$.*

Proof. For any $\epsilon > 0$ we have

$$\mathrm{Energy}(\{y\}^{c(t-\epsilon)};0) = 0.$$

We conclude that $u_t(y,s)$ vanishes from $s=0$ to $s=t-2\epsilon$, and hence $u(y,t-2\epsilon)=0$. Now we let $\epsilon\to 0$ and use the continuity of u. □

Some immediate corollaries of this are:

Corollary 4.4. *If u, w are two solutions to the wave equation such that u and w agree at time $t=0$ on all points within distance ct to y, and the same for u_t and w_t, then $u(y,t)=w(y,t)$. In other words, the value of $u(y,t)$ depends only on the value of u at a fixed time in the "space-time cone of speed c at y". In other words, this wave equation has finite speed of wave propagation bounded by c.*

Corollary 4.5. *Fixing $u(\,\cdot\,,0)$ and $u_t(\,\cdot\,,0)$, there is at most one solution, $u(x,t)$ for $t>0$, to the wave equation.*

4.2. A proof of the energy inequality. Let $I\subset\mathbb{R}$ be an interval. By a *(graph-time) vector field* on $\mathcal{G}\times I$ we mean a pair (G,F) where $F=F(t)$ is an integrating factor on $\mathcal{G}$ and $G=G(t)$ is vector field on $\mathcal{G}$ both depending on $t\in I$. By its *divergence* we mean

$$\nabla_{\mathrm{gt}}\cdot(G,F) = \nabla\cdot G + F_t,$$

which is an integrating factor that depends on time, t, where by F_t we mean the partial derivative of F with respect to t, i.e., we differentiate F's $d\mathcal{V}$ component and its $d\mathcal{E}$ component with respect to time.

Proposition 4.6. *Consider a divergence free graph-time vector field, (G, F), i.e., $\nabla \cdot G + F_t = 0$ as a boundary inclusive equality. Write $F = F_V\, d\mathcal{V} + F_E\, d\mathcal{E}$. Assume that $F_V \geq 0$ at all vertices and that $cF_E \geq |G|$ on $(\mathcal{G} \setminus V) \times I$. If A is any open set with A^{ct_0} of finite type, and if $0, t_0 \in I$, then*

$$\mathcal{I}(t) = \int_{A^{c(t_0-t)}} F(t)$$

is a nonincreasing function in $t \in [0, t_0]$.

The energy inequality in Section 4.1 follows almost at once by taking

$$F = \gamma\mathcal{E}(\nabla u)^2 + (\alpha\mathcal{V} + \beta\mathcal{E})u_t^2 \quad \text{and} \quad G = -2\gamma u_t \nabla u,$$

adding $\gamma\sigma u^2 \mathcal{V}|_{K_2}$ to F for the mixed boundary condition.

Proof. Let $S(t) = A^{c(t_0-t)}$. If $\partial S(t)$ contains a vertex, then $\mathcal{I}(t)$ is right continuous at t, and has a jump (if any) from the left of

$$\mathcal{I}(t-0) - \mathcal{I}(t) = \sum_{x \in \mathring{V} \cap \partial S(t)} F_V(x)\mathcal{V}(x).$$

Next partition $\partial S(t)$ into interior and boundary points,

$$\mathring{B} = \partial S(t) \cap \mathring{\mathcal{G}} \qquad \text{and} \qquad B^{\partial} = \partial S(t) \cap \partial\mathcal{G}.$$

$\mathring{B}$ will contain no vertices for all but finitely many t. If $\mathring{B}$ contains no vertex, then using Proposition 2.21 we see that

$$\mathcal{I}'(t) = -\sum_{x \in \mathring{B}, e \ni x} F_E(x,t) c a_e + \int_{S(t) \cup B^{\partial}} F_t(t)$$

(where F_t contains both a $d\mathcal{V}$ term and a $d\mathcal{E}$ term). By taking $S(t)$ and adding to it $\partial S(t)$ as new vertices each of weight 1, we get a graph, $\mathcal{G}_t$; taking $F_V = 0$ at all new vertices, we may write:

$$\begin{aligned} \int_{S(t) \cup B^{\partial}} F_t(t) &= \int_{\mathcal{G}_t \setminus \mathring{B}} F_t(t) = -\int_{\mathcal{G}_t \setminus \mathring{B}} \nabla \cdot G(t) \\ &= \int_{\mathring{B}} \widetilde{\mathbf{n}} \cdot G(t) = \sum_{x \in \mathring{B}, e \ni x} \mathbf{n}_{e,x} \cdot G(x,t) a_e, \end{aligned}$$

using the divergence theorem.

Recalling that $\mathbf{n}_{e,x}$ is a unit vector, we have

$$-F_E(x,t)c + \mathbf{n}_{e,x} \cdot G(x,t) \leq -F_E(x,t)c + |G(x,t)| \leq 0$$

for all x, t. Hence

$$\mathcal{I}'(t) = \sum_{x \in \mathring{B}, e \ni x} a_e\big(-F_E(x,t)c + \mathbf{n}_{e,x} \cdot G(x,t)\big) \leq 0.$$

We have shown that $\mathcal{I}(t)$ is nonincreasing at all but finitely many points (i.e., the t's with $\mathring{B}$ containing a vertex), and at these finitely many points we know that $\mathcal{I}(t)$ is continuous or has a decreasing jump. Hence $\mathcal{I}(t)$ is nonincreasing. □

The proposition is usually proven (in analysis) by invoking a space-time divergence theorem on a truncated cone in space-time (or graph-time here), such as those (x,t) with $\mathrm{dist}(x,A) < c(t_0-t)$ (see, for example, [**Smo83**] Chapter 4 or [**CH89**]). While this approach also works, setting up a graph-time divergence theorem seems like more trouble than it's worth for our purposes at this point.

4.3. More general wave equations. One can generalize the uniqueness results for the wave equation, i.e., Corollaries 4.4 and 4.5, to the same results for a wave equation of the form

$$(\alpha\, d\mathcal{V} + \beta\, d\mathcal{E})u_{tt} = \nabla\cdot(\gamma\nabla u) + \delta\cdot\nabla u + \epsilon u,$$

where δ is a C^0 vector field and ϵ is a C^0 function, and we assume β never vanishes. The proof is a simple adaptation of the analysis proof given in, for example, Smoller's book [**Smo83**]. We define the energy exactly as before (ignoring δ and ϵ), but now prove

$$e^{Kt_0}\mathrm{Energy}(A^{ct_0};0) \geq \mathrm{Energy}(A;t_0) \tag{4.1}$$

for some constant K depending on δ,ϵ, provided that u, u_t vanish on A^{ct_0} at $t=0$. Uniqueness with "propagation speed" at most c follows as before.

We shall outline the proof of Equation (4.1) when $\delta = 0$; the general case is a bit messier but similar (see [**Smo83**] for details). First we notice that if all is the same as in Proposition 4.6 except that $\nabla\cdot G + F_t$ does not necessarily vanish (where by F and G we mean $F = \gamma\mathcal{E}(\nabla u)^2 + (\alpha\mathcal{V}+\beta\mathcal{E})u_t^2$ and $G = -2\gamma u_t\nabla u$), then we have

$$\mathcal{I}(t) \leq \mathcal{I}(0) + \int_0^t \int_{A^{c(t_0-s)}} (\nabla\cdot G + F_t)\, ds.$$

Hence setting

$$E(t) = \mathrm{Energy}(A^{c(t_0-t)};t)$$

for $0 \leq t \leq t_0$, we have for such t

$$E(t) \leq E(0) - 2\int_0^t \int_{A^{c(t_0-s)}} \epsilon u(x,s)u_t(x,s)\, d\mathcal{E}(x)\, ds. \tag{4.2}$$

The integral on the right-hand side can be bounded by

$$\left|\int_0^t \int_{A^{c(t_0-s)}} uu_t\right| \leq \int\int u^2/2 \;+\; \int\int u_t^2/2.$$

The $u_t^2/2$ term integrated over space is bounded by a constant times $E(t)$ (since β vanishes nowhere), and the $u^2/2$ can be bounded by a constant times t^2 times a u_t^2 integral by Poincaré's inequality (see [**Smo83**]). It easily follows that the double integral in Equation (4.2) is bounded by a constant times $\int_0^t E(s)\,ds$, and hence

$$E(t) \le E(0) + K \int_0^t E(s)\,ds$$

for a constant K. Gronwall's inequality implies $E(t) \le e^{Kt}E(0)$, which is just Equation (4.1).

4.4. Differentiability and the wave equation. In this subsection we show how to prove the existence of a solution to the wave equation for sufficiently "differentiable" initial conditions. On the circle, i.e., $\mathbb{R}/(2\pi\mathbb{Z})$, there is a rough correspondence between differentiability and having Fourier coefficients decaying. We shall use the same for graphs, to give a nice description of when we are sure that the wave equation has a solution.

Let $f_1, f_2, \ldots$ be the eigenfunctions of Δ_E with eigenvalues $\lambda_1, \lambda_2, \ldots$ with some boundary conditions of any type specified before. Let D^k (which depends on the boundary conditions) be those formal sums $\sum_i a_i f_i$ with $\sum_i i^k |a_i| < \infty$ (here k is any real, although typically a nonnegative integer). Let B^k be the same, except that the a_i's are subject to the weaker condition that $i^k|a_i|$ is bounded in i. Let Diff^k be those functions $f \in C^k$ such that for $j = 0, \ldots, k-1$ we have:

(1) $f^{(j)}(v,e)$ depends only on v if j is even, where $f^{(j)}(v,e)$ is the j-th derivative of f at v along e.
(2) The sum of $f^{(j)}(v,e)$ over all e for fixed v is zero for all v, if j is odd.

It is not hard to prove the following facts:

Proposition 4.7. *We have natural inclusions $B^{k+1+\epsilon} \subset D^k \subset B^k$ for any k and $\epsilon > 0$, and inclusions $D^k \subset \mathrm{Diff}^k \subset B^k$ for any nonnegative integer k.*

We remark that Diff^k's compatibility with D^k, B^k makes it, in some sense, a better notion of differentiability than C^k defined in Section 2.

Proof. The first inclusions are straightforward. The inclusion $D^k \subset \mathrm{Diff}^k$ follows by viewing the formal sum as an absolutely convergent sum of functions whose derivatives up to k-th order also form an absolutely convergent sum (here we use the periodic nature of the eigenpairs). Finally the inclusion $\mathrm{Diff}^k \subset B^k$ follows by integration by parts of the inner product (f, f_i) for an $f \in \mathrm{Diff}^k$ along each edge: $\int_0^1 f(x)A_e \cos(\omega x + B_e)$ is proportional to the integral of $\omega^{-k} f^{(k)}(x)$ times sine or cosine of plus or minus $\omega x + B_e$ plus boundary terms. The boundary terms at $v \in \mathring{V}$ are proportional to sums over e of $f^{(j)}(v,e)A_e$ times sine or cosine $\omega x + B_e$. Knowing that $A_e \cos(B_e)$ is independent of e (for a given v, with $x = 0$ corresponding to v), and knowing that the sum of $A_e \sin(B_e)$ vanishes (since $\Delta_V f_i = 0$ for all i), we see that the boundary terms at $\mathring{V}$ disappear; similarly we see that boundary terms at boundary vertices vanish. Now we use Weyl's law and the fact that $f \in C^k$ to see that $i^k(f, f_i)$ is bounded. $\square$

We now state an existence theorem in terms of D^2; it follows from the above proposition that our theorem also applies to the class Diff^4, which is easier to understand, in a sense, than D^2.

Proposition 4.8. *Let $g, h \in D^2$. If $g = \sum a_i f_i$ and $h = \sum b_i f_i$, then*

$$u(x,t) = \sum_i f_i\Big(a_i \cos(\sqrt{\lambda_i}\, t) + \frac{b_i}{\sqrt{\lambda_i}} \sin(\sqrt{\lambda_i}\, t)\Big) \tag{4.3}$$

is a solution to the wave equation with $u(\,\cdot\,,0) = g$ and $u_t(\,\cdot\,,0) = h$.

Proof. We need to know that u_{tt} exists, and if $g, h \in D^2$ then the sum of twice differentiated terms is absolutely convergent and this u_{tt} exists. The rest is an easy verification. $\square$

Any g, h in L^2, say, will have eigenfunction expansions. It makes sense to define $u(x,t)$ by the formal sum above (in Equation (4.3)), which for fixed t will always lie in L^2 (although u_{tt} need not exist).

4.5. Chebyshev polynomials and the wave operator. We again assume that $\mathcal{G}$ is finite in this subsection. Let f_i, λ_i be as in the Section 4.4, and set Summ to be D^0 in the notation of the previous section, i.e.,

$$\mathrm{Summ} = \Big\{ \sum_i a_i f_i \;\Big|\; \sum |a_i| < \infty \Big\}.$$

Elements of Summ may be viewed as formal sums, or we may identify them with the bounded function on $\mathcal{G}$ to which they converge (since the f_i are uniformly bounded). Summ is the set of functions with a summable eigenfunction coefficient series; it is easy to see that it contains, for example, $H^1(\mathcal{G})$. The map sending f_i to its restriction on vertices extends to a continuous map from Summ to $L^\infty(\mathcal{G}, \mathcal{V})$. Since $\mathcal{G}$ is finite this gives rise to a continuous map $M_{E\to V} : \mathrm{Summ} \to L^2(\mathcal{G}, \mathcal{V})$.

If $g \in \mathrm{Summ}$ with $g = \sum a_i f_i$, then

$$\big(\cos\sqrt{\Delta_E}\,\big) g = \sum a_i \big(\cos\sqrt{\lambda_i}\,\big) f_i$$

lies in Summ, and we know

$$M_{E\to V}\big(\cos\sqrt{\Delta_E}\,\big) g = \sum a_i \big(\cos\sqrt{\lambda_i}\,\big) M_{E\to V} f_i.$$

Since $1 - \cos\sqrt{\lambda_i}$ is the Δ_V eigenvalue corresponding to $M_{E\to V} f_i$ and $\widetilde{A} = I - \Delta_V$, we have

$$\widetilde{A} M_{E\to V} = M_{E\to V} \cos\sqrt{\Delta_E}$$

as maps from Summ to $L^2(\mathcal{G}, \mathcal{V})$, assuming all edge lengths are 1. It follows that for any polynomial, P, we have

$$P\big(\widetilde{A}\big)M_{E\to V} = M_{E\to V}P\big(\cos\sqrt{\Delta_E}\,\big).$$

If T_k is the k-th Chebyshev polynomial, given by

$$T_k(\cos x) = \cos(kx),$$

we have

$$T_k\big(\widetilde{A}\big)M_{E\to V} = M_{E\to V}\cos\big(k\sqrt{\Delta_E}\,\big).$$

We conclude the following theorem:

Theorem 4.9. *Let all edge lengths be* 1*. Let* $g \in$ Summ*, and let* u *be the formal solution to the wave equation as in Equation* (4.3) *with* $h = 0$*. Then for any integer* k *and interior vertex* v *we have*

$$u(v, k) = T_k\big(\widetilde{A}\big)M_{E\to V}g.$$

So if we wish to know this wave equation solution at vertices and at only integral times, we need only know the initial condition at the vertices. The values of the solution there are given in term of Chebyshev polynomials of the normalized adjacency matrix. Notice that knowing the values of u at nonintegral times requires knowing f along the edges.

Notice that Theorem 4.9 is valid for infinite (locally finite) graphs, since the wave equation solution is determined locally for any finite time, and the equality in Theorem 4.9 is a local statement as well.

We finish this subsection by remarking that if we ignore the map $M_{E\to V}$, we can say that $\widetilde{A}$ "acts like" $\cos\sqrt{\Delta_E}$. Noting that $\widetilde{A}$ is $I - \Delta_V$ for an appropriate graph theoretic Laplacian, Δ_V, we can see that $I - \Delta_V$ "acts like" $\cos\sqrt{\Delta_E}$.

4.6. Wave propagation through vertices. Using Theorem 4.9 it is not hard to see what happens when a wave is sent through a vertex. More precisely, let G be the infinite d-regular star,[5] i.e., G's vertices are v_0 union $v_{i,j}$ with $i = 1, \ldots, d$ and j a positive integer, and G's edges are $\{v_0, v_{i,1}\}$ for all i, and $\{v_{i,j}, v_{i,j+1}\}$ for all i and for all positive j (see Figure 2).

Taking g to be a function which is zero on all vertices except $v_{1,j}$ for some $j \geq 2$, we apply Theorem 4.9 to see that of the "wave" traveling towards v_0, we have $(2/d) - 2$ of the wave comes back along the $i = 1$ edge, and $2/d$ of it travels down each $i > 1$ edge.

This motivates the following theorem. We state this theorem in terms of the length 1 d-regular star, by which we mean the graph, G, as above,

[5]Note that the wave equation effectively ignores vertices of degree 2, i.e., one gets the same equation if one treats the two edges incident with the vertex as one longer edge. So the infinite d-regular star is equivalent to a graph with one degree d vertex and d edges of infinite length.

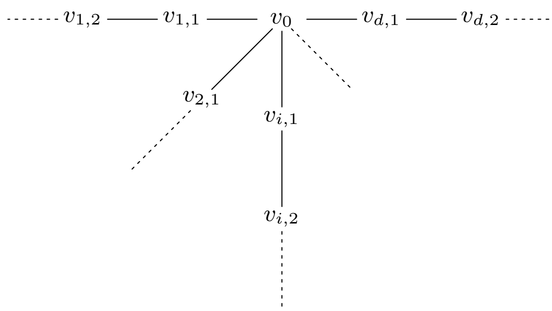

Figure 2.

except we restrict j to take on only the value 1. We endow the edges of G with standard coordinates $x_1, \ldots, x_d$ where $x_i(0) = v_0$ and $x_i(1) = v_{i,1}$.

Theorem 4.10. *Let $u(x,t)$ be the solution to the wave equation for $0 \le t \le 1/4$ given by*

$$u(x_i,t) = \begin{cases} f(x_1+t) & \text{for } i=1, \\ 0 & \text{otherwise,} \end{cases}$$

where f is any twice differentiable function supported on $(1/4, 3/4)$. Then the solution for $0 \le t \le 5/4$ to the wave equation exists and is given by

$$\widetilde{u}(x_i,t) = \begin{cases} f(x_1+t) + \big((2/d)-1\big) f(t-x_1) & \text{for } i=1, \\ (2/d) f(t-x_i) & \text{otherwise.} \end{cases}$$

Notice that this theorem tells us how waves propagate through vertices. Also, notice that we know that $\widetilde{u}$ is unique.

Proof. Clearly $\widetilde{u}$ satisfies the wave equation on edge interiors. At v_0, the only vertex of interest for $t \le 5/4$, we have $\widetilde{u}$ is continuous (taking the limiting value $2f(t)/d$ along each edge at v_0) and satisfies $\Delta_V \widetilde{u} = 0$ there. Hence $\widetilde{u}$ satisfies the wave equation. □

4.7. Finite propagation speed of wave operators. There are a number of operators on $L^2(\mathcal{G}, \mathcal{E})$ that arise from the wave operator, which have a "finite speed of propagation". We mention one classical one, and a generalization of it. First we need some definitions.

By the support of a function, f, in $L^2(\mathcal{G}, \mathcal{E})$ we mean the complement of the union of those open sets, U, for which $f = 0$ almost everywhere in U.

Definition 4.11. Let A_t be a family of bounded (everywhere defined) operators on $L^2(\mathcal{G}, \mathcal{E})$ indexed on $t \ge 0$. We say that A_t have *speed of propagation at most c* if $(A_t f, g) = 0$ for any $f, g \in L^2(\mathcal{G}, \mathcal{E})$ and t with the supports of f and g a distance at least ct apart.

Definition 4.12. A subset, D, of $L^2(\mathcal{G},\mathcal{E})$ is called *supportingly dense* if for any $f \in L^2(\mathcal{G},\mathcal{E})$ and $\epsilon > 0$ there is $\widetilde{f} \in D$ such that $\|f - \widetilde{f}\|_2 < \epsilon$ and (each point of) the support of $\widetilde{f}$ is within distance ϵ of the support of f.

In our propagation speed definition, rather than requiring $f, g \in L^2(\mathcal{G},\mathcal{E})$, it suffices to take $f, g \in D$ where D is any supportingly dense subset of $L^2(\mathcal{G},\mathcal{E})$. We also remark that standard mollification arguments show that for any k, Diff^k is supportingly dense; it follows that B^k and D^k are, as well.

Consider the operator

$$W_t = \cos\bigl(t\sqrt{\Delta}\,\bigr).$$

By the spectral theorem this can be viewed as an operator on $L^2(\mathcal{G},\mathcal{E})$ whose norm is bounded by 1. We know by Proposition 4.8 that W_t restricted to $D^2 \cap L^2(\mathcal{G},\mathcal{E})$ has propagation speed at most 1. Since D^2 is supportingly dense in $L^2(\mathcal{G},\mathcal{E})$ we see that W_t has speed of propagation ≤ 1.

Next for any $a \in \mathbb{R}$ consider the operator:

$$W_{t,a} = h\bigl(t^2(\Delta - a)\bigr),$$

where

$$h(x) = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \cdots = \begin{cases} \cos\sqrt{x} & \text{if } x \ge 0, \\ \cosh\sqrt{-x} & \text{if } x < 0. \end{cases}$$

So $W_{t,0}$ is just W_t as above. Since Δ is positive semidefinite, $\|W_{t,a}\| \le \cosh\bigl(t\sqrt{a}\,\bigr)$. The analogue of Proposition 4.8 for the wave equation $u_{tt} = -\Delta u + au$ is easily verified, and we conclude (using Subsection 4.3) as we did for W_t that:

Proposition 4.13. *For fixed a, the propagation speed of $W_{t,a}$ is ≤ 1.*

5. Applications

In this section we give examples of how to apply our edge-based approach to get graph theoretic results from analysis results. We give a new bound on eigenvalues based on set distances in graph theory; the bound it gives on diameters can be better or worse than the well-known bound of Chung, Faber, and Manteuffel (in [**CFM94**]); our technique is very simple and works in analysis (to give the result of Friedman and Tillich in [**FT**]). We also show, for example, that the graph diameter inequalities of Chung, Grigor'yan, and Yau in [**CGY96**, **CGY97**], which mildly generalize that of Chung, Faber, and Manteuffel (in [**CFM94**]), is optimal to first-order, in a certain sense, for small Laplacian eigenvalue. We give other results that illustrate our ability to translate from analysis to graph theory, although these other results do not improve the best known graph theory results.

5.1. Distances and diameter: using old results. A number of articles have inequalities relating distances to sets or diameters of a space to Laplacian eigenvalues, both for manifolds and graphs (see [**AM85**, **Moh91**, **LPS88**, **CFM94**, **CGY96**, **BL97**, **CGY97**]). In [**FT**] the philosophy that $I - \Delta_G$ "acts like" $\cos\sqrt{\Delta_E}$ of the preceding section is used to apply graph theoretic techniques (namely those of [**CGY97**]) to analysis and get an improved eigenvalue bound based. Here we go the other way, taking analysis results or more general results, and apply them to get graph theoretic results. The result we obtained is not the best known, but it is better than some results, especially one that can be derived from the same technique using the vertex-based Laplacian. However, in the next subsection we give a distances/eigenvalues technique that yields new results in graph theory.

Consider the result of Bobkov and Ledoux, which states that for any metric probability space, (M, ρ, μ), and disjoint Borel sets X, Y we have

$$\sqrt{\lambda}\,\rho(X,Y) \leq -\log(\mu X\ \mu Y) \tag{5.1}$$

where $\rho(X,Y)$ is the distance from X to Y and λ is the optimal constant in the Poincaré inequality (in other words, λ is the first nonzero Neumann eigenvalue in the case of a compact manifold or finite graph). Let us apply this to bounding the diameter of a graph.

One way is to directly apply Equation (5.1) to the vertex-based situations, with $\lambda = \lambda_V$, the vertex-based Laplacian eigenvalue. We get $\mu X = |X|/n$, and we take X, Y to consist of single points of distance D where D is the diameter. We conclude that $\sqrt{\lambda_V}\ D \leq \log(n^2) = 2\log n$, or

$$D \leq \frac{2\log n}{\sqrt{\lambda_V}}.$$

Here λ_V is the first nonzero eigenvalue of the Laplacian $I - \widetilde{A}$ where $\widetilde{A}$ is the normalized adjacency matrix. Notice that if our graph is d-regular, then $\lambda_V = \lambda_T/d$ where λ_T is the first nonzero eigenvalue of the traditional graph theoretic Laplacian. We conclude:

$$D \leq 2\log n\sqrt{d/\lambda_T}.$$

Alternatively we may apply Equation (5.1) to the edge-based Laplacian. Notice that the similar and slightly weaker inequalities of [**CGY96**, **CGY97**] require the Laplacian, Δ, to have $\cos\bigl(t\sqrt{-\Delta}\bigr)$ extend supports of functions by a distance at most t. So our edge-based Laplacian could be used with these earlier results, whereas the vertex-based Laplacian could not. We take X, Y to be balls of size $1/2$ about two points of distance D, the diameter. We conclude:

$$\sqrt{\lambda_E}\ (D-1) \leq \log(n^2) = 2\log n.$$

Seeing as $\sqrt{\lambda_E} = \cos^{-1}(1-\lambda_V)$, we have

$$D - 1 \leq \frac{2\log n}{\cos^{-1}(1-\lambda_V)}.$$

But it is easy to see that $\cos^{-1}(1-a) \geq \sqrt{2a}$ for any a. Hence we have

$$D - 1 \leq \sqrt{2/\lambda_V}\ \ \log n.$$

In the d-regular case this gives

$$D - 1 \leq \sqrt{(2d)/\lambda_T}\ \ \log n. \tag{5.2}$$

We conclude:

(1) With the same technology, i.e., using Equation (5.1), the edge-based technique does better than the vertex-based technique by roughly a factor of $\sqrt{2}$.
(2) In previous Laplacian bounds (e.g., [**CGY96**, **CGY97**]) the edge-based technique can be applied whereas the vertex-based technique cannot (by support extending restrictions that are equivalent to the wave equation having propagation speed 1).

It is interesting to note that the edge-based diameter bound we derived (in Equation (5.2)) improves the original Alon–Milman bound[6] of

$$D \leq 2\lfloor \sqrt{(2d/\lambda_T)}\ \log_2 n \rfloor$$

(see [**AM85**]), in the case where d/λ_T and n are large. Similarly we have improved Mohar's improvement (see [**Moh91**]) of the Alon–Milman result, taking $\lambda_\infty \leq 2d$ (the highest Laplacian eigenvalue)[7]). But the result of Chung, Faber, and Manteuffel (see [**CFM94**]) improves on our edge-based result by a factor of 2. In fact, it is precisely this factor of 2 that we gain in applying the result in Chung, Faber, and Manteuffel to analysis, in [**FT**], using the relation in this paper between graph Laplacians and analysis-like (e.g., edge-based) Laplacians.

5.2. Distances and diameters: new results. In this subsection we give a new way of proving distance/eigenvalues or diameter/eigenvalue results. Graph theoretically this yields a new result, which is an edge-based analogue of the Chung, Faber, and Manteuffel (see [**CFM94**]) result; our new results cannot be compared to that of Chung, Faber, and Manteuffel — it can yield better or worse results. This proof also carries over to analysis, where it gives a different (and in some sense shorter) proof of the Friedman–Tillich result in [**FT**]. But however different our proofs or results are, they can be

[6]It is interesting that this bound is often misquoted, as the authors use $[a]$ to denote $\lfloor a \rfloor$ (the greatest integer $\leq a$) without ever explicitly saying so in the paper. Many authors incorrectly guess the meaning of $[a]$.

[7]Since we are thinking of d/λ_T as being large, it seems reasonable to expect that λ_∞ should not be far from $2d$.

viewed as variants of older results stemming from the same basic technique used in [**CGY97**] and [**FT**].

Let X_0 denote the constants in $L^2(\mathcal{G}, \mathcal{E})$, and X_1 the orthogonal complement of X_0; for $i = 0, 1$ let π_i denote the projection onto X_i (so $\pi_0 + \pi_1$ is the identity). Let $W_{t,a}$ be the operators defined at the end of Section 4, and fix $a = \lambda_E$ to be the first nonzero eigenvalue of Δ.

Proposition 5.1. *If f, g are two functions whose supports are at a distance of at least d, we have*

$$\cosh\bigl(d\sqrt{\lambda_E}\,\bigr) \geq \frac{\|\pi_1 f\|\|\pi_1 g\|}{\|\pi_0 f\|\|\pi_0 g\|}.$$

Proof. By Proposition 4.13 we have

$$0 = (W_{d,a} f, g) = (W_{d,a}\pi_0 f, \pi_0 g) + (W_{d,a}\pi_1 f, \pi_1 g).$$

Now

$$(W_{d,a}\pi_0 f, \pi_0 g) = \pm\cosh\bigl(d\sqrt{\lambda_E}\,\bigr)\|\pi_0 f\|\|\pi_0 g\|$$

since $\pi_0 f, \pi_0 g$ are both constants. Since $W_{d,a}$ restricted to X_1 has norm at most one, we have

$$\bigl|(W_{d,a}\pi_1 f, \pi_1 g)\bigr| \leq \|\pi_1 f\|\|\pi_1 g\|,$$

and the proposition follows. □

Corollary 5.2. *Let X, Y be disjoint measurable subsets of distance $\geq d$. Then*

$$d\sqrt{\lambda_E} \leq \cosh^{-1}\left(\sqrt{\frac{\mathcal{E}(X^{\mathrm{c}})\mathcal{E}(Y^{\mathrm{c}})}{\mathcal{E}(X)\mathcal{E}(Y)}}\right).$$

Proof. Take f, g to be the respective characteristic functions of X, Y. □

We finish this subsection with a discussion of the above proposition and its corollary.

First of all, these proofs carry right over to analysis, where they yield the results of Friedman and Tillich (in [**FT**]) with a considerably simpler proof; however, the basic idea that $(W_{d,a} f, g) = 0$ based on the supports of f, g the the speed of propagation of W appears before.

Second, the above proposition and corollary can be generalized to k functions or k sets for any $k \geq 2$, with the same technique that appears in [**CGY96**], also used in [**CGY97**, **FT**]. In the analysis case we recover the results for k functions or sets that appear in [**FT**]. For graphs, our corollary would read that if $X_1, \ldots, X_k$ were disjoint measurable subsets any two of which had supports of distance $\geq d$, then

$$d\sqrt{\lambda_E} \leq \min_{i \neq j} \cosh^{-1}\left(\sqrt{\frac{\mathcal{E}(X_i^{\mathrm{c}})\mathcal{E}(X_j^{\mathrm{c}})}{\mathcal{E}(X_i)\mathcal{E}(X_j)}}\right).$$

Our proposition would read similarly.

Finally, we remark that the above corollary yields a diameter result that can be better (or worse) than that of Chung, Faber, and Manteuffel (see [**CFM94**]). Similarly our corollary can be better or worse than the comparable theorem in [**CGY97**]. For example, the Chung, Faber, and Manteuffel result states that in a regular graph

$$(D-1)\cosh^{-1}\left(1+\frac{2\lambda_V}{\lambda_n-\lambda_V}\right)\leq\cosh^{-1}(n-1),$$

where D is the diameter, $n=|V|$, and λ_V,λ_n are respectively the smallest and largest positive Δ_V eigenvalues. Taking balls of radius $\delta/2$ about two points of distance D and applying Corollary 5.2 we get

$$(D-\delta)\sqrt{\lambda_E}\leq\cosh^{-1}(2n/\delta-1)$$

for any $\delta\leq 2$. It follows that the result obtained here, taking $\delta=2$, is better than the result in [**CGY97**], provided that λ_V and $2-\lambda_n$ are both $\leq c/\log n$ for a constant, c (using $\cos\sqrt{\lambda_E}=1-\lambda_V$).

5.3. Invariants. A typical spectral invariant studied in the analysis literature is the wave invariant

$$W(t)=\mathrm{Trace}\big(\cos t\sqrt{\Delta_E}\,\big)=\sum_j\big(\cos t\sqrt{\lambda_j}\,\big),$$

with λ_j running through all Laplacian eigenvalues. This sum can be understood in several ways; here we think of

$$\widetilde{W}(t)=\sum_j\big(e^{it\sqrt{\lambda_j}}\,\big)$$

as a complex analytic function defined on the subset of complex numbers with positive imaginary part, and then we extend $\widetilde{W}$ analytically to the whole complex plane; W is just the real part of $\widetilde{W}$.

It is well-known that in analysis the real singularities of W are at $t=0$ and t being (plus or minus) the length of a closed geodesic. If all edges of a graph have length one, then we know we have $\alpha_1,\dots,\alpha_{2|E|}$ such that the edge-based eigenvalues (with their respective multiplicities) are precisely those squares of $\alpha_j+2\pi\mathbb{Z}_{\geq 0}$. We have

$$\widetilde{W}(t)=\frac{1}{1-e^{2\pi it}}\sum_{j=1}^{2|E|}e^{it\alpha_j}.$$

This has a pole at $t\mathbb{R}$ precisely when $t=0$ or t is an integer with $\sum e^{it\alpha_j}\neq 0$.

To understand the vanishing or not of $\sum e^{it\alpha_j}$, consider that the α_j's are of two types; one type is a $(\pi,2\pi)$ pair coming from an edge and interior vertex count, and such α_j's cancel in the sum $\sum\cos(it\alpha_j)$ for t an odd integer and contribute 2 when t is even; the other type is a $\alpha, 2\pi - \alpha$ pair with

$$e^{it\alpha} + e^{it(2\pi-\alpha)} = 2\cos(t\alpha)$$

for t an integer. Hence this sum is essentially the trace of the t-th Chebyshev polynomial in $\widetilde{A}$.

It follows that the first odd $t > 0$ for which $\widetilde{W}$ has a pole is the length of the smallest odd cycle. However it doesn't seem like such a simple statement holds for higher odd values of t or even values, and so the invariant $\widetilde{W}$ is not entirely analogous to its analysis counterpart.

It is natural to ask if any spectral, edge-based invariants (such as those whose analysis analogues are interesting, for example) yield new and interesting graph invariants. Such invariants would, in particular, include traces of Chebyshev polynomials of $\widetilde{A}$ when the edge lengths are one.

5.4. Cheeger's inequality. In this section we mention that Cheeger's inequality holds for the edge-based Laplacian as well as the vertex-based, but the second one, at least for r-regular graphs yields the first one. We will require some notions from [**FT99**].

Consider an open subset of $A \subset \mathcal{G}$ whose boundary contains no vertices, and let $\mathcal{A}(\partial A)$ be the "area" of A's boundary (see [**FT99**]); this is just the sum of a_e for each boundary point of A lying on e. Also $\mathcal{E}(A)$ is just the total $\mathcal{E}$ measure of A, and we set

$$h_E = \min_{\mathcal{E}(A)\leq\mathcal{E}(\mathcal{G})/2} \frac{\mathcal{A}(\partial A)}{\mathcal{E}(A)}.$$

The co-area formula of [**FT99**] and the arguments to prove Cheeger's inequality immediately carry over here to yield:

$$\lambda_E \geq h_E^2/4. \tag{5.3}$$

We can compare this to Cheeger's inequality for the vertex-based case (i.e., Dodziuk's inequality, see [**Dod84**, **FT99**]),

$$\lambda_V \geq h_V^2/2 \tag{5.4}$$

for the 1-regular graphs (see Section 2), where

$$h_V = \min_{B\subset V,\, \mathcal{V}(B)\leq\, \mathcal{V}(V)/2} \frac{\mathcal{E}(E(B, B^{\mathrm{c}}))}{\mathcal{V}(B)},$$

where $E(B, B^{\mathrm{c}})$ denotes the set of edges with one endpoint in B and one in B^{c}.

Let us compare these two Cheeger's inequalities, in case the graph is d-regular in the traditional sense (each vertex is the endpoint of d edges) with unit edge lengths and weights. We consider now the graph $\mathcal{G}$ derived from it, where $\mathcal{V}$ at any vertex is taken to be d so as to make the graph 1-regular.

We shall need a simple lemma:

Lemma 5.3. *For a* 1*-regular graph* $h_V \geq h_E/2$.

Proof. Let B be the subset of vertices of size smaller than $\mathcal{V}(V)/2$ for which

$$h_V = \frac{\mathcal{E}(E(B, B^{\mathrm{c}}))}{\mathcal{V}(B)}.$$

For a subset X of vertices we denote by X_t the set of points lying on edges that have both endpoints on B or are within a distance t from X. Note that

$$\mathcal{E}(B_{1/2}) + \mathcal{E}({B^{\mathrm{c}}}_{1/2}) = \mathcal{E}(\mathcal{G}).$$

It follows that the measure of one of these two sets is smaller than or equal to $\mathcal{V}(\mathcal{G})/2$.

Case 1: $\mathcal{E}(B_{1/2}) \leq \mathcal{V}(\mathcal{G})/2$. Here $\frac{\mathcal{A}(\partial B_{1/2})}{\mathcal{E}(B_{1/2})} \geq h_E$. Since $\mathcal{E}(E(B, B^{\mathrm{c}})) = \mathcal{A}(\partial B_{1/2})$ and $\mathcal{V}(B) \leq 2\mathcal{E}(B_{1/2})$, we obtain

$$h_V = \frac{\mathcal{E}(E(B, B^{\mathrm{c}}))}{\mathcal{V}(B)} \geq \frac{\mathcal{A}(\partial B_{1/2})}{2\mathcal{E}(B_{1/2})} \geq h_E/2.$$

Case 2: $\mathcal{V}({B^{\mathrm{c}}}_{1/2}) \leq \mathcal{E}(\mathcal{G})/2$. This implies that

$$\mathcal{V}(V)/4 = \mathcal{E}(\mathcal{G})/2 \geq \mathcal{V}({B^{\mathrm{c}}}_{1/2}) \geq \mathcal{V}(B^{\mathrm{c}})/2,$$

so B^{c} is also of measure smaller than or equal to $\mathcal{V}(V)/2$, and therefore $\mathcal{V}(B) = \mathcal{V}(B^{\mathrm{c}}) = \mathcal{V}(V)/2$. This means that

$$h_V = \frac{\mathcal{E}(E(B, B^{\mathrm{c}}))}{\mathcal{V}(B^{\mathrm{c}})}.$$

By using the same arguments as in the previous case (with B^{c} replacing B) we also get $h_V \geq h_E/2$. ☐

Now notice that

$$\lambda_E = (\cos^{-1}(1 - \lambda_V))^2 \geq 2\lambda_V.$$

Hence

$$\lambda_E \geq 2\lambda_V \geq h_V^2 \geq h_E^2/4.$$

In other words: for 1-regular graphs Cheeger's inequality for vertices, (5.4), implies the Cheeger inequality for edges, (5.3).

5.5. Optimal distance bounds. In this section we prove the following theorem:

Theorem 5.4. *Let* C, C_1, C_2 *be constants such that the following holds: for any graph with edge lengths one, whose diameter* D *is realized by two vertices,* u, v*, we have*

$$D - 1 \leq \frac{C}{\sqrt{2\lambda_V} + C_1\lambda_V} \log\left(\frac{C_2\mathcal{V}(V)^2}{\mathcal{V}(u)\mathcal{V}(v)}\right). \tag{5.5}$$

Then $C \geq 1/2$*. The same is true if we insist that the graph is* 1*-regular.*

The bound of Chung, Faber, and Manteuffel (in [**CFM94**]) implies Equation (5.5) with $C = 1/2$ assuming $\mathcal{V}$ is constant; the generalization to general $\mathcal{V}$ is implied by the results in [**CGY96**, **CGY97**] (with different constants C_2 in the two articles). Actually, there they assume the edge weights are integral; but this assumption clearly implies the same for rational edge weights, and therefore arbitrary real edge weights. So regarding λ_V as small, these inequalities are optimal to first-order.

Our proof is based on a standard metric probability space, the "exponential distribution on the nonnegative reals," or on a standard Riemannian manifold, the surface of revolution of $y = e^{-x}$ with x nonnegative. We model this on a graph by taking a sequence of edges with edge weight exponentially decreasing. These analysis examples (the exponential distribution and the surface of revolution of $y = e^{-x}$) prove a bound in analysis that is analogous to the bound $C \geq 1/2$ in the theorem above (see [**FT**]).

Proof. Fix a small $\eta \in (0, 1)$, and let $r = 1 - \eta$. Fix an integer $D > 0$. Consider the graph, $\mathcal{G}$, whose vertices are the integers, $V = \{0, 1, \ldots, D\}$, with an edge from i to $i+1$ of weight r^i for all nonnegative integers $i < D$.

Making $\mathcal{G}$ into a 1-regular graph is done by taking $\mathcal{V}(i) = r^{i-1} + r^i$ for $i \neq 0, D$, dropping one of these summands when $i = 0$ or $i = D$. Here the vertices furthest from each other are 0 and D, and therefore

$$\log\left(\frac{\mathcal{V}(V)^2}{\mathcal{V}(u)\mathcal{V}(v)}\right) = \log\left(\frac{4\left(\frac{1-r^{D+1}}{1-r}\right)^2}{1 \cdot r^D}\right) \leq K(r) - D\log r.$$

We obtain a lower bound on λ_V by using Cheeger's inequality (5.4). Obviously to find Cheeger's constant h_V we just need to consider vertex sets that are connected, i.e., of the form $B = \{a, a+1, \ldots, b\}$.

We have to distinguish between two cases: $a > 0$ and $a = 0$. In the first case

$$\frac{\mathcal{E}(E(B, B^{\mathrm{c}}))}{\mathcal{V}(B)} = \frac{r^{a-1} + \mathbf{1}_{\{b \neq D\}} r^b}{r^{a-1} + 2\sum_{i=a}^{b-1} r^i + \mathbf{1}_{\{b \neq D\}} r^b} \geq \frac{1}{2(1 + r + r^2 + \cdots)} = \frac{1-r}{2}.$$

In the second case we get the same lower bound with a slightly more tedious calculation. First we note that in this case $b \neq D$ (since $\mathcal{V}(B) \leq \mathcal{V}(V)/2$) and therefore

$$\mathcal{V}(B) = r^b + 2\sum_{i=0}^{b-1} r^i \leq \mathcal{V}(V)/2 = \frac{1 - r^{D+1}}{1 - r}. \tag{5.6}$$

This implies that

$$r^b \geq \frac{1 + r^{D+1}}{1 + r}. \tag{5.7}$$

We use the previous upper bound on $\mathcal{V}(B)$, and this lower bound on r^b to obtain

$$\frac{\mathcal{E}(E(B,B^{\mathrm{c}}))}{\mathcal{V}(B)} = \frac{r^b}{\mathcal{V}(B)} \geq \frac{(1+r^{D+1})(1-r)}{(1+r)(1-r^{D+1})} \geq \frac{1-r}{2}.$$

By using (5.4) it follows that

$$\lambda_V \geq (1-r)^2/8 = \eta^2/8,$$

and therefore

$$D - 1 \leq \frac{C}{\eta/2 + C_1\eta^2/8} \left(K(r) - D\log r\right).$$

Taking $D \to \infty$ we conclude

$$1 \leq -\frac{C}{\eta/2 + C_1\eta^2/8} \log(1-\eta).$$

Taking $\eta \to 0^+$ yields $1 \leq 2C$. □

References

[AM85] N. Alon and V.D. Milman, λ_1, *isoperimetric inequalities for graphs, and superconcentrators*, J. Combin. Theory Ser. B, **38**(**1**) (1985), 73–88, MR 0782626 (87b:05092), Zbl 0549.05051.

[BL97] S. Bobkov and M. Ledoux, *Poincaré's inequalities and Talagrand's concentration phenomenon for the exponential distribution*, Probab. Theory Related Fields, **107**(**3**) (1997), 383–400, MR 1440138 (98e:60026), Zbl 0878.60014.

[CFM94] F.R.K. Chung, V. Faber and T.A. Manteuffel, *An upper bound on the diameter of a graph from eigenvalues associated with its Laplacian*, SIAM J. Discrete Math., **7**(**3**) (1994), 443–457, MR 1285582 (95j:05115), Zbl 0808.05072.

[CGY96] F.R.K. Chung, A. Grigor'yan, and S.-T. Yau, *Upper bounds for eigenvalues of the discrete and continuous Laplace operators*, Adv. Math., **117**(**2**) (1996), 165–178, MR 1371647 (96m:58254), Zbl 0844.53029.

[CGY97] F.R.K. Chung, A. Grigor'yan, and S.-T. Yau. *Eigenvalues and diameters for manifolds and graphs*, in 'Tsing Hua lectures on geometry & analysis' (Hsinchu, 1990–1991), 79–105, Internat. Press, Cambridge, MA, 1997, MR 1482032 (98h:58206), Zbl 0890.58093.

[CH89] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Vol. II, Partial differential equations, Wiley, New York, 1962, MR 1013360 (90k:35001), Zbl 0729.35001.

[Dod84] J. Dodziuk, *Difference equations, isoperimetric inequality and transience of certain random walks*, Trans. Amer. Math. Soc., **284** (1984), 787–794, MR 0743744 (85m:58185), Zbl 0512.39001.

[Fri69] A. Friedman, *Partial Differential Equations*. Holt, Rinehart and Winston, 1969, MR 0445088 (56 #3433), Zbl 0224.35002.

[Fri93] J. Friedman, *Some geometric aspects of graphs and their eigenfunctions*, Duke Math. J., **69**(**3**) (1993), 487–525, MR 1208809 (94b:05134), Zbl 0785.05066.

[FT] J. Friedman and J.-P. Tillich, *Laplacian eigenvalues and distances between subsets of a manifold*, J. Differential Geom., **56**(**2**) (2000), 285–299, MR 1863018 (2002g:58050), Zbl 1032.58015.

[FT99] J. Friedman and J.-P. Tillich. *Calculus on graphs*, preprint, October 1999, arXiv: cs.DM/0408028.

[GT83] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Grundlehren der Math. Wissenschaften, **224**, Springer-Verlag, 2nd edition, 1983, MR 0737190 (86c:35035), Zbl 0562.35001.

[Hur00] N.E. Hurt, *Mathematical Physics of Quantum Wires and Devices: From Spectral Resonances to Anderson Localization*, Math. and its Appl., **506**, Kluwer, Dordrecht, 2000, MR 1862155 (2002h:81004).

[KZ01] P. Kuchment and H. Zeng, *Convergence of spectra of mesoscopic systems collapsing onto a graph*, J. Math. Anal. Appl., **258**(**2**) (2001), 671–700, MR 1835566 (2002d:35158), Zbl 0982.35076.

[LPS88] A. Lubotzky, R. Phillips and P. Sarnak, *Ramanujan graphs*, Combinatorica, **8**(**3**) (1988), 261–277, MR 0963118 (89m:05099), Zbl 0661.05035.

[Moh91] B. Mohar, *Eigenvalues, diameter, and mean distance in graphs*, Graphs Combin., **7**(**1**) (1991), 53–64, MR 1105467 (92d:05105), Zbl 0771.05063.

[RS01] J. Rubinstein and M. Schatzman, *Variational problems on multiply connected thin strips.* I. *Basic estimates and convergence of the Laplacian spectrum*, Arch. Ration. Mech. Anal., **160**(**4**) (2001), 271–308, MR 1869667 (2003b:35007), Zbl 0997.49003.

[Smo83] J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, Grundlehren der Mathematischen Wissenschaften, **258**, Springer-Verlag, New York, 1983, MR 0688146 (84d:35002), Zbl 0508.35002.

[Wey12] H. Weyl, *Der asymptotische verteilungsgesetz der eigenwerte linearer partieller differentialgleichunger*, Math. Ann., **71** (1912), 441–469, JFM 43.0436.01.

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