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Let A be a prime ring whose symmetric Martindale quotient ring contains a nontrivial idempotent. Generalized skew derivations of A are characterized by acting on zero products. Precisely, if $g, \delta\colon A \to A$ are additive maps such that $\sigma(x)g(y) + \delta(x)y = 0$ for all $x, y \in A$ with $xy = 0$, where σ is an automorphism of A, then both g and δ are characterized as specific generalized σ-derivations on a nonzero ideal of A.

1. Results

Let B be a ring with a subring A. An additive map $\delta\colon A \to B$ is called a derivation if $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in A$. In a recent paper Jing, Lu and Li proved the following result [**6**, Theorem 6]:

Let $\mathcal{B}$ be a standard operator algebra in a Banach space X containing the identity operator I and let $\delta\colon \mathcal{B} \to \mathcal{B}$ be a linear map such that $\delta(AB) = \delta(A)B + A\delta(B)$ for any pair $A, B \in \mathcal{B}$ with $AB = 0$. Then $\delta(AB) = \delta(A)B + A\delta(B) - A\delta(I)B$ for all $A, B \in \mathcal{B}$. If in addition $\delta(I) = 0$, then δ is a derivation.

The result says that an additive map on a standard operator algebra is almost a derivation if it satisfies the expansion formula of derivations on pairs of elements with zero product. Since standard operator algebras involve many idempotents, from this point of view Chebotar, Ke and P.-H. Lee studied maps acting on zero products in the context of prime rings [**2**]. To state their results precisely we must first fix some notation.

Throughout, unless specially stated, A will denote a prime ring with center Z, extended centroid C and symmetric Martindale quotient ring Q. The maximal right and left quotient rings of A will be denoted by Q_{mr} and Q_{ml}, respectively. See [**1**] for details. Theorem 2 of [**2**] says this:

Let $\delta\colon A \to A$ be an additive map such that $\delta(x)y + x\delta(y) = 0$ for $x, y \in A$ with $xy = 0$. Suppose that Q contains a nontrivial idempotent e such that $eA \cup Ae \subseteq A$.

(I) *If $1 \in A$, then $\delta(xy) = \delta(x)y + x\delta(y) - \lambda xy$ for all $x, y \in A$, where $\lambda = \delta(1) \in Z$. In particular, if $\delta(1) = 0$, then δ is a derivation of A.*

(II) *If* $\deg A \geq 3$, *there exists* $\lambda \in C$ *such that* $\delta(xy) = \delta(x)y + x\delta(y) - \lambda xy$ *for all* $x, y \in A$.

Generalized derivations and σ-*derivations* (or *skew derivations*) are two natural generalizations of derivations, and are defined as follows. Let σ be an automorphism of A. An additive map $\delta\colon A \to Q_{ml}$ is called a σ-*derivation* if $\delta(xy) = \sigma(x)\delta(y) + \delta(x)y$ for all $x, y \in A$. Basic examples are derivations and $\sigma - 1$. Given $b \in A$, the map $\delta\colon x \in A \mapsto \sigma(x)b - bx$ obviously defines a σ-derivation, called the *inner* σ-*derivation* defined by b.

An additive mapping $g\colon A \to Q_{ml}$ is a *generalized derivation* if there exists a derivation $d\colon A \to Q_{ml}$ such that $g(xy) = xg(y) + d(x)y$ for all $x, y \in A$. As basic examples we mention derivations, generalized inner derivations (maps $x \mapsto ax + xb$ for $a, b \in A$) and left A-module mappings from A into itself. From this one easily sees that a map δ as in Theorem 2 of [**2**] (see bottom of previous page) is indeed a generalized derivation. In this paper we will generalize that theorem from a different point of view. We start with a definition of generalized skew derivations, generalizing both skew derivations and generalized derivations.

An additive map $g\colon A \to Q_{ml}$ is called a *generalized* σ-*derivation*, where σ be an automorphism of A, if there exists an additive map $\delta\colon A \to Q_{ml}$ such that $g(xy) = \sigma(x)g(y) + \delta(x)y$ for all $x, y \in A$. It is clear that δ is uniquely determined by g, which is called the associated additive map of g. It is easy to check that δ is always a σ-derivation (see [**10**]). We are now in a position to state our main result:

Theorem 1.1. *Let A be a prime ring with symmetric Martindale quotient ring Q. Suppose that Q contains a nontrivial idempotent and that g, $\delta\colon A \to A$ are additive maps. If $\sigma(x)g(y) + \delta(x)y = 0$ for all $x, y \in A$ with $xy = 0$, where σ is an automorphism of A, there exist a nonzero ideal N of A and a σ-derivation $d\colon A \to Q$ such that*

$$g(x) = d(x) + \sigma(x)b \quad \textit{and} \quad \delta(x) = d(x) + ax$$

for all $x \in N$, where $b \in Q_{ml}$ and $a \in Q_{mr}$. In addition, we can take $N = A$ if $eA \cup Ae \subseteq A$ for some nontrivial idempotent $e \in Q$.

The proof of the theorem depends on both the Lie structure of rings and the theory of functional identities, and will be given in the next section. As an immediate consequence of Theorem 1.1 we have the following generalization of Chebotar, Ke and P.-H. Lee's theorem [**2**, Theorem 2]:

Corollary 1.2. *Let A be a prime ring with extended centroid C and symmetric Martindale quotient ring Q. Suppose that Q contains a nontrivial idempotent. If $\delta\colon A \to A$ is an additive map such that $x\delta(y) + \delta(x)y = 0$ for $x, y \in A$ with $xy = 0$, then there exists a nonzero ideal N of A such that*

$\delta(x) = d(x) + \lambda x$ for all $x \in N$, where $\lambda \in C$ and $d\colon A \to Q$ is a derivation. In addition, we can take $N = A$ if $eA \cup Ae \subseteq A$ for some nontrivial idempotent $e \in Q$.

The following corollary gives Jing, Lu and Li's theorem [**6**, Theorem 6] in the context of prime rings:

Corollary 1.3. *Let A be a prime ring with extended centroid C and $c \in A$. Suppose that A possesses a nontrivial idempotent. If $\delta\colon A \to A$ is an additive map such that $x\delta(y) + \delta(x)y + xcy = 0$ for $x, y \in A$ with $xy = 0$, then there exist a derivation $d\colon A \to AC$ and $\mu \in C$ such that $\delta(x) = d(x) + (\mu - c)x$ for all $x \in A$.*

Proof. By assumption, we have $x(\delta(y)+cy)+\delta(x)y = 0$ for all $x, y \in A$ with $xy = 0$. In view of Theorem 1.1, there exist a derivation $d\colon A \to Q$, $a \in Q_{mr}$ and $\mu \in Q_{ml}$ such that $\delta(x) = d(x) + ax$ and $\delta(x) + cx = d(x) + x\mu$ for all $x \in A$. Choose a dense right ideal ρ and a dense left ideal λ of A such that $a\rho \subseteq A$ and $\lambda\mu \subseteq A$. Let $x \in \rho$, $z \in A$ and $y \in \lambda$. Then $xzy \in \rho A\lambda \subseteq \rho \cap \lambda$. Thus $(a + c)x, y\mu \in A$ and $((a + c)x)zy = xz(y\mu)$. By Martindale's Lemma [**11**], y and $y\mu$ are C-dependent for $y \in \lambda$. It is now easy to prove that $\mu \in C$. Thus $a = \mu - c$ follows and so $\delta(x) = d(x) + (\mu - c)x$ for all $x \in A$. In particular, we have $d(A) \subseteq AC$, proving the corollary.

The next application is to generalized polynomial identities. An additive map $f\colon A \to A$ is called an *elementary operator* if there exist finitely many $a_i, b_i \in AC$ such that $f(x) = \sum_i a_i x b_i$ for all $x \in A$.

Corollary 1.4. *Let A be a prime ring with extended centroid C. Suppose that its symmetric Martindale quotient ring Q contains a nontrivial idempotent. If f and g are two elementary operators of A satisfying $xf(y) + g(x)y = 0$ for $x, y \in A$ with $xy = 0$, then there exist $a, b, q \in AC$ and such that $f(y) = [q, y] + yb$ and $g(x) = [q, x] + ax$ for all $x, y \in A$.*

Proof. Since f, g are elementary operators, there exist finitely many a_i, b_i, $c_i, d_i \in AC$ such that $f(x) = \sum_i a_i x b_i$ and $g(x) = \sum_j c_j x d_j$ for all $x \in A$. In view of Theorem 1.1, there exist a nonzero ideal N of A, a derivation $d\colon A \to Q$ and elements $a \in Q_{mr}$, $b \in Q_{ml}$ such that

$$\textstyle\sum_i a_i y b_i = d(y) + yb \quad\text{and}\quad \sum_j c_j x d_j = d(x) + ax$$

for all $x, y \in N$. It is well-known that d can be uniquely extended to a derivation of Q_{ml} into Q_{ml} and $\sum_i a_i y b_i = d(y) + yb$ for all $y \in Q_{ml}$ (see, for instance, [**9**, Theorem 2]). In particular, we set $y = 1$, implying that $b = \sum_i a_i b_i \in AC$. An analogous argument proves $a \in AC$, so $d(AC) \subseteq AC$. Applying Kharchenko's Theorem ([**7**, Lemma 1] or [**8**, Theorem 2]), we conclude that d is X-inner; that is, there exists $q \in Q$ such that $d(y) = [q, y]$ for all $y \in A$. Thus $qy - yq = \sum_i a_i y b_i - yb$ for all $y \in A$. Applying

Martindale's Lemma [**11**], q lies in the C-linear span of the elements b_i's, b and 1. Thus $q \in AC + C$. Since, for $\beta \in C$, $[q+\beta, y] = [q, y]$ for all $y \in A$, we may take $q \in AC$, proving the corollary.

The following example shows that the existence of nontrivial idempotents in Q is essential to Theorem 1.1.

Example 1.5. Let A be a prime ring, not a domain, with center Z and let

$$\mathcal{M}_A = \{a \in A \mid xay = 0 \text{ whenever } x, y \in A \text{ with } xy = 0\}.$$

Suppose that $\mathcal{M}_A$ is noncentral. Choose an element $c \in \mathcal{M}_A \setminus Z$. Let $f, g\colon A \to A$ be additive maps defined by $f(x) = cxc$ and $g(x) = x$ for all $x \in A$. Then $xf(y) + g(x)y = 0$ for $x, y \in A$ with $xy = 0$. We claim that f, g cannot assume the forms given in Theorem 1.1. Indeed, suppose there exist a derivation $d\colon A \to Q$ and a nonzero ideal N of A such that

$$f(x) = d(x) + xb \quad \text{and} \quad g(x) = d(x) + ax$$

for all $x \in N$, where $a \in Q_{mr}$ and $b \in Q_{ml}$. Then $d(x) = (1-a)x$ for all $x \in A$, implying $d = 0$. Thus $xb = cxc$ for all $x \in A$. Applying Martindale's Lemma [11], we see that $c \in C$, the extended centroid of A, and so $c \in Z$, a contradiction.

Such prime rings do exist. One example is due to Dubrovin [**3**]. Another is $A = K\{x, y\}/(x^2)$ [**5**, pp. 105–108], where $K\{x, y\}$ is the free algebra over a field K in two noncommuting indeterminates x and y. In this example, A is a prime ring and $Kx + xAx + (x^2)/(x^2)$ coincides with the set of all elements of A with square zero. Let $\bar{x} = x + (x^2) \in A$. Then $a\bar{x}b = 0$ whenever $a, b \in A$ with $ab = 0$. Thus $\bar{x}$ lies in $\mathcal{M}_A$ and is noncentral.

2. Proof of Theorem 1.1

Lemma 2.1. *Let I be a nonzero ideal of A and let $f\colon I \to Q_{ml}$ be a left I-module map. Then there exists $q \in Q_{ml}$ such that $f(x) = xq$ for all $x \in I$.*

Proof. Notice that $Q_{ml}I$ is a dense left ideal of Q_{ml}. We define the map $\widetilde{f}\colon Q_{ml}I \to Q_{ml}$ by $\sum_i x_i a_i \mapsto \sum_i x_i f(a_i)$, where $x_i \in Q_{ml}, a_i \in I$. Then $\widetilde{f}$ is well-defined. Indeed, let $\sum_i x_i a_i = 0$, where $x_i \in Q_{ml}, a_i \in I$. Choose a dense left ideal J of A such that $Jx_i \subseteq I$ for each i. Then, for $y \in J$, we have

$$0 = f\big(y \textstyle\sum_i x_i a_i\big) = f\big(\sum_i (yx_i)a_i\big) = \sum_i yx_i f(a_i) = y\big(\sum_i x_i f(a_i)\big),$$

implying $\sum_i x_i f(a_i) = 0$. Thus $\widetilde{f}$ is well-defined. It is clear that $\widetilde{f}$ is a left Q_{ml}-module map extending f. We remark that the maximal left quotient ring of Q_{ml} coincides with itself. Thus there exists a $q \in Q_{ml}$ such that $\widetilde{f}(z) = zq$ for all $z \in Q_{ml}I$. In particular, $f(x) = xq$ for all $x \in I$. This proves the lemma.

Lemma 2.2. *Let I be a nonzero ideal of A and let $f, g\colon I \to Q_{ml}$ be two additive maps. Suppose that $f(x)y + \sigma(x)g(y) = 0$ for all $x, y \in I$, where σ is an automorphism of A. Then there exists $a \in Q_{ml}$ such that $f(x) = \sigma(x)a$ and $g(y) = -ay$ for all $x, y \in I$.*

Proof. Let $x, y, z \in I$. By assumption,

$$f(zx)y + \sigma(zx)g(y) = 0 \quad \text{and} \quad \sigma(z)(f(x)y + \sigma(x)g(y)) = 0.$$

Thus $(f(zx) - \sigma(z)f(x))y = 0$ and so $f(zx) = \sigma(z)f(x)$ since A is prime. Note that σ can be uniquely extended to an automorphism of Q_{ml}. Consider the map $\phi\colon I \to Q_{ml}$ defined by $\phi(x) = \sigma^{-1}(f(x))$ for $x \in I$. Then ϕ is a left I-module map. By Lemma 2.1, there exists $b \in Q_{ml}$ such that $\phi(x) = xb$ for all $x \in I$. That is, $f(x) = \sigma(x)a$ for all $x \in I$, where $a = \sigma(b) \in Q_{ml}$. It is clear that $g(y) = -ay$ for all $y \in I$. This proves the lemma.

Although the next lemma has a more general version, for our purposes we need only the following special form:

Lemma 2.3. *Let $d\colon M \to Q$ be a σ-derivation, where M is a nonzero ideal of A and σ is an automorphism of A. Assume that there exists a nonzero ideal J of A such that $d(m)J + Jd(m) \subseteq A$ for all $m \in M$. Then d can be uniquely extended to a σ-derivation from A into Q.*

Proof. Replacing J by $M \cap J$, we may assume from the start that $J \subseteq M$. Let $a \in A$. Define a map $\psi_a\colon MJ \to A$ by the rule

$$\psi_a\big(\textstyle\sum_i m_i x_i\big) = \sum_i d(am_i)x_i - \sum_i \sigma(a)d(m_i)x_i \in A,$$

where $m_i \in M$ and $x_i \in J$. We claim that ψ_a is well-defined. Indeed, $\sum_i m_i x_i = 0$ implies $\sum_i (am_i)x_i = 0$, so

$$\begin{aligned} 0 = \textstyle\sum_i d((am_i)x_i) &= \textstyle\sum_i (\sigma(am_i)d(x_i) + d(am_i)x_i) \\ &= \textstyle\sum_i d(am_i)x_i - \sum_i \sigma(a)d(m_i)x_i, \end{aligned}$$

since

$$\begin{aligned} \textstyle\sum_i \sigma(am_i)d(x_i) &= \textstyle\sum_i \sigma(a)\sigma(m_i)d(x_i) \\ &= \textstyle\sum_i \sigma(a)(d(m_i x_i) - d(m_i)x_i) \\ &= \sigma(a)d\left(\textstyle\sum_i m_i x_i\right) - \sum_i \sigma(a)d(m_i)x_i \\ &= -\textstyle\sum_i \sigma(a)d(m_i)x_i. \end{aligned}$$

The claim is proved. It is clear that ψ_a is a right A-module map. Thus ψ_a is defined by an element $\widetilde{d}(a) \in Q_r$, the right Martindale quotient ring of A. That is, $\widetilde{d}\colon A \to Q_r$ has the following property: $\widetilde{d}(a)m = d(am) - \sigma(a)d(m)$ for $a \in A$ and $m \in M$. Let $a, b \in A$ and $m \in M$. Then $\widetilde{d}(ab)m = d(abm) - \sigma(ab)d(m)$. On the other hand, since $bm \in M$ we have

$$\begin{aligned} (\sigma(a)\widetilde{d}(b) + \widetilde{d}(a)b)m &= \sigma(a)(d(bm) - \sigma(b)d(m)) + d(abm) - \sigma(a)d(bm) \\ &= d(abm) - \sigma(ab)d(m). \end{aligned}$$

Thus $\widetilde{d}(ab) = \sigma(a)\widetilde{d}(b) + \widetilde{d}(a)b$, proving that $\widetilde{d}$ is a σ-derivation. Clearly, $\widetilde{d}(m) = d(m)$ for all $m \in M$, so $\widetilde{d}$ is an extension of d. Notice that $J\sigma(M)$ is a nonzero ideal of A. Let $a \in A$ and $m \in M$. Then $d(ma) = \sigma(m)\widetilde{d}(a)+d(m)a$ and so

$$J\sigma(M)\widetilde{d}(a) \subseteq Jd(M) + Jd(M)A \subseteq A,$$

implying that $\widetilde{d}(a) \in Q$. Hence, $\widetilde{d}\colon A \to Q$ and the lemma is proved.

We are now ready to prove the main theorem stated in §1.

Theorem 1.1. *Let A be a prime ring with symmetric Martindale quotient ring Q. Suppose that Q contains a nontrivial idempotent and that g, $\delta\colon A \to A$ are additive maps. If $\sigma(x)g(y) + \delta(x)y = 0$ for all $x, y \in A$ with $xy = 0$, where σ is an automorphism of A, there exist a nonzero ideal N of A and a σ-derivation $d\colon A \to Q$ such that*

$$g(x) = d(x) + \sigma(x)b \quad \textit{and} \quad \delta(x) = d(x) + ax$$

for all $x \in N$, where $b \in Q_{ml}$ and $a \in Q_{mr}$. In addition, we can take $N = A$ if $eA \cup Ae \subseteq A$ for some nontrivial idempotent $e \in Q$.

Proof. Let e be a nontrivial idempotent of Q. Choose a nonzero ideal I of A such that $Ie+eI \subseteq A$. We consider the additive subgroup E of Q generated by the set $\{f \in Q \mid If + fI \subseteq A \text{ and } f^2 = f\}$. Then $e \in E$. We claim that $0 \neq I^2[E,E]I^2 \subseteq E + E^2$.

We follow Herstein's argument [**4**, Proof of Lemma 1.3]. Let $f \in E$ and $x \in I$. Then $f + fx(1-f)$, $f + (1-f)xf \in E$ and so

$$[f, x] = (f + fx(1-f)) - (f + (1-f)xf) \in E.$$

Thus $[E, I] \subseteq E$. Indeed, $[E,E]I^2 \subseteq [E, EI^2] + E[E, I^2] \subseteq E + E^2$, and so

$$I^2[E,E]I^2 \subseteq [I^2, [E,E]I^2] + [E,E]I^4 \subseteq [I^2, E + E^2] + E + E^2.$$

Since $[I^2, E + E^2] \subseteq E + E^2$, we see that $I^2[E,E]I^2 \subseteq E + E^2$. Finally, $0 \neq [e,\, e + eI^2(1-e)] \subseteq [E,E]$, proving our claim.

Let $x, y \in I$ and $f = f^2 \in E$. Then xf, $(1-f)y$, $x(1-f)$, fy belong to A. By assumption,

$$\begin{aligned} &\delta(xf)(1-f)y + \sigma(xf)g((1-f)y) = 0 \qquad \text{and} \\ &\delta(x(1-f))fy + \sigma(x(1-f))g(fy) = 0. \end{aligned} \tag{1}$$

Solve the two equations by Lemma 2.2: there exist two unique elements $u, v \in Q_{ml}$, depending on f, such that

$$\begin{aligned} &\delta(xf)(1-f) = \sigma(x)u, \quad \sigma(f)g((1-f)y) = -uy \qquad \text{and} \\ &\delta(x(1-f))f = \sigma(x)v, \quad \sigma(1-f)g(fy) = -vy. \end{aligned} \tag{2}$$

Thus, by (2), we see that

$$\delta(x)f - \delta(xf) = \sigma(x)(v-u) \quad \text{and} \quad \sigma(f)g(y) - g(fy) = (v-u)y. \tag{3}$$

By (3) and the definition of E, there exists an additive map $d\colon E \to Q_{ml}$ such that

$$\delta(x)m - \delta(xm) = -\sigma(x)d(m) \quad \text{and} \quad \sigma(m)g(y) - g(my) = -d(m)y \tag{4}$$

for all $x, y \in I$ and all $m \in E$. Let $m_1, m_2 \in E$; then

$$\delta(x)m_2 - \delta(xm_2) = -\sigma(x)d(m_2) \text{ and } \delta(x)m_1 - \delta(xm_1) = -\sigma(x)d(m_1) \tag{5}$$

for all $x \in I$. Let $x \in I^2$; then $xm_1 \in I$. By (5) we have

$$\begin{aligned} \delta(xm_1)m_2 - \delta(xm_1m_2) &= -\sigma(xm_1)d(m_2) \qquad \text{and} \\ -\delta(xm_1)m_2 + \delta(x)m_1m_2 &= -\sigma(x)d(m_1)m_2. \end{aligned}$$

Thus

$$\delta(x)m_1m_2 - \delta(x(m_1m_2)) = -\sigma(x)(\sigma(m_1)d(m_2) + d(m_1)m_2). \tag{6}$$

On the other hand, let $y \in I^2$; then $m_2y \in I$. By (4) we have

$$\begin{aligned} \sigma(m_1)g(m_2y) - g(m_1m_2y) &= -d(m_1)m_2y, \\ \sigma(m_1)\sigma(m_2)g(y) - \sigma(m_1)g(m_2y) &= -\sigma(m_1)d(m_2)y, \end{aligned}$$

implying that

$$\sigma(m_1m_2)g(y) - g((m_1m_2)y) = -(d(m_1)m_2 + \sigma(m_1)d(m_2))y. \tag{7}$$

This means that d can be extended to $E + E^2$ in such a way that

$$\delta(x)m - \delta(xm) = -\sigma(x)d(m) \quad \text{and} \quad \sigma(m)g(y) - g(my) = -d(m)y$$

for all $x, y \in I^2$ and all $m \in E + E^2$. Moreover, $d(m_1m_2) = \sigma(m_1)d(m_2) + d(m_1)m_2$ for all $m_1, m_2 \in E$. Repeating the argument above, we can extend d to $E + E^2 + E^3 + E^4$ in such a way that

$$\delta(x)m - \delta(xm) = -\sigma(x)d(m) \quad \text{and} \quad \sigma(m)g(y) - g(my) = -d(m)y \tag{8}$$

for all $x \in I^4$ and all $m \in E + E^2 + E^3 + E^4$. Moreover,

$$d(uv) = \sigma(u)d(v) + d(u)v \tag{9}$$

for all $u, v \in E + E^2$. Let $M = I^2[E, E]I^2 \neq 0$. Then M is a nonzero ideal of A contained in $E + E^2$.

Let $m \in M$. By (8) we see that $\sigma(I^4)d(m) \subseteq A$ and $d(m)I^4 \subseteq A$. Thus $d(m) \in Q$ follows, showing that $d\colon M \to Q$ is a σ-derivation satisfying

$$\delta(x)m - \delta(xm) = -\sigma(x)d(m) \quad \text{and} \quad \sigma(m)g(y) - g(my) = -d(m)y \tag{10}$$

for all $x, y \in I^4$ and all $m \in M$. Set $J = \sigma(I^4) \cap I^4$. Then J is a nonzero ideal of A and, moreover, $d(m)J + Jd(m) \subseteq A$ for all $m \in M$. In view of

Lemma 2.3, d can be uniquely extended to a σ-derivation from A into Q. Thus we can rewrite (10) as

$$\begin{aligned} \delta(xm) - d(xm) &= (\delta(x) - d(x))m, \\ \sigma(m)(g(y) - d(y)) &= g(my) - d(my) \end{aligned} \tag{11}$$

for all $x, y \in I^4$ and all $m \in M$. Consider the map $\phi \colon I^4 \to Q$ defined by

$$x \in I^4 \mapsto \phi(x) = \delta(x) - d(x).$$

By (11), ϕ is a right M-module map. Thus it is a right A-module map by the primeness of A. By Lemma 2.1, there exists $a \in Q_{mr}$ such that $\phi(x) = ax$ and so $\delta(x) = d(x) + ax$ for all $x \in I^4$. By an analogous argument, there exists $b \in Q_{ml}$ such that $g(x) = d(x) + \sigma(x)b$ for all $x \in I^4$. We are now done by setting $N = I^4$.

Suppose, in addition, that $eA \cup Ae \subseteq A$ for some nontrivial idempotent $e \in Q$. Then $I = A$ in our construction. Moreover, $EA + AE \subseteq A$. Therefore, (11) remains true for all $x \in A$. So the map $\phi \colon I^4 \to Q$ can be replaced by $\phi \colon A \to Q$. Now our conclusion holds trivially. This proves the theorem.

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