

*Pacific
Journal of
Mathematics*

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Volume 216 No. 2

October 2004

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A recent problem in dynamics is to determine whether an attractor Λ of a C^r flow X is C^r robust transitive. By an *attractor* we mean a transitive set to which all positive orbits close to it converge. An attractor is C^r robust transitive (or C^r *robust* for short) if it has a neighborhood U such that the set $\bigcap_{t>0} Y_t(U)$ is transitive for every flow Y that is C^r close to X . We give sufficient conditions for robustness of attractors based on the following definitions: an attractor is *singular-hyperbolic* if it has singularities, all of which are hyperbolic, and is partially hyperbolic with volume expanding central direction (Morales, Pacifico and Pujals, 1998). An attractor is C^r *critically robust* if it has a neighborhood U such that $\bigcap_{t>0} Y_t(U)$ is in the closure of the closed orbits of every flow Y C^r close to X . We show that on compact 3-manifolds all C^r critically robust singular-hyperbolic attractors with only one singularity are C^r robust.

1. Introduction

A recent problem in dynamics is to determine whether an attractor Λ of a C^r flow X is C^r robust transitive or not. By an *attractor* we mean a transitive set to which all positive orbits close to it converge. An attractor is C^r robust transitive (or C^r *robust* for short) if it has a neighborhood U such that the set $\bigcap_{t>0} Y_t(U)$ is transitive for every flow Y that is C^r close to X . We give sufficient conditions for robustness of attractors based on the following definitions: an attractor is *singular-hyperbolic* if it has singularities, all of which are hyperbolic, and is partially hyperbolic with volume expanding central direction [MPP]. An attractor is C^r *critically robust* if it has a neighborhood U such that $\bigcap_{t>0} Y_t(U)$ is in the closure of the closed orbits of every flow Y C^r close to X . We show that, for flows on compact 3-manifolds, all C^r critically robust singular-hyperbolic attractors with only one singularity are C^r robust.

Let us state our result in a precise way. Hereafter X_t is a flow induced by a C^r vector field X on a compact 3-manifold M . The ω -limit set of $p \in M$ is the accumulation point set $\omega_X(p)$ of the positive orbit of p . An invariant set is *transitive* if it equals $\omega_X(p)$ for some point p on it.

Definition 1.1. A compact set in M is an *attracting set* of X if it can be written in the form $\bigcap_{t>0} X_t(U)$ for some neighborhood U . An *attractor* is a transitive attracting set.

See [Mi] where several definitions of attractors are discussed. The central definition of this paper is the following:

Definition 1.2. An attractor of a C^r flow X is C^r robust transitive (or C^r robust for short) if it has a neighborhood U such that $\bigcap_{t>0} Y_t(U)$ is a transitive set of Y for every flow Y that is C^r close to X .

Recently the problem of finding sufficient conditions for robustness of attractors was introduced in [B] and [P]. To deal with it we introduce the following definitions: a compact invariant set Λ of X is *partially hyperbolic* if there are an invariant splitting $T\Lambda = E^s \oplus E^c$ and positive constants K, λ such that:

1. E^s is contracting, namely

$$\|DX_t/E_x^s\| \leq Ke^{-\lambda t}, \quad \forall x \in \Lambda, \forall t > 0.$$

2. E^s dominates E^c , namely

$$\|DX_t/E_x^s\| \cdot \|DX_{-t}/E_{X_t(x)}^c\| \leq Ke^{-\lambda t}, \quad \forall x \in \Lambda, \forall t > 0.$$

The central direction E^c of Λ is said to be *volume expanding* if the additional condition

$$|J(DX_t/E_x^c)| \geq Ke^{\lambda t}$$

holds for all $x \in \Lambda$ and $t > 0$, where $J(\cdot)$ means the Jacobian.

Definition 1.3 ([MPP]). An attractor is *singular-hyperbolic* if it has singularities, all of which are hyperbolic, and is partially hyperbolic with volume expanding central direction.

The most representative example of a C^r robust singular-hyperbolic attractor is the geometric Lorenz attractor [GW]. The main result in [MPP] claims that C^1 robust nontrivial attractors on compact 3-manifolds are singular-hyperbolic. The converse is false: there are singular-hyperbolic attractors on compact 3-manifolds that are not C^r robust [MPu]. The following definition gives a further sufficient condition for robustness of singular-hyperbolic attractors:

Definition 1.4. An attractor of a C^r flow X is C^r *critically robust* if it has a neighborhood U such that $\bigcap_{t>0} Y_t(U)$ is in the closure of the closed orbits of Y , for every flow Y that is C^r close to X .

Hyperbolic attractors on compact manifolds are C^r robust and C^r critically robust for all r . The geometric Lorenz attractor [GW] is an example of a singular-hyperbolic attractor with only one singularity which is also C^r robust and C^r critically robust. In general singular-hyperbolic attractors

with only one singularity may be neither C^r robust nor C^r critically robust [MPu]. Nevertheless we shall prove that on compact 3-manifolds C^r critically robustness implies C^r robustness among singular-hyperbolic attractors with only one singularity. More precisely:

Theorem A. *A C^r critically robust singular-hyperbolic attractor with only one singularity on compact 3-manifolds is C^r robust.*

This gives explicit sufficient conditions for the robustness of attractors but they *depend on the perturbed flow*. E. Pujals is interested in conditions *depending on the unperturbed flow only*. It would also be interesting to determine whether the conclusion of Theorem A holds if we interchange the roles of robust and critically robust in the statement.

The proof of Theorem A relies on recent work [MP2]. We reproduce the necessary results in Section 2 for completeness. The proof of Theorem A is in Section 3.

2. Singular-hyperbolic attracting sets

In this section we describe the results in [MP2], omitting some proofs; see [MP2] for details. Hereafter X is a C^r flow on a closed 3-manifold M . The closure of B will be denoted by $\text{Cl}(B)$. If A is a compact invariant set of X we denote by $\text{Sing}_X(A)$ the set of singularities of X in A , and by $\text{Per}_X(A)$ the union of the periodic orbits of X in A . A compact invariant set H of X is *hyperbolic* if the tangent bundle over H has an invariant decomposition $E^s \oplus E^X \oplus E^u$ such that E^s is contracting, E^u is expanding and E^X is generated by the direction of X [PT]. Stable Manifold Theory [HPS] asserts the existence of the stable manifold $W_X^s(p)$ and the unstable manifold $W_X^u(p)$ associated to $p \in H$. These manifolds are respectively tangent to the subspaces $E_p^s \oplus E_p^X$ and $E_p^X \oplus E_p^u$ of T_pM . In particular, $W_X^s(p)$ and $W_X^u(p)$ are well-defined if p belongs to a hyperbolic periodic orbit of X . If O is an orbit of X we write $W_X^s(O) = W_X^s(p)$ and $W_X^u(O) = W_X^u(p)$ for some $p \in O$. We observe that $W_X^{s(u)}(O)$ does not depend on $p \in O$. When $\dim E^s = \dim E^u = 1$ we say that H is of *saddle type*. In this case $W_X^s(p)$ and $W_X^u(p)$ are two-dimensional submanifolds of M . The maps $p \in H \rightarrow W_X^s(p)$ and $p \in H \rightarrow W_X^u(p)$ are continuous (on compact parts). Moreover, a compact, singular, invariant set Λ of X is *singular-hyperbolic* if all its singularities are hyperbolic and the tangent bundle over Λ has an invariant decomposition $E^s \oplus E^c$ such that E^s is contracting, E^s dominates E^c and E^c is volume expanding (i.e., the Jacobian of DX_t/E^c grows exponentially as $t \rightarrow \infty$). Again, Stable Manifold Theory asserts the existence of the strong stable manifold $W_X^{ss}(p)$ associated to $p \in \Lambda$. This manifold is tangent to the subspace E_p^s of T_pM . For all $p \in \Lambda$ we define $W_X^s(p) = \bigcup_{t \in \mathbf{R}} W_X^{ss}(X_t(p))$. If p is regular (i.e., $X(p) \neq 0$) then $W_X^s(p)$ is a

well-defined two-dimensional submanifold of M . The map $p \in \Lambda \rightarrow W_X^s(p)$ is continuous (on compact parts) at the regular points p of Λ . A singularity σ of X is *Lorenz-like* if its eigenvalues $\lambda_1, \lambda_2, \lambda_3$ are real and satisfy

$$\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1$$

up to some reordering of the eigenvalues. A Lorenz-like singularity σ is hyperbolic, so $W_X^s(\sigma)$ and $W_X^u(\sigma)$ do exist. Moreover, the eigenspace of λ_2 is tangent to a one-dimensional invariant manifold $W_X^{ss}(\sigma)$. This manifold is called the *strong stable manifold* of σ . Clearly $W_X^{ss}(\sigma)$ splits $W_X^s(\sigma)$ into two connected components. We denote by $W_X^{s,+}(\sigma)$ and $W_X^{s,-}(\sigma)$ the two connected components of $W_X^s(\sigma) \setminus W_X^{ss}(\sigma)$.

Let Λ be a singular-hyperbolic set with dense periodic orbits of a three-dimensional flow. It follows from [MPP] that every $\sigma \in \text{Sing}_X(\Lambda)$ is Lorenz-like and satisfies $\Lambda \cap W_X^{ss}(\sigma) = \{\sigma\}$. It follows also from [MPP] that any compact invariant subset without singularities of Λ is hyperbolic of saddle type. If in addition Λ is attracting, there is for every $p \in \text{Per}_X(\Lambda)$ a $\sigma \in \text{Sing}_X(\Lambda)$ such that

$$W_X^u(p) \cap W_X^s(\sigma) \neq \emptyset.$$

This follows from the methods in [MP1].

For every singular-hyperbolic set Λ of a three-dimensional flow X and every Lorenz-like singularity $\sigma \in \text{Sing}_X(\Lambda)$ we define

$$\begin{aligned} P^+ &= \{p \in \text{Per}_X(\Lambda) : W_X^u(p) \cap W_X^{s,+}(\sigma) \neq \emptyset\}, \\ P^- &= \{p \in \text{Per}_X(\Lambda) : W_X^u(p) \cap W_X^{s,-}(\sigma) \neq \emptyset\}, \\ H_X^+ &= \text{Cl}(P^+), \\ H_X^- &= \text{Cl}(P^-). \end{aligned}$$

These sets will play an important role.

Lemma 2.1. *Let Λ be a connected, singular-hyperbolic, attracting set with dense periodic orbits and only one singularity σ . Then $\Lambda = H_X^+ \cup H_X^-$.*

Next we state a technical lemma to be used later. If S is a submanifold we denote by $T_x S$ the tangent space at $x \in S$. A *cross-section* of X is a compact submanifold Σ transverse to X and diffeomorphic to the two-dimensional square $[0, 1]^2$. If Λ is a singular-hyperbolic set of X and $x \in \Sigma \cap \Lambda$, then x is regular and so $W_X^s(x)$ is a two-dimensional submanifold transverse to Σ . In this case we denote by $W_X^s(x, \Sigma)$ the connected component of $W_X^s(x) \cap \Sigma$ containing x . We shall be interested in a special cross-section described as follows: let Λ be a singular-hyperbolic set of a three-dimensional flow X and let $\sigma \in \text{Sing}_X(\Lambda)$. Suppose that the closed orbits contained in Λ are dense in Λ . Then σ is Lorenz-like [MPP], and therefore one can describe the flow using the Grobman–Hartman Theorem [dMP]. Indeed, we can assume that the flow of X around σ is the linear flow $\lambda_1 \partial_{x_1} + \lambda_2 \partial_{x_2} + \lambda_3 \partial_{x_3}$

in a suitable coordinate system $(x_1, x_2, x_3) \in [-1, 1]^3$ around $\sigma = (0, 0, 0)$. A cross-section Σ of X is *singular* if it corresponds to the submanifolds $\Sigma^+ = \{x_3 = 1\}$ or $\Sigma^- = \{x_3 = -1\}$ in the coordinate system (x_1, x_2, x_3) . We denote by l^+ and l^- the curves obtained by intersecting $\{x_2 = 0\}$ with Σ^+ and Σ^- , respectively; these curves are contained in $W_X^{s,+}(\sigma)$ and $W_X^{s,-}(\sigma)$ respectively. We state without proof the following straightforward lemma:

Lemma 2.2. *Let Λ a singular-hyperbolic set with dense periodic orbits of a three-dimensional flow X , and fix $\sigma \in \text{Sing}_X(\Lambda)$. There are singular cross-sections Σ^+, Σ^- as above such that every orbit of Λ passing close to some point in $W_X^{s,+}(\sigma)$ (resp. $W_X^{s,-}(\sigma)$) intersects Σ^+ (resp. Σ^-). If $p \in \Lambda \cap \Sigma^+$ is close to l^+ , then $W_X^s(p, \Sigma^+)$ is a vertical curve crossing Σ^+ . If $p \in \text{Per}_X(\Lambda)$ and $W_X^u(p) \cap W_X^{s,+}(\sigma) \neq \emptyset$, then $W_X^u(p)$ contains an interval $J = J_p$ intersecting l^+ transversally; and the same is true if we replace $+$ by $-$.*

To prove transitivity we shall use two lemmas:

Lemma 2.3 (Birkhoff’s criterion). *Let T be a compact, invariant set of X such that for all open sets U, V intersecting T there is $s > 0$ such that $X_s(U \cap T) \cap V \neq \emptyset$. Then T is transitive.*

Lemma 2.4. *Let Λ be a connected, singular-hyperbolic, attracting set with dense periodic orbits and only one singularity σ . Let U, V be open sets, $p \in U \cap \text{Per}_X(\Lambda)$ and $q \in V \cap \text{Per}_X(\Lambda)$. If $W_X^u(p) \cap W_X^{s,+}(\sigma) \neq \emptyset$ and $W_X^u(q) \cap W_X^{s,+}(\sigma) \neq \emptyset$, there exist $t > 0$ and $z \in W_X^u(p)$ arbitrarily close to $W_X^u(p) \cap W_X^{s,+}(\sigma)$ such that $X_t(z) \in V$. The same is true if we replace $+$ by $-$.*

Let Λ be a singular-hyperbolic set of $X \in \mathcal{X}^r$ satisfying:

1. Λ is connected.
2. Λ is attracting.
3. The closed orbits contained in Λ are dense in Λ .
4. Λ has only one singularity σ .

We note that condition 3 implies

$$(H1) \quad \Lambda = \text{Cl}(\text{Per}_X(\Lambda)).$$

Proposition 2.5. *Suppose that, for any given $p, q \in \text{Per}_X(\Lambda)$, either*

1. $W_X^u(p) \cap W_X^{s,+}(\sigma) \neq \emptyset$ and $W_X^u(q) \cap W_X^{s,+}(\sigma) \neq \emptyset$, or
2. $W_X^u(p) \cap W_X^{s,-}(\sigma) \neq \emptyset$ and $W_X^u(q) \cap W_X^{s,-}(\sigma) \neq \emptyset$.

Then Λ is transitive.

Proof. By Birkhoff’s criterion we only need prove that for all open sets U, V intersecting Λ there exists $s > 0$ such that $X_s(U \cap \Lambda) \cap V \neq \emptyset$. For this we proceed as follows: by (H1) there are $p \in \text{Per}_X(\Lambda) \cap U$ and $q \in \text{Per}_X(\Lambda) \cap V$.

First suppose that alternative 1 holds. By Lemma 2.4, there are $z \in W_X^u(p)$ and $t > 0$ such that $X_t(z) \in V$. Since $z \in W_X^u(p)$, we have $w = X_{-t'}(z) \in U$ for some $t' > 0$. Since Λ is an attracting set, $w \in \Lambda$. If $s = t + t' > 0$ we conclude that $w \in (U \cap \Lambda) \cap X_{-s}(V)$ and so $X_s(U \cap \Lambda) \cap V \neq \emptyset$. If alternative 2 of the corollary holds we can find $s > 0$ such that $X_s(U \cap \Lambda) \cap V \neq \emptyset$ in a similar way (replacing $+$ by $-$). The result follows. \square

Proposition 2.6. *If there is a sequence $p_n \in \text{Per}_X(\Lambda)$ converging to some point in $W_X^{s,+}(\sigma)$ such that $W_X^u(p_n) \cap W_X^{s,-}(\sigma) \neq \emptyset$ for all n , then Λ is transitive. The same is true interchanging $+$ and $-$.*

Proof. Let $p, q \in \text{Per}_X(\Lambda)$ be fixed. Suppose that $W_X^u(p) \cap W_X^{s,+}(\sigma) \neq \emptyset$ and $W_X^u(q) \cap W_X^{s,-}(\sigma) \neq \emptyset$. By Lemma 2.2 we can fix a cross-section $\Sigma = \Sigma^+$ through $W_X^{s,+}(\sigma)$ and an open arc $J \subset \Sigma \cap W_X^u(p)$ intersecting $W_X^{s,+}(\sigma)$ transversally. Again by Lemma 2.2 we can assume that $p_n \in \Sigma$ for every n . Because the direction E^s of Λ is contracting, the size of $W_X^s(p_n)$ is uniformly bounded away from zero. It follows that there is n large so that J intersects $W_X^s(p_n)$ transversally. Applying the Inclination Lemma [dMP] to the saturation of $J \subset W_X^u(p)$, and the assumption $W_X^u(p_n) \cap W_X^{s,-}(\sigma) \neq \emptyset$, we conclude that $W_X^u(p) \cap W_X^{s,-}(\sigma) \neq \emptyset$. So alternative 2 of Proposition 2.5 holds; it follows from that proposition that Λ is transitive. \square

Proposition 2.7. *If there is $a \in W_X^u(\sigma) \setminus \{\sigma\}$ such that $\sigma \in \omega_X(a)$, then Λ is transitive.*

Proof. Without loss of generality we can assume that there exists z in $\omega_X(a) \cap W_X^+(\sigma)$. If $W_X^u(q) \cap W_X^{s,-}(\sigma) = \emptyset$ for all $q \in \text{Per}_X(\Lambda)$, then $W_X^u(q) \cap W_X^{s,+}(\sigma) \neq \emptyset$ for all $q \in \text{Per}_X(\Lambda)$; see [MP1]. Then Λ is transitive by Proposition 2.5 since alternative 1 holds for all $p, q \in \text{Per}_X(\Lambda)$. So we can assume that there is $q \in \text{Per}_X(\Lambda)$ such that $W_X^u(q) \cap W_X^{s,-}(\sigma) \neq \emptyset$. It follows from the dominating condition of the singular-hyperbolic splitting of Λ that the intersection $W_X^u(q) \cap W_X^{s,-}(\sigma)$ is transversal. This allows us to choose a point in $W_X^u(q)$ arbitrarily close to $W_X^{s,-}(\sigma)$ on the side of $W_X^u(q) \cap W_X^{s,-}(\sigma)$ accumulating a . Since Λ is attracting and satisfies (H1), one can find a sequence $p_n \in \text{Per}_X(\Lambda)$ converging to $z \in W_X^{s,+}(\sigma)$ such that for all n there is p'_n in the orbit of p_n such that the sequence p'_n converges to some point in $W_X^{s,-}(\sigma)$. Now suppose for a contradiction that Λ is not transitive. Then Proposition 2.6 implies

$$W_X^u(p'_n) \cap W_X^{s,+}(\sigma) = \emptyset \quad \text{and} \quad W_X^u(p_n) \cap W_X^{s,-}(\sigma) = \emptyset$$

for n large. But $W_X^u(p_n) = W_X^u(p'_n)$ since p'_n and p_n are in the same orbit of X . So $W_X^u(p_n) \cap (W_X^{s,+}(\sigma) \cup W_X^{s,-}(\sigma)) = \emptyset$. However

$$W_X^u(p_n) \cap W_X^s(\sigma) = \emptyset,$$

a contradiction since $\text{Sing}_X(\Lambda) = \{\sigma\}$. We conclude that Λ is transitive. \square

Theorem 2.8. *If Λ is not transitive, there is for all $a \in W_X^u(\sigma) \setminus \{\sigma\}$ a periodic orbit O of X with positive expanding eigenvalues such that $a \in W_X^s(O)$.*

Proof. Fix $a \in W_X^u(\sigma) \setminus \{\sigma\}$, and assume that $\omega_X(a)$ is not a periodic orbit. We will obtain a contradiction once we prove that if $p, q \in \text{Per}_X(\Lambda)$ then p, q satisfy one of the two alternatives in Proposition 2.5. To prove this we proceed as follows: as noted before, both $W_X^u(p)$ and $W_X^u(q)$ intersect $W_X^s(\sigma)$ (see [MP1]). Then we can assume

$$W_X^u(p) \cap W_X^{s,+}(\sigma) \neq \emptyset \quad \text{and} \quad W_X^u(q) \cap W_X^{s,-}(\sigma) \neq \emptyset.$$

By using this and the linear coordinate around σ , it is easy to construct an open interval $I = I_a$, contained in a suitable cross-section $\Sigma = \Sigma_a$ of X containing a , and such that $I \setminus \{a\}$ is formed by two intervals $I^+ \subset W_X^u(p)$ and $I^- \subset W_X^u(q)$. Observe that the tangent vector of I is contained in $E^c \cap T\Sigma_a$. Proposition 2.7 implies that $\sigma \notin \omega_X(a)$, since Λ is not transitive. It follows that $H = \omega_X(a)$ is a hyperbolic set of saddle type; see [MPP]. As in [M] one proves that H is one-dimensional, so Bowen’s Theory of hyperbolic one-dimensional sets [Bw] applies. In particular we can choose a family of cross-sections $\mathcal{S} = \{S_1, \dots, S_r\}$ of small diameter such that H is the flow-saturate of $H \cap \text{int } S'$, where $S' = \bigcup S_i$ and $\text{int } S'$ is the interior of S' . Also, $I \subset \Lambda$ since Λ is attracting. Recall that the tangent direction of I is contained in E^c . Since E^c is volume expanding and H is nonsingular, the Poincaré map induced by X on S' is expanding along I . As in [MP1, p. 371] we can find $\delta > 0$ and a open arc sequence $J_n \subset S'$ in the positive orbit of I with length bounded away from 0 such that there is a_n in the positive orbit of a contained in the interior of J_n . We can fix $S = S_i \in \mathcal{S}$ in order to assume that $J_n \subset S$ for every n . Let $x \in S$ be a limit point of a_n . Then $x \in H \cap \text{int } S'$. Because I is tangent to E^c , the interval sequence J_n converges to an interval $J \subset W_X^u(x)$ in the C^1 topology ($W_X^u(x)$ exists since $x \in H$ and H is hyperbolic). J is not trivial since the length of J_n is bounded away from 0. If a_n were in $W_X^s(x)$ for n large we would conclude that x is periodic by [MP1, Lemma 5.6], a contradiction since $\omega_X(a)$ is not periodic. We conclude that for infinitely many values of n , $a_n \notin W_X^s(x)$. Since $J_n \rightarrow J$ and Λ has strong stable manifolds of uniformly size, there exists

$$z_n \in (W_X^s(a_{n+1}) \cap S) \cap (J_n \setminus \{a_n\})$$

for all n large. For every n let J_n^+ and J_n^- be the two connected components of $J_n \setminus \{a_n\}$, with J_n^+ in the positive orbit of I^+ and J_n^- in the positive orbit of I^- . Clearly, either $z_n \in J_n^+$ or $z_n \in J_n^-$.

If $z_n \in J_n^+$ there is $v_{n+1} \in \text{Per}_X(\Lambda) \cap S$ close to a_{n+1} such that

$$W_X^s(v_{n+1}) \cap J_n^+ \neq \emptyset \quad \text{and} \quad W_X^s(v_{n+1}) \cap J_{n+1}^- \neq \emptyset.$$

Since v_{n+1} is periodic, it follows from [MPP] that $W_X^u(v_{n+1})$ intersects $W_X^{s,+}(\sigma)$ or $W_X^{s,-}(\sigma)$. The choice of v_{n+1} implies that its orbit passes close

to a point in $W_X^{s,-}(\sigma)$. Since Λ is not transitive we conclude that $W_X^u(v_{n+1})$ intersects $W_X^{s,-}(\sigma)$. Since $W_X^s(v_{n+1}) \cap J_n^+$ is transversal, the Inclination Lemma then implies $W_X^u(p) \cap W_X^{s,-}(\sigma) \neq \emptyset$. Hence

$$W_X^u(p) \cap W_X^{s,-}(\sigma) \neq \emptyset \quad \text{and} \quad W_X^u(q) \cap W_X^{s,-}(\sigma) \neq \emptyset.$$

If $z_n \in J_n^-$ we can prove by similar arguments that

$$W_X^u(p) \cap W_X^{s,+}(\sigma) \neq \emptyset \quad \text{and} \quad W_X^u(q) \cap W_X^{s,+}(\sigma) \neq \emptyset.$$

These alternatives yield the desired contradiction. Therefore $\omega_X(a) = O$ for some periodic orbit O of X . To finish we prove that the expanding eigenvalue of O is positive. Suppose by contradiction that it is not. Fix a cross-section Σ intersecting O in a single point p_0 . This section defines a Poincaré map $\Pi : \text{Dom } \Pi \subset \Sigma \rightarrow \Sigma$ of which p_0 is a hyperbolic fixed-point. The assumption implies that $D\Pi(p_0)$ has negative expanding eigenvalue. Because $p_0 \in \text{Per}_X(\Lambda)$, it follows from [MPP] that $W_X^u(p_0)$ intersects $W_X^{s,+}(\sigma)$ or $W_X^{s,-}(\sigma)$. We shall assume the former case since the proof in the latter is similar. We claim that $W_X^u(p) \cap W_X^{s,+}(\sigma) \neq \emptyset$ for all $p \in \text{Per}_X(\Lambda)$. Indeed, let $p \in \text{Per}_X(\Lambda)$ be fixed. Again $W_X^u(p)$ intersects $W_X^{s,+}(\sigma)$ or $W_X^{s,-}(\sigma)$. In the first case we are done. So we can assume that $W_X^u(p) \cap W_X^{s,-}(\sigma) \neq \emptyset$. By flow-saturating this intersection we obtain an interval $K \subset W_X^u(p) \cap \Sigma$ intersecting $W_X^s(p_0, \Sigma)$ transversally. At the same time, there is an interval $J \subset W_X^{s,+}(\sigma) \cap \Sigma$ intersecting $W_X^u(p_0, \Sigma)$ transversally. Since the expanding eigenvalue of $D\Pi(p_0)$ is negative the Inclination Lemma implies that the backward iterates $\Pi^{-n}(J)$ of J accumulate on $W_X^s(p_0, \Sigma)$ in both sides. Because K has transversal intersection with $W_X^s(p_0, \Sigma)$ we conclude that one such backward iterate intersects K , and this yields $W_X^u(p) \cap W_X^{s,+}(\sigma) \neq \emptyset$ as desired, proving the claim. The claim together with Proposition 2.5 implies that Λ is transitive, yielding the contradiction needed to complete the proof of theorem. \square

Hereafter we shall assume that Λ is not transitive. Let $a \in W_X^s(\sigma) \setminus \{\sigma\}$ be fixed. By Theorem 2.8, $a \in W_X^s(O)$ for some periodic orbit O with positive expanding eigenvalue. This last property implies that the unstable manifold $W_X^u(O)$ of O is a cylinder with generating curve O . Then O separates $W_X^u(O)$ into two connected components, which we denote by $W^{u,+}, W^{u,-}$ according to the following convention (see Figure 1): there is an interval $I = I_a$, contained in a suitable cross-section of X and containing a , such that if I^+, I^- are the connected components of $I \setminus \{a\}$ then $I^+ \subset W_X^u(p)$ and $I^- \subset W_X^u(q)$ for some periodic points $p, q \in \Lambda$ (recall that Λ is not transitive). In addition I is tangent to the central direction E^c of Λ (see Figure 1). Since $a \in W_X^s(O)$ and I is tangent to E^c , the flow of X carries I to an interval I' transverse to $W_X^s(O)$ at a . Note that the flow carries I^+ and I^- into I_0^+ and I_0^- respectively.

Definition 2.9. We denote by $W^{u,+}$ the connected component of $W^u \setminus O$ that is accumulated (via the Inclination Lemma and the Strong λ Lemma [dMP, D]) by the positive orbit of I_0^+ . We denote $W^{u,-}$ the connected component of $W_X^u(O) \setminus O$ accumulated by the positive orbit of I_0^- .

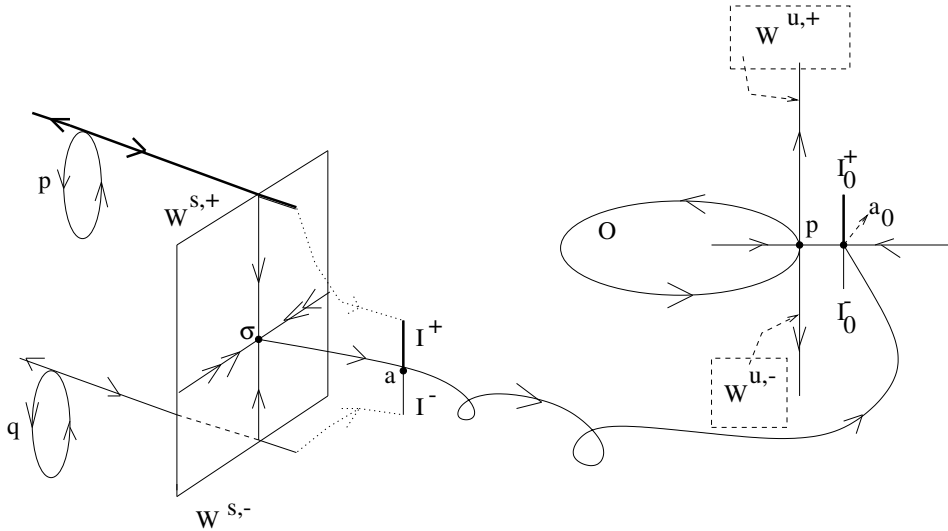


Figure 1. Definition of $W^{u,+}$ and $W^{u,-}$.

It can easily be proved using the Strong λ Lemma [D] that this definition does not depend on p, q, J_p, J_q .

Proposition 2.10. $W^{u,+} \cap W_X^{s,-}(\sigma) = \emptyset$ and $W^{u,+} \cap W_X^{s,+}(\sigma) \neq \emptyset$. The same is true interchanging $+$ and $-$.

Proof. For simplicity set $W = W^{u,+}$. First we prove that $W \cap W_X^{s,-}(\sigma) = \emptyset$. Suppose for a contradiction that $W \cap W_X^{s,-}(\sigma) \neq \emptyset$. Since this last intersection is transversal, there is an interval $J \subset W_X^{s,-}(\sigma)$ intersecting W transversally. Now, fix a cross-section $\Sigma = \Sigma^+$ as in Lemma 2.2 and let $p \in \text{Per}_X(\Lambda)$ be such that $W_X^u(p) \cap W_X^{s,+}(\sigma) \neq \emptyset$. Then there is a small interval $I \subset W_X^u(p) \cap \Sigma$ transversal to $\Sigma \cap W_X^{s,+}(\sigma)$. By the definition of $W = W^{u,+}$ (Definition 2.9), the positive orbit of I accumulates on W . Since J is transversal to W the Inclination Lemma implies that the positive orbit of I intersects J . This proves $W_X^u(p) \cap W_X^{s,-}(\sigma) \neq \emptyset$ for all $p \in \text{Per}_X(\Lambda)$. It follows that alternative 2 of Proposition 2.5 holds for all p, q , which contradicts the nontransitivity of Λ and proves that $W \cap W_X^{s,-}(\sigma) = \emptyset$, as desired. Now suppose for a contradiction that $W \cap W_X^{s,+}(\sigma) = \emptyset$. Since $W \cap W_X^{s,-}(\sigma) = \emptyset$ we obtain $W \cap W_X^s(\sigma) = \emptyset$ (see [MPP]). But the denseness of periodic orbits and the Inclination Lemma imply that $W \cap W_X^s(\sigma) \neq \emptyset$.

This is a contradiction, which proves that $W \cap W_X^{s,+}(\sigma) \neq \emptyset$. The result follows. \square

Proposition 2.11. $H^+ = \text{Cl}(W^{u,+})$ and $H^- = \text{Cl}(W^{u,-})$.

Proof. Fix $q \in P^+$, i.e., $W_X^u(q) \cap W_X^{s,+}(\sigma) \neq \emptyset$. Note that $W^{u,+} \cap W_X^{s,+}(\sigma)$ is nonempty by Lemma 2.10. Using (H1) and the Inclination Lemma it is not hard to prove that $W^{u,+}$ accumulates on q ; therefore $H^+ \subset \text{Cl}(W^{u,+})$. Conversely let $x \in W^{u,+}$ be fixed. By (H1) and $W^{u,+} \subset \Lambda$ there is z in $\text{Per}_X(\Lambda)$ near x . Choosing z close to x we ensure that $W_X^s(z) \cap W^{u,+} \neq \emptyset$, because stable manifolds have size uniformly bounded away from zero. If $W_X^u(z) \cap W_X^{s,-}(\sigma) \neq \emptyset$, the Inclination Lemma and the fact that $W_X^s(z)$ intersects $W^{u,+}$ imply that $W^u \cap W_X^{s,-}(\sigma) \neq \emptyset$. This contradicts Proposition 2.10, so $W_X^u(z) \cap W_X^{s,-}(\sigma) = \emptyset$. By [MP1] we obtain $z \in P^+$, proving that $x \in H^+$ and the lemma. \square

Proposition 2.12. If $z \in \text{Per}_X(\Lambda)$ and $W_X^s(z) \cap W^{u,+} \neq \emptyset$, then

$$\text{Cl}(W_X^s(z) \cap W^{u,+}) = \text{Cl}(W^{u,+}).$$

The same is true if we replace $+$ by $-$.

Proof. We have shown that $\text{Cl}(W_X^{s,+}(\sigma) \cap W^{u,+}) = \text{Cl}(W^{u,+})$. Fix $x \in W^{u,+}$. By (H1) there is $w \in \text{Per}_X(\Lambda)$ close to x . In particular $W_X^s(w)$ intersects $W^{u,+}$. If $W_X^u(w) \cap W_X^{s,+}(\sigma) = \emptyset$ then $W_X^u(w) \cap W_X^{s,-}(\sigma) \neq \emptyset$ by [MP1]. It follows from the Inclination Lemma that $W^{u,+} \cap W_X^{s,-}(\sigma) \neq \emptyset$, contradicting Proposition 2.10. We conclude that $W_X^u(w) \cap W_X^{s,+}(\sigma) \neq \emptyset$. Note that $W_X^s(w) \cap W^{u,+} \neq \emptyset$ gets close to x as $w \rightarrow x$. Since the intersection $W_X^u(w) \cap W_X^{s,+}(\sigma) \neq \emptyset$ is transversal we can apply the Inclination Lemma to find a transverse intersection $W^{u,+} \cap W_X^{s,+}(\sigma)$ close to x . This proves $\text{Cl}(W_X^{s,+}(\sigma) \cap W^{u,+}) = \text{Cl}(W^{u,+})$. Finally we prove $\text{Cl}(W_X^s(z) \cap W^{u,+}) = \text{Cl}(W^{u,+})$. Choose $x \in W^{u,+}$. Since $\text{Cl}(W_X^{s,+}(\sigma) \cap W^{u,+}) = \text{Cl}(W^{u,+})$ there is an interval $I_x \subset W^{u,+}$ arbitrarily close to x such that $I_x \cap W_X^{s,+}(\sigma) \neq \emptyset$. The positive orbit of I_x first passes through a and then accumulates on $W^{u,+}$. But $W_X^s(z) \cap W^{u,+} \neq \emptyset$ by assumption. Since this intersection is transversal the Inclination Lemma implies that the positive orbit of I_x intersects $W_X^s(z)$. By taking the backward flow of the last intersection we get $W_X^s(z) \cap I_x \neq \emptyset$. This proves the lemma. \square

Given $z \in \text{Per}_X(\Lambda)$, let $H_X(z)$ be the homoclinic class associated to z .

Proposition 2.13. If $z \in \text{Per}_X(\Lambda)$ is close to a point in $W^{u,+}$, then

$$H_X(z) = \text{Cl}(W^{u,+}).$$

The same is true if we replace $+$ by $-$.

Proof. Let $z \in \text{Per}_X(\Lambda)$ be a point close to one in $W^{u,+}$. It follows from the continuity of the stable manifolds that $W_X^s(z) \cap W^{u,+} \neq \emptyset$. We claim that $H_X(z) = \text{Cl}(W^{u,+})$. Indeed $W^{u,+} \cap W_X^{s,-}(\sigma) = \emptyset$ by Proposition 2.10. This equality and the Inclination Lemma imply that $W_X^u(z) \cap W_X^s(\sigma) \subset W_X^{s,+}(\sigma)$. By Proposition 2.12, $W_X^s(z) \cap W^{u,+}$ is dense in $W^{u,+}$ since $W_X^s(z) \cap W^{u,+} \neq \emptyset$. Let Σ be a cross-section containing p_0 and fix $x \in W^{u,+}$. We can assume $x, z \in \Sigma$. Since $W_X^u(z) \cap W_X^s(\sigma) \neq \emptyset$ and $W_X^u(z) \cap W_X^s(\sigma) \subset W_X^{s,+}(\sigma)$ there is an interval $I \subset W_X^u(z)$ intersecting $W_X^{s,+}(\sigma)$. Then the positive orbit of I yields an interval J close to σ intersecting $W_X^{s,+}(\sigma)$. In addition, the positive orbit of J yields an interval K whose positive orbit *accumulates* $W^{u,+}$ (recall Definition 2.9). Since $W_X^s(z) \cap W^{u,+}$ is dense in $W^{u,+}$ and $x \in W^{u,+}$, the orbit $W_X^s(z)$ passes close to x . The Inclination Lemma applied to the positive orbit of K yields a homoclinic point z' associated to z which is close to x . This proves that $x \in H_X(z)$, so $\text{Cl}(W^{u,+}) \subset H_X(z)$. The opposite inclusion is a direct consequence of the Inclination Lemma applied to $W_X^s(z) \cap W^{u,+} \neq \emptyset$. We conclude that $\text{Cl}(W^{u,+}) = H_X(z)$ as desired. □

Theorem 2.14. *Let Λ be a singular-hyperbolic set of a C^r flow X on a closed three-manifold, where $r \geq 1$. Suppose that the following properties hold:*

1. Λ is connected.
2. Λ is attracting.
3. The closed orbits contained in Λ are dense in Λ .
4. Λ has a unique singularity σ .
5. Λ is not transitive.

Then H^+ and H^- are homoclinic classes of X .

Proof. Let Λ be a singular-hyperbolic set of X satisfying the theorem's conditions. To prove that H^+ is a homoclinic class it suffices by Proposition 2.11 to prove that $\text{Cl}(W^{u,+})$ is a homoclinic class. By condition 3 of the Theorem we can choose $z \in \text{Per}_X(\Lambda)$ arbitrarily close to a point in $W^{u,+}$. Then $\text{Cl}(W^{u,+}) = H_X(z)$ by Proposition 2.13 and the result follows. □

3. Proof of Theorem A

First we introduce some notations. Hereafter M is a compact 3-manifold and \mathcal{X}^r is the space of C^r flows in M equipped with the C^r topology, $r \geq 1$. The *nonwandering set* of $X \in \mathcal{X}^r$ is the set $\Omega(X)$ of points $p \in M$ such that for all neighborhood U of p and $T > 0$ there is $t > T$ such that $X_t(U) \cap U \neq \emptyset$. An attracting Λ with isolating block U has a continuation $\Lambda(Y)$ for $Y \in \mathcal{X}^r$ close to X defined by $\Lambda(Y) = \bigcap_{t>0} Y_t(U)$. This continuation is then defined when Λ is an attractor. A compact invariant set is *nontrivial* if it is not a

closed orbit of X . Transitive sets for flows are always connected. The proof of Theorem A is based on the following result:

Theorem 3.1. *Let Λ be a singular-hyperbolic set of $X \in \mathcal{X}^r$, $r \geq 1$. Suppose that the following properties hold:*

1. Λ is connected.
2. Λ is attracting.
3. The closed orbits contained in Λ are dense in Λ .
4. Λ has only one singularity.
5. Λ is not transitive.

Then for every neighborhood U of Λ there is a flow Y that is C^r close to X and such that

$$\Lambda(Y) \not\subset \Omega(Y).$$

To prove this theorem we shall use the following definitions and facts: let $X \in \mathcal{X}^r$ and let Λ be a singular-hyperbolic set of X satisfying the conditions of the theorem. Let σ be the unique singularity of Λ . As mentioned on page 330, σ is Lorenz-like. As in Section 2, $W_X^{ss}(\sigma)$ divides $W_X^s(\sigma)$ into two connected components, which we denote by $W_X^{s,+}(\sigma)$ and $W_X^{s,-}(\sigma)$, or $W^{s,+}$, $W^{s,-}$ for short. Recall that $\text{Per}_X(\Lambda)$ denotes the union of the periodic orbits of X in Λ . Fix such $a \in W_X^u(\sigma) \setminus \{\sigma\}$. By Theorem 2.8, $\omega_X(a) = O$ for some periodic orbit with positive expanding eigenvalues of X . In particular, $W^{u,+}$ and $W^{u,-}$ are defined (Definition 2.9).

Lemma 3.2. $\text{Cl}(W^{u,+}) \cap W^{s,-} = \emptyset$.

Proof. Suppose for a contradiction that $\text{Cl}(W^{u,+}) \cap W^{s,-} \neq \emptyset$. By Lemma 2.2 there is a singular cross-section Σ^- such that every orbit of Λ passing close to some point in $W_X^{s,-}(\sigma)$ intersects Σ^- . Let $q \in \Lambda$ be periodic such that $W_X^u(q) \cap W_X^{s,-}(\sigma) \neq \emptyset$. Since $\text{Cl}(W^{u,+}) \cap W^{s,-} \neq \emptyset$, we have $\text{Cl}(W^{u,+}) \cap \Sigma^- \neq \emptyset$. Because closed orbits are dense we can prove that $q \in \text{Cl}(W^{u,+})$. It follows that $H^- \subset \text{Cl}(W^{u,+})$, so $\Lambda = \text{Cl}(W^{u,+})$ by Lemma 2.1. Also, since Λ is not transitive, $H^+ = \text{Cl}(W^{u,+})$ (Proposition 2.11) and H^+ is a homoclinic class (Theorem 2.14). Since homoclinic classes are transitive sets we conclude that Λ is transitive, a contradiction. This proves the result. □

Lemma 3.3. *Let D be a fundamental domain of $W_X^{uu}(p_0)$ contained in $W^{u,+}$. There exist a neighborhood V of D and a cross-section Σ^- of X intersecting $W^{s,-}$ satisfying the following properties:*

1. Every X -orbit's sequence in Λ converging to a point in $W^{s,-}$ intersects Σ^- .
2. No positive X -orbit with initial point in V intersects Σ^- .

Proof. Fix a fundamental domain F of $W_X^s(\sigma)$ and define $F^- = F \cap W_X^{s,-}(\sigma)$. Then there is a compact interval $F' \subset F^-$ such that $\Lambda \cap F^- \subset F'$. By

Lemma 3.2 there is $\epsilon > 0$ such that $B_\epsilon(\text{Cl}(W^{u,+})) \cap B_\epsilon(F') = \emptyset$. Clearly we can choose a cross-section Σ^- of X inside $B_\epsilon(F')$ such that every X -orbit's sequence in Λ converging to some point in $W^{s,-}$ intersects Σ^- . Since $W^{u,+}$ is invariant and $\text{Cl}(W^{u,+}) \cap B_\epsilon(F') = \emptyset$, every positive orbit with initial point in D cannot intersect Σ^- . By using the contracting foliation of Λ we have the same property for every positive trajectory with initial point in a neighborhood V of D . This proves the result. \square

Now we define a perturbation (pushing) close to a point $a \in W_X^u(\sigma) \setminus \{\sigma\}$. To this end we fix the following cross-sections:

1. Σ_a , containing a in its interior.
2. $\Sigma' = X_1(\Sigma)$.
3. Σ_0 , intersecting O in a single interior point.
4. Σ^+, Σ^- , which intersect $W^{s,+}, W^{s,-}$, respectively, and point toward the side of a .

Every X -orbit intersecting $\Sigma^+ \cup \Sigma^-$ will intersect Σ . Note that there is a well-defined neighborhood \mathcal{O} given by

$$\mathcal{O} = X_{[0,1]}(\Sigma_a).$$

This neighborhood will be the support of the pushing described in the Figures 2 and 3. The pushing in \mathcal{O} yielding the perturbed flow Y of X is obtained in the standard way (see [dMP]).

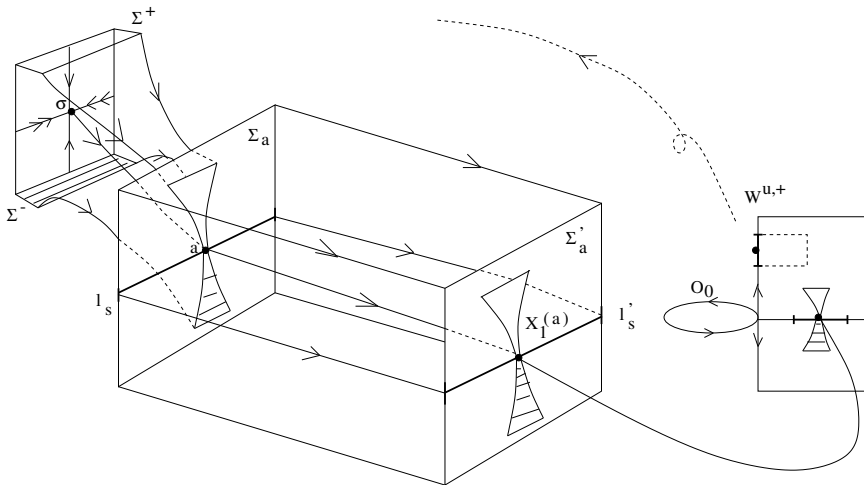


Figure 2. Unperturbed flow X .

We have to prove that $\Lambda(Y) \not\subset \Omega(Y)$ for the perturbed flow Y . For this purpose we observe that by 5 of Theorem 3.1 and Proposition 2.5 we can assume that there q periodic in U such that $W_X^u(q) \cap W_X^{s,-}(\sigma) \neq \emptyset$. We obtain in this way an interval K in $\Sigma^- \cap W_X^u(q)$ crossing Σ^- as in

Y that is C^r close to X . Clearly $\Lambda(Y)$ is a singular-hyperbolic set of Y for all Y close to X . Because Λ has a unique singularity, so does $\Lambda(Y)$. Now recall that Λ is an attractor by assumption. In particular, Λ is transitive (recall Definition 1.1). It follows that Λ is connected and so the neighborhood U above can be arranged to be connected. Then $\Lambda(Y)$ is connected as well. Summarizing, $\Lambda(Y)$ is a singular-hyperbolic set of Y satisfying conditions 1–4 of Theorem 3.1. If Λ is not C^r robust, we can find a Y that is C^r close to X and such that $\Lambda(Y)$ is not transitive. Then $\Lambda(Y)$ satisfies all conditions of Theorem 3.1, and we can find a Y' that is C^r close to Y and such that $\Lambda(Y') \not\subset \Omega(Y')$. This is a contradiction, since $\Lambda(Y') \subset \Omega(Y')$ (recall that $\Lambda(Y') \cap C(Y')$ is dense in $\Lambda(Y')$). This contradiction proves the result. \square

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Received April 24, 2003. The work was partially supported by CNPq, FAPERJ and PRONEX/DYN. SYS.

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