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We estimate the Hausdorff measure and dimension of Cantor sets in terms of a sequence given by the lengths of the bounded complementary intervals. The results provide the relation between the decay rate of this sequence and the dimension of the associated Cantor set.

It is well-known that not every Cantor set on the line is an s-set for some $0 \leq s \leq 1$. However, if the sequence associated to the Cantor set C is nonincreasing, we show that C is an h-set for some continuous, concave dimension function h. We construct the function h from the sequence associated to the set C.

1. Introduction

A Cantor set is a compact, perfect, totally disconnected subset of the real line. In this article we will consider only Cantor sets of Lebesgue measure zero. The complement of a Cantor set is a countable union of disjoint open intervals. We will use the term *gap* for any bounded convex component of the complement of a Cantor set.

Every Cantor set is completely determined by its gaps. Since the gaps are disjoint, the sum of their lengths equals the diameter of the Cantor set.

There is a natural way to associate to each summable sequence of positive numbers a unique Cantor set having gaps with lengths equal to the terms of the sequence. In this correspondence the order of the sequence is important. Different rearrangements can lead to different Cantor sets. Of course, if two sequences lead to the same Cantor set, one is a rearrangement of the other.

In the first part of this paper we will concentrate on finding the Hausdorff measure of a Cantor set in terms of the decay of the sequence of the lengths of the gaps. In particular we will show that the Hausdorff dimension depends totally on this behavior.

We establish an equivalence relation between sequences and show that Cantor sets in the same equivalence class have the same dimension.

Since the Cantor set depends on the order of the sequence, one expects that the dimension of the resulting set also depends on the order. This is true, and moreover, the arrangement of the sequence in monotone non-increasing order yields the Cantor set with the largest dimension out of all Cantor sets with the same set of gap lengths (see also [**BT54**]).

Let $0 \leq s \leq 1$. An s-set is a set on the line of Hausdorff dimension s and whose Hausdorff s-measure is finite and positive. Let h be a nondecreasing, right-continuous function taking the value zero at the origin. The Hausdorff h-measure $\mathcal{H}^h$ is defined in the same way as the Hausdorff s-measure but replaces the function x^s by $h(x)$ (see [**Rog98**], [**Hau19**]):

$$(1) \quad \mathcal{H}^h(A) = \lim_{\delta \to 0} \inf \left\{ \sum h(\mathrm{diam} E_i) : E_i \text{ open}, \bigcup E_i \supset A, \, \mathrm{diam} E_i \leq \delta \right\}.$$

A set $A \subset \mathbb{R}$ is an h-set if $0 < \mathcal{H}^h(A) < +\infty$.

Given $0 \leq s \leq 1$, it is not difficult to construct a Cantor set that is an s-set. It is also known that not every Cantor set of dimension s is an s-set. So should a set of dimension s but Hausdorff measure zero or infinity be considered s-dimensional?

Hausdorff proposed the h-measure that bears his name to further the investigation of non-s-sets. In this paper we prove that every Cantor set C on the line associated to a sequence of nonincreasing positive real numbers is an h-set for some continuous concave function h. We explicitly construct h in terms of the sequence that defines the Cantor set. In other words, for *every* sequence, the set with the largest Hausdorff dimension is also an h-set for some appropriate function.

The study of Cantor sets through the decay of the complementary intervals was initiated by Borel in 1948 [**Bor49**] and continued by Besicovitch and Taylor in their seminal paper [**BT54**]. The present paper extends some of their results.

Tricot [**Tri81**] and Falconer [**Fal97**] obtained results associating properties of the gaps of a Cantor set with its box dimension. (See also [**Tri95**]). In [**CMPS03**] the particular case of the sequence x^p was thoroughly analyzed.

Throughout the paper, we will use the notation $\dim A$ for the Hausdorff dimension of a set A, since it is the only concept of dimension that we are considering. The Hausdorff s-measure of a set A will be denoted by $\mathcal{H}^s(A)$.

2. Cantor sets associated to a sequence

We will now assign to each summable sequence of positive numbers a unique Cantor set with gaps whose lengths correspond to the terms of this sequence. Let $a = \{a_k\}$, for $k = 1, 2, \ldots$, be a sequence of positive real numbers such that $\sum a_k = S_a < \infty$. Let I be an interval of length $|I| = S_a$. We first remove from I an open interval of length a_1, whose position will be clear in a moment. We next remove from the left remaining interval an interval of length a_2 and from the right an interval of length a_3. We continue in this way, removing at the i-th step 2^{i-1} intervals from left to right. It is easy to see that we end up with a Cantor set, which we will call C_a. Since $\sum a_k = |I|$, there is only one choice for the location of each interval to be removed in the construction, and C_a is well-defined.

The gap of C_a associated with the term a_k will be denoted g_{a_k}. If g and g' are gaps, we will say that $g < g'$ if all $x \in g$, $y \in g'$ satisfy $x < y$. Given a sequence a and its associated Cantor set C_a, we define a *cut* of C_a to be a partition of $\mathbb{N} = L \cup R$ such that

$$g_{a_\ell} < g_{a_r} \quad \text{for all } \ \ell \in L, \ r \in R.$$

We will allow L or R to be empty. The following lemma is an immediate consequence of the definitions.

Lemma 1. *Every point in C_a defines a cut and, conversely, every cut of C_a defines a unique point of C_a.*

Let C_a and C_b be Cantor sets associated to sequences a and b respectively. As a result of the definition of C_a and C_b it is clear that, for any $n, m \in \mathbb{N}$,

$$g_{a_n} < g_{a_m} \quad \text{implies} \quad g_{b_n} < g_{b_m}.$$

This implies that if (L, R) defines a cut of C_a, it also defines a cut of C_b.

If $x \in C_a$ is defined by a cut (L, R), then $x = \sum_{n \in L} |g_{a_n}|$.

3. Equivalences of Cantor sets

The previous considerations allow us to define a natural map π_{ab} from C_a into C_b, assigning to the point $x \in C_a$ the point $y \in C_b$ defined by the same cut associated to x, i.e., if $L_a(x) = \{n \in \mathbb{N} : g_{a_n} \subset [0, x]\}$, then

$$y = \pi_{ab}(x) = \sum_{n \in L_a(x)} |g_{b_n}|.$$

Observe that y can be written also as

$$y = \sum_{n \in L_b(y)} |g_{b_n}|, \quad \text{with } L_b(y) = \{n \in \mathbb{N} : g_{b_n} \subset [0, y]\}.$$

The map $\pi_{ab} : C_a \to C_b$ is one-to-one and onto. It can be extended linearly to a one-to-one map from $[0, S_a]$ into $[0, S_b]$, by mapping the gap g_{a_n} linearly into the gap g_{b_n}.

Note that π is an increasing function, since given $x, y \in C_a$ with $x < y$, we have

$$\pi_{ab}(y) - \pi_{ab}(x) = \sum_{n \in L_a(y)} b_n - \sum_{n \in L_a(x)} b_n = \sum_{n \in L_a(y) \setminus L_a(x)} b_n = \sum_{\{n \,:\, g_{a_n} \subset [x,y]\}} b_n > 0.$$

Thus π_{ab} is increasing on C_a. This implies that π_{ab} is increasing on $[0, S_a]$. Since $\pi_{ab} : [0, S_a] \to [0, S_b]$ is onto, it must be continuous and consequently $\pi_{ab}^{-1} : [0, S_b] \to [0, S_a]$ is also continuous.

We have therefore proved the following proposition:

Proposition 1. *If C_a and C_b are the Cantor sets associated to arbitrary sequences a and b, then the map $\pi_{ab} : [0, S_a] \to [0, S_b]$ is increasing, one to one, onto and bicontinuous. Furthermore $\pi_{ab}(C_a) = C_b$.*

Definition 1. We define an order relation $\prec$ between summable sequences of positive terms as follows: if a and b are two such sequences, we set

$$a \prec b \quad \text{if there exists } k > 0 \text{ such that } \frac{a_n}{b_n} < k \text{ for all } n \in \mathbb{N}.$$

In this case we say that a *is of lower order than* b. If $a \prec b$ and $b \prec a$ we say that a and b are of *the same order* and we write $a \sim b$. Note that

$$a \sim b \iff k_1 < \frac{a_n}{b_n} < k_2 \text{ for all } n \in \mathbb{N},$$

for some constants $k_1, k_2 > 0$.

We will need the following result from [**CMPS03**]:

Proposition 2. *Let $a = \{a_k\}_{k\in\mathbb{N}}$ be defined by $a_k = \left(\frac{1}{k}\right)^p$, with $p > 1$. Then $\dim(C_a) = 1/p$, and moreover, C_a is a $(1/p)$-set; precisely,*

$$\frac{1}{8}\left(\frac{2^p}{2^p - 2}\right)^{1/p} \leq \mathcal{H}^{1/p}(C_a) \leq \left(\frac{1}{p-1}\right)^{1/p}.$$

The following notation is convenient in the proofs below.

Notation 1. We write $\lambda^{(p)}$ for the sequence whose n-th term is n^{-p}.

Theorem 1. *Let C_a and C_b be Cantor sets associated to the sequences a and b.*

1. *If $a \prec b$ then $\dim(C_a) \leq \dim(C_b)$; thus $\dim(C_a) = \dim(C_b)$ if $a \sim b$.*
2. *There exist sequences $a = \{a_n\}$ and $b = \{b_n\}$ such that*

$$\liminf \frac{a_n}{b_n} = 0 \quad \textit{and} \quad \dim(C_a) = \dim(C_b).$$

Proof of theorem. For part (1), if $a \prec b$, we will show that the map π_{ba} defined above is Lipschitz. Given $x, y \in C_b$ with $x < y$, we have

$$\pi_{ba}(y) - \pi_{ba}(x) = \sum_{\{n\,:\, g_{b_n} \subset [x,y]\}} a_n \leq k \sum_{\{n\,:\, g_{b_n} \subset [x,y]\}} b_n = k(y - x).$$

Then $\dim(C_a) = \dim(\pi_{ba}(C_b))$. By an elementary property of Hausdorff dimension we obtain $\dim(\pi_{ab}(C_b)) \leq \dim(C_b)$, proving (1).

For part (2), consider a sequence $a = \{a_n\}$ such that for some fixed $p > 1$

$$\lim_{n\to\infty} \frac{a_n}{n^{-p}} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{n^{-q}}{a_n} = 0, \quad \text{for all } q > p.$$

The maps $\pi_{\lambda^{(p)}a} : C_{\lambda^{(p)}} \to C_a$ and $\pi_{a\lambda^{(q)}} : C_a \to C_{\lambda^{(q)}}$ are Lipschitz, as can be seen by means of an argument similar to that of part (1). This implies that $q - 1 \leq \dim(C_a) \leq p - 1$ for all $q > p$. Then $\dim(C_a) = 1/p$. ☐

4. Computation of Hausdorff dimensions

In this section we define some indices associated with a summable sequence. These numbers can be considered as a measure of the decay rate of the sequence. We then compare their values with the dimension of the associated Cantor set.

For a sequence $a = \{a_n\}$, define

$$\begin{aligned}\beta(a) &= \inf\{s : 0 < s,\ a \prec \lambda^{(1/s)}\},\\ \gamma(a) &= \sup\{s : 0 < s,\ \lambda^{(1/s)} \prec a\},\\ \delta(a) &= \inf\left\{s : 0 < s \le 1,\ \textstyle\sum_n a_n^s < \infty\right\}.\end{aligned}$$

Of these constants, only δ is invariant under rearrangements; β and γ are not. Since we know that for the sequence $\lambda^{(p)}$ rearrangements can indeed change the dimension (see [**CMPS03**]), we have to discard the intuition that $\delta(a) = \dim(C_a)$.

A historical survey of various indices associated with the decay of gaps (when a_n decreases) and the box dimension is given by Tricot in [**Tri81**], together with more complete results. In particular he shows that

$$\gamma(a) = \underline{\lim}\, \frac{-\log n}{\log a_n} \qquad \text{and} \qquad \beta(a) = \overline{\lim}\, \frac{-\log n}{\log a_n},$$

and if $a = \{a_n\}$ is monotonic decreasing, then $\delta(a) = \beta(a)$.

Proposition 3. *Let a be a summable sequence of positive terms.*

(1) $\gamma(a) \le \dim(C_a) \le \beta(a)$.
(2) $\gamma(a) \le \delta(a) \le \beta(a)$.

Proof. Part (1) is a consequence of Theorem 1 and the definition of $\gamma(a)$ and $\beta(a)$.

For part (2), choose $s > 0$ such that $a \prec \lambda^{(1/s)}$, which is to say

$$a_n \le \frac{c}{n^{1/s}} \quad \text{for some } c > 0 \text{ and every } n.$$

Then, for some other constant c',

$$a_n^{s+\epsilon} \le \frac{c'}{n^{(s+\epsilon)/s}} \quad \text{for every } n,$$

which implies that $s+\epsilon \ge \delta(a)$ for all $\epsilon > 0$; thus $\delta(a) \le \beta(a)$. Furthermore, for each $\epsilon \ge 0$ we have

$$c\, n^{-\frac{1}{\gamma(a)-\epsilon}} \le a_n \quad \text{for some } c \text{ and every } n,$$

and so

$$\sum_n a_n^{\gamma(a)-\epsilon} = +\infty,$$

which implies that $\delta(a) \geq \gamma(a) - \epsilon$. Since ϵ is arbitrary, we conclude that $\delta(a) \geq \gamma(a)$. □

A consequence of the proposition and Tricot's result is that if a is a monotone nonincreasing summable sequence of positive terms and $\widetilde{a}$ is any rearrangement of a then $\beta(a) = \delta(a) = \delta(\widetilde{a}) \leq \beta(\widetilde{a})$.

Another immediate consequence of the definition of $\gamma(a)$ and $\beta(a)$ is:

Property. Let a be a summable sequence of positive terms. If $0 < b < \beta(a)$ then $\overline{\lim}_{n\to\infty} n^{1/b} a_n = +\infty$, and if $\gamma(a) \leq b$ then $\underline{\lim}_{n\to\infty} n^{1/b} a_n = 0$.

This tells us that if we take a rearrangement $\widetilde{a}$ of a monotone nonincreasing sequence a such that $\beta(a) \neq \beta(\widetilde{a})$ (so that $\beta(a) < \beta(\widetilde{a})$), then $\overline{\lim}_{n\to\infty} n^{1/\beta(a)} \widetilde{a}_n = +\infty$. Therefore, if $\widetilde{a}$ is a rearrangement of a monotonic nonincreasing sequence a, then $\dim(C_{\widetilde{a}}) \leq \beta(a) < \beta(\widetilde{a})$.

4.1. Monotone nonincreasing sequences. For a nonincreasing sequence a, we already know that $\delta(a) = \beta(a)$. In addition, by Proposition 3, we know that $\gamma(a) \leq \dim(C_a) \leq \beta(a)$. Therefore, if $\lim(\log a_n / \log n) = \ell$, we have $\dim(C_a) = -1/\ell$.

This result extends the result of Falconer [**Fal97**, p. 55]. Moreover, that author shows that if that limit does not exist then the upper and lower box-dimensions disagree.

In this case, however, we still want to determine the dimension of C_a. To this end, we introduce two new constants associated to the sequence a. Set $r_n = \sum_{j\geq n} a_j$. Using an argument analogous to the one used in [**CMPS03**], one can see that the s-Hausdorff measure of C_a is bounded by

$$H^s(C_a) \leq c \,\underline{\lim}\, n \left(\frac{r_n}{n}\right)^s .$$

We therefore define two constants associated to the sequence a:

$$\tau(a) = \inf\left\{ s > 0 : \underline{\lim}\, n \left(\frac{r_n}{n}\right)^s < +\infty \right\},$$

$$\alpha(a) = \underline{\lim}\, \alpha_n, \quad \text{where } n \left(\frac{r_n}{n}\right)^{\alpha_n} = 1.$$

Note. The constant α associated to a monotone sequence a was introduced in [**BT54**]. The authors show that $\dim(C_{\widetilde{a}}) \leq \alpha(a)$, where $\widetilde{a}$ is any rearrangement of a.

It is interesting to remark that $\overline{\lim}\, \alpha_n$ was introduced already in 1948 by Emil Borel with the name of *logarithmic density.*

From results in the seminal paper by Besicovitch and Taylor [**BT54**], one can conclude that $\dim(C_a) = \alpha(a)$ for a monotonic nonincreasing sequence (see [**CHM03**]), and that for each t and β with $0 \leq t \leq \beta$ there is a monotone nonincreasing sequence $a = \{a_n\}$ such that $\beta(a) = \beta$ and $\alpha(a) = t$. Our next proposition, however, expresses the surprising result that if $\gamma(a)$ is *strictly smaller* than $\beta(a)$, then $\alpha(a)$ has a smaller than expected bound.

Proposition 4. *With notation as above, for a nonincreasing sequence a,*

$$\alpha(a) = \tau(a) \quad \textit{and} \quad \alpha(a) \leq \frac{\gamma(a)}{1-\beta(a)+\gamma(a)}.$$

Proof. We first show that $\alpha(a) \leq \tau(a)$. Let $s > 0$ be such that $\underline{\lim}\, n\left(\frac{r_n}{n}\right)^s < +\infty$. Then

$$n\left(\frac{r_n}{n}\right)^s = n\left(\frac{r_n}{n}\right)^{\alpha_n}\left(\frac{r_n}{n}\right)^{s-\alpha_n} = \left(\frac{r_n}{n}\right)^{s-\alpha_n}.$$

So $\underline{\lim}\left(\frac{r_n}{n}\right)^{s-\alpha_n} < +\infty$. Since $\lim\left(\frac{r_n}{n}\right)^{-1/k} = +\infty$ for each fixed $k > 0$, there must exist a subsequence α_{n_k} such that $\alpha_{n_k} < s + 1/k$ for all k. We have $\alpha(a) = \underline{\lim}\,_n \alpha_n \leq \underline{\lim}\, \alpha_{n_k} \leq s$, and therefore $\alpha(a) \leq \tau(a)$.

For the converse, $\tau(a) \leq \alpha(a)$, assume that $\alpha(a) < \tau(a)$, and consider s such that $\alpha(a) < s < \tau(a)$. Let $\{a_{n_k}\}$ be such that $\lim_k a_{n_k} = \alpha(a)$ and $a_{n_k} < s$ for all k. Then

$$+\infty = \underline{\lim}\,_n n\left(\frac{r_n}{n}\right)^s = \underline{\lim}\,_k n_k\left(\frac{r_{n_k}}{n_k}\right)^s = \underline{\lim}\,_k\left(\frac{r_{n_k}}{n_k}\right)^{s-\alpha_{n_k}} = 0$$

(since $s - \alpha_{n_k} > c > 0$ for some c and for all k). This contradiction shows that $\alpha(a) = \tau(a)$.

For the other inequality, note that if $\gamma(a) = \beta(a)$, then

$$\frac{\gamma(a)}{1-\beta(a)+\gamma(a)} = \gamma(a)$$

and there is nothing to prove. However, if $\gamma(a) < \beta(a)$, then

$$\frac{\gamma(a)}{1-(\beta(a)-\gamma(a))} < \beta(a).$$

To show that $\alpha(a)$ satisfies the desired inequality, we prove

$$\alpha(a) \leq \frac{\gamma(a)+\varepsilon}{1-(\beta(a)-\gamma(a)-\varepsilon)} \quad \text{for each } \varepsilon > 0.$$

To this end, we will show that for each $\varepsilon > 0$, there is a subsequence $\{a_{n_k}\}_k$ of $\{a_n\}_n$ for which r_{n_k} is at most $O\left(n_k^{-\frac{1-\beta(a)}{(\gamma(a)+\varepsilon)}}\right)$.

Fix $\beta(a) - \gamma(a) \geq \varepsilon > 0$, and set $\gamma_\varepsilon = \gamma(a) + \varepsilon$. We see immediately from the definition of $\gamma(a)$ that there is a subsequence n_k such that $a_{n_k} \leq n_k^{-1/\gamma_\varepsilon}$. This is the subsequence that we desire.

Since a_n is monotone, we can estimate r_{n_k} from above. Fix n_k. Define a new sequence $\{b_n\}_n$ in the following way:

$$b_j = \begin{cases} a_j & \text{for } j \leq n_k, \\ n_k^{-1/\gamma_\varepsilon} & \text{for } n_k \leq j < \lceil n_k^{\beta(a)/\gamma_\varepsilon} \rceil, \\ j^{-1/\beta(a)} & \text{for all larger } j. \end{cases}$$

Here as usual $\lceil x \rceil$ stands for the smallest integer that is larger or equal than x. So we have that $a_j \le b_j$ for all j, and therefore $\sum_{j \ge n_k} a_j \le \sum_{j \ge n_k} b_j$.

We can estimate that

$$\sum_{j=n_k}^{\lceil n_k^{\beta/\gamma_\varepsilon} \rceil} b_j = \frac{\lceil n_k^{\beta(a)/\gamma_\varepsilon} \rceil - n_k}{n_k^{1/\gamma_\varepsilon}} \sim n_k^{(\beta(a)-1)/\gamma_\varepsilon} \quad \text{for } k \text{ large enough,}$$

and, using an integral comparison, we see that

$$\sum_{j \ge \lceil n_k^{\beta(a)/\gamma_\varepsilon} \rceil} b_j = C\big(n_k^{\beta(a)/\gamma_\varepsilon}\big)^{(\beta(a)-1)/\beta(a)}.$$

Since both of these terms are $O(n_k^{(\beta(a)-1)/\gamma_\varepsilon})$, we have

$$\alpha(a) \le \frac{\gamma(a)+\varepsilon}{1-(\beta(a)-\gamma(a)-\varepsilon)} \quad \text{for every } \varepsilon. \qquad \square$$

In [**Tri95**] it is proved that

$$\beta(a) = \overline{\lim} \; -\frac{\log n}{\log a_n} = \overline{\lim}\, \alpha_n.$$

Proposition 4 shows this is false, in general, for the $\underline{\lim}$. Moreover, we know that there are no sequences a, with $\gamma(a) < \beta(a)$ and

$$\frac{\gamma(a)}{1-\beta(a)+\gamma(a)} < \dim(C_a) \le \beta(a).$$

So the question now is whether there exists a sequence a such that

$$\gamma(a) \le \dim(C_a) \le \frac{\gamma(a)}{1-\beta(a)+\gamma(a)}.$$

The next proposition answers this question completely and emphasizes the asymmetry between the $\overline{\lim}$ and the $\underline{\lim}$.

Proposition 5. *Let $0 < \gamma \le \beta \le 1$ be given. For any number t such that $\gamma \le t \le \gamma/(1-\beta+\gamma)$, there is a monotonic nonincreasing sequence a such that $\dim(C_a) = t$ and*

$$\gamma(a) = \gamma \quad \textit{and} \quad \beta(a) = \beta. \qquad (*)$$

Proof. Let $0 \le s \le 1$, and define

$$f(s) = \frac{\gamma(1-s\beta)}{1-\beta+\gamma(1-s)}.$$

For each s we construct a monotonic nonincreasing sequence $a^{(s)}$ satisfying $\dim(C_{a^{(s)}}) = f(s)$, $\gamma(a^{(s)}) = \gamma$, and $\beta(a^{(s)}) = \beta$. Since f is decreasing, $f(0) = \frac{\gamma}{1-\beta+\gamma}$ and $f(1) = \gamma$, there exists for any $t \in \big[\gamma, \frac{\gamma}{1-\beta+\gamma}\big]$ an s_t so that $\dim(C_{a(s_t)}) = t$.

To construct such a sequence, let

$$R = \frac{1-\gamma s}{1-\beta s}\,\frac{\beta}{\gamma}$$

and define $p_n = 2^{R^n}$, for $n = 0, 1, 2, \ldots$. Define the sequence $a^{(s)} = \{a_n\}$ as follows: $a_0 = a_1 = 1$ and

$$a_j = (p_n)^{-(1-s\gamma)/\gamma} j^{-s} \qquad \text{when } p_n \le j < p_{n+1}.$$

Notice that $a_{p_n} = p_n^{-1/\gamma}$ and

$$a_{(p_{n+1}-1)} = p_n^{-(1-s\gamma)/\gamma}(p_n^R - 1)^{-s} \sim p_{n+1}^{-1/\beta}.$$

Furthermore, $n^{-1/\gamma} \le a_n \le n^{-1/\beta}$. Hence $\gamma(a^{(s)}) = \gamma$ and $\beta(a^{(s)}) = \beta$, so a satisfies the desired conditions. In addition $a^{(s)}$ satisfies

$$\alpha(a^{(s)}) = \frac{\gamma(1-s)}{(1-\beta)+\gamma(1-s)} = f(s).$$

To show this, we estimate r_{p_n}. We see that

$$r_{p_n} = \sum_{p_n \le j < p_{n+1}} a_j + \sum_{j \ge p_{n+1}} a_j \sim C p_n^{-\frac{1-s\gamma}{1-s\beta}\frac{1-\beta}{\gamma}}, \tag{2}$$

so that

$$\alpha(a^{(s)}) \le \frac{\gamma(1-s\beta)}{(1-\beta)+\gamma(1-s)}.$$

To see the other inequality observe that for $i \in \mathbb{N}$ with $p_n < i < p_{n+1}$, we have

$$\alpha_i = \frac{\ln(1/i)}{\ln(r_i/i)} \ge \frac{\ln(1/p_n)}{\ln(r_{p_n}/p_n) = \alpha_{p_n}}.$$

This estimate is obtained by noting that if τ is such that $i = p_n^\tau$ $(1 < \tau < R)$, then

$$r_i \approx p_n^{-\frac{1-s\gamma}{\gamma}}\left(p_n^{R(1-s)} - p_n^{\tau(1-s)}\right) + p_n^{R^2(1-s)-\frac{R}{\gamma}},$$

and since $1 < \tau < R$, by (2) asymptotically we have that $r_i/r_{p_n} \to 1$. Thus, for large enough values, we know that $1 < \ln r_i / \ln r_{p_n} < \tau$, which is equivalent to the desired inequality. Therefore $\dim(C_{a^{(s)}}) = f(s)$. □

We summarize in the next theorem the main results of this section.

Theorem 2. *Let $a = \{a_n > 0\}$ be a summable sequence.*

(1) $0 \le \gamma(a) \le \dim(C_a) \le \alpha(a) \le \dfrac{\gamma(a)}{1-\beta(a)+\gamma(a)} \le \beta(a)$. *In particular, when the sequence a is nonincreasing we have* $\dim(C_a) = \alpha(a)$.

(2) *Given numbers α, β and γ with*

$$0 \le \gamma \le \alpha \le \frac{\gamma}{1-\beta+\gamma} \le \beta \le 1,$$

there exists a summable sequence a (which can be chosen to be nonincreasing) such that $\gamma(a)=\gamma$, $\alpha(a)=\alpha$ and $\beta(a)=\beta$.

Given a nonincreasing sequence a it could happen that the $\alpha(a)$-Hausdorff measure of the associated Cantor set C_a is zero or infinite. In the next section we will see that we can still say something in this case.

5. Dimension function

To analyze this situation it will be useful to refine the notion of dimension, in the spirit of Hausdorff's original work. Throughout this section we fix a monotonic nonincreasing sequence $a = \{a_k\}$ of positive terms such that $\sum a_k = 1$.

We associate to a another nonincreasing sequence:

$$b = \{b_n\} \quad \text{with} \quad b_n = \frac{r_n}{n}, \quad \text{where } r_n = \sum_{j=n}^{\infty} a_j \text{ as before.}$$

Fix a decreasing function $f : [1, +\infty) \to \mathbb{R}$ such that $f(k) = b_k$, for example

$$f(x) = b_k(k+1-x) + b_{k+1}(x-k), \quad x \in [k, k+1).$$

Then define

$$h(t) = \begin{cases} 1/f^{-1}(t), & t \in (0, b_1], \\ 0, & t = 0. \end{cases}$$

Then h is a nondecreasing, concave function and $h(b_k) = 1/k$. This function will be useful for determining the dimension of the Cantor set C_a.

We will need some auxiliary results and (more!) notation.

Let W denote the set of binary words of finite length:

$$W = \{e\} \cup \{w_1 \cdots w_r : w_i \in \{0,1\},\ r \in \mathbf{N}\},$$

where e denotes the empty word. If $w, w' \in W$ let ww' be the concatenation of w and w', and $|w|$ the length w, with $|e| = 0$. Let W^* denote the set of words of positive length. Given w, either an infinite binary word or a finite binary word of length at least k, we will denote by $w(k)$ the truncation $w_1 \cdots w_k$.

It is convenient to use the elements of W to describe the intervals of our Cantor set C_a. Let I_e denote the initial interval. ($I_e = I_0^0$). If $w \in W$, $|w| = k$ and I_w is an interval of step k in the construction, denote by I_{w0} and I_{w1} the left and right intervals obtained by removing the open interval from I_w.

In this way, if I_w is an interval of step $|w|$, with

$$I_w = I^{|w|}_{\sum_{j=1}^{|w|} w_j 2^{k-j}},$$

and if $w' \in W$, we see that $I_{ww'}$ is an interval of step $|ww'|$, which we say is *related* to I_w.

It is worthwhile to note at this stage that in the case of a monotonic nonincreasing sequence, the lengths of I_w also form a nonincreasing sequence.

For the sequence b_n defined on the previous page we will now denote by b_w the element of the sequence corresponding to b_ℓ, with $\ell = 2^k + \sum_{j=1}^{k} w_j 2^{k-j}$ and $k = |w|$.

In particular,

$$\text{if} \quad b_w = b_{2^k+l} \quad \text{then} \quad b_{ww'} = b_{2^{k'}(2^k+l)+s}, \tag{3}$$

where $l = \sum_{j=1}^{k} w_j 2^{k-j}$ with $k = |w|$ and $s = \sum_{j=1}^{k'} w'_j 2^{k'-j}$ with $k' = |w'|$.

Lemma 2. *With the above notation, for every $k \geq 1$, and $w, \widetilde{w}$ of length k, and any w',*

$$\frac{1}{2}\,\frac{h(b_{ww'})}{h(b_w)} \;\leq\; \frac{h(b_{\widetilde{w}w'})}{h(b_{\widetilde{w}})} \;\leq\; 2\,\frac{h(b_{ww'})}{h(b_w)}.$$

In particular, for any w' we have $h(b_{ww'}) \;\leq\; 4\; h(b_w)$.

Proof. Recall that $h(b_\ell) = 1/\ell$ and let $k' = |w'|$. Define

$$l = \sum_{j=0}^{k} w_j 2^{k-j}, \quad r = \sum_{j=0}^{k} \widetilde{w_j} 2^{k-j} \quad \text{and} \quad s = \sum_{j=0}^{k'} w'_j 2^{k'-j}.$$

Then, by (3),

$$\frac{h(b_{ww'})}{h(b_w)} \;=\; \frac{2^k + l}{2^{k'}(2^k + l) + s} \quad \text{and} \quad \frac{h(b_{\widetilde{w}w'})}{h(b_{\widetilde{w}})} \;=\; \frac{2^k + r}{2^{k'}(2^k + r) + s}.$$

Now noting that

$$\frac{1}{2} \;\leq\; \frac{2^k + r}{2^k + l} \;\leq\; 2,$$

we obtain the desired result.

For the second inequality just note that h is nondecreasing and therefore the right-hand side is less or equal than 2 for *any* w'. ☐

These bounds of the ratios of $h(b_k)$ will be useful for defining a measure on C_a. Since the construction of this Cantor set relies on the size of the gaps, it will be useful to define a measure depending on the size of the gaps.

Proposition 6. *There exists a probability measure μ_h supported on C_a such that, for every $k \geq 1$ and $0 \leq \ell \leq 2^k - 1$,*

$$\tfrac{1}{4}\, h(b_{2^k+\ell}) \;\leq\; \mu_h(I^k_\ell) \;\leq\; 2\; h(b_{2^k+\ell}). \tag{4}$$

Proof. For $m \geq 1$ consider the probability measure μ_m supported on the intervals I_ℓ^m of level m and such that

$$\mu_m(I_t^m) = \frac{h(b_{2^m+t})}{\sum_{j=0}^{2^m-1} h(b_{2^m+j})}.$$

If $k \leq m$ and $w = w_1 \dots w_k$ is such that $\sum_{j=0}^{k} w_j 2^{k-j} = t$, we have

$$\mu_m(I_t^k) = \mu_m(I_w) = \sum_{|w'|=m-k} \mu_m(I_{ww'}),$$

and hence

$$\left(\sum_{j=0}^{2^m-1} h(b_{2^m+j})\right) \mu_m(I_t^k) \ = \sum_{|w'|=m-k} h(b_{ww'}). \tag{5}$$

But by the bounds found in Lemma 2,

$$h(b_{ww'}) \ \leq \ 2\, h(b_w)\, \frac{h(b_{\widetilde{w}w'})}{h(b_{\widetilde{w}})}, \quad \text{for all } \widetilde{w} \text{ such that } |\widetilde{w}| = |w| = k.$$

Hence, recalling the definition of w, we obtain (from (5)), that for all $\widetilde{w}$ such that $|\widetilde{w}| = k$,

$$\left(h(b_{\widetilde{w}}) \sum_{j=0}^{2^m-1} h(b_{2^m+j})\right) \mu_m(I_t^k) \leq 2\, h(b_w) \sum_{|w'|=m-k} h(b_{\widetilde{w}w'}),$$

and therefore

$$\begin{aligned}\left(\sum_{|\widetilde{w}|=k} h(b_{\widetilde{w}}) \sum_{j=0}^{2^m-1} h(b_{2^m+j})\right) \mu_m(I_t^k) &\leq 2\, h(b_w) \left(\sum_{|\widetilde{w}|=k} \sum_{|w'|=m-k} h(b_{\widetilde{w}w'})\right) \\ &= 2\, h(b_w) \sum_{j=0}^{2^m-1} h(b_{2^m+j}),\end{aligned}$$

which yields

$$\mu_m(I_\ell^k) \leq 2\, \frac{h(b_{2^k+\ell})}{\sum_{j=0}^{2^k-1} h(b_{2^k+j})}, \quad k \leq m.$$

But noting that

$$\frac{1}{2} \leq \sum_{j=0}^{2^m-1} h(b_{2^m+j}) \ \leq 1$$

and using the other inequality of Lemma 2, we finally obtain

$$\tfrac{1}{2}\, h(b_{2^k+\ell}) \leq \mu_m(I_\ell^k) \leq 4\, h(b_{2^k+\ell}) \quad \text{for every } 1 \leq k \leq m,\ 0 \leq \ell \leq 2^k - 1.$$

Now let μ_h be the weak*-limit of μ_m. then (see for example [**Mat95**]) for every $1 \le k$, $0 \le \ell \le 2^k - 1$,

$$\tfrac{1}{2}\, h(b_{2^k+\ell}) \le \mu_h(I_\ell^k) \le 4\, h(b_{2^k+\ell}). \qquad \square$$

We are now ready to prove our main result. Recall that an h-set was defined in Equation (1) of the introduction.

Theorem 3. *Let $a = \{a_k\}$ be a nonincreasing sequence of positive terms such that $\sum a_k = 1$ and C_a the associated Cantor set. Then C_a is an h-set. Moreover*

$$\tfrac{1}{32} \le \mathcal{H}^h(C_a) \le 1,$$

where $\mathcal{H}^h$ is the Hausdorff measure associated to h, and h is the dimension function defined on page 54.

Proof. For the upper bound, fix $\delta > 0$ and let n_0 be such that $n \ge n_0$, $r_n = \sum_{j \ge n} a_j < \delta$. Then the intervals $E_1, \dots, E_n$ remaining after the gaps associated to $a_1, \dots, a_{n-1}$ are removed form a δ-covering of C_a, and since h is concave, we have

$$\sum_{i=1}^{n} h(|E_i|) \le nh\left(\frac{|E_1| + \cdots + |E_n|}{n}\right) = nh\left(\frac{r_n}{n}\right) = 1.$$

Therefore $\mathcal{H}^h(C_a) \le 1$.

For the lower bound, the idea is to try to use the measure μ_h and apply a generalized version of the *mass transfer principle.* To this end, let U be any open set with $\operatorname{diam}(U) = \rho < 1$. Let $k \ge 1$ and $0 \le \ell \le 2^k - 2$ be such that $b_{2^k+\ell+1} \le \rho < b_{2^k+\ell}$ (the case $b_{2^{k+1}} \le \rho < b_{2^{k+1}-1}$ will be considered separately). Then, because the lengths of the intervals I_l^k form a nonincreasing sequence,

$$\rho < b_{2^k} = \frac{|I_0^k| + \cdots + |I_{2^k-1}^k|}{2^k} < |I_0^k|.$$

Then U can intersect at most two consecutive intervals of step $k-1$. Hence, for all positive $t \le 2^k - 2$,

$$\begin{aligned} \mu_h(U) &\le (\mu_h(I_t^{k-1}) + \mu_h(I_t^{k-1})), \\ &\le 2h(b_{2^{k-1}+t}) + 2h(b_{2^{k-1}+t}) \quad \text{by Proposition 6} \\ &\le 8h(b_{2^{k-1}}) \le 32\, h(b_{2^k+\ell+1}) \quad \text{by Lemma 2.} \end{aligned}$$

Since h is nondecreasing, $\mu_h(U) \le 32h(b_{2^k+\ell+1}) \le 32h(\operatorname{diam}(U))$.

Now assume ρ satisfies $b_{2^{k+1}} \le \rho < b_{2^{k+1}-1}$. Since $\rho < b_{2^k}$, we still have

$$\mu_h(U) \le 8\, h(b_{2^{k-1}}) = 8\, \frac{1}{2^{k-1}} = 32\, \frac{1}{2^{k+1}} = 32\, h(b_{2^{k+1}}),$$

and so again, $\mu_h(U) \le 32\, h(\operatorname{diam}(U))$.

Therefore, if $\{U_k\}$ is a δ-covering of C_a, we have

$$\sum_k h(\operatorname{diam}(U_k)) \geq \frac{1}{32} \sum_k \mu_h(U_k) \geq \frac{1}{32}\, \mu_h(C_a).$$

Since this is true for every δ-covering, we obtain

$$\mathcal{H}^h_\delta(C_a) \geq \frac{1}{32}\, \mu_h(C_a),$$

and therefore $\mathcal{H}^h(C_a) \geq \frac{1}{32}$. □

One can also establish a certain equivalence relation among dimension functions: $h \equiv g$ if there exist constants c_1 and c_2 such that

$$c_1 \leq \underline{\lim}_{x \to 0+} \frac{h(x)}{g(x)} \leq \overline{\lim}_{x \to 0+} \frac{h(x)}{g(x)} \leq c_2.$$

The following result relates the function h to $\alpha(a)$.

Proposition 7. *If $a \sim n^{-1/s}$ then $h \equiv x^s$.*

Proof. Since $a \sim n^{-1/s}$, we have $\gamma(a) = \beta(a) = s$, and hence there exist $c > 0$ and $d > 0$ such that

$$cn^{-1/s} \leq a_n \leq dn^{-1/s},$$

and therefore

$$Cn^{-1/s} \leq \frac{r_n}{n} \leq Dn^{-1/s}.$$

Hence

$$0 < c_1 \leq \underline{\lim}_{x \to 0+} \frac{h(x)}{x^s} \leq \overline{\lim}_{x \to 0+} \frac{h(x)}{x^s} \leq c_2 < +\infty.$$ □

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