Pacific Journal of Mathematics

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We study not necessarily self-similar Dirichlet forms on the Sierpiński gasket that can be described as limits of compatible resistance networks on the sequence of graphs approximating the gasket. We describe the compatibility conditions in detail, and we also present an alternative description, based on just 3 conductance values and the 3-dimensional space of harmonic functions. In addition, we show how to parameterize all the Dirichlet forms by a set of independent variables.

1. Introduction

The central object in the theory of analysis on fractals, as it has developed so far, is the Dirichlet form, analogous to $\int |\nabla f|^2 \, dx$ in the theory of Riemannian manifolds. Great effort has gone into the construction of self-similar Dirichlet forms on self-similar fractals, with the successful theory of Kigami [**Ki2**] covering a class of finitely ramified fractals called *post-critically finite* (pcf). This class of fractals includes the familiar Sierpiński gasket (SG), and Kigami's approach is based on his original construction for SG in [**Ki1**], which was completed by Sabot [**Sa**] with the classification of all self-similar Dirichlet forms on SG (see also [**Me**, **SST**, **Te**]). These Dirichlet forms have the property that points have positive capacity. In general this property is true for regular harmonic structures and is not true for nonregular harmonic structures. We will discuss this in more detail in Section 7.

The goal of this paper is to describe the class of *all* Dirichlet forms on SG — we drop the self-similarity requirement on the form, though of course SG retains its topological self-similarity. It should be understood that the geometric structure of SG inherited from the standard realization in the plane plays no role in this theory. Instead, in Section 6 we use the harmonic coordinates realization of SG from [**Ki3**]. However, the question of what is the "correct" geometry for SG, or what might be the analog of a Riemannian metric, is not resolved at present. Following our motivational analogy a step further, we might imagine that the class of pcf self-similar fractals plays the role of symmetric spaces in Riemannian geometry, and SG plays the role of the sphere. The self-similarity condition on the Dirichlet form might be analogous to group invariance of the Riemannian metric on a symmetric space. This suggests that an understanding of the special case of SG might

yield useful insights into the problem of describing all Dirichlet forms on the general finitely ramified fractal (again there is no generally accepted definition of this class of fractals).

In a crude sense, there already exists a solution to our problem. Consider the sequence of graphs $\{\Gamma_n\}$, where Γ_0 is the complete graph on the vertices $V_0 = \{q_1, q_2, q_3\}$, the boundary points of the triangle containing SG, and Γ_n is defined inductively to contain the vertices

$$V_n = \bigcup_{i=1}^{3} F_i V_{n-1},$$

where $F_i x = \frac{1}{2}(x + q_i)$ and the edge relation $x \sim_j y$ holds if there exists F_i and $x', y' \in V_{n-1}$ with $x' \sim_{j-1} y'$ and $x = F_i x'$, $y = F_i y'$. The sets of vertices V_n are contained in SG, and SG is the closure of $V_* = \bigcup V_n$ in a suitable topology. We will say that a Dirichlet form is *point-sensitive* if it assigns positive capacity to all points in V_*. Given any Dirichlet form $\mathcal{E}$ on SG with domain $\operatorname{dom}\mathcal{E}$, we can define its restriction to Γ_n as follows: for any function f on V_n define its n-harmonic extension $\widetilde{f}$ to SG to be the element in $\operatorname{dom}\mathcal{E}$ satisfying $\widetilde{f}|_{V_n'} = f$ and minimizing $\mathcal{E}(\widetilde{f}, \widetilde{f})$. Note that it is important that the Dirichlet form is point-sensitive for this definition to make sense. (For example, in a planar region for the usual Dirichlet form, no such minimizer exists.) The restriction $\mathcal{E}_n$ is defined by $\mathcal{E}_n(f, f) = \mathcal{E}(\widetilde{f}, \widetilde{f})$, and similarly for the bilinear form $\mathcal{E}_n(f, g) = \mathcal{E}(\widetilde{f}, \widetilde{g})$. Then $\mathcal{E}_n$ is a Dirichlet form on Γ_n and so can be written

$$(1.1) \qquad \mathcal{E}_n(f, f) = \sum_{x \sim_n y} c(x, y)(f(x) - f(y))^2$$

for certain positive conductance coefficients $c(x, y)$. The reciprocal of a conductance is a resistance, and the graph Γ_n equipped with the Dirichlet form (1.1) is called a *resistance network*. In this paper we study point-sensitive Dirichlet forms in terms the approximating sequence of Dirichlet forms $\mathcal{E}_n$.

The Dirichlet forms $\mathcal{E}_n$ satisfy a compatibility condition: the restriction of $\mathcal{E}_n$ to Γ_{n-1} must be equal to $\mathcal{E}_{n-1}$. This may be expressed as an algebraic condition relating the conductances on V_n and V_{n-1}. In fact it suffices to understand this condition on the first level, $n = 1$, since it is a local condition on each cell $F_w V_0$, where $F_w = F_{w_1} \cdots F_{w_n}$ for any word $w_1 \cdots w_n$ of length $|w| = n$ from the alphabet $\{1, 2, 3\}$. For $n = 1$ this becomes a polynomial equation in the 9 conductances on V_1 in terms of the 3 conductances on V_0. We will show that for each choice of V_0 conductances the solution set is a 6-dimensional manifold by giving an explicit parameterization in Section 5. We give a geometric interpretation in Section 6. Nevertheless, the algebraic structure of the compatibility conditions is still somewhat mysterious.

A new description of Dirichlet forms that we present involves the harmonic functions. These are just the harmonic extensions from Γ_0, and so form a 3-dimensional space $\mathcal{H}$, each h in $\mathcal{H}$ being determined by its values on V_0, which is regarded as the boundary of SG. (In general, harmonic extensions from Γ_n give functions that are harmonic everywhere except at the points of V_n.) Of course $\mathcal{H}$ contains the constants, since these have zero energy. A basis for $\mathcal{H}$ consists of the functions h_k such that $h_k(q_j) = \delta_{kj}$, and $h_1 + h_2 + h_3 = 1$.

One easy consequence of this description is that $\mathcal{E}$ is determined by the space $\mathcal{H}$ and the 3 conductance coefficients $d_1 = c(q_2, q_3)$, $d_2 = c(q_1, q_3)$, $d_3 = c(q_1, q_2)$. We will demonstrate this in a constructive manner by deriving an algebraic formula for the 9 conductances on V_1 in terms of this data (more precisely, the coefficients d_k and the values of h_k at the points in V_1). The same formula may be used inductively to find all coefficients of the restrictions $\mathcal{E}_n$, using the values of h_k on points in V_n.

To complete the description it would be desirable to have necessary and sufficient conditions on the data to be associated with a Dirichlet form, but we have been unable to do this. We will exhibit some necessary conditions on the data already on the level V_1 that involve both the initial conductances and the values of h_k. The analogous conditions on all levels V_n are also necessary, but since they are expressed in terms of the conductances on V_{n-1}, which are derived from the data in a complicated way, we have been unable to write them down explicitly. Nevertheless, we are able to show that there are no "dead ends", in the sense that every resistance network on Γ_n is the restriction of a point-sensitive regular Dirichlet form on SG. Note that this says more than just that there are no algebraic obstructions to extending from Γ_{n-1} to Γ_n, since it involves the analytic property that the conductances tend to infinity in the limit.

In Section 4 we show that the functions in $\mathcal{H}$ already determine the initial conductances (up to a constant multiple) under a seemingly reasonable hypothesis that has the form of a local Harnack inequality. We also give an example when the initial conductances are not determined by $\mathcal{H}$.

Given a compatible sequence of Dirichlet forms $\{\mathcal{E}_n\}$ there exists a limiting Dirichlet form $\mathcal{E}$ on SG. For any function f on SG the sequence $\mathcal{E}_n(f|_{V_n}, f|_{V_n})$ is monotone increasing, so

$$(1.2) \qquad \mathcal{E}(f, f) = \lim_{n \to \infty} \mathcal{E}_n(f|_{V_n}, f|_{V_n})$$

is always defined (allowing $+\infty$) for a function f on V_*, and we may define $\mathrm{dom}\,\mathcal{E}$ as all functions with $\mathcal{E}(f, f) < \infty$. It is not hard to see that $\mathcal{E}_n$ is the restriction of $\mathcal{E}$ to Γ_n. In what follows we always assume that the conductances $c(x, y)$ are positive.

It is evident that $\mathcal{E}$ is point-sensitive, as demonstrated in Proposition 7.1 (Theorem 7.2 gives another proof). Many questions related to such sequences of Dirichlet forms $\mathcal{E}_n$ are studied in [**Ki4**] in detail. An important tool in this study is an effective resistance metric R to be defined in Section 7. If Ω is the R-completion of V_*, any function in dom $\mathcal{E}$ is a restriction of an R-continuous function on Ω. We will show in Section 7 that any point of Ω has positive capacity, and relate it to some properties of Green's function.

In the case when harmonic functions are continuous, Ω can be naturally considered to be a subset of SG. Under this assumption an important question is whether the harmonic structure is *regular*, that is $\Omega = $ SG. The answer is positive if all the conductances tend to infinity, but this condition is not necessary, as demonstrated by an example in Section 7. We also give an example where harmonic functions are continuous but $\Omega \neq$ SG.

Another set of objects naturally associated with $\mathcal{E}$ is the set of energy measures ([**Ku**]). For every $f \in$ dom $\mathcal{E}$ there is a positive measure ν_f such that $\mathcal{E}(f, f) = \nu_f(\text{SG})$ and more generally there is a signed measure $\nu_{f,g}$ for each pair $f, g \in$ dom $\mathcal{E}$ (with $\nu_{f,f} = \nu_f$) such that

$$\mathcal{E}(f, g) = \nu_{f,g}(\text{SG}).$$

It is conceivable that it determines $\mathcal{E}$ but we have been unable to prove this.

To put our results into perspective, let us review the situation when the underlying space is the unit interval. Given a Dirichlet form, let h denote the harmonic function satisfying the boundary conditions $h(0) = 0$, $h(1) = 1$. The reciprocals of the conductances determine a finite positive measure μ of total mass c^{-1}, where c is the conductance coefficient $c(0, 1)$, and $h(x) = c\mu([0, x])$. The only conditions on the measure are that it be continuous and assign positive mass to nontrivial intervals. The only conditions on the function h are that it be continuous and strictly increasing. Note that μ determines h but h only determines μ up to a constant multiple. The Dirichlet form is expressible in terms of c and h as

$$\mathcal{E}(f, f) = c \int_0^1 \left| \frac{d}{dx}(f \circ h^{-1}(x)) \right|^2 dx,$$

with dom $\mathcal{E}$ consisting of all functions f for which $f \circ h^{-1}$ belongs to the Sobolev space L_1^2. In this case the energy measure for h satisfies $\nu_h = c^2\mu$.

The reader can find an extensive exposition of the analysis on fractals in [**B**, **Ki4**]. These books also contain lengthy bibliographies.

2. Basic equations

We consider graphs Γ_0 and Γ_1 as shown in Figure 2.1. Γ_0 has vertices q_1, q_2, q_3 with conductances d_1, d_2, d_3 on the opposite edge. Γ_2 has an additional 3 vertices x, y, z and 9 edges, with conductances $c_1, \ldots, c_9$ as shown. The boundary of Γ consists of the vertices q_1, q_2, q_3. We refer to c_2, c_5, c_8 as
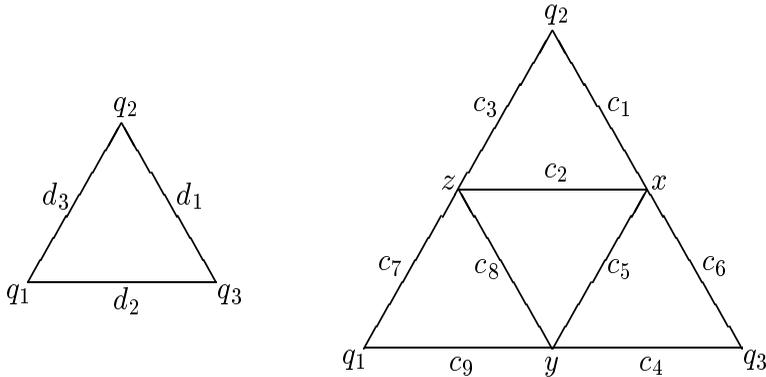
Figure 2.1. The graphs Γ_0 and Γ_1 with vertices and conductances on edges labeled.

inner conductances since the edges do not connect to the boundary vertices, and the others as *outer* conductances.

We let h_j denote the harmonic function on Γ_1 with boundary values $h_j(q_k) = \delta_{jk}$. Note that $h_1 + h_2 + h_3 = 1$. We write $x_j = h_j(x)$, $y_j = h_j(y)$ and $z_j = h_j(z)$.

The equations that express the fact that h_j is harmonic at the 3 vertices x, y, z can be expressed in terms of a matrix X which we write as a sum $X = X_o + X_i$, with

$$(2.1) \qquad X_o = \begin{pmatrix} c_1 + c_6 & 0 & 0 \\ 0 & c_4 + c_9 & 0 \\ 0 & 0 & c_3 + c_7 \end{pmatrix},$$

$$(2.2) \qquad X_i = \begin{pmatrix} c_2 + c_5 & -c_5 & -c_2 \\ -c_5 & c_5 + c_8 & -c_8 \\ -c_2 & -c_8 & c_2 + c_8 \end{pmatrix},$$

corresponding to the contributions from the outer and inner conductances, respectively. We also need matrices

$$(2.3) \qquad H = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}$$

and

$$(2.4) \qquad J = \begin{pmatrix} 0 & c_1 & c_6 \\ c_9 & 0 & c_4 \\ c_7 & c_3 & 0 \end{pmatrix}.$$

The harmonic equation is then

$$(2.5) \qquad XH = J.$$

Since

$$(2.6) \qquad \det J = c_1 c_4 c_7 + c_3 c_6 c_9,$$

which is positive, it follows that X and H are invertible. It is also clear by inspection that $\det X$ is positive since all terms except $-2c_2 c_5 c_8$ are positive, and these cancel $2c_2 c_5 c_8$ from the diagonal. A lengthy computation reveals that $\det X$ is given by

$$(2.7) \quad c_1 c_4 c_7 + c_3 c_6 c_9 + c_1 c_3 (c_4 + c_9) + c_4 c_6 (c_3 + c_7) + c_7 c_9 (c_1 + c_6)$$
$$+ c_2 (c_4 + c_9)(c_1 + c_3 + c_6 + c_7) + c_5 (c_3 + c_7)(c_1 + c_4 + c_6 + c_9)$$
$$+ c_8 (c_1 + c_6)(c_3 + c_4 + c_7 + c_9)$$
$$+ (c_2 c_5 + c_5 c_8 + c_2 c_8)(c_1 + c_3 + c_4 + c_6 + c_7 + c_9).$$

We can use (2.5) to solve $H = X^{-1} J$ to obtain the harmonic function values explicitly in terms of the conductances, but we will wait until the next section to give the details. Note, however, that (2.5) implies $\det H > 0$. Also, the condition $h_1 + h_2 + h_3 = 1$ becomes $H1 = 1$, where 1 denotes the vector of 1's.

Another interesting observation from (2.5) is that if we know H and all the outer conductances, we can find all the inner conductances via

$$(2.8) \qquad X_i = -X_o + JH^{-1},$$

since J only involves outer conductances.

Now we consider the consistency condition for the Dirichlet forms on Γ_1 and Γ_0. There are many ways to express this condition, but we concentrate on one that involves only the outer conductances and H, namely

$$(2.9) \quad \begin{pmatrix} d_2 + d_3 & -d_3 & -d_2 \\ -d_3 & d_1 + d_3 & -d_1 \\ -d_2 & -d_1 & d_1 + d_2 \end{pmatrix} = \begin{pmatrix} c_7 + c_9 & 0 & 0 \\ 0 & c_1 + c_3 & 0 \\ 0 & 0 & c_4 + c_6 \end{pmatrix} - J^* H.$$

Using the off-diagonal terms, we obtain the set of equations

$$(2.10) \quad \begin{cases} \begin{pmatrix} x_3 & z_3 \\ x_1 & z_1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_3 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_3 \end{pmatrix}, \\[2mm] \begin{pmatrix} y_1 & x_1 \\ y_2 & x_2 \end{pmatrix} \begin{pmatrix} c_4 \\ c_6 \end{pmatrix} = \begin{pmatrix} d_2 \\ d_1 \end{pmatrix}, \\[2mm] \begin{pmatrix} z_2 & y_2 \\ z_3 & y_3 \end{pmatrix} \begin{pmatrix} c_7 \\ c_9 \end{pmatrix} = \begin{pmatrix} d_3 \\ d_2 \end{pmatrix}. \end{cases}$$

We may easily solve (2.10) for the outer conductances, obtaining

$$(2.11) \quad \begin{cases} c_1 = \dfrac{\begin{vmatrix} d_1 & z_3 \\ d_3 & z_1 \end{vmatrix}}{\begin{vmatrix} x_3 & z_3 \\ x_1 & z_1 \end{vmatrix}}, & c_4 = \dfrac{\begin{vmatrix} d_2 & x_1 \\ d_1 & x_2 \end{vmatrix}}{\begin{vmatrix} y_1 & x_1 \\ y_2 & x_2 \end{vmatrix}}, & c_7 = \dfrac{\begin{vmatrix} d_3 & y_2 \\ d_2 & y_3 \end{vmatrix}}{\begin{vmatrix} z_2 & y_2 \\ z_3 & y_3 \end{vmatrix}}, \\[4ex] c_3 = \dfrac{\begin{vmatrix} x_3 & d_1 \\ x_1 & d_3 \end{vmatrix}}{\begin{vmatrix} x_3 & z_3 \\ x_1 & z_1 \end{vmatrix}}, & c_6 = \dfrac{\begin{vmatrix} y_1 & d_2 \\ y_2 & d_1 \end{vmatrix}}{\begin{vmatrix} y_1 & x_1 \\ y_2 & x_2 \end{vmatrix}}, & c_9 = \dfrac{\begin{vmatrix} z_2 & d_3 \\ z_3 & d_2 \end{vmatrix}}{\begin{vmatrix} z_2 & y_2 \\ z_3 & y_3 \end{vmatrix}}. \end{cases}$$

In the next section we will show that all the denominators are positive. We then use (2.8) to obtain

$$(2.12) \quad \begin{cases} c_2 = \left(c_1 \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} - c_6 \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \right) \Big/ \det H, \\[3ex] c_5 = \left(c_1 \begin{vmatrix} x_3 & x_1 \\ z_3 & z_1 \end{vmatrix} - c_6 \begin{vmatrix} z_1 & z_2 \\ x_1 & x_2 \end{vmatrix} \right) \Big/ \det H, \\[3ex] c_8 = \left(c_9 \begin{vmatrix} x_3 & x_2 \\ y_3 & y_2 \end{vmatrix} - c_4 \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \right) \Big/ \det H. \end{cases}$$

By substituting (2.11) into (2.12) and simplifying we obtain

$$(2.13) \quad \begin{cases} c_2 = \begin{vmatrix} x_1 & y_1 & z_1 \\ d_1 x_1 & d_2 y_2 & d_3 z_3 \\ x_3 & y_3 & z_3 \end{vmatrix} \Big/ \left(\begin{vmatrix} x_3 & z_3 \\ x_1 & z_1 \end{vmatrix} \det H \right), \\[5ex] c_5 = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ d_1 x_1 & d_2 y_2 & d_3 z_3 \end{vmatrix} \Big/ \left(\begin{vmatrix} y_1 & x_1 \\ y_2 & x_2 \end{vmatrix} \det H \right), \\[5ex] c_8 = \begin{vmatrix} d_1 x_1 & d_2 y_2 & d_3 z_3 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \Big/ \left(\begin{vmatrix} z_2 & y_2 \\ z_3 & y_3 \end{vmatrix} \det H \right). \end{cases}$$

We may summarize these computations in the following theorem:

Theorem 2.1. *Given the conductances $\{d_j\}$ on Γ_0 and the harmonic values H on Γ_1 satisfying $H1 = 1$, there is at most one choice of conductances $\{c_j\}$ on Γ_1 which yields these values, and there is such a choice, with values given by (2.11) and (2.13), if and only if these values are all positive.*

Proof. We deduced (2.11) and (2.13) under the assumption that such conductances exist, so this proves uniqueness. Conversely, assume all values

given in (2.11) and (2.13) are positive. From (2.11) we obtain (2.10). This is equivalent to (2.9) under the assumption $H1 = 1$; this yields the diagonal terms. From (2.13) and (2.11) we get (2.12), and also the other 3 expressions for c_2, c_5, c_8, which together yield (2.8), or equivalently (2.5). But (2.5) says that H gives the correct values for the harmonic functions determined by the conductances $\{c_j\}$, and then (2.5) and (2.9) tell us that the restriction of the Dirichlet form on Γ_1 to Γ_0 gives the conductances $\{d_j\}$. □

To use these expressions iteratively it is useful to put them in a form that is relative to any choice of 3 linearly independent harmonic functions, not just the choice (h_1, h_2, h_3) made above. Let $(\widetilde{h}_1, \widetilde{h}_2, \widetilde{h}_3)$ denote such functions. For any vertices (p_1, p_2, p_3) of Γ_1 define the 3×3 matrix

$$(2.14) \qquad \widetilde{h}(p_1, p_2, p_3) = \{\widetilde{h}_j(p_k)\}.$$

Corollary 2.2. *Suppose $(\widetilde{h}_1, \widetilde{h}_2, \widetilde{h}_3)$ is any set of 3 linearly independent harmonic functions on Γ_1 for conductances $\{c_j\}$, and suppose the restriction to Γ_0 yields conductances $\{d_j\}$. Then the outer conductances are given by*

$$(2.15) \quad \begin{cases} c_1 = \dfrac{d_1 \det \widetilde{h}(q_2, q_3, z) + d_3 \det \widetilde{h}(q_2, q_1, z)}{\det \widetilde{h}(q_2, x, z)}, \\[3mm] c_3 = \dfrac{d_1 \det \widetilde{h}(q_2, x, q_3) + d_2 \det \widetilde{h}(q_2, x, q_1)}{\det \widetilde{h}(q_2, x, z)}, \\[3mm] c_4 = \dfrac{d_1 \det \widetilde{h}(q_3, q_2, x) + d_2 \det \widetilde{h}(q_3, q_1, x)}{\det \widetilde{h}(q_3, y, x)}, \\[3mm] c_6 = \dfrac{d_1 \det \widetilde{h}(q_3, y, q_2) + d_2 \det \widetilde{h}(q_3, y, q_1)}{\det \widetilde{h}(q_3, y, x)}, \\[3mm] c_7 = \dfrac{d_2 \det \widetilde{h}(q_1, q_3, y) + d_3 \det \widetilde{h}(q_1, q_2, y)}{\det \widetilde{h}(q_1, z, y)}, \\[3mm] c_9 = \dfrac{d_2 \det \widetilde{h}(q_1, z, q_3) + d_3 \det \widetilde{h}(q_1, z, q_2)}{\det \widetilde{h}(q_1, z, y)}, \end{cases}$$

and the inner conductances by

$$(2.16)$$
$$\begin{cases} c_2 = \\ \dfrac{d_1 \det \widetilde{h}(x,q_2,q_3) \det \widetilde{h}(q_2,y,z) - d_2 \det \widetilde{h}(q_1,y,q_3) \det \widetilde{h}(q_2,x,z) + d_3 \det \widetilde{h}(q_1,q_2,z) \det \widetilde{h}(q_2,x,y)}{\det \widetilde{h}(q_1,q_2,q_3) \det \widetilde{h}(q_2,x,z)} \\[3mm] c_5 = \\ \dfrac{d_1 \det \widetilde{h}(x,q_2,q_3) \det \widetilde{h}(q_3,y,z) + d_2 \det \widetilde{h}(q_1,y,q_3) \det \widetilde{h}(q_3,x,z) - d_3 \det \widetilde{h}(q_1,q_2,z) \det \widetilde{h}(q_3,y,x)}{\det \widetilde{h}(q_1,q_2,q_3) \det \widetilde{h}(q_3,y,x)} \\[3mm] c_8 = \\ \dfrac{-d_1 \det \widetilde{h}(x,q_2,q_3) \det \widetilde{h}(q_1,z,y) + d_2 \det \widetilde{h}(q_1,y,q_3) \det \widetilde{h}(q_1,x,z) + d_3 \det \widetilde{h}(q_1,q_2,z) \det \widetilde{h}(q_1,x,y)}{\det \widetilde{h}(q_1,q_2,q_3) \det \widetilde{h}(q_1,z,y)}. \end{cases}$$

Proof. First we observe that for the initial choice $\widetilde{h} = h$, (2.15) is the same as (2.11) and (2.16) is the same as (2.13), so the result is valid in that case. But in general $\widetilde{h}_j = \sum_k B_{jk} h_k$ for some invertible matrix B, hence $\widetilde{h}(p_1, p_2, p_3) = h(p_1, p_2, p_3) B^*$. Thus ratios of quantities of the form $\det \widetilde{h}(p_1, p_2, p_3)$ are independent of the choice of $\widetilde{h}$, and (2.15) and (2.16) are composed entirely of such ratios. □

It is also possible to express these identities in terms of resistances rather than conductances. If we write $R_j = d_j^{-1}$ and $r_j = c_j^{-1}$, we have

$$(2.17) \qquad r_1 = R_1 R_3 \frac{\begin{vmatrix} x_3 & z_3 \\ x_1 & z_1 \end{vmatrix}}{\begin{vmatrix} R_3 & z_3 \\ R_1 & z_1 \end{vmatrix}}, \quad \text{etc.}$$

and

$$(2.18) \qquad r_2 = R_1 R_2 R_3 \det H \frac{\begin{vmatrix} x_3 & z_3 \\ x_1 & z_1 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & z_1 \\ R_2 R_3 x_1 & R_1 R_3 y_3 & R_1 R_2 z_3 \\ x_3 & y_3 & z_3 \end{vmatrix}}, \quad \text{etc.}$$

Corollary 2.3. *Two Dirichlet forms on* SG (*giving points positive capacity*) *having the same harmonic functions and the same initial conductances on* Γ_0 *must be equal. In particular, the space of Dirichlet forms may be parameterized by a subset of the space of 6-tuples* $(d_1, d_2, d_3, h_1, h_2, h_3)$ *where* $d_j > 0$, *and* h_j *are continuous functions* SG $\to \mathbb{R}$ *satisfying* $h_j(q_k) = \delta_{jk}$ *and* $h_1 + h_2 + h_3 = 1$.

Proof. Applying Theorem 2.1 inductively we see that the two Dirichlet forms have the same conductances at each level, hence by (1.2) they are equal. □

In the next section we will find some necessary conditions that the 6-tuple must satisfy. However, we have not been able to find really explicit necessary and sufficient conditions for a 6-tuple to correspond to a Dirichlet form.

3. Positivity conditions

Our first goal is to show that the denominators in (2.11) are all positive. To do this we return to the explicit solution of (2.5) for H in terms of the conductances $\{c_j\}$. In (2.7) we have expressed $\det X$ as a sum of 50 positive terms, each a product of 3 distinct conductances. We may describe all the terms in a scheme that allows three types. There are 8 of type I, a product of 3 outer conductances corresponding to edges in the graph that are disconnected. There are 24 of type II, a product of one inner conductance and 2 outer conductances where one of the outer edges is connected to the

inner edge, and the other outer edge is not connected to the inner edge. There are 18 of type III, a product of 2 inner and one outer conductance without any other restrictions.

Each entry of H is a ratio with denominator $\det X$ and numerator equal to a sum of terms selected from the 50 terms in $\det X$, as follows:

$$(3.1) \begin{cases} x_1 \det X = (c_2c_5+c_2c_8+c_5c_8)(c_7+c_9) + c_7c_9(c_2+c_5) \\ \qquad\quad + c_2c_4c_7+c_3c_5c_9 \\ y_1 \det X = (c_2c_5+c_2c_8+c_5c_8)(c_7+c_9) + c_7c_8(c_1+c_6) \\ \qquad\quad + (c_3+c_7)c_9(c_2+c_5) + c_9(c_1+c_6)(c_2+c_3+c_7+c_8) \\ z_1 \det X = (c_2c_5+c_2c_8+c_5c_8)(c_7+c_9) + c_9c_8(c_1+c_6) \\ \qquad\quad + (c_4+c_9)c_7(c_2+c_5) + c_7(c_1+c_6)(c_4+c_5+c_8+c_9) \\ x_2 \det X = (c_2c_5+c_2c_8+c_5c_8)(c_1+c_3) + c_2c_3(c_4+c_9) \\ \qquad\quad + c_1(c_3+c_7)(c_5+c_8) + c_1(c_4+c_9)(c_2+c_3+c_7+c_8) \\ y_2 \det X = (c_2c_5+c_2c_8+c_5c_8)(c_1+c_3) + c_1c_3(c_5+c_8) \\ \qquad\quad + c_1c_5c_7 + c_3c_6c_8 \\ z_2 \det X = (c_2c_5+c_2c_8+c_5c_8)(c_1+c_3) + c_1c_2(c_4+c_9) \\ \qquad\quad + c_3(c_1+c_6)(c_5+c_8) + c_3(c_4+c_9)(c_1+c_2+c_5+c_6) \\ x_3 \det X = (c_2c_5+c_2c_8+c_5c_8)(c_4+c_6)+c_4c_5(c_3+c_7) \\ \qquad\quad + c_6(c_4+c_9)(c_2+c_8) + c_6(c_3+c_7)(c_4+c_5+c_8+c_9) \\ y_3 \det X = (c_2c_5+c_2c_8+c_5c_8)(c_4+c_6) + c_5c_6(c_3+c_7) \\ \qquad\quad + c_4(c_1+c_6)(c_2+c_8)+c_4(c_3+c_7)(c_1+c_2+c_4+c_5) \\ z_3 \det X = (c_2c_5+c_2c_8+c_5c_8)(c_4+c_6) + c_4c_6(c_2+c_8) \\ \qquad\quad + c_2c_6c_9 + c_1c_4c_8. \end{cases}$$

Note that $x_1 \det X$, $y_2 \det X$ and $z_3 \det X$ consist of 10 terms each, while the others consist of 20 terms.

Theorem 3.1. *Each of the determinants* $\begin{vmatrix} y_1 & x_1 \\ y_2 & x_2 \end{vmatrix}$, $\begin{vmatrix} z_2 & y_2 \\ z_3 & y_3 \end{vmatrix}$ *and* $\begin{vmatrix} x_3 & z_3 \\ x_1 & z_1 \end{vmatrix}$ *is positive.*

Proof. We give two proofs. The first is computational, while the second is conceptual. For the first proof we show

$$(3.2) \qquad (y_1 \det X)(x_2 \det X) - (y_2 \det X)(x_1 \det X) > 0$$

by showing that it is a sum of 300 positive terms. Note that the first product consists of 400 terms and the second product consists of 100 terms. We claim that all 100 terms in the second product are contained among the 400 terms of the first product. In fact, 9 of the 10 terms in $x_1 \det X$, namely all except $c_2c_4c_7$, are among the 20 terms of $y_1 \det X$. Similarly, 9 of the 10 terms in $y_2 \det X$, all except $c_3c_6c_8$, are among the 20 terms of $x_2 \det X$. This already takes care of 81 terms of the product. It remains to check

the 19 terms of the product that have either $c_3c_6c_8$ or $c_2c_4c_7$ as one of the factors. This may be seen by inspection. For example, $(c_3c_6c_8)(c_2c_4c_7)$ appears as $(c_2c_3c_4)(c_6c_7c_8)$, and $(c_3c_6c_8)(c_2c_7c_9)$ appears in 2 different ways, $(c_2c_3c_9)(c_6c_7c_8)$ or $(c_2c_3c_8)(c_6c_7c_9)$. The positivity of the other 2 determinants follows by symmetry.

The second proof is based on the following interpretation of the result. Note that

$$\begin{vmatrix} y_1 & x_1 \\ y_2 & x_2 \end{vmatrix} = \begin{vmatrix} y_1 & x_1 & 0 \\ y_1 & x_2 & 0 \\ y_3 & x_3 & 1 \end{vmatrix} = \det h(y, x, q_3),$$

and (y, x, q_3) are the vertices of a small triangle in Γ_1 traversed in the same order as (q_1, q_2, q_3) traverses the large triangle. Since the determinant is a continuous function of the conductances, and it is clearly positive for the choice of equal conductances $(y_1 = x_2 = \frac{2}{5}, \ y_2 = x_1 = \frac{1}{5})$, it suffices to show that it is never zero. The determinant being zero means that there is a nonzero harmonic function whose restriction to the points (x, y, q_3) is zero. We now show that this is impossible.

Suppose h is a harmonic function on Γ_1 satisfying $h(y) = h(x) = h(q_3) = 0$. If h is not identically zero we may arrange to have $h(q_1) > 0$, without loss of generality. Since h is harmonic at y we must have $h(z) < 0$. Since h is harmonic at x we then must have $h(q_2) > 0$. But then h will not be harmonic at z.   □

The unique extension property for harmonic functions from small triangles proved above is not usually valid for graphs that arise in the study of self-similar fractals. It fails for the pentagasket ([**ASST**]) and hexagasket fractals, for examples.

The next result is a relativized version of the result, analogous to Corollary 2.2. Note that $h(q_1, q_2, q_3)$ is the identity matrix so $\det h(q_1, q_2, q_3) = 1$.

Corollary 3.2. *Let $(\widetilde{h}_1, \widetilde{h}_2, \widetilde{h}_3)$ be linearly independent harmonic functions for which $\det \widetilde{h}\,(q_1, q_2, q_3) > 0$. Then $\det \widetilde{h}\,(y, x, q_3)$, $\det \widetilde{h}(z, q_2, x)$ and $\det \widetilde{h}(q_1, z, y)$ are all positive.*

Proof. As in the proof of Corollary 2.2 we write $\widetilde{h} = hB^*$. Since each matrix is multiplied by B^*, we must have $\det B^* > 0$ from the hypothesis. Then multiplication by B^* preserves the sign of the other determinants.   □

Since the denominators in (2.11) are all positive, so are the numerators. These inequalities may be summarized as

$$(3.3) \qquad \frac{z_1}{z_3} > \frac{d_3}{d_1} > \frac{x_1}{x_3}, \qquad \frac{x_2}{x_1} > \frac{d_1}{d_2} > \frac{y_2}{y_1}, \qquad \frac{y_3}{y_2} > \frac{d_2}{d_3} > \frac{z_3}{z_2}.$$

Note that the positivity of the determinants in Theorem 3.1 means that the inequalities in (3.3) are valid without the middle term. Also, by multiplying the 3 rows of (3.3) the middle term becomes 1, and we obtain

$$(3.4) \qquad x_1 y_2 z_3 < x_2 y_3 z_1, \qquad x_1 y_2 z_3 < x_3 y_1 z_2.$$

It would also be possible to give a computational proof of (3.4) along the lines of the first proof of Theorem 3.1.

Theorem 3.3. *Given positive conductances $\{d_j\}$ and a matrix H with positive entries satisfying $H1 = 1$, a necessary and sufficient condition for the existence of positive conductances $\{c_j\}$ such that H is the matrix of harmonic values and $\{d_j\}$ is the restriction to Γ_0 is that the following 13 determinants all be positive:*

$$(3.5) \qquad \det H,$$

$$(3.6) \qquad \begin{vmatrix} y_1 & x_1 \\ y_2 & x_2 \end{vmatrix}, \ \begin{vmatrix} z_2 & y_2 \\ z_3 & y_3 \end{vmatrix}, \ \begin{vmatrix} x_3 & z_3 \\ x_1 & z_1 \end{vmatrix},$$

$$(3.7) \qquad \begin{vmatrix} d_1 & z_3 \\ d_3 & z_1 \end{vmatrix}, \ \begin{vmatrix} x_3 & d_1 \\ x_1 & d_3 \end{vmatrix}, \ \begin{vmatrix} d_2 & x_1 \\ d_1 & x_2 \end{vmatrix}, \ \begin{vmatrix} y_1 & d_2 \\ y_2 & d_1 \end{vmatrix}, \ \begin{vmatrix} d_3 & y_2 \\ d_2 & y_3 \end{vmatrix}, \ \begin{vmatrix} z_2 & d_3 \\ z_3 & d_2 \end{vmatrix},$$

$$(3.8) \qquad \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ d_1 x_1 & d_2 y_2 & d_3 z_3 \end{vmatrix}, \ \begin{vmatrix} x_1 & y_1 & z_1 \\ d_1 x_1 & d_2 y_2 & d_3 z_3 \\ x_3 & y_3 & z_3 \end{vmatrix}, \ \begin{vmatrix} d_1 x_1 & d_2 y_2 & d_3 z_3 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}.$$

Proof. We have already observed the necessity of the positivity of (3.5), (3.6) and (3.7), and the condition $H1 = 1$. The positivity of the inner conductances (2.13) then requires the positivity of (3.8). Conversely, if all 13 determinants are positive then (2.11) and (2.13) show that all the conductances $\{c_j\}$ are positive, and the result follows from Theorem 2.1.   □

We now turn to the problem of characterizing all possible matrices H of harmonic values. This is just the problem of finding all H such that there exists a choice of $\{d_j\}$ so that the 13 determinants are positive.

Theorem 3.4. *Given a matrix H with positive entries satisfying $H1 = 1$, there exists a set $\{c_j\}$ of positive conductances on Γ_1 having H as its matrix of harmonic values, if and only if the 4 determinants (3.5) and (3.6) are positive, and there exists an open half-space in $\mathbb{R}^3$ containing all 12 of the following vectors:*

$$(3.9) \qquad (1,0,0), \ (0,1,0), \ (0,0,1),$$

$$(3.10) \quad (x_1,y_1,z_1) \times (x_2,y_2,z_2), \ (x_2,y_2,z_2) \times (x_3,y_3,z_3), \ (x_3,y_3,z_3) \times (x_1,y_1,z_1),$$

$$(3.11) \quad (0,1,0) \times (x_1,y_1,z_1), \ (0,0,1) \times (x_2,y_2,z_2), \ (1,0,0) \times (x_3,y_3,z_3),$$
$$(x_1,y_1,z_1) \times (0,0,1), \ (x_2,y_2,z_2) \times (1,0,0), \ (x_3,y_3,z_3) \times (0,1,0).$$

Moreover, if these conditions hold, (d_1, d_2, d_3) may be chosen so that the vector $(d_1 x_1, d_2 y_2, d_3 z_3)$ has positive inner product with all 12 of the vectors above.

Proof. By choosing $\{d_j\}$ appropriately, we can make the vector

$$v = (d_1 x_1, d_2 y_2, d_3 z_3)$$

equal to any vector with positive entries. This means the inner products of v with the vectors (3.9) are all positive. The positivity of the determinants (3.8) is clearly equivalent to the positivity of the inner product of v and the vectors (3.10). We claim the same equivalence with (3.7) and (3.11). For example,

$$\begin{vmatrix} d_1 & z_3 \\ d_3 & z_1 \end{vmatrix} > 0 \iff (d_1 x_1) z_1 - (d_3 z_3) x_1 > 0,$$

and the left-hand side of this last inequality equals $v \cdot \big((0, 1, 0) \times (x_1, y_1, z_1)\big)$. The other equivalences follow similarly. The result now follows from the observation that the existence of an open half-space $\{u : u \cdot v > 0\}$ containing a set of vectors is equivalent to the existence of a vector with positive inner product with all of them. $\square$

The next result shows that we will never hit a "dead end" in extending a Dirichlet form from Γ_n to SG.

Theorem 3.5. *Every resistance network on Γ_{n-1} is the restriction of a Dirichlet form on SG which is a self-similar regular Dirichlet form on every small triangle in Γ_n.*

Proof. It suffices to give a proof for Γ_0. Let positive conductances d_1, d_2, d_3 be fixed. We will show that, for any small enough $\varepsilon > 0$, the network is the restriction of a network on Γ_1 such that $c_9 = c_7 = \varepsilon c_8 = 1/s_1$, $c_1 = c_3 = \varepsilon c_2 = 1/s_2$, and $c_4 = c_6 = \varepsilon c_5 = 1/s_3$.

Let R_j be the effective resistances in the network Γ_0 and let $F_j(s_1, s_2, s_3, \varepsilon)$ be the effective resistances between points in V_0 in the network Γ_1. We have to show that the equations $R_j = F_j(s_1, s_2, s_3, \varepsilon)$, $j = 1, 2, 3$, have positive solutions s_1, s_2, s_3 for all small enough $\varepsilon > 0$.

Note that we can take $\varepsilon = 0$, since in this case we have a well-defined network with zero inner resistances. Considering all indices mod 3, we have

$$F_j(s_1, s_2, s_3, 0) = \tfrac{1}{2}(s_{j-1} + s_{j+1}).$$

It is easy to see that if $\varepsilon = 0$ then the equations have unique positive solutions s_1, s_2, s_3 since R_1, R_2, R_3 satisfy the triangle inequalities. One can also see that the 3×3 matrix

$$\left\{ \frac{\partial F_j}{\partial s_i}(s_1, s_2, s_3, 0) \right\}_{i,j=1,2,3}$$

is nonsingular. The functions F_j are rational, and so for any small enough $\varepsilon > 0$, positive solutions exist by the implicit function theorem and the continuity of the derivatives.

Then each of the three triangles in Γ_1 has conductances proportional to $(1, 1, 1/\varepsilon)$. Each such triangle of conductances is the restriction of a regular self-similar Dirichlet form for any $\varepsilon > 0$, as shown in [**Sa**] (see also [**Ki4**], Exercise 3.1). □

4. Uniqueness of the initial conductances

We now show that, under certain assumptions, the initial conductances d_1, d_2, d_3 are uniquely determined up to a constant multiple by the space of harmonic functions $\mathcal{H}$. These assumptions have the form of local Harnack inequalities at the boundary points. In particular, if $\mathcal{H}$ corresponds to a self-similar harmonic structure then the initial conductances are determined uniquely up to a constant multiple.

Let $z^{(n)} = F_1^n(q_2)$ and $y^{(n)} = F_1^n(q_3)$, in particular $z = z^{(1)}$ and $y = y^{(1)}$. Also let $z_j^{(n)} = h_j(z^{(n)})$ and $y_j^{(n)} = h_j(y^{(n)})$.

Lemma 4.1 (monotonicity). *For any $n \geq 0$,*

$$(4.1) \qquad \frac{z_3^{(n-1)}}{z_2^{(n-1)}} < \frac{z_3^{(n)}}{z_2^{(n)}} < \frac{d_2}{d_3} < \frac{y_3^{(n)}}{y_2^{(n)}} < \frac{y_3^{(n-1)}}{y_2^{(n-1)}} .$$

Proof. Let

$$M^{(n)} = \begin{pmatrix} z_2^{(n)} & y_2^{(n)} \\ z_3^{(n)} & y_3^{(n)} \end{pmatrix} .$$

One can show that for any $m \geq 0$ we have $\det M^{(n)} > 0$ and

$$(4.2) \qquad \frac{z_3^{(m)}}{z_2^{(m)}} < \frac{d_2}{d_3} < \frac{y_3^{(m)}}{y_2^{(m)}} .$$

The argument is similar to (3.3) because $M^{(n)}$ satisfies a relation analogous to (2.10). Set

$$A_n = \begin{pmatrix} a_{11}^{(n)} & a_{12}^{(n)} \\ a_{21}^{(n)} & a_{22}^{(n)} \end{pmatrix} = (M^{(n-1)})^{-1} M^{(n)}.$$

It is clear from the definition that, for a harmonic function h with $h(q_1) = 0$, A_n^T transforms the values of h at $z^{(n-1)}$, $y^{(n-1)}$ into its values at $z^{(n)}$, $y^{(n)}$. That is,

$$(4.3) \qquad \begin{pmatrix} h(z^{(n)}) \\ h(y^{(n)}) \end{pmatrix} = \begin{pmatrix} a_{11}^{(n)} & a_{21}^{(n)} \\ a_{12}^{(n)} & a_{22}^{(n)} \end{pmatrix} \begin{pmatrix} h(z^{(n-1)}) \\ h(y^{(n-1)}) \end{pmatrix}.$$

Hence a_n, b_n, c_n, d_n are all positive. Then the required inequalities

$$(4.4) \qquad \frac{z_3^{(n-1)}}{z_2^{(n-1)}} < \frac{z_3^{(n)}}{z_2^{(n)}} \qquad \text{and} \qquad \frac{y_3^{(n)}}{y_2^{(n)}} < \frac{y_3^{(n-1)}}{y_2^{(n-1)}}$$

are easy to check by induction since $M^{(n)} = M^{(n-1)} A_n$. Together, (4.2) and (4.4) yield (4.1). $\qquad \square$

Corollary 4.2. *If*

$$(4.5) \qquad \lim_{n\to\infty} \frac{y_3^{(n)}}{y_2^{(n)}} - \frac{z_3^{(n)}}{z_2^{(n)}} = 0$$

then $\dfrac{d_2}{d_3}$ is uniquely determined and

$$(4.6) \qquad \lim_{n\to\infty} \frac{y_3^{(n)}}{y_2^{(n)}} = \lim_{n\to\infty} \frac{z_3^{(n)}}{z_2^{(n)}} = \frac{d_2}{d_3}.$$

Theorem 4.3. *Suppose the following local form of the Harnack inequality at q_1 holds: there exists $\delta > 0$ such that if h is harmonic and satisfies*

$$h(q_1) = 0, \quad h(y^{(n-1)}) > 0, \quad \text{and} \quad h(z^{(n-1)}) > 0,$$

then

$$(4.7) \qquad \delta < \frac{h(z^{(n)})}{h(y^{(n)})} < \frac{1}{\delta}$$

for all $n \geq 0$. Then the ratio $\dfrac{d_3}{d_2}$ is uniquely determined by the space $\mathcal{H}$.

Proof. Our aim is to show that Corollary 4.2 applies. Let $M^{(n)}$ and A_n be as defined in the proof of Lemma 4.1. Then it is easy to see that both $a_{11}^{(n)}/a_{12}^{(n)}$ and $a_{21}^{(n)}/a_{22}^{(n)}$ are in $(\delta, 1/\delta)$. Then

$$(4.8) \quad \frac{y_3^{(n)}}{y_2^{(n)}} - \frac{z_3^{(n)}}{z_2^{(n)}} = \frac{\det M^{(n)}}{y_2^{(n)} z_2^{(n)}}$$

$$= \frac{\det M^{(n-1)} \det A_n}{\left(z_2^{(n-1)} a_{11}^{(n)} + y_2^{(n-1)} a_{21}^{(n)} \right) \left(z_2^{(n-1)} a_{12}^{(n)} + y_2^{(n-1)} a_{22}^{(n)} \right)}$$

$$\leq \frac{\det M^{(n-1)}}{z_2^{(n-1)} y_2^{(n-1)}} \frac{\det A_n}{a_{11}^{(n)} a_{22}^{(n)}} \leq \left(\frac{y_3^{(n-1)}}{y_2^{(n-1)}} - \frac{z_3^{(n-1)}}{z_2^{(n-1)}} \right) (1 - \delta^2),$$

which implies (4.6). $\qquad \square$

Example 4.4. We will now see that the space of harmonic functions need not determine the initial conductances uniquely. The harmonic functions in this example are not continuous, however. On each level n all the conductances are positive, but some tend to zero as $n \to \infty$.

Let

$$H_\varepsilon = \begin{pmatrix} \varepsilon^2 & \varepsilon - \varepsilon^2 & 1 - \varepsilon \\ 1 - \varepsilon & \varepsilon^2 & \varepsilon - \varepsilon^2 \\ \varepsilon - \varepsilon^2 & 1 - \varepsilon & \varepsilon^2 \end{pmatrix}.$$

If $H = H_\varepsilon$ then, as $\varepsilon \to 0$,

$$c_1 = d_1 + d_1 O(\varepsilon) + d_3 O(\varepsilon),$$

$$c_3 = \frac{1}{\varepsilon} d_3 + d_1 O(1) + d_3 O(1),$$

$$c_2 = \varepsilon d_3 + d_1 O(\varepsilon^2) + d_2 O(\varepsilon^2) + d_3 O(\varepsilon^2).$$

There are similar formulas for the other conductances since H_ε is rotationally invariant. One can see that there exist $K, K' > 0$ such that if

(4.9) $$\max_{i,j=1,2,3} \frac{d_i}{d_j} < \frac{K}{\varepsilon}$$

then

$$\max_{k,\ell=1,2,3} \frac{c_k}{c_\ell} < \frac{K'}{\varepsilon^2}.$$

We suppose that for each $w \in W_n$ the matrix H_w that defines the continuation of a harmonic function from $F_w\{q_1, q_2, q_3\}$ to $F_w\{x, y, z\}$ is H_{ε_n}, where the $\varepsilon_n > 0$ are to be chosen later. In terms of (2.14) this means that

(4.10) $$H_w = h\big(F_w(x), F_w(y), F_w(z)\big)\big(h(F_w(q_1), F_w(q_2), F_w(q_3))\big)^{-1} = H_{\varepsilon_n}.$$

Then all the conductances for all n are positive if (4.9) is satisfied with small enough $\varepsilon = \varepsilon_0$ and $\varepsilon_{n+1} \leq K\varepsilon_n^2 / K'$ for all $n \geq 0$.

5. A parameterization of compatible Dirichlet forms

In this section we describe the compatibility conditions in going from conductances on Γ_0 to conductances on Γ_1 in parametric form. Specifically, for each choice of conductances $\{d_j\}$ we give an algorithm that constructs all compatible Dirichlet forms on Γ_1 based on the choice of 6 parameters chosen without restriction from $(0, 1)^6$. The correspondence between parameters and Dirichlet forms is one-to-one. We use the Δ–Y transform and its inverse ([**Ki4**]) several times.

Algorithm 5.1. Given any positive conductances $\{d_j\}$ on Γ_0, for each choice of $(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) \in (0, 1)^6$ perform the following steps to obtain a distinct compatible Dirichlet form on Γ_1:
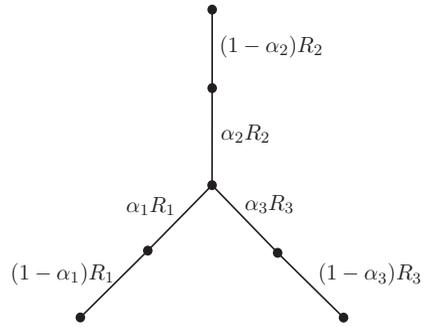
Step 1. Compute initial resistances $R_1, R_2, R_3 > 0$ for the Δ–Y transformed triangle in Figure 5.1, so that

(5.1) $$R_j = \frac{d_j}{d_1 d_2 + d_2 d_3 + d_1 d_3} \quad \text{and} \quad d_j = \frac{R_{j-1} R_{j+1}}{R_1 + R_2 + R_3}.$$
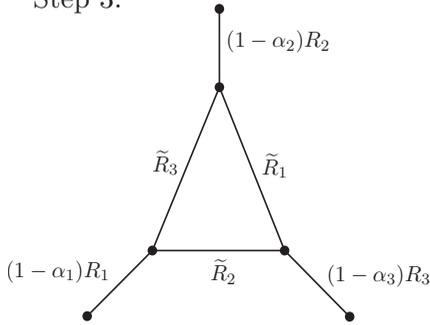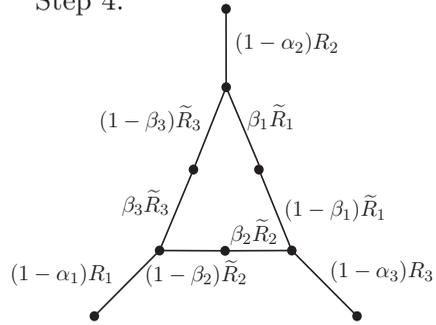
Step 1. Initial network
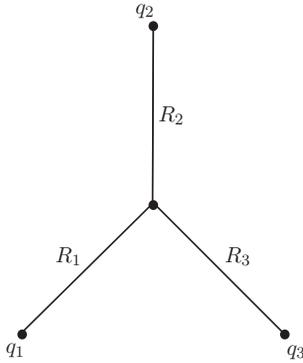
Step 2.

Figure 5.1. Pictorial description of Steps 1–4.

For convenience we identify indices mod 3, so $R_0 = R_3$, etc.

Step 2. Choose any three numbers $\alpha_1, \alpha_2, \alpha_3 \in (0, 1)$ to split the resistances R_j into $\alpha_j R_j$ and $(1 - \alpha_j)R_j$.

Step 3. Do a Y–Δ transform for resistances $(\alpha_1 R_1, \alpha_2 R_2, \alpha_3 R_3)$:

$$(5.2) \qquad \widetilde{R}_j = \frac{\alpha_1 \alpha_2 R_1 R_2 + \alpha_1 \alpha_3 R_1 R_3 + \alpha_2 \alpha_3 R_2 R_3}{\alpha_j R_j}.$$

Step 4. Choose any three numbers $\beta_1, \beta_2, \beta_3 \in (0, 1)$ to split the resistances $\widetilde{R}_1, \widetilde{R}_2, \widetilde{R}_3$. Then for each $j = 1, 2, 3$ do a Y-Δ transform on the triple of resistances

$$(5.3) \qquad (R_1^j, R_2^j, R_3^j) = \big((1 - \alpha_j)R_j, (1 - \beta_{j+1})\widetilde{R}_{j+1}, \beta_{j-1}\widetilde{R}_{j-1}\big).$$

These 4 steps are summarized in Figure 5.1. The results give resistances on Γ_1, whose reciprocals define the compatible Dirichlet form.

Remarks. To simplify computations in this algorithm one can consider the resistances R_j as the initial data and avoid the Y–Δ transform in Step 4.

The algorithm can easily yield an explicit but complicated formula for the conductances $c_1, \ldots, c_9$ as a rational function of the initial conductances and the parameters. The points where $\widetilde{R}_j$ are divided into $\beta_j \widetilde{R}_j$ and $(1 - \beta_j)\widetilde{R}_j$ are the points x, y, z. Thus we can also recover the values of the harmonic functions at x, y, z, although this is not necessary for the algorithm.

To obtain a parametric description of all compatible Dirichlet forms on Γ_k we just use the algorithm iteratively on each of the triangles on each level up to k; the choices of the 6 parameters can be made independently.

Corollary 5.2. *The space of all Dirichlet forms on Γ_1 compatible with a fixed Dirichlet form on Γ_0 is a manifold of dimension* 6. *More generally, the space of all Dirichlet forms on Γ_k compatible with a fixed Dirichlet form on Γ_0 is a manifold of dimension* $3(3^k - 1)$.

Given any choice of the 6 parameters $\alpha_1^w, \alpha_2^w, \alpha_3^w, \beta_1^w, \beta_2^w, \beta_3^w \in (0, 1)^6$ for each word w of finite length, we obtain a compatible sequence of Dirichlet forms on $\{\Gamma_n\}$. This yields a Dirichlet form on SG with all points having positive capacity exactly when all the conductances tend to infinity (resistances tend to 0). We have not been able to find necessary and sufficient conditions on the parameters for this, but we will give some related results in Section 7 and we can give sufficient conditions for harmonic functions to be continuous.

Next we will show that harmonic functions are continuous if the parameters are uniformly separated from 0 and 1, and if the parameters are chosen at random. First we need the following lemmas:

Lemma 5.3. *Let* $M(h) = \max_{x \in \mathrm{SG}} h(x) - \min_{x \in \mathrm{SG}} h(x)$ *for a harmonic function* h. *Then, for any* $j = 1, 2, 3$,

$$(5.4) \qquad\qquad M(h \circ F_j) \leq (1 - m)M(h),$$

where m *is the minimum of the off-diagonal elements of matrix* H.

Proof. By the maximum principle $M(h) = \max_{i=1,2,3} h(q_i) - \min_{i=1,2,3} h(q_i)$. It is enough to give a proof in the case $j = 1$ and $h(q_1) = 0$. Then we have

$$(y_2 + y_3) \min_{i=1,2,3} h(q_i) \leq h(y) \leq (y_2 + y_3) \max_{i=1,2,3} h(q_i)$$

and $y_2 + y_3 = 1 - y_1 \leq 1 - m$. Therefore

$$(1 - m) \min_{i=1,2,3} h(q_i) \leq h(y) \leq (1 - m) \max_{i=1,2,3} h(q_i),$$

since $\min_{i=1,2,3} h(q_i) \leq 0$ and $\max_{i=1,2,3} h(q_i) \geq 0$. A similar estimate holds for $h(z)$, and the result follows. $\qquad\square$

Lemma 5.4. *Let* $\delta = \delta(\alpha_1, \cdots, \beta_3) = \min_{i=1,2,3} \min \{\alpha_i \beta_{i+1}, \alpha_i(1 - \beta_{i-1})\}$. *Then* $\delta \leq m$.

Proof. Since $h_1 \geq 0$, we have $z_1 = h_1(z) \geq \alpha_1(1 - \beta_3)$. Similarly $y_1 \geq \alpha_1\beta_2$ etc. One can obtain these inequalities, for example, from harmonic equations $h_1(a_1) = \alpha_1 + h(a)(1 - \alpha_1)$ and $h_1(z) = (1 - \beta_3)h(a_1) + \beta_3 h(a_2)$, where $a, a_1, \ldots, a_3$ are defined in Section 6. □

We will denote $\delta(\alpha_1^{w_1 \cdots w_k}, \ldots, \beta_3^{w_1 \cdots w_k})$ by $\delta_{w_1 \cdots w_k}$.

Theorem 5.5.

(1) *If for any infinite word $w_1, w_2, \ldots$*

$$\sum_{k=0}^{\infty} \delta_{w_1 \cdots w_k} = \infty, \tag{5.5}$$

then harmonic functions are continuous.

(2) *Suppose*

$$\delta_{\inf} = \inf_{w \in W_*} \delta_w > 0. \tag{5.6}$$

Then harmonic functions are Hölder continuous with Hölder exponent $1 - \delta_{\inf}$.

Proof. By (5.4) and Lemma 5.3 we have, for any finite word $w = w_1 \cdots w_n$,

$$M(h \circ F_w) \leq M(h) \prod_{k=0}^{n-1} (1 - \delta_{w_1 \cdots w_k}). \tag{5.7}$$

□

Strictly speaking, the algorithm above defines harmonic functions on V_* only. However, the continuity on V_* implies the existence of a unique continuation to SG.

Corollary 5.6.

(1) *Suppose that $\xi_w = (\alpha_1^w, \alpha_2^w, \alpha_3^w, \beta_1^w, \beta_2^w, \beta_3^w) \in (0, 1)^6$ are independent identically distributed random 6-dimensional vectors indexed by the words w of finite length. Then with probability one harmonic functions are continuous.*

(2) *Suppose that there is $\varepsilon > 0$ such that $\alpha_j^w, \beta_j^w \in [\varepsilon, 1-\varepsilon]$ for all w, j. Then harmonic functions are Hölder continuous with Hölder exponent $1 - \varepsilon^2$.*

Proof. Under the assumptions of part (1) of the corollary, the δ_w are independent identically distributed random variables which are strictly positive. Therefore the conclusion holds by (5.7) and the Law of Large Numbers.

Part (ii) follows from Theorem 5.5 and the definition of δ_w. □

6. Geometric interpretation

Algorithm 5.1 can be uniquely presented (up to a constant factor) in geometric terms. This presentation is based on an idea of Kigami to consider a pair of harmonic functions as coordinates for SG [**Ki3**].

To each point $p \in$ SG there corresponds a point $(h_3(p), h_2(p)) \in \mathbb{R}^2$. If harmonic functions separate points — and we always assume this — then the correspondence is one-to-one. In this section we identify p and $(h_3(p), h_2(p))$. This yields a different embedding SG $\subset \mathbb{R}^2$. We have $q_1 = (0,0)$, $q_2 = (0,1)$, $q_3 = (1,0)$. Also, we assume that a harmonic function changes linearly along each resistor, and so a resistor corresponds to a line segment in $\mathbb{R}^2$.

For $A, B, C \in \mathbb{R}^2$ we denote by $T(A,B,C)$ the triangle with vertices A, B, C.

It is easy to see that choosing the matrix H is equivalent to choosing three points x, y, z inside of $T(q_1, q_2, q_3)$. The positivity of the determinants in Theorem 3.3 is equivalent to the fact that none of the line segments shown in Figure 6.1 intersect or coincide.
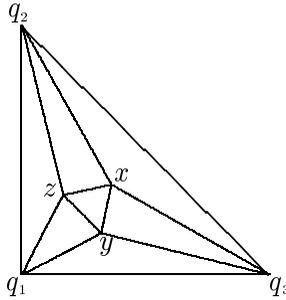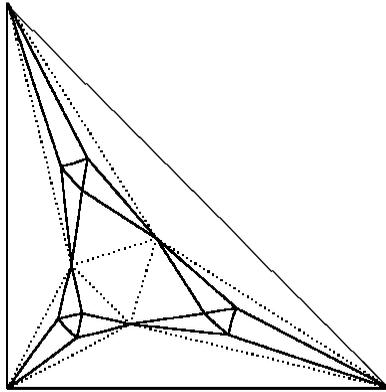

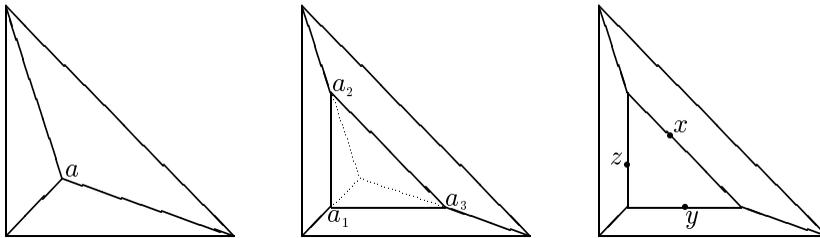**Figure 6.1.** Geometric interpretation of conditions of Theorem 3.3.

The Monotonicity Lemma 4.1 corresponds to the fact that the triangles of level $n+1$ are nested inside of those of level n (Figure 6.2).

Going back to Section 5, choosing conductances R_1, R_2, R_3 up to a constant factor corresponds to choosing a point a inside of $T(q_1, q_2, q_3)$. The coordinates of this point a can be thought of as the values of the harmonic functions h_3, h_2 at the imaginary point a, the point of intersection of the resistances R_1, R_2, R_3. This point is imaginary in the sense that it may not correspond to any point in SG. The point a may be thought of as the center of mass of SG. Thus Step 1 is choosing a point a inside of $T(q_1, q_2, q_3)$. Step 2 is choosing a_j in each of the line segments $[q_j, a]$, $j = 1, 2, 3$, and drawing a new triangle $T(a_1, a_2, a_3)$ instead of line segments $[a_j, a]$. Step 3 is choosing points x, y, z inside of the line segments $[a_j, a_{j+1}]$.

We summarize the preceding discussion as follows:

Figure 6.2. Geometric interpretation of the Monotonicity Lemma 4.1 for $n = 2$.

Figure 6.3. Geometric interpretation of the construction in §5.

Proposition 6.1. *Let distinct points $x, y, z \in T(q_1, q_2, q_3)$ correspond to a matrix H as described above. Then H corresponds to positive conductances $c_1, \ldots, c_9$ if and only if there are three points a_j inside of triangles $T(q_1, y, z), T(q_2, x, z), T(q_3, x, y)$ such that:*

(1) *$x \in [a_2, a_3]$, $y \in [a_1, a_3]$, $z \in [a_2, a_3]$,*
(2) *three straight lines from q_j to a_j intersect in a single point a.*
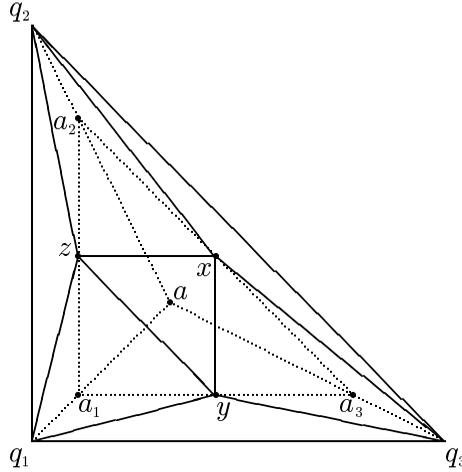
Figure 6.4. Geometric interpretation of Proposition 6.1.

7. Effective resistance metric, Green's function
and capacity of points

We first recall from [**Ki4**] some facts about limits of resistance networks. Although we state all the results of this section for the Sierpiński gasket, they can be applied to general pcf fractals with only minor changes.

Let $\mathcal{E}(f, f)$ be defined by (1.2) for any function f on V_*, where $\mathcal{E}_n$ is a compatible sequence of Dirichlet forms on Γ_n.

Proposition 7.1. *Every point of $V_* = \bigcup_{n \geq 0} V_n$ has positive capacity.*

Proof. Let $x \in V_*$. Then $x \in V_n$ for some n. The capacity of $\{x\}$ with respect to $\mathcal{E}$ is the same as that with respect to $\mathcal{E}_n$ because of the compatibility of the sequence of networks. The latter capacity is positive because V_n is a finite set. $\qquad\square$

The effective resistance is defined for any $x, y \in V_*$ by

$$(7.1) \qquad R(x, y) = \left(\min_u \{\mathcal{E}(u, u) : u(x) = 1, u(y) = 0\}\right)^{-1}.$$

Here the minimum is taken over all functions on V_*. Note that $x, y \in V_n$ for large enough n and that (7.1) does not change if $\mathcal{E}$ is replaced by $\mathcal{E}_n$, because of the compatibility condition (see [**Ki4**], Proposition 2.1.11). By Theorem 2.1.14 in [**Ki4**], $R(x, y)$ is a metric on V_*. In what follows we will write R-continuity, R-closure etc. for continuity, closure etc. with respect to the effective resistance metric R. It is known that if $\mathcal{E}(u, u) < \infty$ then u is R-continuous ([**Ki4**], Theorem 2.2.6(1)). The main ingredient in the proof of this fact is the inequality

$$(7.2) \qquad\qquad |u(x) - u(y)|^2 \leq R(x, y)\mathcal{E}(u, u).$$

Let Ω be the R-completion of V_*. We can conclude from (7.2) that if u is a function on V_* such that $\mathcal{E}(u, u) < \infty$ then u has a unique continuation

to Ω that is R-continuous. We will denote this continuation by the same symbol u and the set of such functions by $\mathcal{F}$.

Remark. It is proved in Proposition 3.3.2 of [**Ki4**] that if harmonic functions are continuous then there is a continuous injective map $\theta : \Omega \to$ SG which is the identity on V_*. Therefore in this case we can (and will) consider Ω as a subset of SG. Then Ω is the R-closure of V_*. In a sense, Ω is the set where the Dirichlet form lives. If Ω is not just an abstract completion then the name "Dirichlet form on the SG" is more justified. Strictly speaking, Proposition 3.3.2 in [**Ki4**] is formulated for self-similar harmonic structures, but self-similarity is not used in the proof.

An important question is whether Ω is equal to SG. The answer is positive if all the conductances tend to infinity. This happens, for example, in the case of a regular self-similar harmonic structure. Thus it is natural to say that a harmonic structure is regular if $\Omega =$ SG and nonregular otherwise. It is easy to see that a harmonic structure is regular if all the conductances tend to infinity, but in Example 7.5 we show that the converse is not true. In Example 7.4 we describe a nonregular harmonic structure on SG where harmonic functions are Hölder continuous.

The proof of the next theorem is a part of the proof of Theorem 3.5.6 in [**Ki4**].

Theorem 7.2. *If $x \in \Omega$ then $\{x\}$ has positive capacity.*

Proof. Let $\mathcal{F}_0 = \{u \in \mathcal{F} : u|_{V_0} = 0\}$. Then $(\mathcal{F}_0, \mathcal{E})$ is a Hilbert space. If $x \in \Omega$, $u \in \mathcal{F}_0$ and $p \in V_0$, then $|u(x)|^2 \leq R(x,p)\mathcal{E}(u,u)$ and so $u \mapsto u(x)$ is a continuous functional on the Hilbert space $(\mathcal{F}_0, \mathcal{E})$. Hence there exists $h \in F_0$ such that $u(x) = \mathcal{E}(u,h)$ for any $u \in \mathcal{F}_0$. This implies the result by [**Fu**] or [**FOT**]. $\square$

The converse of this statement is proved in [**Ki4**] for any self-similar harmonic structure. If harmonic functions are continuous then Ω is just the R-closure of V_* in SG.

To define the Green's function we use the construction invented by Kigami for self-similar harmonic structures. Let $G = X^{-1}$, where $X = X_o + X_i$ is defined by (2.1) and (2.2). We assume that the elements G_{pq} of G are indexed by $p, q \in V_1 \setminus V_0$ (this set was denoted by $\{x, y, z\}$ in the previous sections). Let ψ_p be the unique piecewise harmonic function such that $\psi_p(q) = \delta_{pq}$ for any $p, q \in V_1$ and $\psi_p \circ F_j$ is harmonic for any $j = 1, 2, 3$. We define

$$\Psi(x,y) = \sum_{p,q \in V_1 \setminus V_0} G_{pq}\psi_p(x)\psi_q(y).$$

For any $p \in V_1 \setminus V_0$ and any $u \in \mathcal{F}_0$ we have

(7.3) $$\mathcal{E}(\Psi(p,\cdot), u) = u(p)$$

(see Lemma 3.5.4 in [**Ki4**]). Let for any $w \in W_*$ we define a Dirichlet form $\mathcal{E}_w(u,u)$ on SG by

$$\mathcal{E}_w(u,u) = \lim_{n\to\infty} \sum_{x,y\in V_n\cap F_w(\mathrm{SG})} c(x,y)\big(u(F_w^{-1}(x)) - u(F_w^{-1}(y))\big)^2$$

(the sum here is a part of the sum in (1.1)). Then we define $\Psi_w(x,y)$ for $\mathcal{E}_w(u,u)$ in the same way as $\Psi(x,y)$ is defined for $\mathcal{E}(u,u)$. (This definition differs by a factor of r_w from the one in [**Ki4**].) Finally, Green's function is

$$(7.4) \qquad g(x,y) = \sum_{w\in W_*} \Psi_w(x,y).$$

Since all the terms in this sum are nonnegative, it either converges or is $+\infty$. Moreover, if x and y are distinct, all but a finite number of terms vanish. Hence if harmonic functions are continuous then $g(x,y)$ is continuous on the set $\{(x,y) : x,y \in \mathrm{SG}, x \neq y\}$.

Theorem 7.3. *If $x \in \Omega$ then $g(x,x) < \infty$, $g(x,\cdot) \in \mathcal{F}_0$,*

$$(7.5) \qquad \mathcal{E}(g(x,\cdot),u) = u(x)$$

for any $u \in \mathcal{F}_0$, and $\mathcal{E}(g(x,\cdot),h) = 0$ for any harmonic function h.

Proof. Let

$$g_n(x,y) = \sum_{k=0}^{n} \sum_{w\in W_k} \Psi_w(x,y).$$

Then $g_n(x,\cdot) \in \mathcal{F}_0$ and for any $u \in \mathcal{F}_0$

$$\mathcal{E}(g_n(x,\cdot),u) = u_n(x),$$

where u_n is the unique piecewise harmonic function that coincides with u on V_n.

It is easy to see that $\mathcal{E}(u-u_n,u_n) = 0$ and $\mathcal{E}(u_n,u_n) \to \mathcal{E}(u,u)$ as $n \to 0$. Hence $\mathcal{E}(u-u_n,u-u_n) \to 0$ as $n \to 0$. Therefore $u_n(x) \to u(x)$ as $n \to 0$ by (7.2). This implies (7.5) since $g_{n+1}(x,\cdot) - g_n(x,\cdot)$ is an $\mathcal{E}$-orthogonal sequence and $u \mapsto u(x)$ is a bounded functional on $\mathcal{F}_0$. Therefore $g(x,\cdot)$ lies in $\mathcal{F}_0$ and $g(x,x) = \mathcal{E}(g(x,\cdot),g(x,\cdot)) < \infty$. $\qquad\square$

This theorem and its converse are proved in [**Ki4**] for any self-similar harmonic structure. Again, if harmonic functions are continuous the result holds for points in the R-closure of V_* in SG.

Example 7.4. It is easy to construct an example of a nonregular self-similar harmonic structure on SG using a modified self-similar structure. Consider the set of 27 contractions $\{F_w : |w| = 3\}$, each with contraction ratio $\frac{1}{8}$. We choose constants $r_w : |w| = 3$ (see [**Ki4**]) as follows: we choose two numbers r, r' and then set $r_{ijk} = r$ if all indices i,j,k are different and $r_{ijk} = r'$ otherwise. It is easy to see that, because of symmetry, for any choice of $r > 0$ there is an $r' > 0$ that gives a self-similar harmonic structure.

(The structure is dihedral-3 symmetric and so has the spectral decimation property, although of a complicated form. See [**MT**].)

This harmonic structure is nonregular if we choose $r \geq 1$. However it is evident that the parameters described in Section 5 satisfy the condition of Corollary 5.6(2) for any self-similar harmonic structure. Therefore harmonic functions are Hölder continuous.

Example 7.5. We show that conductances need not tend to infinity for a regular harmonic structure. We construct a Dirichlet form $\mathcal{E}$ which is not self-similar, although it is "piecewise self-similar". It is clear from the construction that harmonic functions are continuous and the R-topology is equivalent to the standard one.

First, we define an auxiliary family of Dirichlet forms $\mathcal{E}^\delta$ as follows: let $\mathcal{E}_0^\delta$ be the standard symmetric Dirichlet form on Γ_0 for which all three conductances are equal to one. For $\mathcal{E}_1^\delta$ let the inner conductances be equal to $\delta > 0$ and the outer conductances be such that $\mathcal{E}_1^\delta$ is compatible with $\mathcal{E}_0^\delta$. Clearly, this is possible for any choice of positive δ. Then we extend $\mathcal{E}_1^\delta$ to a regular Dirichlet form $\mathcal{E}^\delta$ on SG as in Theorem 3.5.

To construct $\mathcal{E}$ we choose an infinite sequence of finite words $\{w(n)\}_{n \geq 1}$ such that $F_{w(n)}(\text{SG}) \bigcap F_{w(m)}(\text{SG}) = \varnothing$ if $n \neq m$. Then we choose a sequence of positive numbers $\{\delta_n\}_{n \geq 1}$ such that $\delta_n \leq (\frac{3}{5})^{|w(n)|}$. Finally, we take the standard Dirichlet form on SG and replace it on each triangle $F_{w(n)}(\text{SG})$ by $(\frac{5}{3})^{|w(n)|} \mathcal{E}^{\delta_n} \left(f \circ F_{w(n)}^{-1}, f \circ F_{w(n)}^{-1} \right)$.

Note added in proof. In [**Ki5**], a work submitted after our paper was completed, Kigami proves some extensions of results from his book [**Ki4**] to a wider context which includes all the Dirichlet forms considered in this paper.

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