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For a compact closed n-dimensional manifold, we derive the Calderón–Zygmund inequality for the Hodge Laplacian, with constants depending only on bounds on the injectivity radius, volume and the curvature operator. We obtain the Poincaré–Sobolev inequality for forms as a consequence.

Introduction

On a compact Riemannian manifold M without boundary, we give $L^{2,q}$ estimates for differential forms ϕ in terms of $\Delta\phi$, where Δ is the Hodge Laplacian. Calderón–Zygmund theory and its techniques on $\mathbb{R}^n$ treat certain singular integral operators of which Green's operator for the Laplacian is a basic example. Here we generalize these techniques to apply to forms on a manifold. Our goal is to obtain a Calderón–Zygmund inequality where the constant depends only on geometric quantities. The main result of this article is the following:

Theorem 0.1. *Let (M^n, g), $n \geq 3$, be a compact, connected, oriented Riemannian manifold without boundary. Suppose that*

$$(*) \qquad \operatorname{inj}(M) \geq i_0, \quad \operatorname{Vol}(M) \leq V, \quad \textit{and} \quad |\operatorname{Ric}| \leq \Lambda,$$

and that the curvature operator S is bounded from below with $S \geq -K$, for some constant $K \geq 0$. For $1 < q < \infty$, there exists a constant $C_q = C(q, n, \Lambda, K, i_0, V)$, such that for any m-form $\phi \in H_m^{\perp}$,

$$\|\nabla^2\phi\|_{L^q(M)} \leq C_q \, \|\Delta\phi\|_{L^q(M)},$$

where H_m is the space of harmonic m-forms.

Remarks 0.2.

(i) This work extends some of the results in [**H**] for functions, and gives new estimates for forms.

(ii) In the case of 1-forms, the condition on the curvature operator is not necessary, and Theorem 0.1 holds with an assumption on the Ricci curvature only.

(iii) With lower bounds on the Ricci curvature and the injectivity radius, one can also replace the condition $\operatorname{Vol}(M) \leq V$ in $(*)$ by $\operatorname{Diam}(M) \leq d$.

Theorem 0.1 leads to a Poincaré–Sobolev inequality:

Corollary 0.3. *With M^n as in Theorem 0.1, for $1 < q < \infty$, there exists a constant A_q, depending only on n, q, Λ, K, i_0, V, such that for all coclosed m-forms $\psi \in H_m^\perp$,*

$$\left(\int_M |\psi|^{\frac{np}{n-p}}\right)^{\frac{n-p}{np}} \leq A_q \left(\int_M |d\psi|^p\right)^{\frac{1}{p}}, \quad \textit{where } \frac{1}{p} = \frac{1}{q} - \frac{1}{n}.$$

0.4. . We summarize the methods and organization of this article. To prove Theorem 0.1, we derive estimates of the Green form on $H_m^\perp$ under the conditions given in ($*$) — this is done in Section 2. Our estimates depend on the lower bound on $\lambda_{1,m}$, the first eigenvalue of the Hodge Laplacian, following from the work of Chanillo and Treves [**CT**]. We represent the solution of $\Delta\phi = \omega$ in $H_m^\perp$ as a singular integral using the Green form constructed in suitable coordinates. Given $1 < q < \infty$, we choose p so that $p > \max\{q, n\}$, and fix a $C^{1,\alpha}$ harmonic coordinate atlas with $\alpha = 1 - \frac{n}{p}$. We use regularity of the metric in these coordinates to obtain estimates of the Green form in terms of the harmonic radius. In Section 3, we derive Theorem 0.1 for $p = 2$ from the Weitzenbock formula and lower bound of the curvature operator. To obtain estimates of $\nabla^2\phi$ in L^p, we introduce a Calderón–Zygmund decomposition for forms on a manifold (Section 4) and prove a weak-type inequality (Section 5). The proof of Theorem 0.1 is given in Section 6. We conclude in Section 7 with applications of Theorem 0.1 in proving the Poincaré–Sobolev inequality and in the study of locally conformally flat manifolds.

1. Construction of the Green form for the Hodge Laplacian

The result on the C^α-compactness of Riemannian manifolds that we will use is the following theorem of Anderson. Our method requires the following $C^{1,\alpha}$ harmonic coordinates:

Theorem 1.1 (Anderson, [**An**, 2.2])**.** *Let M^n be a compact manifold with*

$$|\mathrm{Ric}| \leq \Lambda, \quad \mathrm{inj}(M) \geq i_0, \quad \mathrm{Vol}(M) \leq V.$$

Given $\alpha \in (0,1)$, there exists $r_0 = C(n, \alpha, i_0, \Lambda) > 0$ and a finite atlas of harmonic coordinate charts $\{u_\lambda, B_{x_\lambda}(r_0)\}_{\lambda=1,\ldots,\mathcal{N}}$ such that:

(i) *The coordinate functions $u^i : B_x(r_0) \to \mathbb{R}$ are harmonic.*
(ii) *The sets $B_{x_\lambda}(r_0/2)$ cover M, and the number $\mathcal{N}$ of coordinates is bounded from above by a constant depending on V, Λ, i_0, n.*
(iii) *The metric components in this coordinate system,*

$$g_{ij}(y) = g\left(\frac{\partial}{\partial u^i}\Big|_y, \frac{\partial}{\partial u^j}\Big|_y\right),$$

satisfy

$$g_{ij}(x) = \delta_{ij}, \quad \partial g_{ij}(x) = 0, \tag{1.2a}$$

$$\tfrac{1}{4}\delta_{ij} \le g_{ij}(y) \le 4\delta_{ij} \quad \textit{for } y \in B_x(r_0), \quad \textit{and}$$

$$\|\partial^2 g_{ij}\|_{L^p(B_x(r_0))} \le r_0^{-1-\alpha}, \tag{1.2b}$$

where $p > n$ is defined by $\alpha = 1 - \frac{n}{p}$.

1.3. . The above convergence of Riemannian metrics is actually in $L^{2,p}$ [**An**]. By the Sobolev embedding $L^{2,p} \subset C^{1,\alpha}$,

$$\sup_{B(r_0)} |\partial g| \le C(n,p)\, r_0^\alpha \, \|\partial^2 g\|_{L^p(B(r_0))} \le C(n,p)\, r_0^{-1}. \tag{1.4}$$

Notation 1.5. We fix $q \in (0,\infty)$ and assume throughout that $|\mathrm{Ric}| \le \Lambda$. We choose $p > \max\{q,n\}$, and fix a $C^{1,\alpha}$ harmonic coordinate atlas with $\alpha = 1 - \frac{n}{p}$. Without explicit statements, all constants shall depend on n, the dimension of M, and m, the degree of the forms. Indices in capital letters are multi-indices: $I = (i_1 \cdots i_m)$, and $g_{IJ} = \det(g_{i_\alpha j_\beta})$ forms an $\binom{n}{m} \times \binom{n}{m}$ matrix. Greek letter indices always run from 1 to m, while Latin letter indices such as j or j_α run from 1 to n regardless of subscripts.

1.6. . In $C^{1,\alpha}$ harmonic coordinates, define a distance function on $B_\lambda(r_0)$ by

$$\rho_\lambda^2(x,y) = \sum_{i,j} g_{ij}(y)(x^i - y^i)(x^j - y^j),$$

and a distance function on M by $\rho^2(x,y) = \sum_\lambda \chi_\lambda(y)\rho_\lambda^2(x,y)$, where $\{\chi_\lambda\}$ is a partition of unity subordinate to the coordinate atlas.

We use this distance function to construct the parametrix of the Green form. (See [**dR**].) Let

$$h(x,y) = \big((n-2)\omega_{n-1}\big)^{-1} \rho(x,y)^{2-n} f(x,y),$$

where ω_{n-1} is the volume of the standard sphere S^n and $f(x,y)$ is a smooth cut-off function equal to 1 if $d(x,y) \le r_0/4$ and to 0 if $d(x,y) \ge r_0/2$.

Define a double m-form $A \in \bigwedge^m(M \times M)$ by taking m-fold exterior products in x and in y:

$$A(x,y) = \bigwedge_x^m \bigwedge_y^m \big(d_x d_y(-\tfrac{1}{2}\rho(x,y)^2)\big).$$

In coordinates, the components of $A(x,y)$ are determinants of $m \times m$ matrices,

$$A_{IJ}(x,y) = \det(A_{i_\alpha, j_\beta}),$$

where

$$A_{i,j}(x,y) = g_{ij}(y) + \sum_k \frac{\partial g_{ik}(y)}{\partial y^j}(x^k - y^k). \tag{1.7}$$

Finally, define the double m-form $H(x,y) = h(x,y) \wedge A(x,y)$.

1.8. . As immediate consequences of the choice of coordinates, we have

$$|H(x,y)| \leq C_p\, d(x,y)^{2-n}.$$

Since $\partial_{x^k} A_{i,j} = \partial_j g_{ik}(y)$ and $\left|\Gamma^i_{jk}(x)\right| \leq C_p\, r_0^{-1}$ by (1.4), we have

$$\left|\nabla_x A(x,y)\right| \leq C_p\, r_0^{-1}. \tag{1.9}$$

Hence,

$$\left|\nabla H(x,y)\right| \leq C_p\, r_0^{-1}\, d(x,y)^{1-n}.$$

1.10. . It follows from (1.2b) that $\|\partial\Gamma\|_{L^p(B_\lambda(r_0))} \leq C_n\, r_0^{-1-\alpha}$, so that we have estimates for $\nabla_i \nabla_j A_{IJ}$ in L^p. We in fact have

$$\left\|\nabla_{x^i}\nabla_{x^j} A(x,y)\right\|_{L^p(M)} \leq C_p(\Lambda, r_0^{-1}, V). \tag{1.11}$$

1.12. . In harmonic coordinates, it is well-known that the Laplace operator on functions reduces to $\Delta f = -\sum_{i,j} g^{ij}\partial_i\partial_j$, which does not involve derivatives of the metric. We show that the Laplacian on forms likewise simplifies in harmonic coordinates. More precisely, we show that second derivatives of the metric, $\partial^2 g_{ij}$, do not appear. This is the key that allows us to give a pointwise estimate of ΔH in terms of r_0 (Lemma 1.16). In fact, by direct computation in coordinates, we have

$$\begin{aligned} \Delta\omega_{i_1\cdots i_m} = {} & -g^{ij}\partial_i\partial_j\omega_{i_1\cdots i_m} + (2\, g^{ij}\Gamma^{k_\alpha}_{ji_\alpha} - \partial_{i_\alpha} g^{k_\alpha i})\partial_i\omega_{i_1\cdots k_\alpha\cdots i_m} \\ & - g^{ij}\Gamma^{k_\alpha}_{ii_\alpha}\Gamma^{l_\beta}_{ji_\beta}\omega_{i_1\cdots k_\alpha\cdots l_\beta\cdots i_m} + g^{ij}\partial_i\Gamma^{k_\alpha}_{ji_\alpha}\omega_{i_1\cdots k_\alpha\cdots i_m} \\ & - g^{ik_\alpha}\partial_i\Gamma^{l_\beta}_{i_\beta i_\alpha}\omega_{i_1\cdots k_\alpha\cdots l_\beta\cdots i_m} + \partial_{i_\alpha}\big(g^{k_\alpha l}\Gamma^{l_\beta}_{i_\beta l}\big)\omega_{i_1\cdots l_\beta\cdots k_\alpha\cdots i_m}. \end{aligned} \tag{1.13}$$

By exchanging summations $\sum_{\alpha=1}^m$ and $\sum_{\beta=1}^m$, antisymmetry causes cancellation in the $\partial\Gamma$ terms in (1.13). Hence, we have

$$\begin{aligned} \Delta\omega_{i_1\cdots i_m} = {} & -g^{ij}\partial_i\partial_j\omega_{i_1\cdots i_m} + (2\, g^{ij}\Gamma^{k_\alpha}_{ji_\alpha} - \partial_{i_\alpha} g^{k_\alpha i})\partial_i\omega_{i_1\cdots k_\alpha\cdots i_m} \\ & - g^{ij}\Gamma^{k_\alpha}_{ii_\alpha}\Gamma^{l_\beta}_{ji_\beta}\omega_{i_1\cdots k_\alpha\cdots l_\beta\cdots i_m} + \partial_{i_\alpha}\big(g^{jl}\Gamma^{k_\alpha}_{jl}\big)\omega_{i_1\cdots k_\alpha\cdots i_m}. \end{aligned}$$

However, in harmonic coordinates, $\sum_{j,k=1}^n g^{jk}\Gamma^l_{jk} = 0$ for each $l = 1, \ldots, n$. Therefore,

$$\Delta\omega_I = -g^{ij}(x)\partial_i\partial_j\omega_I(x) + \sum_{j,J} b_{i,J}(x)\partial_i\omega_J(x) + \sum_K c_K(x)\omega_K(x), \tag{1.14}$$

where $b_{i,J} = 2g^{ij}\Gamma^{k_\alpha}_{ji_\alpha} - \partial_{i_\alpha} g^{k_\alpha i}$ and $c_K = -g^{ij}\Gamma^{k_\alpha}_{ii_\alpha}\Gamma^{l_\beta}_{ji_\beta}$, $J = (i_1 \cdots k_\alpha \cdots i_m)$ and $K = (i_1 \cdots k_\alpha \cdots l_\beta \cdots i_m)$. Formula (1.14) shows that the coefficients $b_{i,J}$ and c_K in $\Delta\omega$ are controlled in $C^{1,\alpha}$ harmonic coordinates.

An immediate consequence of (1.14) is

$$\big|\Delta_x A(x,y)\big| \le C_p\, r_0^{-2}, \tag{1.15}$$

which follows directly from (1.2), (1.4), and (1.7).

Lemma 1.16. *In a $C^{1,\alpha}$ harmonic coordinate atlas with harmonic radius r_0, we have*

$$\big|\Delta H(x,y)\big| \le C_p\, r_0^{-\alpha}\, d(x,y)^{\alpha-n}. \tag{1.17}$$

Proof. Write

$$\Delta H(x,y) = \Delta h(x,y) \wedge A(x,y) + h(x,y) \wedge \Delta A(x,y) - 2\beta(x,y),$$

where $\beta_{IJ}(x,y) = \nabla^i h(x,y)\nabla_i A_{IJ}$. In [**H**], Proposition 3.2, it is shown that the function h in C^α-harmonic coordinates satisfies

$$\big|\Delta_x h(x,y)\big| \le C(n,p,m)\, r_0^{-\alpha} d(x,y)^{\alpha-n}. \tag{1.18}$$

For completeness, its proof is outlined here. The terms in $\omega_{n-1}\Delta_x h(x,y) = -\omega_{n-1} g^{ij}(x)\partial_{x^i}\partial_{x^j} h(x,y)$ of order $d(x,y)^{-n}$ are computed:

$$\begin{aligned}
&-\frac{1}{n-2}\, g^{ij}(x)\partial_{x^i}\partial_{x^j}\rho^{2-n}(x,y)\\
&= g^{ij}(x) g_{ij}(y)\rho^{-n}(x,y)\\
&\quad - n g^{ij}(x)\rho^{-n-2}(x,y)\Big(\sum_k g_{jk}(y)(x^k-y^k)\Big)\Big(\sum_l g_{il}(y)(x^l-y^l)\Big) + \cdots .
\end{aligned}$$

Writing $g^{ij}(x) = g^{ij}(y) + \big(g^{ij}(x) - g^{ij}(y)\big)$, this becomes

$$\begin{aligned}
&\sum_{i,j} \big(g^{ij}(x) - g^{ij}(y)\big) g_{ij}(y)\rho^{-n}(x,y)\\
&\qquad\qquad - n\rho^{-n-2}\sum \big(g^{ij}(x) - g^{ij}(y)\big) g_{jk}(y) g_{il}(y)(x^k-y^k)(x^l-y^l),
\end{aligned}$$

which is bounded by

$$\begin{aligned}
r_0^{-\alpha} d(x,y)^\alpha |g_{ij}|\rho^{-n}(x,y) - n\rho^{-n-2} r_0^{-\alpha} d(x,y)^\alpha |g_{jk}||g_{il}|(x^k-y^k)(x^l-y^l)&\\
\le C_n r_0^{-\alpha} d(x,y)^{\alpha-n}&.
\end{aligned}$$

The terms of next higher order in $\Delta_x h(x,y)$ are bounded by $C(n)\, r_0^{-1}\rho^{1-n}$, so that (1.18) holds. Lemma 1.16 follows from (1.9), (1.15), and (1.18). □

1.19. . From Lemma 1.16, one obtains Green's formula: for all m-forms $\phi \in C^2$,

$$\phi(x) = \int_M H(y,x) \wedge \star\Delta\phi(y) - \int_M \Delta_y H(y,x) \wedge \star\,\phi(y), \tag{1.20}$$

so that $H(x,y) = h(x,y) \wedge A(x,y)$ is a parametrix for the Hodge Laplacian. The proof of (1.20) follows the same argument as that of Green's formula for a parametrix constructed in normal coordinates (see for example [**dR**]), and shall be omitted here.

1.21. . The Green form can then be constructed from $H(x,y)$ by iterating Green's formula. (See [**Au**].) We fix some notation to be used later. Set

$$\Omega\phi(x) = \int_{y\in M} H(x,y) \wedge \star\,\phi(y),$$
$$Q\phi(x) = -\int_{y\in M} \Delta_x H(x,y) \wedge \star\,\phi(y).$$

Green's formula (1.20) can be written as $I = \Omega^\star\Delta + Q^\star$, where $(\Omega\phi,\psi) = (\phi,\Omega^\star\psi)$. Taking metric transposes, we obtain the second Green's formula

$$I = \Delta\Omega + Q, \tag{1.22}$$

Let

$$G(x,y) = H(x,y) + \sum_{i=1}^{N} F_i(x,y) + R(x,y),$$

where $\Gamma_1(x,y) := -\Delta_x H(x,y)$,

$$\Gamma_{i+1}(x,y) := \int_{z\in M} \Gamma_1(x,z) \wedge \star\,\Gamma_i(z,y),$$
$$F_i(x,y) := \int_{z\in M} H(x,z) \wedge \star\,\Gamma_i(z,y).$$

We define $R(x,y)$ as the unique solution of

$$\Delta_x R(x,y) = \Gamma_{N+1}(x,y), \tag{1.23}$$

since we shall consider forms in $H_m^\perp$. Also, let

$$Q^i\omega(x) = \int_M \Gamma_i(x,y) \wedge \star\,\omega(y).$$

With these notations, $F_i = \Omega Q^i$. We have

$$\Delta G = \Delta\Omega + \Delta\sum_{i=1}^{N} F_i + \Delta\mathcal{R} = I - Q + \sum_{i}^{N} \Delta\Omega Q^i + Q^{N+1}$$
$$= I - Q + \sum_{i=1}^{N} \left(Q^i - Q^{i+1}\right) + Q^{N+1} = I.$$

Hence, the $G(x,y)$ defined above gives the Green form on $H_m^\perp$.

2. Estimating the Green form

2.1. . As is well-known (see [**Au**], [**dR**]), we obtain the following bounds from the smoothing property of $Q = I - \Delta\Omega$. In the following, Λ is the (lower) bound on Ricci curvature.

For each $i = 1, \dots, N$,

$$\left|\Gamma_i(x,y)\right| \le C_p(\Lambda, V)\, r_0^{-i\alpha}\, d(x,y)^{i\alpha - n}. \tag{2.2}$$

This follows from Lemma 1.16, wherein $\left|\Gamma_1(x,y)\right| \le C_p\, r_0^{-\alpha}\, d(x,y)^{\alpha-n}$. Subsequent estimates for $\left|\Gamma_i(x,y)\right|$ are obtained inductively, and using the lower bound on the Ricci curvature. Details can be found in [**Au**].

In particular, if $N \ge n/\alpha$, then for all $x, y \in M$,

$$\left|\Gamma_{N+1}(x,y)\right| \le C_p(\Lambda, V) r_0^{-(N+1)\alpha}. \tag{2.3}$$

By (2.2), there exists C_i (depending on p, Λ, r_0^{-1}, V) that is an upper bound for both $\int_{x\in M}\left|\Gamma_i(x,y)\right| dv_g(x)$ for a.e. y and $\int_{y\in M}\left|\Gamma_i(x,y)\right| dv_g(y)$ for a.e. x. Therefore, for $1 < q < \infty$, we have

$$\left\|Q^i\omega(x)\right\|_{L^q(M)} \le C_i \left\|\omega\right\|_{L^q(M)}. \tag{2.4}$$

It follows from 1.8 and (2.2) that

$$\left|F_i(x,y)\right| \le C_p(\Lambda, V)\, r_0^{-i\alpha}\, d(x,y)^{2+i\alpha-n}. \tag{2.5}$$

2.6. . It follows from the C^α compactness of manifolds ([**AC**]; see also [**An**]) and from the continuous dependence of $\lambda_{j,m}$ on the metric, proved by Cheeger and Dodziuk [**Do**], that there is a lower bound for $\lambda_{j,m}$ in terms of the geometric quantities in $(*)$. In fact, in our fixed harmonic coordinate cover $\{u_\lambda, B_{x_\lambda}(r_0)\}$, under condition $(*)$, the pullback under the coordinate map of the canonical measure of (M,g), written $d\tau_\lambda = (u_\lambda^{-1})^\star v_g$, satisfies

$$B^{-1} \le \left|\frac{d\tau_\lambda}{d\mu}\right| \le B$$

($d\mu$ being Lebesgue measure), for some constant B depending on i_0, V, Λ, n. We let $\{\chi_\lambda\}$ be a C^∞ partition of unity satisfying $\|d\chi_\lambda\|_{L^\infty} \le C_n r_0^{-1}$. The method of Chanillo and Treves [**CT**] applied to this cover, with L^2 estimates, gives the lower bound $\lambda_{1,m} \ge C_p(i_0, \Lambda, V, r_0^{-1})$.

Lemma 2.7. *Let*

$$\mathcal{R}\omega(x) = \int_{y\in M} R(x,y) \wedge \star\,\omega(y),$$

where $R(x,y)$ is as given in (1.23). Then there exists a constant $C_q(\Lambda, r_0^{-1}, V)$ such that

$$\|\nabla^2 \mathcal{R}\omega\|_{L^q(M)} \leq C_q \|\omega\|_{L^q(M)}. \tag{2.8}$$

Proof. Given $1 < q < \infty$, we work with harmonic coordinates as in (1.2), with $p > \max\{q, n\}$. It suffices to show that

$$\left\|\nabla_x^2 R(x,y)\right\|_{L^p(x;M)} \leq C_p(\Lambda, r_0^{-1}, V), \tag{2.9}$$

independently of $y \in M$, where $\alpha = 1 - \frac{n}{p}$.

Assuming (2.9), and using the inequality

$$\|\nabla^2 \mathcal{R}\omega\|_{L^q}^q \leq V^{q(p-1)/p} \int_M \left\|\nabla_x^2 R(x,y)\right\|_{L^p(x;M)}^q \left|\omega(y)\right|^q dV_g(y),$$

we obtain (2.8). There remains to prove (2.9).

Since $\Delta_x R(x,y) = \Gamma_{N+1}(x,y)$, with $N \geq \frac{n}{\alpha}$, by (2.3) we have for a.e. y,

$$\begin{aligned} \left\|R(x,y)\right\|_{L^2(x;M)} &\leq \frac{1}{\lambda_{1,m}} \left\|\Gamma_{N+1}(x,y)\right\|_{L^2(x;M)} \\ &\leq C_p(i_0, \Lambda, V, r_0^{-1})\, r_0^{-(N+1)\alpha}, \end{aligned} \tag{2.10}$$

the last inequality coming from 2.6 and (2.3). A pointwise estimate for $|R(x,y)|$ can be derived by iterating N times Green's formula (1.20):

$$\begin{aligned} R(x,y) = \int_{z\in M} \left(H(z,x) + \sum_{i=1}^{N} F_i(z,x)\right) \wedge \star \Delta_z R(z,y) \\ + \int_{z\in M} \Gamma_{N+1}(z,x) \wedge \star\, R(z,y). \end{aligned}$$

Therefore,

$$\begin{aligned} \left|R(x,y)\right| \leq \left\|(\Delta_z R(z,y))\right\|_{L^\infty(z;M)} \int_M \left(\left|H(z,x)\right| + \sum_i \left|F_i(z,x)\right|\right) dV_g(z) \\ + \left\|\Gamma_{N+1}(z,x)\right\|_{L^2(z;M)} \left\|R(z,y)\right\|_{L^2(z;M)}. \end{aligned}$$

For the right-hand side of this equation, we have

$$\int_M \left(\left|H(z,x)\right| + \sum_i \left|F_i(z,x)\right|\right) dV_g(z) \leq C_p(\Lambda, V)\, r_0^{-N\alpha},$$

which along with (2.3) and (2.10) gives,

$$\left|R(x,y)\right| \leq C_p(\Lambda, V)\, r_0^{-2(N+1)\alpha} \quad \text{for a.e. } y \in M. \tag{2.11}$$

Fix $y \in M$ and consider R as an m-form in x. In $B(r_0)$, $\Delta R = \Gamma_{N+1}$ in harmonic coordinates is an $\binom{n}{m}$ elliptic system of the form (1.14). A Schauder estimate in $B(r_0)$ gives

$$r_0 \sup_{B(r_0/2)} |\partial R_I| + r_0^2 \sup_{B(r_0/2)} |\partial^2 R_I| \leq C_0\Big(\sup_{B(r_0)} |R_I| + \big\|(\Gamma_{N+1})_I\big\|_{C^\alpha(B(r_0))}\Big), \tag{2.12}$$

with C_0 depending on the $C^\alpha(B(r_0))$-norms of the coefficients b_J^j and c_K. By (1.2) and (1.4),

$$\begin{aligned} \|b_{i,J}\|_{C^\alpha(B(r_0))} &\leq C(n,m)\, r_0^{-1-\alpha}, \\ \|c_K\|_{C^\alpha(B(r_0))} &\leq C(n,p,m)\, r_0^{-2-\alpha}. \end{aligned} \tag{2.13}$$

By (2.3), we in fact have $(\Gamma_{N+1})_I \in C^1$. Therefore, (2.11) and (2.3) applied to (2.12) give

$$\sup_{B(r_0/2)} |\partial_x^2 R_{IJ}| \leq C_p(\Lambda, r_0^{-1}, V).$$

Finally, the L^p bound of $\nabla^2 R$ stated in (2.9) is obtained from the regularity of the metric given in (1.2b). □

3. The Weitzenböck formula and the L^2 inequality

The L^2 case of Theorem 0.1 is a Gårding type inequality, which follows from the Weitzenböck formula when the curvature operator is bounded from below. We show this in the following lemma:

Lemma 3.1. *Suppose $S \geq -K$, with $K \geq 0$. Then*

$$\|\nabla^2\phi\|_{L^2(M)} \leq C_p(K, \Lambda, r_0^{-1}) \left(\|\Delta\phi\|_{L^2(M)} + \|\phi\|_{L^2(M)}\right). \tag{3.2}$$

Proof. From the Weitzenböck formula we have $\Delta = \nabla^\star\nabla + \mathcal{E}$, where $\mathcal{E}$ is the curvature term, and $S \geq -K$ implies $\langle \mathcal{E}(\phi), \phi\rangle \geq -K\, m\,(n-m)\,|\phi|^2$ for m-forms ϕ. (For details, see [**L**].)

Hence, with $S \geq -K$,

$$\begin{aligned} \big\|(d+\delta)\,\omega\big\|_{L^2(M)}^2 &= \big((d+\delta)\,\omega,\, (d+\delta)\,\omega\big) = (\Delta\omega, \omega) \\ &= (\nabla^\star\nabla\omega + \mathcal{E}\omega, \omega) = (\nabla\omega, \nabla\omega) + (\mathcal{E}\omega, \omega) \\ &\geq \|\nabla\omega\|_{L^2(M)}^2 - K\, m\,(n-m)\,\|\omega\|_{L^2(M)}^2, \end{aligned}$$

so that for any form ω,

$$\|\nabla\omega\|_{L^2(M)}^2 \leq \big\|(d+\delta)\omega\big\|_{L^2(M)}^2 + K\, m\,(n-m)\,\|\omega\|_{L^2(M)}^2. \tag{3.3}$$

To obtain the second-order inequality, we observe that $(d+\delta)\nabla$ and $\nabla(d+\delta)$ differ by a term involving only first-order derivatives. That is, $(d+\delta)\nabla\phi = \nabla(d+\delta)\phi + a_i\nabla_i\phi$, where the a_i are given in terms of Γ_{ij}^k, which are bounded by n, α, and r_0^{-1} in $C^{1,\alpha}$ harmonic coordinates.

Applying (3.3) twice, we have

$$
\begin{aligned}
(3.4)\ \|\nabla^2\phi\|^2_{L^2(M)} & \\
&\le \|(d+\delta)\nabla\phi\|^2_{L^2(M)} + K\,m\,(n-m)\,\|\nabla\phi\|^2_{L^2(M)}\\
&= \|\nabla(d+\delta)\phi + a_i\nabla_i\phi\|^2_{L^2(M)} + K\,m\,(n-m)\,\|\nabla\phi\|^2_{L^2(M)}\\
&\le \|\nabla(d+\delta)\phi\|^2_{L^2(M)} + C(K,n,m,p,\Lambda,r_0^{-1})\|\nabla\phi\|^2_{L^2(M)}\\
&\le \|(d+\delta)^2\phi\|^2_{L^2(M)} + K\,m\,(n-m)\,\|(d+\delta)\phi\|^2_{L^2(M)} + C_p\|\nabla\phi\|^2_{L^2(M)}\\
&\le \|\Delta\phi\|^2_{L^2(M)} + C_p(K,n,m,\Lambda,r_0^{-1})\|\nabla\phi\|^2_{L^2(M)}.
\end{aligned}
$$

For the last inequality, $(d+\delta)^2\phi = \Delta\phi$, whereas for $(d+\delta)\phi$ we used the following fact (see [**GM**]), which holds for all m-forms ϕ:

$$\frac{1}{n-m+1}\,|\delta\phi|^2 + \frac{1}{m+1}\,|d\phi|^2 \le |\nabla\phi|^2.$$

The estimate of $\|\nabla\phi\|_{L^2}$ in (3.4) follows again by the Weitzenböck formula:

$$
\begin{aligned}
\|\nabla\phi\|^2_{L^2(M)} &\le (\Delta\phi,\phi) + K\,m\,(n-m)\|\phi\|^2_{L^2(M)}\\
&\le \|\Delta\phi\|_{L^2(M)}\|\phi\|_{L^2(M)} + K\,m\,(n-m)\|\phi\|^2_{L^2(M)}\\
&\le \|\Delta\phi\|^2_{L^2(M)} + C(K,n,m)\|\phi\|^2_{L^2(M)}.
\end{aligned}
$$

The last inequality applied to (3.4) proves (3.2). ☐

Lemma 3.5. *There exists a constant $C_1 = C_1(n,p,m,K,\Lambda,r_0^{-1},V)$ such that*

$$\|\nabla^2\Omega\omega\|_{L^2(M)} \le C_1\|\omega\|_{L^2(M)}. \tag{3.6}$$

Proof. Let $\varphi(x) = \Omega\omega(x)$. Since $\Delta\varphi(x) = \omega(x) - Q\omega(x)$,

$$
\begin{aligned}
\|\nabla^2\Omega\omega\|_{L^2(M)} &= \|\nabla^2\varphi\|_{L^2(M)}\\
&\le C_p(K,\Lambda,r_0^{-1})\left(\|\omega\|_{L^2(M)} + \|Q\omega\|_{L^2(M)} + \|\varphi\|_{L^2(M)}\right),
\end{aligned}
$$

where the last inequality is given by (3.2). The estimate of $\|Q\omega\|_{L^2}$ is given by (2.4). The estimate of $\|\varphi\|_{L^2}$ follows from 1.8, since

$$
\begin{aligned}
\|\varphi\|_{L^2(M)} &\le \|H(x,y)\|_{L^1(M\times M)}\|\omega\|_{L^2(M)}\\
&\le C(n,m,p,V)\,\|\omega\|_{L^2(M)}.
\end{aligned}
$$

☐

4. Calderón–Zygmund decomposition for forms

4.1. . The following decomposition of a differential form is in reference to a fixed harmonic coordinate atlas $\{u_\lambda, U_\lambda\}$. However, the constants in the estimates derived from this decomposition will depend only on the harmonic radius r_0, and do not otherwise depend on the particular coordinate atlas. The decomposition of the form ω uses properties of the double m-form $A(x,y)$ defined on page 183.

4.2. **.** Fix an m-form ω, let $a > 0$ be a constant, and let $t_i = \frac{1}{16} 2^{-i} r_0$, where r_0 is the harmonic radius. Let

$$E_0 = \{x \in M : |\omega(x)| \leq a\},$$
$$\widetilde{E}_i = \left\{ x \in M - E_0 : \frac{1}{|B_x(t_i)|} \int_{B_x(t_i)} |\omega(x)| dV_g(x) > a \right\}.$$

Let $E_i = \widetilde{E}_i - \widetilde{E}_{i-1}$, so that the E_i are disjoint sets and $M = E_0 \cup \bigcup_i E_i$ up to a set of measure 0. These sets E_i are not geodesic balls, which makes it difficult to estimate their volume. Starting with E_1, choose a set of points $x_j \in E_1$ such that the geodesic balls $\{B_{x_j}(2t_1)\}$ cover E_1, while the balls $\{B_{x_j}(t_1)\}$ are disjoint. In this way, choose points $\{x_{i_1}, x_{i_2}, \ldots, x_{i_{N(i)}}\}$ from the sets E_i so that the balls $B_{x_i}(t_i)$ are disjoint from each other and from all balls previously chosen from $E_1, \ldots, E_{i-1}$.

By reassigning indices, we have disjoint geodesic balls $\{B_{x_j}(r_j)\}$ whose union covers $\bigcup_i E_i$, where r_j is one of the t_i. And because $x_j \in E_i$ but $x_j \notin E_{i-1}$ for some i, we have

$$\text{(4.3)} \quad \frac{1}{\left|B_{x_j}(2r_j)\right|} \int_{B_{x_j}(2r_j)} \left|\omega(x)\right| dV_g(x) \leq a \leq \frac{1}{\left|B_{x_j}(r_j)\right|} \int_{B_{x_j}(r_j)} \left|\omega(x)\right| dV_g(x).$$

Let

$$Q_k = B_{x_k}(2r_k) - \left(\bigcup_{j \geq k} B_{x_j}(r_j) \cup \bigcup_{i=1}^{k-1} Q_i \right).$$

The sets Q_k are disjoint and

$$\text{(4.4)} \qquad M = E_0 \cup \bigcup_k Q_k, \quad B_{x_k}(r_k) \subset Q_k \subset B_{x_k}(2r_k).$$

4.5. **.** To prove the weak-type inequality, we split ω into a "good" and a "bad" part with respect to the decomposition $\{Q_k\}$. Define a form g by

$$g(x) = \begin{cases} \omega(x) & \text{if } x \in E_0, \\ \dfrac{1}{|Q_k|} \displaystyle\int_{y \in Q_k} A(x,y) \wedge \star\omega(y) & \text{if } x \in Q_k. \end{cases}$$

Let $\omega = g + b$ and $b_k = \chi_{Q_k} b$. We have directly the estimate

$$\text{(4.6)} \qquad \int_M \left|g(x)\right| dv_g \leq C_p \int_M \left|\omega(x)\right| dv_g\,.$$

With a lower bound on the Ricci curvature, the decomposition has the following properties:

4.7. . A lower bound on the Ricci curvature yields a volume doubling condition. By (4.3) and (4.4),

$$\begin{aligned}\frac{1}{|Q_k|}\int_{Q_k}\bigl|\omega(x)\bigr|\,dv_g(x) &\le \frac{1}{\bigl|B_{x_k}(r_k)\bigr|}\int_{B_{x_k}(2r_k)}\bigl|\omega(x)\bigr|\,dv_g(x)\\ &\le C_{n,\Lambda}\frac{1}{\bigl|B_{x_k}(2r_k)\bigr|}\int_{B_{x_k}(2r_k)}\bigl|\omega(x)\bigr|\,dv_g(x)\\ &\le C_{n,\Lambda}\,a.\end{aligned}\tag{4.8}$$

4.9. . Again using volume comparison, (4.3), and (4.4), we get

$$a \le \frac{1}{\bigl|B_{x_k}(r_k)\bigr|}\int_{B_{x_k}(r_k)}\bigl|\omega(x)\bigr|\,dv_g(x) \le \frac{C_{n,\Lambda}}{|B_{x_k}(4r_k)|}\int_{Q_k}\bigl|\omega(x)\bigr|\,dv_g(x).$$

Hence,

$$\sum_k \bigl|B_{x_k}(4r_k)\bigr| \le C_{n,\Lambda}\,\frac{1}{a}\,\|\omega\|_{L^1(M)}.\tag{4.10}$$

4.11. . If $x \in E_0$, then $|g(x)| \le a$. If $x \in Q_k$, we have

$$\bigl|g(x)\bigr| \le \frac{1}{|Q_k|}\int_M \bigl|A(x,y)\bigr|\,\bigl|\omega(y)\bigr|\,dv_g(y) \le C_p(n,m,\Lambda)\,a,$$

on account of the inequality $\bigl|A(x,y)\bigr| \le C(n,p,m)$ and (4.8).

The following lemma gives the cancellation property for b_k.

Lemma 4.12. *Let Q_k and $\omega = g+b$ be as defined in (4.6), where $r_k \le \frac{1}{32}r_0$. Then*

$$\left|\int_{Q_k} A(x,y)\wedge\star\, b_k(y)\right| \le C_p(r_0^{-1})\bigl(2\,r_k + d(x,x_k)\bigr)\int_{y\in Q_k}\bigl|\omega(y)\bigr|\,dv_g(y).$$

Proof. Since $b_k = \chi_{Q_k}\, b = \chi_{Q_k}(\omega - g)$, we have

$$\begin{aligned}&\int_{Q_k} A(x,y)\wedge\star\, b_k(y)\\ &\qquad= \int_{Q_k} A(x,y)\wedge\star(\omega(y) - g(y))\\ &\qquad= \int_{Q_k} A(x,y)\wedge\star\,\omega(y) - \int_{y\in Q_k} A(x,y)\wedge\star\left(\frac{1}{|Q_k|}\int_{z\in Q_k} A(y,z)\wedge\star\,\omega(z)\right)\\ &\qquad= \int_{Q_k} A(x,y)\wedge\star\,\omega(y) - \int_{z\in Q_k}\left(\frac{1}{|Q_k|}\int_{y\in Q_k} A(x,y)\wedge\star A(y,z)\right)\wedge\star\,\omega(z).\end{aligned}$$

We show that

$$\frac{1}{|Q_k|}\int_{y\in Q_k} A(x,y)\wedge\star A(y,z) = A(x,z) + U(x,z),$$

where

$$|U(x,z)| \le C_p\, r_0^{-1}(2r_k + d(x,x_k)) \tag{4.13}$$

for all $x, z \in Q_k$, which will prove the lemma.

From (1.7), we can write $A_{IL}(x,y) = g_{IL}(y) + B_{IL}(x,y)$, where

$$|B_{IL}(x,y)| \le C_p\, r_0^{-1} d(x,y)$$

and $g_{IL} = \det\left(g_{i_\alpha l_\beta}\right)$. We have

$$\begin{aligned}
\frac{1}{|Q_k|}\int_{y\in Q_k} &A(x,y)\wedge \star A(y,z)\\
&= \sum_{I,K}\left(\frac{1}{|Q_k|}\int_{y\in Q_k} A_{IL}(x,y) g^{JL}(y) A_{JK}(y,z)dv_q(y)\right) dx^I dz^K\\
&= A(x,z) + \sum_{I,K} U_{IK}(x,z),
\end{aligned}$$

where

$$\begin{aligned}
U_{IK}(x,z) = \frac{1}{|Q_k|}\int_{y\in Q_k} &(A_{IK}(y,z) - A_{IK}(x,z))\, dv_g(y)\\
&+ \sum_J \frac{1}{|Q_k|}\int_{y\in Q_k} B_I^J(x,y) A_{JK}(y,z) dv_g(y).
\end{aligned}$$

By (1.7),

$$\left|A_{ik}(y,z) - A_{ik}(x,z)\right| \le \sum_j \left|\partial_k g_{ij}(z)\right| |x^j - y^j|.$$

Because of $y \in Q_k$ and (1.4), we obtain

$$\left|A_{ik}(y,z) - A_{ik}(x,z)\right| \le C_p\, r_0^{-1}\big(2r_k + d(x,x_k)\big).$$

Moreover $\left|B_I^J(x,y)\right| \le C_p\, r_0^{-1}\big(2r_k + d(x,x_k)\big)$ for $y \in Q_k$, so $|U_{IK}(x,z)| \le C_p\, r_0^{-1}\big(2r_k + d(x,x_k)\big)$. Consequently,

$$\begin{aligned}
\left|\int_{Q_k} A(x,y)\wedge \star\, b_k(y)\right| &= \left|\int_{Q_k} U(x,z)\wedge \star\,\omega(z)\right|\\
&\le C_p(r_0^{-1}\big(2r_k + d(x,x_k)\big)\int_{z\in Q_k} |\omega(z)| dv_g,
\end{aligned}$$

and the lemma follows. □

5. A weak-type inequality

Lemma 5.1. *If* $\mathrm{Ric} \geq -\Lambda$ *and* $S \geq 0$*, there exists a constant* C_2 *depending on* $n, m, p, \Lambda, V, r_0^{-1}$*, such that for all* $a > 0$*,*

$$\sigma\big\{x \in M : \big|\nabla_i \nabla_j \Omega \omega(x)\big| > a\big\} \leq C_2 \, \frac{1}{a} \, \|\omega\|_{L^1(M)}, \tag{5.2}$$

where σ *is the canonical measure of* (M, g)*.*

Proof. Denote $T = \nabla_i \nabla_j \Omega$. For the 'good' part g, using properties (4.6), (4.10), and (3.6),

$$\sigma\Big\{x \in M : \big|Tg(x)\big| > \frac{a}{2}\Big\} \leq \frac{4}{a^2} \int_M \big|Tg(x)\big|^2 dv_g \leq C_p(\Lambda, r_0^{-1}, V) \, \frac{1}{a} \int_M \big|\omega(x)\big| \, dv_g.$$

For the 'bad' part b, property (4.8) gives

$$\sigma\Big\{x \in M : |Tb(x)| > \frac{a}{2}\Big\} \leq C_{n,\Lambda} \, \frac{1}{a} \|\omega\|_{L^1(M)} + \frac{2}{a} \sum_k \int_{x \notin B_{x_k}(4r_k)} \big|Tb_k(x)\big| \, dv_g.$$

To prove (5.2), it suffices to show that

$$\sum_k \int_{x \notin B_{x_k}(4r_k)} \big|Tb_k(x)\big| \, dv_g \leq C_p(\Lambda, r_0^{-1}) \|\omega\|_{L^1(M)}.$$

For $x \notin B_{x_k}(4r_k)$ and $y \in Q_k \subset B_{x_k}(2r_k)$, we have $d(x, y) \geq 2r_k$, so $H(x, y) \in C^2$. Hence, for such x,

$$Tb_k(x) = \int_{y \in Q_k} \nabla_{x^i} \nabla_{x^j} H(x, y) \wedge \star \, b_k(y).$$

By construction, $Q_k \subset B_{x_k}(2r_k) \subset B_\lambda(r_0)$ because $r_k \leq \frac{1}{32} r_0$. Therefore, each Q_k is contained in some coordinate $B_\lambda(r_0)$, in which we have

$$\nabla_{x^i} \nabla_{x^j} H(x, y) = \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} h(x, y) A(x, y) + h(x, y) \nabla_{x^i} \nabla_{x^j} A(x, y) + K(x, y),$$

where

$$\begin{aligned} K(x, y) = \sum_k \Gamma_{ij}^k(x) \frac{\partial}{\partial x^k} h(x, y) A(x, y) \\ + \nabla_{x^i} h(x, y) \nabla_{x^j} A(x, y) + \nabla_{x^j} h(x, y) \nabla_{x^i} A(x, y). \end{aligned}$$

We estimate each term separately. Let $Tb_k = S_h b_k + S_A b_k + S_0 b_k$, where

$$\begin{aligned} S_h b_k(x) &= \int_{y \in Q_k} \big(\partial_{x^i} \partial_{x^j} h(x, y)\big) A(x, y) \wedge \star \, b_k(y), \\ S_A b_k(x) &= \int_{y \in Q_k} h(x, y) \nabla_{x^i} \nabla_{x^j} A(x, y) \wedge \star \, b_k(y), \\ S_0 b_k(x) &= \int_{y \in Q_k} K(x, y) \wedge \star \, b_k(y). \end{aligned}$$

Since $\left|K(x,y)\right| \le C_p\, r_0^{-1} d(x,y)^{1-n}$, we immediately have

$$\sum_k \int_{x\notin B_{x_k}(4r_k)} \left|S_0 b_k(x)\right| dv_g(x) \le C_p(\Lambda, V)\, r_0^{-1} \|\omega\|_{L^1(M)}. \tag{5.3}$$

An estimate for S_A follows easily from the L^p bound of $\nabla^2 A$. For a.e. y,

$$\begin{aligned}
&\int_{x\notin B_{x_k}(4r_k)} d(x,y)^{2-n} \left|\nabla_{x^i}\nabla_{x^j} A(x,y)\right| dv_g(x) \\
&\qquad \le \left(\int_{x\notin B_{x_k}(4r_k)} d(x,y)^{(2-n)p'} dv_g(x)\right)^{\frac{1}{p'}} \left\|\nabla_x^2 A(x,y)\right\|_{L^p(M)} \\
&\qquad \le C_p(\Lambda, r_0^{-1}, V) \left(\int_{s=4r_k}^{\frac{3}{4}r_0} s^{\frac{p-n+1}{p-1}} ds\right)^{\frac{1}{p'}}.
\end{aligned}$$

Since $p > n$, we have

$$\sum_k \int_{x\notin B_{x_k}(4r_k)} \left|S_A b_k(x)\right| dv_g(x) \le C_p(\Lambda, r_0^{-1}, V) \sum_k \int_{Q_k} \left|b_k(y)\right| dv_g(y). \tag{5.4}$$

Turning to the term S_h, it is of order $d(x,y)^{-n}$. Here we rely on the cancellation behavior of b_k.

From the definition, $h(x,y)$ is 0 for $d(x,y) \ge \frac{1}{2}r_0$. Hence, $S_h b_k(x) = 0$ if $d(x,x_k) \ge \frac{3}{4}r_0$. With $r_k \le \frac{1}{16}2^{-k}r_0$, let $D_1 = \left\{x : 4r_k \le d(x,x_k) \le \frac{1}{8}r_0\right\}$ and $D_2 = \left\{x : \frac{r_0}{8} \le d(x,x_k) \le \frac{3}{4}r_0\right\}$. Consider the integral of $\left|S_h b_k(x)\right|$ over $\left\{x \notin B_{x_k}(4r_k)\right\}$ as an integral over D_1 and D_2.

If $x \in D_2$, $y \in Q_k$, then $d(x,x_k) \ge \frac{1}{8}r_0$ and $d(y,x_k) \le \frac{1}{16}r_0$, so that $d(x,y) \ge \frac{1}{16}r_0$. Hence, for $x \in D_2$, $\left|\partial_i\partial_j h(x,y)\right| \le C_n \rho^{-n}(x,y) \le C_n r_0^{-n}$. It follows easily, after summing over k, that

$$\sum_k \int_{D_2} \left|S_h b_k(x)\right| dv_g(x) \le C_p(\Lambda, r_0^{-1}) \|\phi\|_{L^1(M)}. \tag{5.5}$$

In D_1, write

$$\begin{aligned}
S_h b_k(x) = &\int_{y\in Q_k} \big(\partial_{x^i}\partial_{x^j} h(x,y) - \partial_{x^i}\partial_{x^j} h(x,x_k)\big) A(x,y) \wedge \star\, b_k(y) \\
&+ \partial_{x^i}\partial_{x^j} h(x,x_k) \int_{y\in Q_k} A(x,y) \wedge \star\, b_k(y).
\end{aligned} \tag{5.6}$$

We show that $\partial^2 h(x,y) - \partial^2 h(x,x_k)$ gains r_k^α, compensated by $r_0^{-\alpha}$. That is, for $x \notin B_{x_k}(4r_k)$ and $y \in Q_k \subset B_{x_k}(2r_k)$, we have

$$\begin{aligned}
|\partial_{x^i}\partial_{x^j} h(x,y) - \partial_{x^i}\partial_{x^j} h(x,x_k)| \le{} & C_p\, r_0^{-1-\alpha} r_k d(x,x_k)^{-n} \\
&+ C_p\, r_0^{-1-\alpha} r_k^\alpha \left[d(x, B_{x_k}(2r_k))\right]^{-n} + C_n r_k \left[d(x, B_{x_k}(2r_k))\right]^{-n-1};
\end{aligned} \tag{5.7}$$

to see this, recall that in D_1 the cutoff function is $f(x,y) = 1 = f(x,x_k)$.

The terms of order ρ^{-n} in $(n-2)\omega_{n-1}\partial_{x^i}\partial_{x^j}h(x,y)$ are

$$g_{ij}(y)\rho^{-n}(x,y) - n\rho^{-n-2}(x,y)\sum_{k,l} g_{jk}(y)g_{il}(y)(x^k - y^k)(x^l - y^l). \tag{5.8}$$

For the first term $g_{ij}\rho^{-n}$,

$$\begin{aligned}
&\left|g_{ij}(y)\rho^{-n}(x,y) - g_{ij}(x_k)\rho^{-n}(x,x_k)\right| \\
&\quad\le \left|g_{ij}(y) - g_{ij}(x_k)\right|\rho^{-n}(x,x_k) + \left|g_{ij}(y)\right|\left|\rho^{-n}(x,y) - \rho^{-n}(x,x_k)\right| \\
&\quad\le C_p r_0^{-1-\alpha} d(y,x_k)\rho^{-n}(x,x_k) \\
&\qquad + 4\left|\rho^2(x,x_k) - \rho^2(x,y)\right|\left|\rho^{-2}(x,y)\rho^{-n}(x,x_k) + \rho^{-n}(x,y)\rho^{-2}(x,x_k)\right|.
\end{aligned}$$

Using the bound for g_{ij} in C^α, we have

$$\left|\rho^2(x,x_k) - \rho^2(x,y)\right| \le C_p\, r_0^{-1-\alpha}(2r_k)^\alpha d(x,y)^2 + 4r_k\big(d(x,x_k) + d(x,y)\big),$$

where $d(y,x_k) \le 2r_k$ because $y \in Q_k$. Also,

$$\begin{aligned}
4\left|\rho^2(x,x_k) - \rho^2(x,y)\right|&\left|\rho^{-2}(x,y)\rho^{-n}(x,x_k) + \rho^{-n}(x,y)\rho^{-2}(x,x_k)\right| \\
&\le C_p\, r_0^{-1-\alpha} r_k^\alpha d(x,x_k)^{-n} + C_n r_k d(x,x_k)^{-n-1}.
\end{aligned}$$

Calculation for the second term of (5.8) proceeds similarly.

We now apply (5.7) and Lemma 4.12 to the first and second terms of (5.6), respectively. We have

$$\begin{aligned}
\left|S_h b_k(x)\right| &\le C_p(r_0^{-1})\, r_k\, d\big(x, B_{x_k}(2r_k)\big)^{-n-1}\int_{y\in Q_k}\left|b_k(y)\right| dv_g(y) \\
&\quad + C_p(r_0^{-1})\big(r_k\, d(x,x_k)^{-n} + r_k^\alpha d(x, B_{x_k}(2r_k))^{-n}\big)\int_{y\in Q_k}\left|b_k(y)\right| dv_g(y) \\
&\quad + C_p(r_0^{-1})\big(2r_k + d(x,x_k)\big)d(x,x_k)^{-n}\int_{y\in Q_k}\left|\omega(y)\right| dv_g(y).
\end{aligned}$$

Integrate each of these terms over $x \in D_1$ and use the lower bound of the Ricci curvature. The leading singular terms are

$$r_k\int_{x\in D_1} d\big(x, B_{x_k}(2r_k)\big)^{-n-1} dv_g(x) \le C_{n,\Lambda} r_k\int_{s=4r_k}^{r_0/8} s^{-2} ds \le C_{n,\Lambda}\left(\frac{1}{4} - \frac{8r_k}{r_0}\right)$$

and

$$\begin{aligned}
r_k^\alpha\int_{x\in D_1} d(x,x_k)^{-n} dv_g(x) \le C_{n,\Lambda} r_k^\alpha\int_{s=4r_k}^{r_0/8} s^{-1} ds &= C_{n,\Lambda} r_k^\alpha \log\frac{r_0}{32 r_k} \\
&\le C_{n,\Lambda}\,\frac{r_0}{32}\, e^{-1/\alpha}.
\end{aligned}$$

The lower-order terms are similarly bounded. We have proved that

$$\int_{x\in D_1}\left|S_1 b_k(x)\right| dv_g(x) \le C_p(\Lambda, r_0^{-1})\int_{Q_k}\left|\omega(y)\right| dv_g,$$

where the constant does not depend on r_k. Summed over k,

$$\sum_k \int_{x\in D_1} \left|S_h b_k(x)\right| dv_g(x) \le C_p(\Lambda, r_0^{-1})\|\omega\|_{L^1(M)}. \tag{5.9}$$

This and (5.5) complete the estimate of S_h and the proof of Lemma 5.1. $\square$

6. Proof of Theorem 0.1

6.1. . The operator $T = \nabla^2\Omega$ is bounded on L^2 by means of the constant C_1 of Lemma 3.5, and is of weak type $(1,1)$ with constant C_2 given in Lemma 5.1. Hence, Marcinkiewicz interpolation gives

$$\|T\omega\|_{L^q(M)} \le C_q\|\omega\|_{L^q(M)} \quad \text{for} \quad 1 < q \le 2,$$

where C_q depends on C_1, C_2, not on T or ω.

To prove that T is bounded also for $2 < q < \infty$, we show that

$$\|T^\star\psi\|_{L^q(M)} \le C_q(\Lambda, r_0^{-1}, V)\,\|\psi\|_{L^q(M)} \quad \text{for} \quad 1 < q \le 2.$$

Write $T^\star = \Omega^\star\nabla^\star\nabla^\star$ as $T^\star = T_1^\star + T_2^\star$, where

$$T_2^\star\psi(x) := \int_{y\in M} h(y,x)\nabla_y\nabla_y A(y,x) \wedge \star\,\psi(y).$$

It suffices to show that $T_1^\star$ is bounded on L^q for $1 < q \le 2$, and that $T_2^\star$ is of weak type, since the L^2 inequality for $T^\star$ holds by duality.

Lemma 6.2. *There exists $C_q(\Lambda, r_0^{-1}, V)$ such that*

$$\|T_2^\star\psi\|_{L^q(M)} \le C_q\|\psi\|_{L^q(M)} \quad \textit{for} \quad 1 < q \le 2.$$

Proof. Given $q \in (1,2]$, for $\dfrac{1}{p} + \dfrac{1}{q} + \dfrac{1}{r} = 1$, and $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, we have

$$\begin{aligned}\left|T_2^\star\psi(x)\right| \le \|\nabla_y^2 A(y,x)\|_{L^p(y;M)}\|h(y,x)\|_{L^{p'}(M)}^{p'/r}& \\ \times \left(\int_{y\in M}\left|h(y,x)\right|^{p'}\left|\psi(y)\right|^q dv_g(y)\right)^{\frac{1}{q}}&.\end{aligned}$$

By (1.11),

$$\begin{aligned}&\int_{x\in M}\left|T_2^\star\psi(x)\right|^q dv_g(x) \\ &\qquad \le C_q(\Lambda, r_0^{-1}, V)\int_{y\in M}\left(\int_{x\in M}\left|h(y,x)\right|^{p'} dv_g(x)\right)\left|\psi(y)\right|^q dv_g(y).\end{aligned}$$

$\|h(y,x)\|_{L^{p'}}$ is easily bounded by the same constants and the lemma follows. $\square$

To show that $T_1^\star$ is of weak type $(1,1)$, we consider the decomposition $\psi = g + b$, with:

$$g(x) = \begin{cases} \psi(x) & \text{if } x \in E_0 \\ \dfrac{1}{|Q_k|} \int_{y \in Q_k} A(y,x) \wedge \star\, \psi(y) & \text{if } x \in Q_k. \end{cases}$$

The rest of the proof of the weak-type inequality is similar and we shall omit it here. We only remark that the cancellation of the leading order term here is given by

$$\left|\partial_{y^i}\partial_{y^j} h(y,x) - \partial_{y^i}\partial_{y^j} h(x_k,x)\right| \leq C_n d(x_k,x)\, d\big(x, B_{x_k}(2r_k)\big)^{-n-1}.$$

6.3. . We have proved that there exists $C_q = C_q(\Lambda, r_0^{-1}, V)$, such that

$$\|\nabla^2\, \Omega\, \omega\|_{L^q(M)} \leq C_q \|\omega\|_{L^q(M)} \quad \text{for} \quad 1 < q < \infty. \tag{6.4}$$

Lemma 6.5. *Given $1 < q < \infty$, there exists a constant $C_q(\Lambda, r_0^{-1}, V)$ such that for all m-forms ω, and $i = 1, \ldots, N$,*

$$\|\nabla^2 \Omega Q^i \omega\|_{L^q(M)} \leq C_{i,q} \|\omega\|_{L^q(M)}. \tag{6.6}$$

Proof. The proof is a direct application of (6.4) and (2.4). □

6.7. . The proof of Theorem 0.1 is now complete: Given $1 < q < \infty$, choose $p > \max\{q, n\}$, and for $H_m^\perp$ construct the Green form $G(x,y) = H(x,y) + \sum_i F_i(x,y) + R(x,y)$ in $C^{1,\alpha}$ harmonic coordinates, with $\alpha = 1 - \frac{n}{p}$. Represent the solution of $\Delta\phi = \omega$ in $H_m^\perp$ by the operator $G = \Omega + \sum_i \Omega\, Q^i + \mathcal{R}$. Theorem 0.1 follows from (6.4), (6.6), and (2.8). □

7. Concluding remarks

7.1. . If $\psi \in H_m^\perp$, by Hodge theory we have

$$\psi = d\delta G\psi + \delta d G\psi = d\delta G\psi + \delta G d\psi.$$

In this case, we have the following Poincaré–Sobolev inequality as a consequence of Theorem 0.1.

Corollary 7.2. *Under the same conditions as Theorem 0.1, for $1 < q < \infty$, there exists a constant $A_q(n, m, \Lambda, K, i_0, V)$, such that, for all m-forms ψ,*

$$\|\psi - d\delta G\psi\|_{L^{1,q}(M)} \leq A_q \|d\psi\|_{L^q(M)}.$$

Proof. Let $d\psi = \omega$. We prove

$$\|\nabla \delta G \omega\|_{L^q} \leq A_q \|\omega\|_{L^q}. \tag{7.3}$$

Let $G\omega = \phi$. In an orthonormal base,

$$|\nabla\delta\phi|^2 = \sum_{(i_1\cdots i_{m+1})}\Big(-\sum_j \nabla_{i_1}\nabla_j\phi_{ji_2\cdots i_{m+1}}\Big)^2$$
$$\leq \sum_{(i_1\cdots i_{m+1})}\sum_j \big(\nabla_{i_1}\nabla_j\phi_{ji_2\cdots i_{m+1}}\big)^2 \leq (n-m)|\nabla^2\phi|^2.$$

We apply Theorem 0.1 to $\phi \in H^\perp_{m+1}$. Since $\omega \in H^\perp_{m+1}$, we obtain (7.3). $\square$

7.4. . By Sobolev embedding it follows from 7.1 and 7.2 that if $\delta\psi = 0$, $\psi \in H^\perp_m$, then

$$\left(\int_M |\psi|^{\frac{np}{n-p}}\right)^{\frac{n-p}{np}} \leq C_q\left(\int_M |d\psi|^p\right)^{\frac{1}{p}}, \quad \text{for } \frac{1}{p} = \frac{1}{q} - \frac{1}{n}.$$

7.5. . When M is locally conformally flat, the Weyl curvature tensor vanishes identically. On the other hand, when M is even-dimensional, it was shown by J.-P. Bourguignon ([**B**], Corollary 8.8) that the Laplace operator on $\frac{n}{2}$-forms can be expressed in terms of the Weyl curvature and the scalar curvature:

$$\Delta = \nabla^\star\nabla - W + \frac{1}{(n-1)(n-2)}R,$$

where the manifold is of dimension $n = 2m$, R is the scalar curvature, and

$$W(\phi)_{i_1\cdots i_m} = \sum_{\alpha,\beta=1}^{m}\sum_{j,k=1}^{n} W_{ki_\beta ji_\alpha}\phi_{i_1\cdots\widehat{i_\beta}k\cdots\widehat{i_\alpha}j\cdots i_m}.$$

Hence, when M is locally conformally flat, we can state:

Theorem 7.6. *Let M be an even-dimensional compact Riemannian manifold without boundary with $\operatorname{inj}(M) \geq i_0$, $\operatorname{Vol}(M) \leq V$, $|\mathrm{Ric}| \leq \Lambda$, and scalar curvature $R \geq -K$. Suppose M is locally conformally flat. Then for $1 < q < \infty$, there exists a constant $C_q = C(q, n, \Lambda, K, i_0, V)$ such that*

$$\|\nabla^2\phi\|_{L^q(M)} \leq C_q\|\Delta\phi\|_{L^q(M)},$$

for any $\frac{n}{2}$-form $\phi \in H^\perp_{n/2}$.

The proof of Theorem 7.6 proceeds as for Theorem 0.1, with the exception that (3.2) holds with only the assumption on lower bound of the scalar curvature.

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