A UNIFIED APPROACH TO UNIVERSAL INEQUALITIES FOR EIGENVALUES OF ELLIPTIC OPERATORS

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We present an abstract approach to universal inequalities for the discrete spectrum of a self-adjoint operator, based on commutator algebra, the Rayleigh–Ritz principle, and one set of “auxiliary” operators. The new proof unifies classical inequalities of Payne–Pólya–Weinberger, Hile–Protter, and H.C. Yang and provides a Yang type strengthening of Hook’s bounds for various elliptic operators with Dirichlet boundary conditions. The proof avoids the introduction of the “free parameters” of many previous authors and relies on earlier works of Ashbaugh and Benguria, and, especially, Harrell (alone and with Michel), in addition to those of the other authors listed above. The Yang type inequality is proved to be stronger under general conditions on the operator and the auxiliary operators. This approach provides an alternative route to recent results obtained by Harrell and Stubbe.

1. Introduction

There has been much work dedicated to extending and strengthening the classical gap inequality of Payne, Pólya, and Weinberger [28], [29] (see also [2], [4], [5], [6], [7], [34]). This result states that

\[ \lambda_{m+1} - \lambda_m \leq \frac{4}{n} \sum_{i=1}^{m} \lambda_i, \]

where \( 0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \) designate the eigenvalues of the membrane problem (multiplicities included)

\[ -\Delta u = \lambda u \quad \text{in} \quad \Omega, \]
\[ u = 0 \quad \text{on} \quad \partial \Omega. \]

The set \( \Omega \subset \mathbb{R}^n \) is a bounded domain and the Laplacian \( \Delta \) is given by

\[ \Delta \equiv \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}. \]

On the strengthening side we find the work of Hile and Protter [21], who showed that

\[ \frac{mn}{4} \leq \sum_{i=1}^{m} \frac{\lambda_i}{\lambda_{m+1} - \lambda_i}. \]
We also find the work of H.C. Yang [35] (see also [2], [3], [8]), who showed that

\[ \sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^2 \leq \frac{4}{n} \sum_{i=1}^{m} \lambda_i (\lambda_{m+1} - \lambda_i). \]  

(1.4)

Harrell and Stubbe [20] extended this inequality further to

\[ \sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^p \leq \frac{2p}{n} \sum_{i=1}^{m} \lambda_i (\lambda_{m+1} - \lambda_i)^{p-1} \quad \text{for } p \geq 2 
\]  

(inequality (14) in Thm. 9, p. 1805) and

\[ \sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^p \leq \frac{4}{n} \sum_{i=1}^{m} \lambda_i (\lambda_{m+1} - \lambda_i)^{p-1} \quad \text{for } p \leq 2 
\]  

(inequality (11) in Thm. 5, p. 1801). The Hile–Protter and H.C. Yang inequalities appear as special cases of this latter inequality for \( p = 0 \) and \( 2 \) respectively. It can be shown, however, that (1.6) for \( p < 2 \) is always weaker than (1.4) (i.e., the \( p = 2 \) case of (1.6)). In fact, the bounds (1.6) can be shown to improve monotonically with \( p \) (see [10]).

These inequalities are called universal because they do not involve domain-dependencies [31] (see also [2], [3], for example).

As extensions of the classical PPW and HP results we find applications of the methods to various geometric and physical situations. Cheng [14] produced the first estimates of this type for a compact hypersurface minimally immersed in \( \mathbb{R}^{n+1} \). He also treated the case of an inhomogeneous membrane and subdomains of \( S^2 \). P.C. Yang and S.-T. Yau [36] produced similar estimates for a hypersurface minimally immersed in \( S^n \). Li [26] treated the case of compact homogeneous spaces. Leung [25] corrected the constants in Yang and Yau’s PPW type bound and produced an HP like version of their work in the spirit of Li’s estimate in [26] (see also [18]).

It was Harrell and Davies (see [16]) who first realized that many of the original PPW arguments rely on facts involving operators, their commutators, and traces. In 1988, Harrell [16] first published a fully “algebraicized” version of the PPW argument. Then in 1993, using projections [17], he produced bounds on the eigenvalue gap for the Dirichlet problem for subdomains of a given Riemannian manifold in terms of their geometry. Harrell and Michel [18], [19] produced an algebraic inequality based on two sets of auxiliary operators and a set of spectral projections. They applied their formula to produce various geometric inequalities and bounds for partial differential operators. Their work improved earlier bounds of Harrell [16], [17]. They also significantly improved and simplified the bounds of Cheng [14] (see [27] where the results are displayed explicitly), Li [26], Yang and Yau [36], and Leung [25]. Their HP type inequalities appear in the “natural”
form described above. At the heart of Harrell and Michel’s improvement appears the exploitation of certain symmetry and commutation properties of the eigenfunctions and eigenvalues. Parallel considerations in the spirit of [5] (see also [4]) were used by the authors [9] to produce new domain-dependent versions of (1.1), (1.3), and (1.4). Ideas along these lines were also used by Lee [24] to produce HP type bounds for the eigenvalues of the Laplace–Beltrami operator on p-forms. These bounds extend and generalize the results of Cheng [14] and Yang and Yau [36]. They are extrinsic, meaning they depend on a curvature operator appearing in the Weitzenböck formula and a mean curvature vector field. The domains in Lee’s work are not minimally immersed, but only immersed in $\mathbb{R}^{n+1}$ or $S^n$ isometrically.

Harrell and Michel’s approach is similar in spirit to that of Hook [23], though they used a somewhat different method of proof. Hook, for his part, generalized the original argument of Hile and Protter [21] in an abstract setting and was able to reproduce their result and improve on results of Hile and Yeh [22] in the context of the biharmonic operator with Dirichlet boundary conditions. He applied his abstract framework to various operators of mathematical physics and produced HP type bounds for them (Schrödinger operators with magnetic potential, second-order elliptic operators with constant coefficients, the Sturm–Liouville problem, the Lamé system). We have recently found a new proof [12] of his results that avoids Hile and Protter’s “free parameters” and allows us to develop his conclusions in a fashion paralleling that of the present work.

In this article, we produce a set of algebraic inequalities from which the classical inequalities of Payne, Pólya, and Weinberger, Hile and Protter, and H.C. Yang described earlier follow.

A self-adjoint semibounded operator is given and we study its eigenvalues. A set of auxiliary symmetric operators is introduced. Various inequalities relating the eigenvalues to commutators and projections are proved.

Here, we use one set of auxiliary operators to produce many of Hook’s results, and do without the projections of Harrell et al. We use solely the Rayleigh–Ritz principle and properties of commutators. Using this method we are able to improve on and generalize recent work by Harrell and Stubbe [20].

The approach we present here has a “unifying” aspect. The extensions of Hile–Protter and H.C. Yang will be proved to follow — save for the addition of a key idea — from the same set of principles as the original PPW inequality. New proofs for the HP and H.C. Yang inequalities, in their abstract setting, are provided. These inequalities can be seen as conditions that certain discriminants be nonpositive [12]. We also prove that both of Yang’s inequalities (see [2], [3]) are stronger than both the PPW and HP results, thus supplying an argument left aside in [35]. An alternative proof of this latter result appears in [3] but our proof here is simpler and more direct.
Our work follows the spirit of [2] and [3]. In these works one of us (MSA) produced an argument partially based on the work of H.C. Yang [35], which does away with the “free parameters” of Hile–Protter [21] and Hook [23] (see also [31], [32]). Ashbaugh and Benguria produced the earliest “parameter-free” proof of the HP inequality in [7]. We recommend the article [2] where a discussion of the history is presented and an explicit version for the Dirichlet Laplacian of some of the arguments we generalize here is laid out (see also [6] for more results and conjectures).

In [9], we present versions of domain-dependent estimates of a related type. Then in [10] we develop the Harrell–Stubbe type inequalities given above in the spirit of this abstract formulation. In particular, we show that (1.5) is weaker than H.C. Yang’s (1.4) if $p$ is restricted to integer values $p > 2$. We also show that (1.6) is intermediate between the H. C. Yang and Hile–Protter inequalities and interpolates between them for $0 \leq p \leq 2$. In our work we adopt much of the notation of Hook in [23].

Our abstract formulation allows for Yang type and Harrell–Stubbe type improvements for the various physical and geometric problems described above. This is presented in [11], where we improve on Harrell and Michel’s works [18], [19], and Hook’s work [23]. In that article, we adopt the point of view of Bandle [13] to produce Yang type inequalities for eigenvalues of domains in $S^2$ and $H^2$ to improve results in [14], [17], [18], [19] by treating the eigenvalue problem in these space forms as inhomogeneous membrane problems.

2. The classical inequalities with one set of auxiliary operators

Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$, $A : \mathcal{D} \subset \mathcal{H} \to \mathcal{H}$ a self-adjoint operator defined on a dense domain $\mathcal{D}$ that is bounded below and has a discrete spectrum $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$, 

$$\{B_k : A(\mathcal{D}) \to \mathcal{H}\}_{k=1}^N$$

a collection of symmetric operators leaving $\mathcal{D}$ invariant, and $\{u_i\}_{i=1}^\infty$ the normalized eigenvectors of $A$, $u_i$ corresponding to $\lambda_i$. We may further assume that $\{u_i\}_{i=1}^\infty$ is an orthonormal basis for $\mathcal{H}$. $[A, B]$ denotes the commutator of two operators defined by $[A, B] = AB - BA$, and $\|u\| = \sqrt{\langle u, u \rangle}$.

Let

$$(2.1) \quad \rho_i = \sum_{k=1}^N \langle [A, B_k]u_i, B_ku_i \rangle$$

and

$$(2.2) \quad \Lambda_i = \sum_{k=1}^N \| [A, B_k]u_i \|^2.$$
The following is motivated by the classical PPW, HP, and H.C. Yang inequalities. We will show that they spring from these inequalities and the general set-up of this theorem.

**Theorem 2.1.** The eigenvalues $\lambda_i$ of the operator $A$ satisfy the inequalities

\begin{equation}
\sum_{i=1}^{m} \rho_i \leq \frac{\sum_{i=1}^{m} \Lambda_i}{\lambda_{m+1} - \lambda_m},
\end{equation}

\begin{equation}
\sum_{i=1}^{m} \rho_i \leq \frac{m}{\lambda_{m+1} - \lambda_i},
\end{equation}

and

\begin{equation}
\sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^2 \rho_i \leq \sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i) \Lambda_i.
\end{equation}

The proof of this theorem will be given in Section 3.

**Remark.** Roughly speaking, the quantities $\Lambda_i$ are analogues of the kinetic energy term corresponding to the eigenstate $u_i$. Since $\{u_i\}_{i=1}^\infty$ is complete, one can write, using Parseval’s decomposition

\begin{equation}
[A, B_k]u_i = \sum_{j=1}^\infty b_{ij}^k u_j,
\end{equation}

where $b_{ij}^k = \langle [A, B_k]u_i, u_j \rangle$. This implies that

\[ \|[A, B_k]u_i\|^2 = \sum_{j=1}^\infty |\langle [A, B_k]u_i, u_j \rangle|^2, \]

or yet

\begin{equation}
\Lambda_i = \sum_{k=1}^{N} \sum_{j=1}^{\infty} |\langle [A, B_k]u_i, u_j \rangle|^2.
\end{equation}

When $A$ is a Dirichlet Laplacian or a Schrödinger operator with magnetic potential, (2.7) reduces to Equation (5) on p. 1798 of [20] with a factor of 4, i.e., $\Lambda_i = 4T_i$, the kinetic energy term in Harrell and Stubbe’s notation (the quantity $|\langle [A, B_k]u_i, u_j \rangle|^2$ is similar to $T_{kij}$ in their notation). Identity (2.7) is exploited in [10] to produce an alternative route to some of the results of [20].

**Lemma 2.2.** The quantity $\rho_i$ can be written in the simpler form

\begin{equation}
\rho_i = \frac{1}{2} \sum_{k=1}^{N} \langle [B_k, [A, B_k]]u_i, u_i \rangle.
\end{equation}
Proof. For \( A, B \) as introduced above and \( u \) an eigenvector \( u_i \) of \( A \), one has
\[
\langle [B, [A, B]]u, u \rangle = \langle B[A, B]u, u \rangle - \langle [A, B]Bu, u \rangle
\]
\[
= \langle [A, B]u, Bu \rangle + \langle Bu, [A, B]u \rangle
\]
\[
= 2\Re \langle [A, B]u, Bu \rangle
\]
\[
= 2 \langle [A, B]u, Bu \rangle,
\]
(2.9)
since
\[
\langle [A, B]u, Bu \rangle = \langle ABu, Bu \rangle - \langle BAu, Bu \rangle
\]
(2.10)
is clearly real by the self-adjointness of \( A \) and the fact that \( u \) is an eigenvector of \( A \) (in the above, we have taken \( Au = \lambda u \) and \( \lambda \), as an eigenvalue of \( A \), is necessarily real). Lemma 2.2 follows in view of (2.9) and the definition of \( \rho_i \).
\[\Box\]

**Lemma 2.3.** Let \( A = -\sum_{k=1}^{N} T_k^2 \) where the \( T_k \)'s are skew-symmetric with domains \( \mathcal{D}(T_k) \) such that \( T_k(\mathcal{D}) \subseteq \mathcal{D}(T_k) \) and \( \mathcal{D}(T_k) \supset \mathcal{D}(A) = \mathcal{D} \). If \( [T_\ell, B_k]u = \delta_\ell k u \), for a vector \( u \) an eigenvector of \( A \), then \( [A, B_k]u = -2T_ku \) and \( [B_k, [A, B_k]]u = 2u \).

**Proof.** By the formal commutator identity \( [A, BC] = B[A, C] + [A, B]C \), one has
\[
[A, B_k]u = -\sum_{\ell=1}^{N} [T_\ell^2, B_k]u = \sum_{\ell=1}^{N} [B_k, T_\ell^2]u
\]
\[
= \sum_{\ell=1}^{N} (T_\ell[B_k, T_\ell] + [B_k, T_\ell]T_\ell)u = -2 \sum_{\ell=1}^{N} \delta_\ell k T_\ell u = -2T_k u.
\]
Therefore,
\[
[B_k, [A, B_k]]u = -2 [B_k, T_k]u = 2u,
\]
(2.12)
and the desired result follows. \(\Box\)

We now conclude the generalization of the classical results.

**Corollary 2.4.** If \( A = -\sum_{k=1}^{N} T_k^2 \) where the \( T_k \)'s are skew-symmetric with domains as delineated above, and if \( [T_\ell, B_k]u = \delta_\ell k u \) for \( u \) an eigenvector of \( A \), then \( \rho_i = N, \Lambda_i = 4\lambda_i \), and
\[
\lambda_{m+1} - \lambda_m \leq \frac{4}{Nm} \sum_{i=1}^{m} \lambda_i,
\]
(2.13)
\[
\frac{Nm}{4} \leq \sum_{i=1}^{m} \frac{\lambda_i}{\lambda_{m+1} - \lambda_i},
\]
(2.14)
and

\[ \sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^2 \leq \frac{4}{N} \sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i) \lambda_i. \]  

(2.15)

If \( C \) is a symmetric operator and \( \alpha \in \mathbb{R} \), we write \( C \geq \alpha \) if \( \langle Cu, u \rangle \geq \alpha \langle u, u \rangle \) for all vectors \( u \in D(C) \). We define \( A \geq B \) for symmetric operators \( A \) and \( B \) which are bounded below if \( D(A) \subset D(B) \) and \( A - B \geq 0 \) on \( D(A) \).

**Theorem 2.5.** Suppose there exist \( \gamma, \beta \) such that

\[ 0 < \gamma \leq \| B_k, [A, B_k] \| \]  

(2.16)

and

\[ -N \sum_{k=1}^{N} [B_k, A]^2 \leq \beta A. \]  

(2.17)

Then

\[ \lambda_{m+1} - \lambda_m \leq \frac{2\beta}{mN\gamma} \sum_{i=1}^{m} \lambda_i, \]  

(2.18)

\[ \frac{mN\gamma}{2\beta} \leq \sum_{i=1}^{m} \frac{\lambda_i}{\lambda_m + 1 - \lambda_i}, \]  

(2.19)

and

\[ \sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^2 \leq \frac{2\beta}{N\gamma} \sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i) \lambda_i. \]  

(2.20)

*Proof.* By virtue of (2.16) and (2.17), \( \rho_i \geq \frac{1}{2} N \gamma \) and \( \Lambda_i \leq \beta \lambda_i \). \( \square \)

### 3. Proof of Theorem 2.1

Let \( \phi \) be a trial function for \( \lambda_{m+1} \) in the Rayleigh–Ritz inequality. Then

\[ \lambda_{m+1} \leq \frac{\langle A\phi, \phi \rangle}{\langle \phi, \phi \rangle} \]  

(3.1)

and

\[ \langle \phi, u_j \rangle = 0 \]  

(3.2)

for \( j = 1, \ldots, m \).

Let \( \phi \) be given by

\[ \phi_i = Bu_i - \sum_{j=1}^{m} a_{ij} u_j, \]  

(3.3)
where $B$ is one of the $B_k$’s, $k = 1, \ldots, N$. Condition (3.2) makes $a_{ij} = \langle Bu_i, u_j \rangle$. Since $B$ is symmetric, we have $a_{ji} = a_{ij}$. Moreover,

$$\|\phi_i\|^2 = \langle Bu_i, \phi_i \rangle,$$

and

$$\langle A\phi_i, \phi_i \rangle = \langle ABu_i - m \sum_{j=1}^{m} a_{ij} \lambda_j u_j, \phi_i \rangle = \langle ABu_i, \phi_i \rangle$$

$$= \langle BAu_i, \phi_i \rangle + \langle [A,B]u_i, \phi_i \rangle = \lambda_i \langle Bu_i, \phi_i \rangle + \langle [A,B]u_i, \phi_i \rangle$$

$$= \lambda_i \langle \phi_i, \phi_i \rangle + \langle [A,B]u_i, \phi_i \rangle.$$

Thus, (3.1) reduces to

$$\lambda_{m+1} - \lambda_i \leq \frac{\langle [A,B]u_i, \phi_i \rangle}{\langle \phi_i, \phi_i \rangle}.$$

Now

$$\langle [A,B]u_i, \phi_i \rangle = \langle [A,B]u_i, Bu_i \rangle - \sum_{j=1}^{m} a_{ij} \langle [A,B]u_i, u_j \rangle.$$

Let $b_{ij} = \langle [A,B]u_i, u_j \rangle$. Then

$$\langle [A,B]u_i, \phi_i \rangle = \langle [A,B]u_i, Bu_i \rangle - \sum_{j=1}^{m} a_{ij} b_{ij}.$$

We now observe that

$$b_{ij} = -\overline{b_{ji}} = (\lambda_j - \lambda_i) a_{ij}.$$

This is evident from

$$b_{ij} = \langle [A,B]u_i, u_j \rangle = \langle ABu_i, u_j \rangle - \langle BAu_i, u_j \rangle$$

$$= \langle Bu_i, Au_j \rangle - \langle BAu_i, u_j \rangle = (\lambda_j - \lambda_i) \langle Bu_i, u_j \rangle = (\lambda_j - \lambda_i) a_{ij}.$$

Therefore,

$$\langle [A,B]u_i, \phi_i \rangle = \langle [A,B]u_i, Bu_i \rangle - \sum_{j=1}^{m} (\lambda_j - \lambda_i) |a_{ij}|^2.$$

**Lemma 3.1** ("Optimal" Cauchy–Schwarz). With the notation and choice of $\phi_i$ as given above, we have

$$\frac{\langle [A,B]u_i, \phi_i \rangle}{\langle \phi_i, \phi_i \rangle} \leq \frac{\| [A,B]u_i \|^2 - \sum_{j=1}^{m} |b_{ij}|^2}{\langle [A,B]u_i, \phi_i \rangle}.$$
Proof. Since \( \langle u_j, \phi_i \rangle = 0 \), for \( 1 \leq i, j \leq m \) and \( \langle [A, B]u_i, \phi_i \rangle \) is real (by the self-adjointness of \( A \); see, for example, (3.11) or (3.6)) we have

\[
(\langle [A, B]u_i, \phi_i \rangle)^2 = \left( \langle [A, B]u_i - \sum_{j=1}^{m} b_{ij} u_j, \phi_i \rangle \right)^2 \leq \left\| [A, B]u_i - \sum_{j=1}^{m} b_{ij} u_j \right\| \| \phi_i \| \leq \left( \left\| [A, B]u_i \right\|^2 - \sum_{j=1}^{m} |b_{ij}|^2 \right) \| \phi_i \|^2.
\]

Hence, noting that \( \langle [A, B]u_i, \phi_i \rangle \geq 0 \) (by (3.6), for example), we get the result.

\[\square\]

Remark. Written in more explicit terms, (3.10) reads

\[(3.13) \quad \langle [A, B]u_i, u_j \rangle = (\lambda_j - \lambda_i) \langle Bu_i, u_j \rangle.\]

Domain considerations aside, this gap formula is at the heart of many good estimates for eigenvalues \([16], [19]\). Hence the coefficient \( b_{ij} \) is “natural” in this context. It is the incorporation of the “counterterms” involving the \( b_{ij} \)’s that makes the use of the Cauchy–Schwarz inequality to obtain (3.12) “optimal”. If these counterterms are dropped both the PPW and HP results are obtained (see (3.18) and (3.20) below). The orthogonality of the \( \phi_i \)'s to the \( u_j \)'s is what allows us to include these counterterms. As remarked above in (2.6), the \( b_{ij} \)'s are precisely the components of \( [A, B]u_i \) along the \( u_j \). This choice of components is optimal for minimizing the norm of any expression of the form \( [A, B]u_i - \sum_{j=1}^{m} c_j u_j \). This can be regarded as the key element that allows us to derive Yang’s strengthened version of the classical inequalities, explaining our designation “optimal use of the Cauchy–Schwarz inequality” in connection with our method. This approach was first noted in preliminary form by Ashbaugh and Benguria \([8]\). Yang’s derivation \([35]\) is more circuitous, and in particular does not make explicit use of this crucial element of our proof. A full explanation of the method appeared first in Ashbaugh \([2]\) (see also \([3], [9]\)).

Corollary 3.2. Under the assumptions of the problem, the following inequality holds:

\[(3.14) \quad (\lambda_{m+1} - \lambda_i) \left( \langle [A, B]u_i, Bu_i \rangle - \sum_{j=1}^{m} (\lambda_j - \lambda_i)|a_{ij}|^2 \right) \leq \left\| [A, B]u_i \right\|^2 - \sum_{j=1}^{m} (\lambda_j - \lambda_i)^2 |a_{ij}|^2.\]

Proof. Start with (3.6), and use Lemma 3.1 along with (3.9) and (3.11). \[\square\]
Since $B$ is one of the $B_k$'s, $a_{ij} = a_{kj}^k$. Let

$$A_{ij} \equiv \sum_{k=1}^{N} |a_{ij}^k|^2.$$  \hspace{1cm} (3.15)

We have $A_{ij} = A_{ji} \geq 0$. Summing (3.14) over $k$, for $1 \leq k \leq N$, and incorporating the definitions of $\rho_i$, $\Lambda_i$, and $A_{ij}$, we get

$$\left(\lambda_{m+1} - \lambda_i\right) \left(\rho_i - \sum_{j=1}^{m} (\lambda_j - \lambda_i)A_{ij}\right) \leq \Lambda_i - \sum_{j=1}^{m} (\lambda_j - \lambda_i)^2 A_{ij}.$$  \hspace{1cm} (3.16)

By dropping the last term on the right-hand side, one has

$$\left(\lambda_{m+1} - \lambda_i\right) \left(\rho_i - \sum_{j=1}^{m} (\lambda_j - \lambda_i)A_{ij}\right) \leq \Lambda_i.$$  \hspace{1cm} (3.17)

Remark. The quantity $\rho_i - \sum_{j=1}^{m} (\lambda_j - \lambda_i)A_{ij} \geq 0$ by virtue of (3.6) (see also (3.11)).

The General PPW Bound for $\lambda_{m+1} - \lambda_m$. From (3.17), we pass to

$$\left(\lambda_{m+1} - \lambda_i\right) \left(\rho_i - \sum_{j=1}^{m} (\lambda_j - \lambda_i)A_{ij}\right) \leq \Lambda_i.$$  \hspace{1cm} (3.18)

Summing over $i = 1, \ldots, m$ yields (2.3). The double sum

$$\sum_{i=1}^{m} \sum_{j=1}^{m} A_{ij}(\lambda_j - \lambda_i)$$  \hspace{1cm} (3.19)

vanishes by antisymmetry.

The Hile–Protter Bound. We rewrite (3.17) as

$$\rho_i - \sum_{j=1}^{m} (\lambda_j - \lambda_i)A_{ij} \leq \frac{\Lambda_i}{\lambda_{m+1} - \lambda_i}.$$  \hspace{1cm} (3.20)

Summing on $i$, $1 \leq i \leq m$, yields (2.4).

The H.C. Yang Bound. We rewrite (3.16) as

$$\left(\lambda_{m+1} - \lambda_i\right) \rho_i \leq \Lambda_i + \sum_{j=1}^{m} (\lambda_{m+1} - \lambda_j)(\lambda_j - \lambda_i)A_{ij}.$$  \hspace{1cm} (3.21)

Multiplying by $(\lambda_{m+1} - \lambda_i)$ and summing on $i$ yields (2.5). The double sum

$$\sum_{i=1}^{m} \sum_{j=1}^{m} (\lambda_{m+1} - \lambda_i)(\lambda_{m+1} - \lambda_j)(\lambda_j - \lambda_i)A_{ij}$$  \hspace{1cm} (3.22)

vanishes by antisymmetry.
4. The case of a Schrödinger like operator

Let $H = A + V$ be an operator defined on $\mathcal{D} \subset \mathcal{H}$, where $A$ and $V$ are self-adjoint operators, $A = -\sum_{k=1}^{N} T_k^2$, and the $T_k$'s are skew-symmetric with domains $T_k(\mathcal{D})$ satisfying $T_k(\mathcal{D}) \subset \mathcal{D}(T_k)$ and $\mathcal{D}(V) \supset \mathcal{D}(T_k) \supset \mathcal{D}(A) \equiv \mathcal{D}$. The operator $H$ is modeled after the Schrödinger operator. We assume that the spectrum of $H$ is discrete consisting of eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots$, and we let $\{u_i\}_{i=1}^{\infty}$ be a complete orthonormal basis of eigenvectors corresponding to $\{\lambda_i\}_{i=1}^{\infty}$. We take a family of symmetric operators $\{B_k : \mathcal{H} \to \mathcal{H}\}_{k=1}^{N}$ that leave $\mathcal{D}$ invariant, such that $[T_k, B_k]u_i = \delta_{ik}u_i$. The quantities $\rho_i$ and $\Lambda_i$ are given by

$$
\rho_i = \sum_{k=1}^{N} \langle [H, B_k]u_i, B_ku_i \rangle \quad \text{and} \quad \Lambda_i = \sum_{k=1}^{N} \| [H, B_k]u_i \|^2.
$$

In obvious notation we write $\rho_i = \rho_i^A + \rho_i^V$, corresponding to the decomposition $H = A + V$. We have the following generalization of Corollary 2.4.

**Theorem 4.1.** Suppose $[V, B_k] = 0$ for $1 \leq k \leq N$. Then $\rho_i = N$, $\Lambda_i = 4(\lambda_i - \langle Vu_i, u_i \rangle)$, and

$$
\lambda_{m+1} - \lambda_m \leq \frac{4}{Nm} \sum_{i=1}^{m} (\lambda_i - \langle Vu_i, u_i \rangle),
$$

$$
Nm \leq \sum_{i=1}^{m} \frac{\lambda_i - \langle Vu_i, u_i \rangle}{\lambda_{m+1} - \lambda_i},
$$

$$
\sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^2 \leq \frac{4}{N} \sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)(\lambda_i - \langle Vu_i, u_i \rangle).
$$

**Proof.** For $[V, B_k] = 0$, $\rho_i^V = 0$. Therefore $\rho_i = \rho_i^A = N$, by Corollary 2.4. For an eigenvector $u$, $[H, B_k]u = [A, B_k]u = -2T_ku$, by Lemma 2.3. Hence,

$$
\| [H, B_k]u_i \|^2 = 4\| T_ku_i \|^2
$$

and

$$
\Lambda_i = \sum_{k=1}^{N} \| [H, B_k]u_i \|^2 = 4 \sum_{k=1}^{N} \| T_ku_i \|^2
$$

$$
= 4 \left( \sum_{k=1}^{N} -T_k^2 u_i, u_i \right) = 4 \langle Au_i, u_i \rangle
$$

$$
= 4 \langle (H - V)u_i, u_i \rangle = 4 \left( \lambda_i - \langle Vu_i, u_i \rangle \right).
$$

The result then follows from previous considerations. □
**Remark.** Suppose $V \geq M > 0$. Then the theorem reduces to

\begin{align}
\lambda_{m+1} - \lambda_m &\leq \frac{4}{Nm} \sum_{i=1}^{m} (\lambda_i - M), \quad (4.5) \\
\frac{Nm}{4} &\leq \sum_{i=1}^{m} \frac{\lambda_i - M}{\lambda_{m+1} - \lambda_i}, \quad (4.6) \\
\sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^2 &\leq \frac{4}{N} \sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i) (\lambda_i - M). \quad (4.7)
\end{align}

Therefore, the classical PPW, HP, and H.C. Yang inequalities (without $M$) hold with strict inequalities. This fact (and more) was noted by Ashbaugh and Benguria [7] for the Hile–Protter type inequality and was first observed in the case of the Payne–Pólya–Weinberger inequality by Allegretto [1], Harrell [unpublished] (in [16] the PPW type inequality (4.1) is interpreted as a family of pointwise bounds for the potential $V$), and Singer, Wong, Yau, and Yau [33]. Hook [23] gives the HP type results applying to various second-order elliptic operators. The Yang type bound (4.7) is stronger than that proved in [20] though we essentially use the same assumptions (see Theorem 1 and Theorem 5 therein).

5. Comparing the bounds

In this section, we are interested in comparing the bounds on $\lambda_{m+1}$ that arise from each of the three classical inequalities of Payne–Pólya–Weinberger, Hile–Protter, and H. C. Yang, including some simpler variants of these bounds. In his paper, H. C. Yang [35] stated that the inequality he obtained implied that of an “averaged” version of PPW that in turn implied that of Hile–Protter (which is known to imply the classical PPW inequality). However, while the first point is clear, he did not give specifics for the remaining points, so we find it instructive to do so here. We will in fact provide a proof valid for the general setting of this work, with applicability to the Laplacian and various other operators. Similar methods were used in [10] to treat the Harrell–Stubbe type inequalities given as (1.5) and (1.6). H.C. Yang’s inequality will be proved to be the strongest of the existing classical universal inequalities.

Throughout this section, we assume that the operators $A$ and $B_k$, for $1 \leq k \leq N$, satisfy the conditions (2.16) and (2.17), namely

$$\gamma \leq [B_k, [A, B_k]] \quad \text{and} \quad -\sum_{k=1}^{N} [B_k, A]^2 \leq \beta A.$$
for some $\beta, \gamma > 0$. For simplicity we set

$$\bar{\lambda} = \frac{\sum_{i=1}^{m} \lambda_i}{m}.$$  

We let

$$\sigma_{\text{PPW}} = \lambda_m + \frac{2\beta}{mN\gamma} \sum_{i=1}^{m} \lambda_i = \lambda_m + \frac{2\beta}{N\gamma} \bar{\lambda}.$$  

We define

$$g_m(\sigma) = \sum_{i=1}^{m} \frac{\lambda_i}{\sigma - \lambda_i},$$  

and, for a fixed index $\ell, 1 \leq \ell \leq m$,

$$\bar{g}_{m\ell}(\sigma) = \sum_{i=1}^{\ell} \frac{\lambda_i}{\sigma - \lambda_{\ell}} + \sum_{i=\ell+1}^{m} \frac{\lambda_i}{\sigma - \lambda_m}.$$  

We define $\sigma_{\text{HP}}$ to be the unique solution of the equation

$$g_m(\sigma) = \frac{mN\gamma}{2\beta},$$  

and $\bar{\sigma}_{\text{HP,}\ell}$ the unique solution of

$$\bar{g}_{m\ell}(\sigma) = \frac{mN\gamma}{2\beta},$$  

both on $(\lambda_m, \infty)$.

The uniqueness of $\sigma_{\text{HP}}$ and $\bar{\sigma}_{\text{HP,}\ell}$ follows from the monotonicity of $g_m(\sigma)$ and $\bar{g}_{m\ell}(\sigma)$, respectively, both of which decrease from $\infty$ to zero as $\sigma$ varies on $(\lambda_m, \infty)$. Since $g_m(\sigma_{\text{HP}}) = \bar{g}_{m\ell}(\bar{\sigma}_{\text{HP,}\ell})$ and $g_m(\sigma) \leq \bar{g}_{m\ell}(\bar{\sigma}_{\text{HP,}\ell})$ for all $\sigma > \lambda_m$, we obtain $\bar{g}_{m\ell}(\bar{\sigma}_{\text{HP,}\ell}) \leq \bar{g}_{m\ell}(\bar{\sigma}_{\text{HP,}\ell})$ from which we have $\sigma_{\text{HP}} \leq \bar{\sigma}_{\text{HP,}\ell}$ (since $\bar{g}_{m\ell}$ is decreasing). Hence $\bar{\sigma}_{\text{HP,}\ell}$ is an upper estimate for $\sigma_{\text{HP}}$. On the other hand, since, by Theorem 2.5, $g_m(\sigma_{\text{HP}}) \leq g_m(\lambda_{m+1})$, we have $\lambda_{m+1} \leq \sigma_{\text{HP}}$. In fact $\bar{\sigma}_{\text{HP,}\ell}$ is given explicitly by

$$\bar{\sigma}_{\text{HP,}\ell} = \frac{\lambda_m + \lambda_{\ell}}{2} + \frac{\beta}{N\gamma} \bar{\lambda}$$

$$+ \left( \left( \frac{\lambda_m - \lambda_{\ell}}{2} + \frac{\beta}{N\gamma} \bar{\lambda} \right)^2 - \frac{2\beta}{mN\gamma} (\lambda_m - \lambda_{\ell}) \sum_{i=1}^{\ell} \lambda_i \right)^{1/2}.$$  

Obviously, the case $\ell = m$ gives us the bound $\bar{\sigma}_{\text{HP,}m} = \sigma_{\text{PPW}}$, which is thus weaker than $\sigma_{\text{HP}}$. 


Next we set
\[
(h_m(\sigma)) = m \sum_{i=1}^{m} (\sigma - \lambda_i)^2 - \frac{2\beta}{N\gamma} \sum_{i=1}^{m} (\sigma - \lambda_i)\lambda_i
\]
and let \( \sigma_Y^{\pm} \) denote the two roots of the quadratic equation \( h_m(\sigma) = 0 \) (that \( h_m(\sigma) \) has two real roots follows from the fact that \( h_m(\lambda_{m+1}) \leq 0 \) and
\[
\lim_{\sigma \to \infty} h_m(\sigma) = \infty
\]
(see Prop. 6, p. 1802 of [20] for an alternative proof of this fact)). We have
\[
\sigma_Y^{\pm} = (1 + \frac{\beta}{N\gamma}) \bar{\lambda} \pm \left( \left( (1 + \frac{\beta}{N\gamma}) \bar{\lambda} \right)^2 - \left( 1 + \frac{2\beta}{N\gamma} \right) \frac{1}{m} \sum_{i=1}^{m} \lambda_i^2 \right)^{1/2},
\]
which can be rewritten in the form
\[
\sigma_Y^{\pm} = (1 + \frac{\beta}{N\gamma}) \bar{\lambda} \pm \left( \left( \frac{\beta}{N\gamma} \bar{\lambda} \right)^2 - \left( 1 + \frac{2\beta}{N\gamma} \right) \frac{1}{m} \sum_{i=1}^{m} (\lambda_i - \bar{\lambda})^2 \right)^{1/2}.
\]
Clearly,
\[
\sigma_Y^{+} \leq \bar{\sigma}_Y,
\]
where
\[
\bar{\sigma}_Y \equiv \left( 1 + \frac{2\beta}{N\gamma} \right) \bar{\lambda}.
\]
Since \( h_m(\lambda_{m+1}) \leq 0 \), we conclude that \( \sigma_Y^{-} \leq \lambda_{m+1} \leq \sigma_Y^{+} \). Also, since
\[
h_m(\lambda_m) = h_{m-1}(\lambda_m) \leq 0,
\]
we have \( \sigma_Y^{-} \leq \lambda_m \) (the quadratic \( h_m(\sigma) \) is negative between its roots \( \sigma_Y^{\pm} \)).
The \( \sigma_Y^{+} \) bound on \( \lambda_{m+1} \) gives
\[
\lambda_{m+1} \leq \bar{\sigma}_Y.
\]
Inequality (5.11) can be written in the form
\[
\frac{N\gamma}{2\beta} \leq \frac{\bar{\lambda}}{\lambda_{m+1} - \bar{\lambda}}.
\]
The function \( f(x) = \frac{x}{\lambda_{m+1} - x} \) is convex on \((0, \lambda_{m+1})\) since
\[
f''(x) = 2\lambda_{m+1}(\lambda_{m+1} - x)^{-3} > 0 \quad \text{for} \quad x < \lambda_{m+1}.
\]
Hence,
\[
f(\bar{\lambda}) = f \left( \frac{1}{m} \sum_{i=1}^{m} \lambda_i \right) \leq \frac{1}{m} \sum_{i=1}^{m} f(\lambda_i)
\]
and thus (from (5.12))

\[
\frac{mN\gamma}{2\beta} \leq \sum_{i=1}^{m} \frac{\lambda_i}{\lambda_{m+1} - \lambda_i},
\]

from which we conclude that Yang’s weaker bound implies the statement of the HP bound. An elementary proof of the fact that \( \tilde{\sigma}_Y \leq \sigma_{HP} \) will be given in the proof of Theorem 5.2 below.

Remark. These calculations show, as in the case of the Dirichlet Laplacian \[35\], that the generalized H.C. Yang inequality implies that of Hile–Protter, which in turn implies that of Payne–Pólya–Weinberger.

We now prove a lemma which we will need for the next theorem.

Lemma 5.1 (Reverse Chebyshev Inequality). Suppose \( \{a_i\}_{i=1}^{m} \) and \( \{b_i\}_{i=1}^{m} \) are two real sequences with \( \{a_i\} \) increasing and \( \{b_i\} \) decreasing. Then the following inequality holds:

\[
\sum_{i=1}^{m} a_i \sum_{i=1}^{m} b_i \geq m \sum_{i=1}^{m} a_i b_i.
\]

Proof. We use the simplified notation

\[
\sum a = \sum_{i=1}^{m} a_i, \quad \sum b = \sum_{i=1}^{m} b_i, \quad \sum ab = \sum_{i=1}^{m} a_i b_i.
\]

Starting with the fact that \((a_i - a_j)(b_i - b_j) \leq 0\) for \(i, j = 1, \ldots, m\), we sum over both indices to arrive at

\[
0 \geq \sum_{i} \sum_{j} (a_i - a_j)(b_i - b_j)
\]

\[
= \sum_{i} \sum_{j} (a_i b_i - a_i b_j - a_j b_i + a_j b_j)
\]

\[
= m \sum_{i} a_i b_i - \sum_{i} a_i \sum_{j} b_j - \sum_{j} a_j \sum_{i} b_i + m \sum_{j} a_j b_j
\]

\[
= 2 \left( m \sum ab - \sum a \sum b \right),
\]

from which the result is immediate.

Remark. A weighted version of this inequality can also be proved. See, for example, p. 43 of \[15\].

Theorem 5.2.

\[
\sigma_{Y}^{\pm} \leq \lambda_{m} \leq \lambda_{m+1} \leq \sigma_{Y}^{\pm} \leq \tilde{\sigma}_{Y} \leq \sigma_{HP} \leq \tilde{\sigma}_{HP}, \ell \leq \sigma_{PPW}.
\]
Proof. That $\tilde{\sigma}_Y \leq \lambda_m \leq \lambda_{m+1} \leq \sigma_Y^+$ and $\sigma_{\text{HP}} \leq \tilde{\sigma}_{\text{HP}},r$ have already been proved in the previous discussion. That $\sigma_Y^+ \leq \tilde{\sigma}_Y$ follows from the definitions of $\sigma_Y^+$ and $\tilde{\sigma}_Y$ (Equations (5.9) and (5.10)). That $\tilde{\sigma}_{\text{HP}},r \leq \sigma_{\text{PPW}}$ is an immediate consequence of (5.6). To complete the chain of inequalities we just need to prove that $\tilde{\sigma}_Y \leq \sigma_{\text{HP}}$. This is equivalent to showing that

$$g_m(\tilde{\sigma}_Y) \geq g_m(\sigma_{\text{HP}})$$

(since $g_m$ is decreasing). From (5.10), we have

$$\frac{N\gamma}{2\beta} = \frac{\tilde{\lambda}}{\tilde{\sigma}_Y - \tilde{\lambda}}.$$  

(5.17)

Since

$$g_m(\sigma_{\text{HP}}) = \frac{mN\gamma}{2\beta},$$

we simply need to prove that

$$\sum_{i=1}^{m} \frac{\lambda_i}{\tilde{\sigma}_Y - \lambda_i} \geq \frac{mN\gamma}{2\beta}.$$  

(5.18)

Using (5.17) and incorporating the definition of $\tilde{\lambda}$, this is equivalent to showing that

$$\sum_{i=1}^{m} \frac{\lambda_i}{\tilde{\sigma}_Y - \lambda_i} \geq \sum_{i=1}^{m} \frac{m \sum_{i=1}^{m} \lambda_i}{(\tilde{\sigma}_Y - \lambda_i)}.$$  

(5.19)

or

$$\sum_{i=1}^{m} \frac{\lambda_i}{\tilde{\sigma}_Y - \lambda_i} \sum_{i=1}^{m} (\tilde{\sigma}_Y - \lambda_i) \geq m \sum_{i=1}^{m} \lambda_i.$$  

(5.20)

The sequence $a_i = \frac{\lambda_i}{\tilde{\sigma}_Y - \lambda_i}$ is increasing, while $b_i = \tilde{\sigma}_Y - \lambda_i$ is decreasing. The result of the theorem then follows by applying Lemma 5.1.

Corollary 5.3.

$$\sigma_Y^+ \leq \sigma_{\text{HP}}.$$  

The result is an obvious consequence of Theorem 5.2. We argue below that this statement can be directly proven without recourse to the intermediate inequalities $\sigma_Y^+ \leq \tilde{\sigma}_Y$ and $\tilde{\sigma}_Y \leq \sigma_{\text{HP}}$. Since $g_m(\sigma)$ is decreasing, the statement would follow from the inequality $g_m(\sigma_Y^+) \geq g_m(\sigma_{\text{HP}})$. Since $g_m(\sigma_{\text{HP}}) = mN\gamma/(2\beta)$, the statement is equivalent to

$$g_m(\sigma_Y^+) \geq \frac{mN\gamma}{2\beta}.$$  

(5.21)
But \( h(\sigma_Y^+) = 0 \), i.e.,

\[
(5.22) \quad \sum_{i=1}^{m} (\sigma_Y^+ - \lambda_i)^2 = \frac{2\beta}{N\gamma} \sum_{i=1}^{m} (\sigma_Y^+ - \lambda_i) \lambda_i.
\]

Hence, eliminating \( 2\beta/(N\gamma) \), (5.21) is equivalent to

\[
(5.23) \quad \sum_{i=1}^{m} \frac{\lambda_i}{\sigma_Y^+ - \lambda_i} \geq \frac{m \sum_{i=1}^{m} (\sigma_Y^+ - \lambda_i) \lambda_i}{\sum_{i=1}^{m} (\sigma_Y^+ - \lambda_i)^2},
\]

or

\[
(5.24) \quad \sum_{i=1}^{m} \frac{\lambda_i}{\sigma_Y^+ - \lambda_i} \sum_{i=1}^{m} (\sigma_Y^+ - \lambda_i)^2 \geq m \sum_{i=1}^{m} (\sigma_Y^+ - \lambda_i) \lambda_i.
\]

The sequence \( a_i = \lambda_i / (\sigma_Y^+ - \lambda_i) \) is increasing, while \( b_i = (\sigma_Y^+ - \lambda_i)^2 \) is decreasing and hence statement (5.24) is true by Lemma 5.1, and the alternative proof of the corollary is complete. \( \square \)

References


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In a previous article we showed how one can see the classical result on existence and uniqueness of a fundamental set of factorial series solutions of a regular difference system as the limit, when \( q \) tends to 1, of analogous results for a \( q \)-difference system obtained by some kind of deformation. We extend here this property to singular-regular (or Fuchsian) systems.

Introduction

Il est bien connu qu’une équation aux \( q \)-différences (resp. aux différences de pas \( \varepsilon \)) dégénère en une équation différentielle lorsque \( q \to 1 \) (resp. \( \varepsilon \to 0 \)) et que ce phénomène se traduit au niveau des solutions. Cette confluence, étudiée de longue date, a été reprise récemment dans le cas des \( q \)-différences par J. Sauloy qui en propose une théorie précise et complète dans [7]. L’étude parallèle du cas différence n’est pas faite et semble plus difficile. C’est un autre type, probablement plus abordable, de confluence que nous envisageons ici, la dégénérescence d’une équation aux \( q \)-différences en une équation aux différences. Il s’agit dans les deux cas d’une équation fonctionnelle construite à partir d’une homographie, à deux points fixes dans le premier cas, un seul dans le second cas. Dans [1] nous abordions, dans l’esprit de [7], le cas régulier. Nous y montrions comment on peut voir le classique théorème d’existence et d’unicité d’un système fondamental de solutions développables en séries de factorielles convergentes et à valeur prescrite en +\( \infty \) comme limite, lorsque le paramètre réel \( q \) tend vers \( 1^+ \), de résultats analogues pour un système aux \( q \)-différences qui en est une déformation convenable. Nous éudions ici le cas d’un système aux différences singulier-régulier (ou fuchsien) au sens de [3] ou [6]. Nous utilisons en particulier une famille de “caractères” inspirée de celle qu’utilise J. Sauloy ([7]) dans le cas \( q \)-différence.

Indiquons le plan de l’article. Dans un premier paragraphe, nous rappelons ce qu’on entend par système aux différences fuchsiens puis indiquons comment le déformer en un système aux \( q \)-différences également fuchsien. Le deuxième paragraphe est consacré au cas d’un système à coefficients constants. On y définit en particulier une famille de caractères méromorphes.
dans $\mathbb{C}$. Le dernier paragraphe montre enfin comment modifier et utiliser les résultats de [1] pour ramener le cas fuchsien au cas constant.

1. Le cadre de l'étude

A l'homothétie $\sigma_q(x) = qx$ est associé l'opérateur aux q-différences $\sigma_q(f)(x) = f(qx)$. Si on remplace $\sigma_q$ par l'homographie $\sigma_{q,1}(x) = qx + 1$ puis que l'on fait tendre $q$ vers 1, on obtient la translation de pas 1, $\tau f(x) = f(x + 1)$.

L'homographie $\sigma_{q,1}$ a, comme $\sigma_q$, deux points fixes et une équation fonctionnelle faisant intervenir $\sigma_{q,1}$ peut être transformée en une équation aux q-différences par un changement homographique de la variable. L'un des deux points fixes de $\sigma_{q,1}$ est situé en $1/(1-q)$ et conflue, lorsque $q \to 1$, vers l'autre, situé en $\infty$.

On peut alors étudier les solutions d'un système aux différences linéaire

\[ (*) \quad X(x + 1) = A(x)X(x) \]

où $A(x)$ est une matrice carrée de dimension $\mu$ et $X$ un vecteur de dimension $\mu$, à partir de celles d'une déformation

\[ (*)_q \quad X(qx + 1) = A_q(x)X(x). \]

La forme de $A_q(x)$, vérifiant $\lim_{q \to 1} A_q(x) = A(x)$, est précisée au paragraphe 1.2, après un paragraphe de rappels concernant les systèmes aux différences singuliers réguliers (ou fuchsiens) en $+\infty$.

1.1. Système aux différences fuchsien en $+\infty$. La terminologie choisie est inspirée de celle du cas différentiel. Le parallélisme entre les deux théories est frappant si l'on utilise l'opérateur $(x-1) \Delta_1$ où

\[ \Delta_1 \quad f(x) = f(x) - f(x - 1) \]

et les séries de factorielles dont nous rappelons la définition.

On choisit une norme sur $\mathbb{C}^\mu$ et une norme compatible sur $gl(\mu, \mathbb{C})$, c'est-à-dire que l'on impose, pour $A \in gl(\mu, \mathbb{C})$ et $U \in \mathbb{C}^\mu$, la condition $\|AU\| \leq \|A\| \|U\|$.

Définition 1.1. Pour tout entier $s \geq 0$, on pose

\[ x^{-[s]} = \begin{cases} 
1 & \text{si } s \geq 1 \\
 \frac{1}{x(x + 1) \cdots (x + s - 1)} & \text{si } s = 0.
\end{cases} \]

Soit $A(x) = \sum_{s \geq 0} A_s x^{-[s]}$ où $A_s \in gl(\mu, \mathbb{C})$ (resp. $A_s \in \mathbb{C}^\mu$) une série de factorielles (formelle). Si $C > 0$ et $\lambda > 0$ sont fixés, on dit que la matrice (resp. le vecteur) $A(x)$ vérifie la condition $(C, \lambda)$ lorsque pour tout $s \geq 1$, on a $\|A_s\| \leq C \frac{\Gamma(\lambda + s - 1)}{\Gamma(\lambda)}$. 

La condition \((C,\lambda)\) assure la convergence (absolue) de la série \(A(x)\) dans le demi-plan \(\mathbb{R}x > \lambda\). Sa somme est alors une fonction holomorphe dans le demi-plan de convergence. Toute fonction holomorphe à l’infini admet un développement en série de factorielles convergente dans un demi-plan \(\mathbb{R}x \gg 0\), mais la réciproque est fausse.

**Définition 1.2.** Le système \((\ast)\) est dit “de première espèce” s’il s’écrit
\[
(x-1) \Delta X(x) = A(x)X(x).
\]
ôù \(A(x)\) vérifie une condition \((C,\lambda)\).

Le résultat suivant est classique ([6] ou [3]).

**Proposition 1.3.** Un système de première espèce admet un système fondamental de solutions de la forme
\[
X(x) = F(x)_{\mathbb{R}}
\]
où \(F \) est une matrice constante, \(F(x)_{\mathbb{R}} = \sum_{s \geq 0} F_s x^{-[s]}\) vérifie une condition \((\tilde{C},\tilde{\lambda})\) et \(F_0\) est inversible.

Comme dans le cas différentiel, un système \((\ast)\) peut avoir un système fondamental de solutions de la forme précédente sans être de première espèce. 

On dit alors que \((\ast)\) est *fuchsien* en \(+\infty*.

Notons \(gl(\mu,\mathcal{K}_f)\) l’anneau des matrices qui s’écrivent \(A(x) = A_1(x) + A_2(x)\) où \(A_2(x)\) vérifie une condition \((C,\lambda)\) et \(A_1(x)\) est un polynôme sans terme constant, à coefficients dans \(gl(\mu,\mathbb{C})\). On note \(GL(\mu,\mathcal{K}_f)\) le groupe des éléments inversibles de \(gl(\mu,\mathbb{C})\).

**Définition 1.4.** Soit \(A(x) \in gl(\mu,\mathcal{K}_f)\) et \(T(x) \in GL(\mu,\mathcal{K}_f)\). On définit
\[
A^T(x) = T(x-1)^{-1}(A(x)T(x) - (x-1) \Delta T(x)).
\]

Les systèmes de matrice \(A(x)\) et \(B(x)\) sont dits \(\mathcal{K}_f\)-équivalents s’il existe \(T(x) \in GL(\mu,\mathcal{K}_f)\) telle que \(B(x) = A^T(x)\).

Cette définition traduit le fait que \(X(x) = T(x)Y(x)\) est une solution du système \((\ast)\) de matrice \(A(x)\) si et seulement si \(Y(x)\) est une solution du système \((\ast)\) de matrice \(A^T(x)\).

On vérifie que, si \(T_1, T_2 \in GL(\mu,\mathcal{K}_f)\), \(A^{T_1}T_2(x) = (A^{T_1})^{T_2}(x)\) et que, si \(T \in GL(\mu,\mathbb{C})\), alors \(A^T(x) = T^{-1}A(x)T\).

Toujours de façon parallèle au cas différentiel, le résultat classique qui suit ([6]) montre qu’il suffit d’étudier le cas d’un système de première espèce.

**Proposition 1.5.** Le système \((\ast)\) est fuchsien en \(+\infty\) si et seulement s’il est \(\mathcal{K}_f\)-équivalent à un système de première espèce.

Le système fondamental de solutions indiqué dans la proposition 1.3 fait intervenir à côté des séries de factorielles les mêmes “caractères”, \(x^\lambda\) et \(\ln x\), que ceux que l’on utilise dans le cas différentiel. Pour un opérateur aux
q-différences J. Sauloy a remarqué qu’il était possible d’utiliser des fonctions \textit{méromorphes} dans \( \mathbb{C}^* \), donc sans monodromie. Nous reprenons cette idée pour les équations aux différences et indiquons au paragraphe 2 comment résoudre un système \((\star)\) de matrice \textit{constante} en utilisant des fonctions méromorphes dans \( \mathbb{C} \), définies à partir de la fonction \( \Gamma \) et qui sont limites lorsque \( q \to 1 \) de celles utilisées dans [7].

1.2. Choix de la \( q \)-déformation. Le cas \textit{régulier} correspond à un système de première espèce pour lequel \( A_0 = 0 \). Ce cas est étudié en [1] en supposant que \( q \) est un \textit{réel} \( > 1 \). Le choix de \( q \) réel est probablement technique mais c’est seulement sous cette hypothèse que nous avons obtenu les résultats que nous réutiliserons ici. C’est pourquoi dans toute la suite on suppose \( q > 1 \) et on pose \( p = 1/q \).

Guidé par le cas régulier, lui-même inspiré de la confluence de

\[
\delta_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}
\]

vers \( df/dx \) quand \( q \to 1 \), on déforme \( \textit{(*)} \) en un système de l’un des deux types suivants.

- Dans le plan des \( x \), on considère un système \((\star)_q\)

\[
p(x - 1) \frac{X(px - p) - X(x)}{(p - 1)x - p} = A_q(x)X(x)
\]

où \( A_q(x) \) est une série de \textit{factorielles mixtes}, c’est-à-dire une série

\[
A_q(x) = \sum_{s \geq 0} A_s(q)e^{(q)}_s(x)
\]

avec \( A_s(q) \in gl(\mu, \mathbb{C}) \), \( e^{(q)}_0(x) = 1 \) et, si \( s \geq 1 \),

\[
e^{(q)}_s(x) = \frac{1}{x(qx + 1)(q^2x + [2]_q) \cdots (q^{s-1}x + [s - 1]_q)}.
\]

Dans cette formule, on a utilisé la notation de Jackson

\[
[z]_q = \frac{q^z - 1}{q - 1}.
\]

Une condition suffisante de convergence est (voir [1]) l’existence de \( M > 0 \) et \( \lambda > 0 \) tels que pour tout \( s \geq 0 \), \( \|A_s(q)\| \leq M/e^{(q)}_s(\lambda) \). La série converge alors dans le domaine

\[
\left| x - \frac{1}{1-q} \right| > \lambda - \frac{1}{1-q}.
\]

Remarquons que l’homographie \( \sigma_{p,-p}(x) = px - p \) qui intervient dans \((\star)_q\) est l’inverse de \( \sigma_{q,1} \).
Dans le plan des $t$ défini par le changement de variable (homographique) $x = (t-1)/(q-1)$, ce qui revient à normaliser l’homographie $\sigma_{q,1}$ en $\sigma_q$, le système $(\star)_q$ se transcrit en

$$(\star)_q \quad (pt - 1)\delta_p Y(t) = \tilde{A}_q(t)Y(t)$$

si on a posé $Y(t) = X\left(\frac{t - 1}{q - 1}\right)$.

Cette fois $\tilde{A}_q(t)$ est une série de $q$-factorielles, c’est-à-dire une série

$$\tilde{A}_q(t) = \sum_{s \geq 0} \frac{\tilde{A}_s(q)}{(t; q)_s}$$

avec $(t; q)_0 = 1$ et, pour $s \geq 1$,

$$(t; q)_s = (1 - t)(1 - qt) \cdots (1 - q^{s-1}t).$$

Les séries de $q$-factorielles sont étudiées dans [1] où il est constaté qu’elles sont une autre façon d’écrire les séries convergentes à l’infini. La confluence s’obtient en faisant le changement de variable $t = q^u$ de sorte que $x = [u]_q$ et que l’application définissant le changement de variable du plan des $x$ vers celui des $u$ tend vers l’identité.

2. Système constant

On étudie dans ce paragraphe le cas du système

$$(\star) \quad (x-1) \nabla_1 X(x) = AX(x)$$

où $A \in gl(\mu, \mathbb{C})$, que l’on déforme, selon la première possibilité, en

$$(\star)_q \quad p(x-1)\frac{X(px - p) - X(x)}{(p-1)x - p} = A(q)X(x)$$

où $A(q) \in gl(\mu, \mathbb{C})$ sera précisée ci-dessous et vérifiera $\lim_{q \to 1} A(q) = A$.

On peut aussi écrire l’équation $(\star)_q$ sous la forme

$$(\star)_q \quad X(x) = \left( B(q) - \frac{1}{x} A(q) \right) X(qx + 1)$$

avec $B(q) = I_\mu - (q - 1)A(q)$ qui est une matrice inversible si $q$ est assez proche de 1.

On traitera successivement le cas où la matrice $A$ est semi-simple, le cas où elle possède un seul bloc de Jordan et enfin le cas général.
2.1. Cas semi-simple. Remarquons d’abord que, lorsque \( \mu = 1 \), l’équation (\( \star \)) est l’équation des caractères :

\[
(x-1) \Delta f(x) = \alpha f(x)
\]

où \( \alpha \in \mathbb{C} \). Le cas où \( \alpha \) est entier est étudié dans [1]. On la déforme en (\( \star \))\(_q\) avec \( A(q) = -[-\alpha]_q \). On est ainsi conduit à utiliser la déformation suivante d’une matrice semi-simple.

**Définition 2.1.** Soit \( A \in gl(\mu, \mathbb{C}) \) une matrice semi-simple. On note \( S \in GL(\mu, \mathbb{C}) \) et \( \alpha_1, \cdots, \alpha_\mu \) des nombres complexes non nécessairement distincts tels que \( A = S^{-1} \text{diag}(\alpha_1, \cdots, \alpha_\mu) S \).

Pour \( q \neq 1 \), soit \( S(q) \in GL(\mu, \mathbb{C}) \) telle que \( \lim_{q \to 1} S(q) = S \). On appelle système \( q \)-déformé du système (\( \star \)) le système (\( \star \))\(_q\) de matrice

\[
A(q) = S(q)^{-1} \text{diag}(-[-\alpha_1]_q, \cdots, -[-\alpha_\mu]_q) S(q).
\]

La matrice \( B(q) \) intervenant dans la forme (\( \star \))\(_q\) est alors

\[
B(q) = S(q)^{-1} \text{diag}(q^{-\alpha_1}, \cdots, q^{-\alpha_\mu}) S(q)
\]

qui est inversible pour tout \( q \) assez proche de 1.

Avant d’étudier le système (\( \star \))\(_q\), nous commençons par quelques rappels classiques ou tirés de [7].

Soit \( \alpha \in \mathbb{C} \). Une fonction \( z(x) \) est solution de l’équation

\[
\left(1 - \frac{\alpha}{x}\right) z(qx + 1) = q^\alpha z(x)
\]

si et seulement si la fonction \( g(t) = z\left(\frac{t-1}{q-1}\right) \) vérifie

(\( \diamond \)) \quad \( (t - q^\alpha)g(tq) = q^\alpha(t - 1) g(t) \).

Puisque \( \lim_{t \to 0} q^\alpha(t-1)/(t-q^\alpha) = 1 \), le point 0 est un point régulier pour l’équation (\( \diamond \)) qui admet la solution méromorphe sur \( \mathbb{C} \) :

\[
g_\alpha(t) = \frac{(pt;p)_\infty}{(p^{\alpha+1}t;p)_\infty}
\]

où \((\xi;p)_\infty = \prod_{k=0}^{\infty}(1 - p^k \xi)\) pour \( \xi \in \mathbb{C} \).

Rappelons que la fonction entière \((\xi;p)_\infty\) vérifie la relation fonctionnelle

\[
(g\xi;p)_\infty = (1 - q\xi)(\xi;p)_\infty
\]

et s’annule aux points \( \xi = q^n, n \in \mathbb{N} \).

Si \( \alpha \notin \mathbb{Z} \), la fonction \( g_\alpha(t) \) a des pôles simples aux points \( t = q^{n+\alpha}, n \in \mathbb{N}^* \) et des zéros simples aux points \( t = q^n, n \in \mathbb{N}^* \).

Si \( \alpha \) est un entier positif, \( g_\alpha \) est le polynôme \((1 - pt)(1 - p^2 t) \cdots (1 - p^\alpha t)\) et si \( \alpha \) est un entier négatif ou nul, \( g_\alpha(t) = 1/(t;q)_{-\alpha} \).
On peut aussi étudier \((\phi)\) en considérant le point \(\infty\) qui est singulier régulier puisque \(\lim_{t \to \infty} q^\alpha(t-1)/(t-q^\alpha) = q^\alpha \in \mathbb{C}^*\).

Suivant [7], la solution méromorphe dans \(\mathbb{C}^*\), obtenue “en partant de l’infini”, est \(g_\alpha(t)p(t)\) où ([7] p. 1060)

\[ p(t) = e_{q^{-\alpha}}\left(\frac{1}{t}\right)\frac{q^\alpha \Theta_q(q^{-\alpha} t)}{\Theta_q(t)} \]

où \(\Theta_q\) est la fonction de Jacobi définie par

\[ \Theta_q(t) = \sum_{n \in \mathbb{Z}} (-1)^n q^{a(n-1)} t^n \]

et où, pour \(c \in \mathbb{C}^*\), \(e_c(t) = \frac{\Theta_q(t)}{\Theta_q(c^{-1} t)}\).

La fonction de Jacobi vérifie \(\Theta_q\left(\frac{1}{t}\right) = -\frac{1}{t} \Theta_q(t)\) et donc

\[ p(t) = q^\alpha \Theta_q\left(\frac{1}{t}\right) \Theta_q(q^{-\alpha} t) \frac{\Theta_q(q^{-\alpha} t) \Theta_q\left(\frac{q^{-\alpha}}{t}\right)}{\Theta_q(t) \Theta_q\left(\frac{q^{-\alpha}}{t}\right)} = \frac{q^\alpha\left(-\frac{1}{t}\right)}{\left(-\frac{q^{-\alpha}}{t}\right)} = 1. \]

Autrement dit \(g_\alpha(t)\) est aussi la solution obtenue à partir du point \(\infty\).

En posant \(t = q^u\), et donc \(x = \lfloor u \rfloor q\), on a

\[ g_\alpha(q^u) = (1 - p)^{\alpha} \frac{\Gamma_p(1 + \alpha_j - u)}{\Gamma_p(1 - u)} \]

où \(\Gamma_p\) est la fonction gamma \(p\)-analogue de Jackson ([5] par exemple).

**Lemme 2.2.** Soit \(\alpha \in \mathbb{C}\), \(q\) un réel \(> 1\) et \(p = \frac{1}{q}\). On pose

\[ f_\alpha(x) = \frac{\Gamma(1 + \alpha - x)}{\Gamma(1 - x)} \]

et

\[ z_\alpha^{(q)}(x) = (1 - p)^{-\alpha} \frac{((1-p)x + p;p)_\infty}{(p^\alpha((1-p)x + p);p)_\infty}. \]

Ces fonctions sont méromorphes sur \(\mathbb{C}\) et vérifient

\[ f_\alpha(x) = \left(1 - \frac{\alpha}{x}\right) f_\alpha(x + 1), \]

\[ z_\alpha^{(q)}(x) = \left(q^{-\alpha} + \frac{-\alpha q}{x}\right) z_\alpha^{(q)}(qx + 1). \]

Uniformément sur tout compact de \(\mathbb{C}\setminus \{\text{pôles de } f_\alpha\}\), on a

\[ \lim_{q \to 1^+} z_\alpha^{(q)}(x) = f_\alpha(x). \]
Preuve. On sait ([5]) que lorsque \( p \to 1^- \), la fonction \( \Gamma_p \) tend vers la fonction \( \Gamma \), uniformément sur tout compact de \( \mathbb{C} \setminus -\mathbb{N} \). Or
\[
z_{\alpha}^{(q)}(x) = (1 - p)^{-\alpha} g_{\alpha}((q - 1)x + 1)
\]
et le facteur de normalisation est choisi pour que la limite existe. Les autres affirmations sont claires.

Remarquons que, si \( \alpha \notin \mathbb{Z} \), \( f_\alpha \) a des pôles simples aux points \( x \in \alpha + \mathbb{N}^* \) et des zéros aux points \( x \in \mathbb{N}^* \). Si \( \alpha \) est un entier \( > 0 \), \( f_\alpha(x) = x^{[-\alpha]} \).

Avec les notations de la définition 2.1 et en remarquant que, si \( X(x) \) est solution de \((*)_q\) et donc de \((*)_q\), la \( j \)-ième composante du vecteur \( Z(x) = S(q)X(x) \) vérifie l'équation
\[
\left(1 - \frac{1}{x[\alpha_j]_q}\right)z(qx + 1) = q^{\alpha_j}z(x)
\]
à laquelle on applique le lemme 2.2, on obtient le résultat suivant.

**Proposition 2.3.** Soit \( A \in \text{gl}(\mu, \mathbb{C}) \) une matrice semi-simple et \((*)_q\) un système \( q \)-déformé du système \((*)\) de matrice \( A \). Avec les notations du lemme 2.2, la matrice
\[
E(q)(x) = S(q)^{-1} \text{diag} \left( z_{\alpha_1}^{(q)}(x), \ldots, z_{\alpha_\mu}^{(q)}(x) \right)
\]
est méromorphe dans \( \mathbb{C} \), à pôles simples appartenant à \( \bigcup_{j=1}^{\mu} [\alpha_j + \mathbb{N}^*]_q \). Elle constitue une matrice fondamentale de solutions de \((*)_q\).

Lorsque \( q \to 1^+ \), \( E(q)(x) \) converge, uniformément sur tout compact de \( \mathbb{C} \setminus \bigcup_{j=1}^{\mu} (\alpha_j + \mathbb{N}^*) \), vers la matrice
\[
E(x) = S^{-1} \text{diag} \left( f_{\alpha_1}(x), \ldots, f_{\alpha_\mu}(x) \right)
\]
qui est une matrice fondamentale de solutions du système \((*)\), constituée de fonctions méromorphes dans \( \mathbb{C} \), à pôles simples appartenant à \( \bigcup_{j=1}^{\mu} (\alpha_j + \mathbb{N}^*) \). Si \( \alpha_j \in \mathbb{Z} \), la demi-ligne de pôles correspondante est absente si \( \alpha_j \geq 0 \) et remplacée par un ensemble fini de \(-\alpha_j\) points si \( \alpha_j < 0 \).

**2.2. Cas d'un seul bloc de Jordan.** On suppose maintenant que \( \mu \geq 2 \) et \( A = \alpha I_\mu + N_\mu \) où \( I_\mu \) est la matrice identité et \( N_\mu \) la matrice nilpotente :
\[
N_\mu = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}
\]
de sorte que pour tout entier \( k \), \( N_\mu^k = 0 \) si et seulement si \( k \geq \mu \).

Pour obtenir un système fondamental de solutions du système
\[
(x-1) = (\alpha I_\mu + N_\mu)X(x)
\]
on utilise des “logarithmes” associés à la famille \( f_\alpha(x) \) de caractères définie au paragraphe précédent. Il s’agit \([2]\) de la suite de fonctions définie pour \( k \in \mathbb{N} \) par

\[
\psi_k(x) = \frac{1}{k!} \frac{\Gamma^{(k)}}{\Gamma} (1 - x)
\]

où \( \Gamma^{(k)} \) désigne la dérivée \( k \)-ième de la fonction \( \Gamma \). Cette suite de fonctions vérifie \( \psi_0(x) = 1 \) et, pour tout \( k \in \mathbb{N}^* \),

\[
(x - 1) \Delta_1 \psi_k(x) = \psi_{k-1}(x).
\]

Une translation de la variable conduit au lemme suivant.

**Lemme 2.4.** Pour \( \alpha \in \mathbb{C} \) et \( k \in \mathbb{N} \), posons \( \psi_{k,\alpha}(x) = \psi_k(x - \alpha) \). Alors, pour tout \( k \geq 1 \),

\[
\psi_{k,\alpha}(x + 1) - \psi_{k,\alpha}(x) = \frac{1}{x - \alpha} \psi_{k-1,\alpha}(x + 1)
\]

et la matrice

\[
\mathcal{L}_{\mu,\alpha}(x) = f_{\alpha}(x) \begin{pmatrix}
1 & \psi_{1,\alpha}(x) & \psi_{2,\alpha}(x) & \cdots & \psi_{\mu-2,\alpha}(x) \\
0 & 1 & \psi_{1,\alpha}(x) & \cdots & \psi_{\mu-2,\alpha}(x) \\
: & : & : & \ddots & : \\
0 & 0 & \cdots & 1 & \psi_{1,\alpha}(x) \\
0 & 0 & \cdots & 0 & 1
\end{pmatrix}
\]

est un système fondamental de solutions du système

\[
(x - 1) \Delta_1 X(x) = (\alpha I_{\mu} + N_{\mu}) X(x).
\]

On déforme \( A \) en \( A(q) = [-\alpha]qI_{\mu} + \varphi(q)N_{\mu} \) où \( \lim_{q \to 1^+} \varphi(q) = 1 \). Le vecteur \( V(x) \) défini par l’égalité \( X(x) = z^{(q)}_{\alpha}(x)V(x) \) vérifie l’équation

\[
V(x) = \left( I_{\mu} - \frac{(q - 1)\varphi(q)(((q - 1)x + 1)}{q^{-\alpha}((q - 1)x + 1) - 1}N_{\mu} \right) V(qx + 1).
\]

En choisissant \( \varphi(q) = q^{-\alpha} \ln q / (q - 1) \) et en posant \( x = [u]_q \) et \( G(u) = V([u]_q) \), on obtient l’équation

\[
G(u) = \left( I_{\mu} - \frac{\ln q}{1 - q^{-\alpha}u}N_{\mu} \right) G(u + 1).
\]

Or, on a le résultat suivant \([2]\).

**Lemme 2.5.** La suite de fonctions définie pour \( k \in \mathbb{N} \) par

\[
L_k(u) = (-1)^k \frac{\ln q}{k!} \frac{d^k}{du^k} \frac{\Gamma_{\mu}(1 - u - \frac{\ln c}{\ln q})}{\Gamma_{\mu}(1 - u)} \bigg|_{u=1}
\]
vérifie $L_0(u) = 1$ et pour $k \in \mathbb{N}^*$,

$$L_k(u + 1) - L_k(u) = \frac{\ln q}{1 - q^{-u}} L_{k-1}(u + 1).$$

**Preuve.** La famille de fonctions définie pour $c$ voisin de 1 par

$$g_c(u) = \frac{\Gamma_p(1 - u - \frac{\ln c}{\ln q})}{\Gamma_p(1 - u)}$$

vérifie $g_1(u) = 1$ et

$$(cq^u - 1)g_c(u + 1) = (q^u - 1)g_c(u)$$

qui traduit la relation fonctionnelle de la fonction $\Gamma_p$. En dérivant $k$ fois cette relation par rapport à $c$ puis en donnant à $c$ la valeur 1, on obtient le résultat annoncé. □

On déduit de ce lemme le résultat suivant.

**Proposition 2.6.** Posons

$$u_q(x) = \frac{\ln(1 + (q - 1)x)}{\ln q} \quad \text{et} \quad \ell_{k,\alpha}^{(q)}(x) = L_k(u_q(x) - \alpha).$$

La matrice

$$\mathcal{L}_{\mu,\alpha}^{(q)}(x) = \begin{pmatrix} 1 & \ell_{1,\alpha}^{(q)}(x) & \ell_{2,\alpha}^{(q)}(x) & \cdots & \ell_{\mu-1,\alpha}^{(q)}(x) \\ 0 & 1 & \ell_{1,\alpha}^{(q)}(x) & \cdots & \ell_{\mu-2,\alpha}^{(q)}(x) \\ \vdots & \vdots & \ddots & \ddots & \cdots \\ 0 & 0 & \cdots & 1 & \ell_{1,\alpha}^{(q)}(x) \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

constitue un système fondamental de solutions du système

$$p(x - 1) \frac{X(px - p) - X(x)}{(p - 1)x - p} = \left(-[\alpha]_q I_{\mu} + q^{-\alpha} \frac{\ln q}{q - 1} N_{\mu}\right) X(x)$$

Ajoutons que pour tout compact $K$ de $\mathbb{C}$, il existe un réel $q_K > 1$ tel que si $x \in K$ et $1 < q < q_K$ alors $|(q - 1)x| < 1$. La fonction $u_q(x)$ est alors bien définie et holomorphe au voisinage de $K$ et $\mathcal{L}_{\mu,\alpha}^{(q)}(x)$ est mérormorphe dans $\mathbb{C}$, à pôles d’ordre $\leq \mu$ appartenant à $[\alpha + \mathbb{N}^*]_q$.

Pour étudier le comportement à la limite, on peut comparer cette famille de “logarithmes” à une autre famille, pour laquelle le passage à la limite est classique ($[2]$).

**Lemme 2.7.** La suite de fonctions définie pour $k \in \mathbb{N}$ par

$$\tilde{L}_k(u) = \frac{1}{k!} \left( \frac{d^k}{dc^k} \frac{\Gamma_p(1 - u + c)}{\Gamma_p(1 - u)} \right)_{c=0}$$
vérifie $\tilde{L}_0(u) = 1$ et pour $k \geq 1$,
\[
\tilde{L}_k(u + 1) - \tilde{L}_k(u) = \frac{1}{1 - q^{-u}} \sum_{j=1}^{k} \frac{(-1)^{j-1}}{j!} \ln^j q \tilde{L}_{k-j}(u + 1).
\]
De plus
\[
\lim_{q \to 1^+} \tilde{L}_k(u) = \psi_k(u)
\]
uniformément sur tout compact de $\mathbb{C} \setminus N^*$.

Le lemme suivant donne le lien entre les fonctions $L_k$ et $\tilde{L}_k$.

**Lemme 2.8.** On a $L_1(u) = \tilde{L}_1(u)$ et, pour $k \geq 2$,
\[
L_k(u) = \tilde{L}_k(u) + \sum_{j=1}^{k-1} a_{k,j} \ln^j q \tilde{L}_{k-j}(u)
\]
où les $a_{k,j}$ sont des nombres rationnels indépendants de $q$.

**Preuve.** On dérive $k$ fois par rapport à $\lambda$ la relation
\[
\Gamma_p \left( 1 - u - \frac{\ln \lambda}{\ln q} \right) = \Gamma_p (1 - u + c) \circ c(\lambda)
\]
où la fonction $c(\lambda) = -\ln \lambda / \ln q$ vérifie $c(1) = 0$ et, pour $k \geq 1$,
\[
c^{(k)}(1) = \frac{(-1)^{k}(k-1)!}{\ln q}.
\]
La relation entre les dérivées est du type
\[
\frac{d^k}{d\lambda^k} \Gamma_p \left( 1 - u - \frac{\ln \lambda}{\ln q} \right) = \sum_{j=1}^{k} \frac{d^j}{d\lambda^j} \Gamma_p (1 - u + c) P_j(c', c'', \ldots, c^{(k-j+1)})
\]
où $P_j$ est une somme à coefficients entiers positifs de termes de la forme $(c')^{n_1}(c'')^{n_2} \cdots (c^{(k-j+1)})^{n_{k-j+1}}$ vérifiant
\[
n_1 + n_2 + \cdots n_{k-j+1} = j \quad \text{et} \quad n_1 + 2n_2 + \cdots + (k-j+1)n_{k-j+1} = k.
\]
Le résultat s'obtient en donnant à $\lambda$ la valeur 1. \hfill $\square$

**Corollaire 2.9.** Lorsque $q \to 1^+$, la fonction $\ell_{k,\alpha}^{(q)}(x)$ converge vers $\psi_{k,\alpha}$, uniformément sur tout compact de $\mathbb{C} \setminus (\alpha + N^*)$.

La proposition suivante dresse le bilan du cas où il y a un seul bloc de Jordan.

**Proposition 2.10.** Soit $\alpha \in \mathbb{C}$ et $A = \alpha I_\mu + N_\mu$ avec $\mu \geq 2$. Posons
\[
A(q) = -[-\alpha]q I_\mu + q^{-\alpha} \frac{\ln q}{q-1} N_\mu.
\]
La matrice $L_{\mu, \alpha}^{(q)}(x)$, définie dans la proposition 2.6, est méromorphe dans $\mathbb{C}$, à pôles d'ordre $\leq \mu$ et appartenant à $[\alpha + \mathbb{N}^*]_q$. Elle constitue une matrice fondamentale de solutions du système

$$p(x - 1) \frac{X(px - p) - X(x)}{(p - 1)x - p} = A(q) X(x).$$

Lorsque $q \to 1^+$, la matrice $L_{\mu, \alpha}^{(q)}(x)$ converge, uniformément sur tout compact de $\mathbb{C} \setminus (\alpha + \mathbb{N}^*)$, vers la matrice $L_{\mu, \alpha}(x)$, définie dans le lemme 2.4. C'est une matrice fondamentale de solutions du système

$$(x - 1) \Delta L X(x) = (\alpha I_\mu + N_\mu) X(x)$$

constituée de fonctions méromorphes dans $\mathbb{C}$, à pôles d'ordre $\leq \mu$ et appartenant à $\alpha + \mathbb{N}^*$.

2.3. Cas général. La mise en commun de ces briques de base conduit au théorème suivant dans lequel on utilise les notations du lemme 2.4 et des propositions 2.3 et 2.6.

**Théorème 2.11.** Soit $A \in gl(\mu, \mathbb{C})$. Il existe $S \in GL(\mu, \mathbb{C})$, des complexes $\alpha_1, \ldots, \alpha_r$ et des entiers naturels non nuls $\mu_1, \ldots, \mu_r$ tels que $\sum_{j=1}^r \mu_j = \mu$ et $A = S^{-1} \text{diag}(A^{(1)}, \ldots, A^{(r)}) S$ où, pour $j = 1, \ldots, r$,

$$A^{(j)} = \begin{cases} \alpha_j & \text{si } \mu_j = 1, \\ \alpha_j I_{\mu_j} + N_{\mu_j} & \text{si } \mu_j \geq 2. \end{cases}$$

Soit $(S(q))_{q>1}$ une famille de matrices appartenant à $GL(\mu, \mathbb{C})$ vérifiant $\lim_{q \to 1^+} S(q) = S$. Posons

$$A(q) = S(q)^{-1} \text{diag}(A^{(1)}(q), \ldots, A^{(r)}(q)) S(q)$$

où, pour $j = 1, \ldots, r$,

$$A^{(j)}(q) = \begin{cases} -[\alpha_j]_q & \text{si } \mu_j = 1, \\ -[\alpha_j]_q I_{\mu_j} + q^{-\alpha_j} \frac{\ln q}{q - 1} N_{\mu_j} & \text{si } \mu_j \geq 2. \end{cases}$$

Alors la matrice

$$E^{(q)}_A(x) = S(q)^{-1} \text{diag}(E^{(1)}_q(x), \ldots, E^{(r)}_q(x)),$$

où $E^{(j)}_q(x) = z^{(q)}_{\alpha_j}(x)$ si $\mu_j = 1$ et $E^{(j)}_q(x) = L_{\mu_j, \alpha_j}^{(q)}(x)$ si $\mu_j \geq 2$, constitue une matrice fondamentale de solutions du système

$$p(x - 1) \frac{X(px - p) - X(x)}{(p - 1)x - p} = A(q) X(x).$$

La matrice $E^{(q)}_A(x)$ est formée de fonctions méromorphes dans $\mathbb{C}$ dont les pôles appartiennent à $\bigcup_{j=1}^r ([\alpha_j]_q + \mathbb{N}^*)$. Un pôle de la forme $k + [\alpha_j]_q$ est d'ordre $\leq \mu_j$. 
Lorsque \( q \rightarrow 1^+ \), la matrice \( \mathcal{E}_A^{(q)}(x) \) converge, uniformément sur tout compact de \( \mathbb{C} \setminus \bigcup_{j=1}^{\mu} (\alpha_j + N^*) \), vers la matrice

\[
\mathcal{E}_A(x) = S^{-1} \text{diag} (\mathcal{E}^{(1)}(x), \ldots, \mathcal{E}^{(r)}(x))
\]

où \( \mathcal{E}^{(j)} = f_{\alpha_j}(x) \) si \( \mu_j = 1 \) et \( \mathcal{E}^{(j)} = L_{\mu_j,\alpha_j}(x) \) si \( \mu_j \geq 2 \). La convergence est uniforme sur tout compact de \( \mathbb{C} \setminus \bigcup_{j=1}^{r} (\alpha_j + N^*) \). Si \( \alpha_j \in \mathbb{Z} \) et \( \mu_j = 1 \), la ligne de pôles correspondante est absente si \( \alpha_j \geq 0 \) et remplacée par un ensemble fini de \( -\alpha_j \) points si \( \alpha_j < 0 \). La matrice \( \mathcal{E}_A(x) \) est une matrice fondamentale de solutions du système

\[
(x-1) \Delta X(x) = AX(x)
\]

constituée de fonctions méromorphes dans \( \mathbb{C} \) dont les pôles appartiennent à \( \bigcup_{j=1}^{r} (\alpha_j + N^*) \). Un pôle de la forme \( k + \alpha_j \) est d’ordre \( \leq \mu_j \).

3. Système de première espèce

On établira au paragraphe 3.1 qu’un système \((\star)\) de première espèce de matrice \( A(x) \) est \( K_f \) – équivalent au système de matrice constante \( A_0 = A(\infty) \) lorsque celle-ci est non résonnante. Nous rappelons la définition de ce mot et le résultat, établi par exemple dans [3], qui permet, par une transformation polynomiale en \( 1/x \), de regrouper toutes les valeurs propres différant d’un entier en une seule valeur propre multiple.

Définition 3.1. Un système \((x-1) \Delta X(x) = A(x)X(x)\), dont la matrice \( A(x) = \sum_{s \geq 0} A_s x^{-[s]} \) est une série de factorielles convergente, est dit non résonnant si les différences de deux valeurs propres distinctes de la matrice \( A_0 \) ne sont pas entièreres.

Proposition 3.2. Appelons \( \alpha_1, \ldots, \alpha_r \) les valeurs propres distinctes de la matrice \( A_0 \). Il existe une transformation \( T(x) = T_0 + (1/x) T_1 \), où \( T_0 \) et \( T_1 \) sont des matrices constantes, telle que, si \( \bar{A}(x) = A^T(x) \), les valeurs propres de \( \bar{A}_0 \) sont \( \alpha_1 + 1, \alpha_2, \ldots, \alpha_r \).

Nous supposons donc dans la suite que le système \((\star)\) est de première espèce et non résonnant. Remarquons qu’alors, si \( A_0(q) \) est la déformation de \( A_0 \) définie dans le théorème 2.11 et si \( q \) est assez proche de 1, le quotient de deux valeurs propres distinctes de la matrice

\[
B_0(q) = I_{\mu} - (q - 1) A_0(q)
\]

n’appartient pas à \( q^\mathbb{Z} \).

Ce dernier paragraphe est composé de deux parties. Dans la première nous établissons que tout système non résonnant est équivalent au système constant de matrice \( A(\infty) \). La deuxième partie traite le même problème pour le système \( q \)-déformé et donne un théorème de confluence.
3.1. Solution canonique dans le cas non résonnant. On établit tout d’abord la proposition suivante qui généralise celle établie dans [1] dans le cas régulier.

**Proposition 3.3.** Soit (⋆) un système non résonnant vérifiant la condition \((C, \lambda)\). On suppose que 0 est valeur propre de \(A_0\). Pour tout vecteur \(U_0\) appartenant au noyau de \(A_0\), le système (⋆) admet une unique solution formelle \(X(x) = U_0 + \sum_{s \geq 1} X_s x^{−[s]}\). De plus, si \(b = \sup_{s \geq 1} \| (sI_\mu + A_0)^{−1} \|\), \(X(x)\) vérifie la condition \((bC\|U_0\|, \lambda + bC)\). En particulier \(X(x)\) converge pour \(\Re x > \lambda + bC\).

**Preuve.** Si \(X(x) = U_0 + \sum_{s \geq 1} X_s x^{−[s]}\) avec \(X_s \in \mathbb{C}^m\), alors
\[
(x−1) \Delta x X(x) = −\sum_{s \geq 1} sX_s x^{−[s]}. 
\]
D’autre part
\[
A(x)X(x) = A_0U_0 + \sum_{s \geq 1} \left( A_0X_s + A_sU_0 + \sum_{(j,\ell,k) \in J_s} c_{j,\ell}^{(k)} A_jX_\ell \right) x^{−[s]}.
\]
Puisque \(A_0U_0 = 0\), le problème formel équivaut à la liste de relations, pour \(s \geq 1\):
\[
(sI_\mu + A_0)X_s + A_sU_0 + \sum_{(j,\ell,k) \in J_s} c_{j,\ell}^{(k)} A_jX_\ell = 0
\]
où \(J_1 = \emptyset\) et pour \(s \geq 2\):
\[
J_s = \{j + \ell + k = s, j \geq 1, \ell \geq 1, k \geq 0\}
\]
et
\[
c_{j,\ell}^{(k)} = \frac{(j + k − 1)! (\ell + k − 1)!}{k! (j − 1)! (\ell − 1)!}.
\]
Ces relations définissent de manière unique les \(X_s\) par récurrence puisque, pour tout \(s \geq 1\), la matrice \(sI_\mu + A_0\) est inversible.

On pose \(u_0 = \|U_0\|\) et pour \(s \geq 1\), \(a_s = \|A_s\|\) et \(\xi_s = \|X_s\|\). Par hypothèse la série \(\sum_{s \geq 1} a_s x^{−[s]}\) converge pour \(\Re x > \lambda\). Notons \(a(x)\) sa somme.

La suite \(\|(sI_\mu + A_0)^{−1}\|\) tend vers 0 quand \(s \to \infty\) et donc \(b\) est fini. Le fait que les \(c_{j,\ell}^{(k)}\) sont positifs permet d’obtenir la suite d’inégalités valables pour \(s \geq 1\),
\[
\xi_s \leq b \left( a_s u_0 + \sum_{(j,\ell,k) \in J_s} c_{j,\ell}^{(k)} a_j \xi_\ell \right).
\]
Par récurrence on définit \(\xi_1 = ba_1 u_0\) et, pour \(s \geq 2\),
\[
\xi_s = b \left( a_s u_0 + \sum_{(j,\ell,k) \in J_s} c_{j,\ell}^{(k)} a_j \xi_\ell \right).
\]
de sorte que \(0 \leq \xi_s \leq \tilde{\xi}_s\) pour \(s \geq 1\). Notons \(\chi(x)\) la somme (au moins formelle) de la série \(\sum_{s \geq 1} \xi_s x^{-[s]}\). Elle vérifie l’équation

\[
\chi(x) = b \sum_{s \geq 1} \left( a_s u_0 + \sum_{(j,\ell,k) \in J_s} c_{j,\ell}^{(k)} a_j \xi_{\ell} \right) x^{-[s]} = b \left( u_0 a(x) + a(x) \chi(x) \right)
\]

ou encore

\[
\chi(x) = \frac{bu_0 a(x)}{1 - ba(x)} = -u_0 \left( 1 - \frac{1}{1 - ba(x)} \right).
\]

La proposition 2.1 de [1] fournit alors le résultat. □

**Théorème 3.4.** Soit \((\ast)\) un système non résonnant. Il existe une unique transformation formelle tangente à l’identité

\[
F(x) = I_\mu + \sum_{s \geq 1} F_s x^{-[s]}
\]

telle que \(A^F(x) = A_0\). De plus \(F(x)\) vérifie une condition \((C', \lambda')\) et converge donc dans un demi-plan \(\Re x \gg 0\).

**Preuve.** Établissons d’abord l’existence d’une unique série formelle \(F(x) = I_\mu + \sum_{s \geq 1} F_s x^{-[s]}\) vérifiant l’égalité

\[(\ast) \quad A(x)F(x) - (x-1) \Delta_1 F(x) = F(x-1)A_0.\]

D’une part,

\[
(x-1) \Delta_1 F(x) = - \sum_{s \geq 1} sF_s x^{-[s]}
\]

d’autre part, en utilisant la classique formule de translation, on a

\[
F(x-1) = I_\mu + \sum_{s \geq 1} \left( F_s + (s-1)! \sum_{k=1}^{s-1} \frac{F_k}{(k-1)!} \right) x^{-[s]}.
\]

Pour \(M, N \in gl(\mu, \mathbb{C})\), notons \(\Phi_{M,N}\) l’endomorphisme défini sur \(gl(\mu, \mathbb{C})\) par

\[
\Phi_{M,N}(U) = MU - UN.\]

L’équation \((\ast)\) équivaut alors à la liste de relations

\[
\Phi_{A_0 + I_\mu, A_0}(F_1) = -A_1
\]

e et pour \(s \geq 2,

\[
\Phi_{A_0 + sI_\mu, A_0}(F_s) = -A_s + (s-1)! \sum_{k=1}^{s-1} \frac{F_k A_0}{(k-1)!} - \sum_{(j,\ell,k) \in J_s} c_{j,\ell}^{(k)} A_j F_\ell.
\]

Dans le cas non résonnant, pour \(s \geq 1\), les matrices \(A_0 + sI_\mu\) et \(A_0\) n’ont pas de valeur propre commune et \(\Phi_{A_0 + sI_\mu, A_0}\) est un isomorphisme. On en déduit par récurrence l’existence et l’unicité de la suite \((F_s)\).
Pour établir la convergence de cette série de factorielles, on interprète (\(\ast\)) comme un système de dimension \(\mu^2\) auquel on peut appliquer la proposition 3.3. Pour cela on remarque que (\(\ast\)) s’écrit aussi

\[
(x-1) \Delta \frac{F(x)}{x-1} = (A(x)F(x) - F(x)A_0) \left( I_{\mu} - \frac{A_0}{x-1} \right)^{-1} \\
= \left( A_0F(x) - F(x)A_0 + \sum_{s \geq 1} A_s F(x)x^{-s} \right) \\
\times \left( I_{\mu} + \sum_{s \geq 1} A_0(A_0 + I_\mu) \cdots (A_0 + (s-1)I_\mu)x^{-s} \right) \\
= \sum_{s \geq 0} A_s(F(x))x^{-s}
\]

où \(A_s\) est la suite d’opérateurs linéaires définis sur l’espace vectoriel \(gl(\mu, C)\) par \(A_0(U) = A_0U - UA_0, A_1(U) = A_1U + (A_0U - UA_0)A_0\) et pour \(s \geq 2,\)

\[
A_s(U) = A_sU + (A_0U - UA_0)B_s + \sum_{(j,\ell,k) \in J_s} c_{j,\ell,k}^{(k)} A_jUB_\ell
\]

où on a posé, pour \(s \geq 1, B_s = A_0(A_0 + I_\mu) \cdots (A_0 + (s-1)I_\mu)\).

L’opérateur \(A_0\) admet 0 pour valeur propre et \(U = I_\mu\) est un vecteur propre pour cette valeur propre. L’hypothèse de non résonance implique qu’aucun entier non nul n’est valeur propre de \(A_0\).

D’autre part, si \(A(x)\) vérifie la condition \((C, \lambda)\) et si \(a_0 = \|A_0\|\), la norme de l’opérateur \(A_s\) se majore par

\[
K_s = C \frac{\Gamma(\lambda + s - 1)}{\Gamma(\lambda)} + 2a_0 \frac{\Gamma(a_0 + s)}{\Gamma(a_0)} + C \sum_{(j,\ell,k) \in J_s} c_{j,\ell,k}^{(k)} \frac{\Gamma(a_0 + j)}{\Gamma(a_0)} \frac{\Gamma(\lambda + \ell - 1)}{\Gamma(\lambda)}.
\]

En exprimant que les séries de factorielles obtenues en développant les deux fonctions

\[
1 - \frac{a_0 - \lambda + 1}{x - \lambda} \quad \text{et} \quad 1 + \frac{a_0 - \lambda + 1}{x - a_0 - 1}
\]

sont inverses l’une de l’autre, on obtient la formule ([1] p. 343)

\[
(a_0 - \lambda + 1) \sum_{(j,\ell,k) \in J_s} c_{j,\ell,k}^{(k)} \frac{\Gamma(a_0 + j)}{\Gamma(a_0 + 1)} \frac{\Gamma(\lambda + \ell - 1)}{\Gamma(\lambda)} = \frac{\Gamma(a_0 + s)}{\Gamma(a_0 + 1)} - \frac{\Gamma(\lambda + s - 1)}{\Gamma(\lambda)}.
\]

On en déduit :

\[
K_s = C \frac{1 - \lambda}{a_0 + 1 - \lambda} \frac{\Gamma(\lambda + s - 1)}{\Gamma(\lambda)} + \left( 2a_0 + \frac{C}{a_0 + 1 - \lambda} \right) \frac{\Gamma(a_0 + s)}{\Gamma(a_0)}.
\]

En posant \(\tilde{\lambda} = \max(\lambda, a_0 + 1)\), on prouve l’existence d’une constante \(\tilde{C} > 0\) telle que \(K_s \leq \tilde{C} \Gamma(\lambda + s - 1)/\Gamma(\lambda)\), ce qui permet d’appliquer la proposition 3.3. \(\square\)
Le système $(\star)$ admet donc un système fondamental de solutions de la forme $F(x)\mathcal{E}_{A_0}(x)$ où $F(x)$ est l’unique solution de $AF(x) = A_0$ telle que $F(\infty) = I_\mu$ et $\mathcal{E}_{A_0}(x)$, défini dans le théorème 2.11, dépend de la forme normale de Jordan $J_0$ de $A_0$ mais aussi du choix d’une matrice $S$ conjuguant $A_0$ à $J_0$. La forme normale $J_0$ est unique lorsqu’on a fixé l’ordre de ses blocs et la matrice $\mathcal{E}_{J_0}(x)$ est alors clairement définie. On va voir que $(\star)$ admet un système fondamental de solutions indépendant du choix de la matrice $S$. Cette propriété repose sur les deux lemmes suivants.

**Lemme 3.5.** Soit $S \in \text{GL}(\mu, \mathbb{C})$. La matrice $F(x)S^{-1}\mathcal{E}_{J_0}(x)$ est un système fondamental de solutions de $(\star)$ si et seulement si $J_0S = SA_0$.

**Preuve.** La définition de $\mathcal{E}_{A_0}(x)$ montre que $\mathcal{E}_{A_0}(x) = S^{-1}\mathcal{E}_{J_0}(x)$ si $S \in \text{GL}(\mu, \mathbb{C})$ a été choisie telle que $J_0S = SA_0$. Inversement si $S \in \text{GL}(\mu, \mathbb{C})$ est telle que $F(x)S^{-1}\mathcal{E}_{J_0}(x)$ soit une solution de $(\star)$, alors on doit avoir $A^FS^{-1} = J_0$. Puisque $A^F = A_0$, on en déduit $A_0S^{-1} = J_0$ ou encore $SA_0S^{-1} = J_0$. □

**Lemme 3.6.** Toute matrice constante qui commute avec $J_0$ commute avec $\mathcal{E}_{J_0}(x)$.

**Preuve.** Reprenons les notations du théorème 2.11 et supposons que

$$J_0 = \text{diag}(A^{(1)}, \ldots, A^{(r)})$$

où chaque $A^{(j)}$ est un bloc de Jordan élémentaire de dimension $\mu_j$, de la forme $A^{(j)} = \alpha_j I_{\mu_j}$ si $\mu_j = 1$ et $A^{(j)} = \alpha_j I_{\mu_j} + N_{\mu_j}$ si $\mu_j \geq 2$. Partitionnons toute matrice constante $S$ selon cette décomposition en blocs élémentaires : $S = (S_{j,h})_{1 \leq j, h \leq r}$. On sait que $S$ commute avec $J_0$ si et seulement si chaque bloc vérifie :

- $S_{j,h} = 0$ si $\alpha_j \neq \alpha_h$
- $N_{\mu_j}S_{j,h} = S_{j,h}N_{\mu_j}$ si $\alpha_j = \alpha_h$.

La matrice $\mathcal{E}_{J_0}(x)$ a la même structure en blocs diagonaux que $J_0$ et le bloc d’indice $j$ est

$$\mathcal{E}^{(j)}(x) = f_{\alpha_j}(x)\left(I_{\mu_j} + \sum_{k=1}^{\mu_j-1} \psi_{k,\alpha_j}(x)N_{\mu_j}^k\right).$$

La condition de commutation de $S$ et de $\mathcal{E}_{J_0}(x)$ s’écrit

$$S_{j,h}\mathcal{E}^{(h)}(x) = \mathcal{E}^{(j)}(x)S_{j,h}$$

pour $j, h = 1, \ldots, r$. Ces relations sont vérifiées pour les indices tels que $\alpha_j \neq \alpha_h$ puisqu’alors $S_{j,h} = 0$. Lorsque $\alpha_j = \alpha_h = \alpha$, la condition s’écrit

$$S_{jh} + \sum_{k=1}^{\mu_h-1} \psi_{k,\alpha}(x)S_{jh}N_{\mu_h}^k = S_{jh} + \sum_{k=1}^{\mu_j-1} \psi_{k,\alpha}(x)N_{\mu_j}^kS_{jh}.$$
ou encore en remarquant que $N_{\mu_i}^k = 0$ pour $k \geq \mu_i$ et $N_{\mu_j}^k = 0$ pour $k \geq \mu_j$,
\[
\sum_{k \geq 1} \psi_{k,\alpha}(x) S_{jh} N_{\mu_i}^k = \sum_{k \geq 1} \psi_{k,\alpha}(x) N_{\mu_j}^k S_{jh}
\]
edgalité qui est assurée par les relations $N_{\mu_j} S_{jh} = S_{jh} N_{\mu_i}$.

**Théorème 3.7.** Soit $(\ast)$ un système de première espèce et $A_0 = A(\infty)$. Soit $F(x)$ la solution de $A^F = A_0$ donnée par le théorème 3.4. Soit $J_0$ une forme de Jordan de $A_0$ et $\mathcal{E}_{J_0}(x)$ la matrice qui lui est associée dans le théorème 2.11. La matrice $F(x) S^{-1} \mathcal{E}_{J_0}(x) S$ est une matrice fondamentale de solutions de $(\ast)$, indépendante du choix de $S \in \text{GL}(\mu, \mathbb{C})$ vérifiant $J_0 S = S A_0$.

**Preuve.** Le lemme 3.5 et le fait qu’en multipliant à droite une matrice de solutions par une matrice constante on obtient une matrice de solutions montrent que si $S_1$ et $S_2$ vérifient la condition indiquée, $F(x) S_i^{-1} \mathcal{E}_{J_0}(x) S_i$ ($i = 1, 2$) est une matrice fondamentale de solutions de $(\ast)$. Puisque la matrice $S = S_1 S_2^{-1}$ commute avec $J_0$, le lemme 3.6 permet d’écrire
\[
S_1^{-1} \mathcal{E}_{J_0}(x) S_1 = S_1^{-1} \mathcal{E}_{J_0}(x) S_1 S_2^{-1} S_2 = S_1^{-1} S_1 S_2^{-1} \mathcal{E}_{J_0}(x) S_2 = S_2^{-1} \mathcal{E}_{J_0}(x) S_2
\]
et le résultat s’en déduit par multiplication à gauche par $F(x)$.

**Définition 3.8.** On appelle *solution canonique* du système $(\ast)$ et on note $\mathcal{X}_{\text{can}}(x)$ la matrice fondamentale de solutions décrite dans le théorème 3.7.

**3.2. Système q-déformé et confluence.** On indique maintenant comment choisir un système q-déformé d’un système $(\ast)$ non résonnant. Cette étude est faite dans le plan de la variable $t = (q - 1)x + 1$ où le système aux $q$-différences obtenu est du type étudié dans [7]. Nous aurons cependant besoin de reprendre en partie les résultats classiques de façon à pouvoir traiter la confluence à l’aide des théorèmes de [1].

On déforme le système $(\ast)$ en un système
\[ (\ast)_q \quad (pt - 1) \delta_q Y(t) = A_q(t) Y(t) \]
où $A_q(t) = \sum_{s \geq 0} \frac{A_s(q)}{(t; q)_s}$ est une série de $q$-factorielles convergente.

L’équation $(\ast)_q$ peut aussi s’écrire
\[ (\bar{\ast})_q \quad Y(t) = B_q(t) Y(qt) \]
où
\[ B_q(t) = I_\mu + \frac{(1 - q)t}{t - 1} A_q(qt). \]
Puisque la somme d’une série de $q$-factorielles convergente est holomorphe à l’infini, si la matrice $B_q(\infty) = I_\mu + (1 - q) A_0(q)$ est inversible, le système $(\ast)_q$ est *fuchsien* à l’infini au sens de [7]. Il est *non résonnant*, toujours
au sens de [7], si le quotient de deux valeurs propres distinctes de $B_0(q) = I_\mu - (q - 1)A_0(q)$ n’appartient pas à $q\mathbb{Z}$. Dans ce cas, on peut montrer ([7]) qu’il existe une unique matrice $F_q(t) \in \text{GL}(\mu, \mathbb{C}[\frac{1}{q}])$ telle que $F_q(\infty) = I_\mu$ et $A_q^{F_q}(t) = A_0(q)$ où $A_q^{F_q}(t) = F_q(pt)^{-1}(A_q(t)F_q(t) - (pt - 1)pF_q(t))$ est la matrice du système obtenu à partir de $(\star)_q$ par le changement de fonction inconnue $Y(t) = F_q(t)Z(t)$. De plus la série $F_q(t)$ converge. Cela permet de décrire un système fondamental de solutions de $(\tilde{\star})_q$ pour $q$ fixé, mais pour obtenir des propriétés de confluence, il faut préciser la dépendance en $q$ des coefficients $A_s(q)$.

Supposons que la matrice $A(x) = \sum_{s \geq 0} A_s x^{-[s]}$ du système $(\star)$ à déformer vérifie la condition $(C, \lambda)$. Pour $A_0(q)$, on reprend les notations et hypothèses faites dans le théorème 2.11 en remplaçant $A$ par $A_0$. Si le système $(\star)$ est non résonnant tout système $(\ast)_q$ dont la matrice $A_q(t)$ a pour terme constant $A_0(q)$ est alors non résonnant si $q$ est assez proche de 1. On suppose ensuite que les coefficients $A_s(q)$, $s \geq 1$, vérifient les hypothèses suivantes (voir [1]) :

1) il existe $q_0 > 1$ tel que si $1 < q < q_0$, alors
$$\|A_s(q)\| \leq (q^C - 1) q^{s + \lambda - 1} |(q^\lambda; q)_{s-1}|,$$

2) $\lim_{q \to 1+} (1 - q)^{-s} A_s(q) = A_s$.

On suppose $q_0$ assez petit pour que $B_0(q)$ soit non résonnante pour $1 < q < q_0$ et on résume toutes ces hypothèses en disant que le système $(\tilde{\star})_q$ est obtenu par $q$-déformation du système $(\star)$. On suit une démarche analogue à celle du paragraphe 3.1 et on commence par établir le $q$-analogue suivant de la proposition 3.3, sous des hypothèses restrictives mais suffisantes pour notre étude.

**Proposition 3.9.** Soit $(\tilde{\star})_q$ un système obtenu par $q$-déformation d’un système $(\star)$ non résonnant dont la matrice $A_0$ admet 0 pour valeur propre. On suppose qu’il existe un vecteur $U_0$, indépendant de $q$, appartenant au noyau de $A_0(q)$ pour tout $q$ assez proche de 1. Le système $(\tilde{\star})_q$ admet alors, pour $q$ assez proche de 1, une unique solution formelle

$$Y(t) = U_0 + \sum_{s \geq 1} \frac{Y_s(q)}{(t; q)_s}.$$

De plus il existe $C' > 0$ et $q_0 > 1$ tels que pour tout $s \geq 1$ et tout $q$ tel que $1 < q \leq q_0$, on ait

$$\|Y_s(q)\| \leq \|U_0\| (q^{C'} - 1) q^{s + \lambda + C' - 1} |(q^\lambda + C'; q)_{s-1}|.$$

**Preuve.** On recopie celle de la proposition 3.3, en utilisant la série majorante

$$\chi(t) = -u_0 \left(1 - \frac{1}{1 - b(q)a_q(t)}\right)$$
où \( u_0 = \|U_0\|, a_q(t) = \sum_{s \geq 1} \|A_s(q)\|/(t; q)_s \) et \( b(q) \) majorent la norme de toutes les matrices \( ([s] q I_\mu + A_0(q))^{-1} \). Pour estimer \( b(q) \), on remarque que les valeurs propres de \( ([s] q I_\mu + A_0(q))^{-1} \) sont de la forme \((q-1)/(q^b - q^{-\alpha})\) où \( \alpha \) est une valeur propre de \( A_0 \), et peuvent se majorer, pour tout \( q > 1 \), par 1 si \( \Re \alpha > 0 \), par \((q-1)/(q^b - 1)\) où \( b = d(\Re \alpha, -N^*) \) si \( \Re \alpha \notin -N^* \) et par la même expression avec \( b < |\epsilon| \) et pour \( q \leq q_0 \) si \( \alpha = -s_0 + i\epsilon \) où \( s_0 \in N^* \). On peut ensuite majorer la norme du bloc de taille \( \nu \) correspondant à \( \alpha : (q-1)((q^b - q^{-\alpha})I_\nu + q^{-\alpha} \ln q N_\nu)^{-1} \) par

\[
\frac{q - 1}{q^b - 1} \frac{1 - q^{-\nu \Re \alpha}}{1 - q^{-\Re \alpha}}
\]

si \( \Re \alpha \neq 0 \) et par \( \nu(q-1)/(q^b - 1)\) si \( \Re \alpha = 0 \). Ces estimations et l’hypothèse faite sur \( a_q(t) \) permettent d’utiliser la proposition 4.5 de [1] pour conclure.

Donnons maintenant l’analogue de la proposition 3.4 en indiquant les modifications à apporter à sa preuve.

**Proposition 3.10.** Soit \((*)_q\) un système obtenu par \( q \)-déformation d’un système \((*)\) non résonnant. L’unique transformation tangente à l’identité \( F_q(t) \) telle que \( \tilde{F}_q^q(t) = A_0(q) \) admet un développement en série de \( q \)-factorielles \( F_q(t) = I_\mu + \sum_{s \geq 1} \frac{F_s(q)}{(t;q)_s} \) dont les coefficients vérifient les deux propriétés :

1) il existe \( q_0 > 1 \), \( C', \lambda' > 0 \) tels que pour tout \( q \) avec \( 1 < q < q_0 \) et tout \( s \geq 1 \),

\[
\|F_s(q)\| \leq (q^{C'} - 1) q^{s+\lambda'-1} |(q^\lambda'; q)_{s-1}|,
\]

2) \( \lim_{q \to 1^+} (1-q)^{-s} F_s(q) = F_s \) où \( F(x) = I_\mu + \sum_{s \geq 1} F_s x^{-[s]} \) est la série de factorielles du théorème 3.4.

**Preuve.** Comme dans la preuve du théorème 3.4, la majoration s’obtient en appliquant la proposition 3.9 avec \( U_0 = I_\mu \) au système de dimension \( \mu^2 \) suivant qui exprime la condition \( \tilde{F}_q^q(t) = A_0(q) \) :

\[
(pt - 1)\delta_p F_q(t) = (A_q(t) F_q(t) - F_q(t) A_0(q)) \left( B_0(q) + \frac{1 - q}{pt - 1} A_0(q) \right)^{-1}.
\]

On a \( \left( B_0(q) + \frac{1 - q}{pt - 1} A_0(q) \right)^{-1} = \sum_{s \geq 0} C_s(q)/(t; q)_s \) où \( C_0(q) = B_0(q)^{-1} \) et pour \( s \geq 1 \),

\[
C_s(q) = q^s (1-q)^s B_0(q)^{-s-1} A_0(q) (A_0(q) + I_\mu) \cdots (A_0(q) + [s-1]q I_\mu).
\]
On peut alors écrire

\[(pt - 1)\delta_p F_q(t) = \sum_{s \geq 0} A_s(q)(F_q(t)) \frac{1}{(t; q)_s}\]

où \(A_s(q)\) est l’opérateur linéaire sur \(gl(\mu, C)\) défini par

\[A_0(q)(U) = (A_0(q)U - UA_0(q))B_0(q)^{-1}\]

et, pour \(s \geq 1\),

\[A_s(q)(U) = A_s(q)UB_0(q)^{-1} + (A_0(q)U - UA_0(q))C_s(q)\]

\[+ \sum_{(j, \ell, k) \in J_s} c_{j, \ell}^{(k)}(q)A_j(q)UC_{\ell}(q),\]

où \(J_s\) est l’ensemble d’indices défini dans la preuve de la proposition 3.3 et où

\[c_{j, \ell}^{(k)}(q) = q^{k+j\ell}(1 - q)^k \frac{[j + k - 1]! \ell + k - 1)!}{[k]![j - 1]![\ell - 1]!}\]

si on pose \([0]! = 1\) et, pour \(n \in N, [n]! = [1]_q[2]_q \cdots [n]_q\).

La deuxième hypothèse faite sur la suite \(A_s(q)\) implique que pour \(s \geq 1\),

\[\lim_{q \to 1} (1 - q)^{-s}C_s(q) = B_s\] (notation du théorème 3.4). En utilisant le fait que

\[c_{j, \ell}^{(k)}(q) = c_{j, \ell}^{(k)}(1 - q)^{-k} \to c_{j, \ell}^{(k)}\]

quand \(q \to 1\), on vérifie que l’opérateur \((1 - q)^{-s}A_s(q)\) a pour limite l’opérateur \(A_s\).

Pour majorer la norme de \(A_s(q)\), on procède comme dans la preuve de 3.4 en utilisant le lemme 4.4 de [1]. Pour cela on remarque qu’en posant

\[D_0(q) = A_0(q)B_0(q)^{-1},\]

chaque bloc de Jordan de \(D_0(q)\) est de la forme

\([\alpha]qI_\nu + \frac{q^\alpha}{q - 1} \sum_{i=1}^{\nu-1} \ln^i qN_i\]

où \(\alpha\) est une valeur propre de \(A_0\) et \(\nu\) la taille du bloc de Jordan correspondant. D’autre part, on peut écrire

\[C_s(q) = q^s(1 - q)^sB_0(q)^{-1}\prod_{k=0}^{s-1} (D_0(q) + [k]_qB_0(q)^{-1})\]

et remarquer que chaque bloc de Jordan de \(D_0(q) + [k]_qB_0(q)^{-1}\) est de la forme

\([\alpha + k]qI_\nu + \frac{q^{\alpha+k}}{q - 1} \sum_{i=1}^{\nu-1} \ln^i qN_i\].

En conclusion on énonce un théorème synthétisant l’étude faite dans le plan de la variable \(x\) initiale, ce qui conduit (toujours selon [1]) à remplacer les séries de \(q\)-factorielles par les séries de factorielles mixtes et à modifier
en conséquence les hypothèses demandées à une $q$-déformation. Le résultat final de convergence est une application du théorème 3.1 de [1].

**Théorème 3.11.** Soit

$$(x-1) \frac{\Delta}{\Delta_{1}} X(x) = A(x)X(x)$$

un système aux différences non résonnant dont la matrice

$$A(x) = \sum_{s \geq 0} A_{s}x^{-[s]}$$

vérifie la condition $(C, \lambda)$.

Soit $S \in \text{GL}(\mu, \mathbb{C})$, $\alpha_{1}, \ldots, \alpha_{r} \in \mathbb{C}$ et $\mu_{1}, \ldots, \mu_{r} \in \mathbb{N}^{*}$ tels que $\sum_{j=1}^{r} \mu_{j} = \mu$ et que, en posant, pour $j = 1, \ldots, r$,

$$A^{(j)} = \begin{cases} \alpha_{j} & \text{si } \mu_{j} = 1, \\ \alpha_{j}I_{\mu_{j}} + N_{\mu_{j}} & \text{si } \mu_{j} \geq 2, \end{cases}$$

on ait

$$A_{0} = S^{-1} \text{diag}(A^{(1)}, \ldots, A^{(r)}) S.$$

On note $X_{\text{can}}(x)$ la solution canonique de $(\ast)$.

Soit $(S(q))_{q \geq 1}$ une famille de matrices appartenant à $\text{GL}(\mu, \mathbb{C})$ vérifiant

$$\lim_{q \to 1^{+}} S(q) = S.$$ Posons, pour $q > 1$,

$$A_{0}(q) = S(q)^{-1} \text{diag}(A^{(1)}(q), \ldots, A^{(r)}(q)) S(q)$$

où, pour $j = 1, \ldots, r$,

$$A^{(j)}(q) = \begin{cases} \llbracket -\alpha_{j} \rrbracket_{q} & \text{si } \mu_{j} = 1, \\ \llbracket -\alpha_{j} \rrbracket_{q}I_{\mu_{j}} + q^{-\alpha_{j}} \frac{\ln q}{q-1} N_{\mu_{j}} & \text{si } \mu_{j} \geq 2. \end{cases}$$

Soit $A_{q}(x) = \sum_{s \geq 1} A_{s}(q)e_{s}^{(q)}(x)$ une série de factorialles mixtes telle que pour tout $s \geq 1$ :

1) il existe $q_{0} > 1$ tel que pour $1 < q < q_{0}$, $\|A_{s}(q)\| \leq C_{s} \frac{q^{s-1}}{e_{s-1}^{(q)}(\lambda)}$,

2) $\lim_{q \to 1^{+}} A_{s}(q) = A_{s}$.

Alors, il existe une unique série formelle de factorialles mixtes

$$F_{q}(x) = I_{\mu} + \sum_{s \geq 1} F_{s}(q)e_{s}^{(q)}(x)$$

telle que, si $E_{A_{0}}^{(q)}(x)$ est la matrice définie dans le théorème 2.11, la matrice

$$X_{q}(x) = F_{q}(x)E_{A_{0}}^{(q)}(x)S(q)$$

constitue un système fondamental de solutions du système

$$(\ast)_{q} \quad p(x-1) \frac{X(px - p) - X(x)}{(p-1)x - p} = (A_{0}(q) + A_{q}(x)) X(x).$$
De plus, il existe $\lambda' \geq \lambda$ tel que la série $F_q(x)$ converge pour $|x - \frac{1}{1-q}| > \lambda' - \frac{1}{1-q}$.

Lorsque $q \to 1^+$, $X_q(x)$ converge vers $X_{\text{can}}(x)$, uniformément sur tout compact de $\{\Re x \geq \lambda' + 1\} \setminus \bigcup_{j=1}^r (\alpha_j + N^*)$.

**Exemple.** En dimension 1, l'équation

$$(x-1) \Delta_1 y(x) = \left(a - \frac{\mu}{x-\lambda}\right) y(x)$$

s'écrit

$$y(x+1) = \frac{x(x+\lambda-1)}{(x-\mu_1)(x-\mu_2)} y(x)$$

où $\mu_1 + \mu_2 = \lambda + a - 1$ et $\mu_1 \mu_2 = \mu - a(1-\lambda)$. Le changement de fonction inconnue

$$y(x) = \frac{\Gamma(1+a-x)}{\Gamma(1-x)} z(x)$$

la transforme en

$$z(x+1) = \frac{(x-a)(x+1-\lambda)}{(x-\mu_1)(x-\mu_2)} z(x)$$

dont la solution, holomorphe dans un demi-plan $\Re x \gg 0$ et ayant 1 pour limite quand $x \to \infty$ dans ce demi-plan, est la fonction

$$z_+(x) = \frac{\Gamma(x-a) \Gamma(x+1-\lambda)}{\Gamma(x-\mu_1) \Gamma(x-\mu_2)}.$$

Cette fonction admet un développement en série de factorielles convergente qui peut s'obtenir en remarquant que la formule de Gauss–Kummer permet d'écrire pour $\Re x > \Re \lambda - 1$,

$$z_+(x) = _2 \phi_1(\mu_1-a, \mu_2-a; x-a; 1).$$

Il suffit alors d'appliquer la formule de translation à cette série de factorielles en $x - a$.

La solution canonique de l'équation donnée est donc

$$y_+(x) = \frac{\Gamma(1+a-x)}{\Gamma(1-x)} z_+(x).$$

Un résultat classique rappelé dans [4] permet de voir $z_+(x)$ comme limite quand $q \to 1^+$ de $2 \phi_1(q^{\mu_1-a}, q^{\mu_2-a}, q^{x-a}; q; q)$ où, par définition,

$$2 \phi_1(a, b, c; q; u) = \sum_{s \geq 0} \frac{(a; q)_s (b; q)_s}{(c; q)_s (q; q)_s} u^s$$
est une série convergente pour $|u| < |qc/(ab)|$. De plus, le $q$-analogue de la formule de Gauss–Kummer est la formule suivante (Jacobi et Heine, citée par [4]) :

$$2\phi_1(a, b, c; q; q) = \frac{(a/c; p)_{\infty}((b/c; p)_{\infty}}{(1/c; p)_{\infty}(ab/c; p)_{\infty}}.$$ 

Ces remarques, jointes à la proposition 2.3, conduisent à considérer la fonction

$$g_q(t) = \frac{(pt; p)_{\infty}(q^{\mu_1}/t; p)_{\infty}(q^{\mu_2}/t; p)_{\infty}}{(q^{-a-1}; p)_{\infty}(q^a/t; p)_{\infty}(q^{\lambda-1}; t; p)_{\infty}}$$

qui vérifie

$$g_q(qt) = q^a (t-1)(t-q^{\lambda-1}) (t-q^{\mu_1})(t-q^{\mu_2}) g_q(t).$$

On en déduit

$$(pt-1)\delta_p g_q(t) = A_q(t)g_q(t)$$

où

$$A_q(t) = q^{-a-1}(t - q^{\mu_1+1})(t - q^{\mu_2+1}) - \frac{pt-1}{(p-1)t}.$$ 

En tenant compte du relation $\mu_1 + \mu_2 + 1 - a - \lambda = 0$ dans la décomposition en éléments simples de $A_q(t)$, on trouve

$$A_q(t) = -[\lambda-a]_q + \frac{(q^{\lambda-\mu_1-1}+1)(q^{\lambda-\mu_2-1}+1)}{(p-1)(t-\lambda)}.$$ 

Dans le plan de la variable $x$, la fonction $f_q(x) = g_q((q-1)x + 1)$ vérifie

$$p(x-1)f_q(px - p) - f_q(x) = \left( -[\lambda-a]_q + \frac{[\lambda - \mu_1 - 1]_q[\lambda - \mu_2 - 1]_q}{x - [\lambda]_q} \right) f_q(x).$$ 

On remarque que les conditions sur les paramètres impliquent la relation $(\lambda - \mu_1 - 1)(\lambda - \mu_2 - 1) = \mu$ et l’équation obtenue est clairement une déformation de l’équation initiale.

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CARD SHuffling AND THE DECOMPOSITION OF 
TENSOR PRODUCTS

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Let $H$ be a subgroup of a finite group $G$. We use Markov chains to quantify how large $r$ should be so that the decomposition of the $r$ tensor power of the representation of $G$ on cosets on $H$ behaves (after renormalization) like the regular representation of $G$. For the case where $G$ is a symmetric group and $H$ a parabolic subgroup, we find that this question is precisely equivalent to the question of how large $r$ should be so that $r$ iterations of a shuffling method randomize the Robinson–Schensted–Knuth shape of a permutation. This equivalence is remarkable, if only because the representation theory problem is related to a reversible Markov chain on the set of representations of the symmetric group, whereas the card shuffling problem is related to a nonreversible Markov chain on the symmetric group. The equivalence is also useful, and results on card shuffling can be applied to yield sharp results about the decomposition of tensor powers.

1. Introduction

Let $\chi$ be a faithful character of a finite group $G$. A well-known theorem of Burnside and Brauer [1] states that if $\chi(g)$ takes on exactly $m$ distinct values for $g \in G$, then every irreducible character of $G$ is a constituent of one of the characters $\chi^j$ for $0 \leq j < m$. It is very natural to investigate the decomposition of $\chi^j$, and the results in this paper are a step in that direction.

Let $\text{Irr}(G)$ denote the set of irreducible representations of a finite group $G$. The Plancherel measure on $\text{Irr}(G)$ is a probability measure that assigns mass $\frac{\dim(\rho)^2}{|G|}$ to $\rho$. The symbol $\chi^\rho$ denotes the character associated to the representation $\rho$. The notation $\text{Ind}$, $\text{Res}$ stands for induction and restriction of class functions. We remind the reader that the character of the $r$-fold tensor product of a representation of $G$ is given by raising the character to the $r$-th power. The inner product $<f_1,f_2>$ denotes the usual inner product on class functions of $G$ defined by

$$\frac{1}{|G|} \sum_{g \in G} f_1(g)f_2(g).$$
Thus if \( f_1 \) is an irreducible character and \( f_2 \) any character, their inner product gives the multiplicity of \( f_1 \) in \( f_2 \). We let \( g^G \) denote the conjugacy class of \( g \) in \( G \).

In Section 2 of this paper, we prove the following result:

**Theorem 1.1.** Let \( H \) be a subgroup of a finite group \( G \) and let \( \text{id} \) denote the identity element. Let \( \pi \) denote the Plancherel measure of \( G \). Suppose that \( |G| > 1 \). Let

\[
\beta = \max_{g \neq \text{id}} \frac{|g^G \cap H|}{|g^G|} = \frac{|H|}{|G|} \max_{g \neq \text{id}} \text{Ind}_{H}^{G}(1)[g].
\]

Then

\[
\sum_{\rho \in \text{Irr}(G)} \left| \left( \frac{|H|}{|G|} \right)^r \dim(\rho) \langle \chi_{\rho}, (\text{Ind}_{H}^{G}(1))^r \rangle - \pi(\rho) \right| \leq |G|^{1/2} \beta^r.
\]

Note that if \( \beta < 1 \), the right-hand side approaches 0 as \( r \to \infty \). The quantity \( \beta \) has been carefully studied in the (most interesting) case that \( G \) is simple and \( H \) a maximal subgroup; references and an example where \( H \) is not maximal are given in Section 2.

The idea behind the proof of Theorem 1.1 is to investigate a natural Markov chain \( J \) on the set of irreducible representations of \( G \). This chain is essentially a probabilistic reformulation of Frobenius reciprocity. This chain can be explicitly diagonalized and then Theorem 1.1 follows from spectral theory of reversible Markov chains, with \( 1 - \beta \) having the interpretation of a spectral gap. In fact Theorem 1.1 is a generalization of a result in our earlier paper \([F1]\), where this Markov chain arose for the symmetric group case \( H = S_{n-1} \) and \( G = S_n \) and was combined with Stein’s method to sharpen a result of Kerov on the asymptotic normality of random character ratios of the symmetric group on transpositions.

The main insight of the current paper is that when \( G \) is the symmetric group \( S_n \) and \( H \) is a parabolic subgroup, the bound of Theorem 1.1 can be improved by card shuffling. Let us describe this in detail for the case \( H = S_{n-1} \). In Theorem 1.1, \( \beta = 1 - \frac{2}{n} \), and one can see using Stirling’s approximation for \( n! \) that for

\[
r > \frac{n \log(n) + 2c}{2 \log(\frac{1}{\beta})},
\]

the bound in Theorem 1.1 is at most \((2\pi)^{1/4}e^{-c}\) (and hence small). Note that all logs in this paper are base \( e \). Thus \( r \) slightly more than \( \frac{1}{4}n^2 \log n \) suffices to make the bound small. The bound of Theorem 1.1 is proved by analyzing a certain Markov chain \( J \) on \( \text{Irr}(S_n) \), started at the trivial representation. The irreducible representations of \( S_n \) correspond to partitions of \( n \) (the one row partition is the trivial representation), so \( J \) is a Markov chain.
on partitions. Although we do not need this observation, we remark that viewing partitions as Young diagrams, this Markov chain amounts to removing a single box with certain probabilities and reattaching it somewhere. We show that the distribution on partitions given by taking \( r \) steps according to \( J \) has a completely different description. Namely starting from the identity permutation (viewed as \( n \) cards in order), perform the following procedure \( r \) times: remove the top card and insert it into a uniformly chosen random position. This gives a nonuniform random permutation, and there is a natural map called the Robinson–Schensted–Knuth or RSK correspondence (see [Sa] for background), which associates a partition to a permutation. We will show that applying this correspondence to the permutation obtained after \( r \) iterations of the top to random shuffle gives exactly the same distribution on partitions as that given by \( r \) iterations of the chain \( J \) started at the trivial representation. This will allow us to use facts about card shuffling to sharpen \( \frac{1}{4} n^2 \log n \) to roughly \( n \log n \), and even to see that the \( n \log(n) \) is sharp to within a factor of two. Precise statements and results for more general parabolic subgroups are given in Section 3.

To close the introduction we make some remarks. First, recall that a Markov chain \( M \) on a finite set \( X \) is called reversible with respect to the probability measure \( \mu \) on \( X \) if \( \mu(x)M(x, y) = \mu(y)M(y, x) \) for all \( x, y \) (this implies \( \mu \) is stationary for \( M \), i.e., \( \mu(y) = \sum_x \mu(x)M(x, y) \) for all \( y \)). The top to random shuffle and its cousins that arise in connection with parabolic subgroups are nonreversible chains. Thus it is rather miraculous that the top to random shuffle has real eigenvalues; this observation is the starting point of a general theory [BHR]. And it is doubly surprising that the top to random shuffle should be connected with the reversible chains \( J \). Since Proposition 3.3 shows these chains to have the same set of eigenvalues, this gives an application of the eigenvalue formulas in [BHR]. See [F4] for some other connections between the top to random shuffle and reversible Markov chains. Second, the problem of studying the convergence rate of the RSK shape after iterated shuffles to the RSK shape of a random permutation is of significant interest independent of its application in this paper. It is closely connected with random matrix theory and in some cases with Toeplitz determinants. See [St], [F2], [F3] and the references therein for details. Third, since Solomon’s descent algebra generalizes to finite Coxeter groups, it is likely that the results in this paper can be pushed through to that setting. (However that would require an analog of the RSK correspondence for finite Coxeter groups). Fourth, we note that some of the results in this paper have now been extended to arbitrary real valued characters of finite groups and to spherical functions of Gelfand pairs [F5], [CF]; as an application one obtains a probabilistic proof of a result of Burnside and Brauer on the decomposition of tensor products [F6].
2. General groups

This section proves Theorem 1.1 and gives an example. Throughout this section \( X = \text{Irr}(G) \) is the set of irreducible representations of a finite group \( G \), endowed with Plancherel measure \( \pi_G \). We also suppose that we are given a subgroup \( H \) of \( G \).

To begin, we use \( H \) to construct a Markov chain on \( \text{Irr}(G) \) that is reversible with respect to \( \pi_G \). For \( \rho \) an irreducible representation of \( G \) and \( \tau \) an irreducible representation of \( H \), we let \( \kappa(\tau, \rho) \) denote the multiplicity of \( \tau \) in \( \text{Res}^G_H(\rho) \). By Frobenius reciprocity, this is the multiplicity of \( \rho \) in \( \text{Ind}^G_H(\tau) \).

**Proposition 2.1.** The Markov chain \( J \) on irreducible representations of \( G \) that moves from \( \rho \) to \( \sigma \) with probability

\[
\frac{|H| \dim(\sigma)}{|G| \dim(\rho)} \sum_{\tau \in \text{Irr}(H)} \kappa(\tau, \rho)\kappa(\tau, \sigma)
\]

is in fact a Markov chain (the transition probabilities sum to 1), and is reversible with respect to the Plancherel measure \( \pi_G \).

**Proof.** First let us check that the transition probabilities sum to 1. Indeed,

\[
\sum_{\sigma \in \text{Irr}(G)} \frac{|H| \dim(\sigma)}{|G| \dim(\rho)} \sum_{\tau \in \text{Irr}(H)} \kappa(\tau, \rho)\kappa(\tau, \sigma)
\]

\[
= \frac{|H|}{|G| \dim(\rho)} \sum_{\tau \in \text{Irr}(H)} \kappa(\tau, \rho) \sum_{\sigma \in \text{Irr}(G)} \dim(\sigma)\kappa(\tau, \sigma)
\]

\[
= \frac{1}{\dim(\rho)} \sum_{\tau \in \text{Irr}(H)} \kappa(\tau, \rho) \dim(\tau) = 1.
\]

The second equality follows since the dimension of a representation induced from a subgroup is its original dimension multiplied by the index of the subgroup.

The reversibility with respect to Plancherel measure is immediate from the definitions. \( \square \)

Next we quickly review some facts from Markov chain theory. We consider the space of real valued functions \( \ell^2(\pi) \) with the norm

\[
\|f\|_2 = \left( \sum_x |f(x)|^2\pi(x) \right)^{1/2}.
\]

If \( J(x, y) \) is the transition rule for a Markov chain on a finite set \( X \), the associated operator (also denoted by \( J \)) on \( \ell^2(\pi) \) is given by \( Jf(x) = \sum_y J(x, y)f(y) \). Let \( J^r(x, y) = J^r_x(y) \) denote the chance that the Markov chain started at \( x \) is at \( y \) after \( r \) steps.
If the Markov chain with transition rule \( J(x, y) \) is reversible with respect to \( \pi \) (i.e., \( \pi(x)J(x, y) = \pi(y)J(y, x) \) for all \( x, y \)), then the operator \( J \) is self adjoint with real eigenvalues

\[-1 \leq \beta_{\min} = \beta_{|X| - 1} \leq \cdots \leq \beta_1 \leq \beta_0 = 1.\]

Let \( \psi_i \ (i = 0, \ldots, |X| - 1) \) be an orthonormal basis of eigenfunctions such that \( J\psi_i = \beta_i \psi_i \) and \( \psi_0 \equiv 1 \). Define \( \beta = \max\{\beta_{\min}, |\beta_1|\} \).

The total variation distance between two probability measures \( Q_1, Q_2 \) on a set \( X \) is defined as

\[ \|Q_1 - Q_2\|_{TV} = \frac{1}{2} \sum_{x \in X} |Q_1(x) - Q_2(x)|. \]

It is elementary that \( \|Q_1 - Q_2\|_{TV} = \max_{\mathcal{A} \subseteq X} |Q_1(\mathcal{A}) - Q_2(\mathcal{A})| \). Thus when the total variation distance is small, the \( Q_1 \) and \( Q_2 \) probabilities of any event \( \mathcal{A} \) are close.

The following lemma is well-known; for a proof see [DSa].

**Lemma 2.2.**

1) \( 2\|J^r_x - \pi\|_{TV} \leq \|\frac{J^r_x}{\pi} - 1\|_2. \)

2) \( J^r(x, y) = \sum_{i=0}^{|X| - 1} \beta^r_i \psi_i(x)\psi_i(y)\pi(y). \)

3) \( \|\frac{J^r_x}{\pi} - 1\|_2^2 = \sum_{i=1}^{|X| - 1} \beta^{2r}_i |\psi_i(x)|^2 \leq 1 - \frac{\pi(x)}{\pi(x)} \beta^{2r}. \)

**Proposition 2.3.** Let \( G \) be a finite group and \( H \) a subgroup of \( G \). Then the eigenvalues and eigenfunctions of the operator \( J \) are indexed by conjugacy classes \( C \) of \( G \).

1) The eigenvalue parameterized by \( C \) is \( \frac{|C \cap H|}{|C|}. \)

2) An orthonormal basis of eigenfunctions \( \psi_C \) is defined by

\[ \psi_C(\rho) = \frac{|C|^{\frac{1}{2}} \chi^\rho(C)}{\dim(\rho)}. \]

**Proof.** First, note that the transition probability in the definition of \( J \) can be rewritten as

\[
\frac{|H| \dim(\sigma) \langle \chi^\sigma, \text{Ind}_H^G \text{Res}_H^G (\chi^\rho) \rangle}{|G| \dim(\rho)} = \frac{|H| \dim(\sigma)}{|G| \dim(\rho)} \frac{1}{|H|} \sum_{g \in G} \chi^\sigma(g) \frac{1}{|g^{-1}gt \in H|} \chi^\rho(t^{-1}gt) \]

\[
= \frac{\dim(\sigma)}{\dim(\rho)} \frac{1}{|G|} \sum_{g \in G} \chi^\sigma(g) \chi^\rho(g) \frac{|g^G \cap H|}{|g^G|}.
\]

The first equality used the well-known formula for induced characters [I].
Now to see that $\psi \mathcal{C}$ is an eigenfunction with the asserted eigenvalue, one calculates that

$$\sum_{\sigma \in \text{Irr}(G)} \frac{\dim(\sigma)}{\dim(\rho)} \frac{1}{|G|} \sum_{g \in G} \chi_{\sigma}^G(g) \chi_{\rho}^G(g) \frac{|g^G \cap H|}{|g^G|} |C|^\frac{1}{2} \chi_{\sigma}^G(C) \chi_{\sigma}^\rho(C)$$

$$= \frac{|C|^\frac{1}{2}}{\dim(\rho)} \sum_{g \in G} |g^G \cap H| \chi_{\rho}^G(g) \frac{1}{|g^G|} \sum_{\sigma \in \text{Irr}(G)} \chi_{\sigma}^G(C) \chi_{\sigma}^\rho(C)$$

$$= \frac{|C|^\frac{1}{2} \chi_{\rho}^G(C)}{\dim(\rho)} \frac{|C \cap H|}{|C|}.$$ 

The second inequality used the orthogonality relations of the characters of $G$.

Finally, the fact that $\psi \mathcal{C}$ are orthonormal follows from the orthogonality relations for irreducible characters. They are a basis since the number of irreducible representations of a finite group is equal to its number of conjugacy classes. □

Next we prove Theorem 1.1 from the introduction.

Proof. First note that the equivalence of the definitions of $\beta$ follows from the general formula for induced characters. Now let 1 denote the trivial representation of $G$. From Proposition 2.3 and part 2 of Lemma 2.2,

$$J_1^\rho(\rho) = \dim(\rho) \sum_{C} \left( \frac{|C \cap H|}{|C|} \right)^r \frac{|C| \chi_{\rho}^G(C)}{|G|}$$

$$= \dim(\rho) \frac{1}{|G|} \sum_{g \in G} \left( \frac{|g^G \cap H|}{|g^G|} \right)^r \chi_{\rho}^G(g)$$

$$= \dim(\rho) \left( \frac{|H|}{|G|} \right)^r \langle \chi_{\rho}, (\text{Ind}_{H}^G(1))^r \rangle,$$

where in the third equality we have used the well-known formula for induced characters used in the proof of Proposition 2.3. The theorem now follows from part 1 of Proposition 2.3 and parts 1 and 3 of Lemma 2.2. □

Remarks.

1) The quantity $\beta$ has been well studied in the case that $G$ is simple and $H$ is a maximal subgroup of $G$. See for instance [GK], [LSh] and the references therein. We defer discussion of the case that $G = S_n$ and $H$ is a parabolic subgroup to Section 3. The remarkable paper [GM] classifies all pairs $(G, H)$ where $G$ is a finite group, $H$ is maximal in $G$, and $\beta$ is at least $1/2$. 

2) Observe that if $\beta = 1$ the upper bound of Theorem 1.1 is useless. And it can happen that $\beta = 1$. For instance if $H$ is a nontrivial normal subgroup of $G$, there are conjugacy classes of $G$ contained in $H$. On the representation theory side, suppose for simplicity that $H$ is normal of index 2. Then except in trivial cases, the state space of the Markov chain $J$ isn’t connected, so the the quantity bounded in Theorem 1.1 won’t go to 0 as $r \to \infty$. Indeed, either $\operatorname{Ind}_H^G \operatorname{Res}_H^G(\rho)$ is two copies of $\rho$ or else the sum of $\rho$ and $\rho'$, where the character of $\rho'$ is equal to the character of $\rho$ on $H$ but takes opposite values on $G - H$ [FH, p. 64].

3) If $\beta = 0$, then $|H| = 1$, which implies that the decomposition of $\operatorname{Ind}_H^G(1)$ is given exactly by Plancherel measure. Then the bound in Theorem 1.1 is an equality.

4) Note that Propositions 2.1 and 2.3 involve the idea of first restricting a representation of $G$ to $H$ and then inducing. There is a similar (but less natural) result for inducing and then restricting. Namely the Markov chain in Proposition 2.1 becomes a reversible Markov chain on irreducible representations of $H$ (with respect to the Plancherel measure of $H$), where one moves from $\rho$ to $\sigma$ with probability

$$
\frac{|H| \dim(\sigma)}{|G| \dim(\rho)} \sum_{\tau \in \operatorname{Irr}(G)} \kappa(\rho, \tau) \kappa(\sigma, \tau).
$$

If $G$ conjugacy classes of $H$ coincide with conjugacy classes of $H$, then the eigenvalues are parameterized by conjugacy classes $C$ of $H$: the eigenvalue is $|C|/|C^G|$ (the denominator is the size of the conjugacy class of $C$ in $G$), and the eigenvector is $|C|^{1/2} \chi(\rho)/\dim(\rho)$. For the pair $(S_n, S_{n+1})$ this was applied in [F1] and the proof method is similar to that of Proposition 2.3. However we believe that it is more natural to restrict and then induce as this involves only the internal structure of the group. Hence we do not develop this remark further.

To conclude this section we compute $\beta$ in the case that $G = \operatorname{GL}(n, q)$ and $H = \operatorname{GL}(n-1, q)$ (which is not a maximal subgroup). There are clearly more examples in this direction that can be worked out using Wall’s formulas for conjugacy class sizes [W] — though as in Proposition 2.4 below some (minor) effort is required to determine when $|g^G \cap H|/|g^G|$ is largest for nontrivial $g$. However as we have no need for them we stop here.

**Proposition 2.4.** Suppose that $G = \operatorname{GL}(n, q)$ and $H = \operatorname{GL}(n-1, q)$, and that $n \geq 2$. Then

$$
\beta = \frac{(1 - 1/q^{n-1})}{q^2(1 - 1/q^n)} \quad \text{for } q > 2 \quad \text{and} \quad \beta = \frac{(1 - 1/q^{n-2})}{q^2(1 - 1/q^n)} \quad \text{for } q = 2.
$$

**Proof.** The conjugacy classes $C$ of $\operatorname{GL}(n, q)$ are parameterized by all ways of associating a partition $\lambda_\phi$ to each monic irreducible polynomial $\phi(z)$ with
coefficients in $F_q$ such that $|\lambda| = 0$ and $\sum_{\phi} \deg(\phi)|\lambda_{\phi}| = n$. Here $|\lambda|$ denotes the size of a partition $\lambda$ and $\deg(\phi)$ denotes the degree of the polynomial $\phi$. Moreover the size of the conjugacy class with this data is ([M], p. 181)

$$\frac{|\text{GL}(n, q)|}{\prod_{\phi} \prod_{j \geq 1} q^{\deg(\phi)(\lambda'_{\phi,j})^2} (1 - 1/q^{\deg(\phi)}) \cdots (1 - 1/q^{\deg(\phi)m_j(\lambda_{\phi})})}.$$  

Here $m_j(\lambda_{\phi})$ is the number of parts of $\lambda_{\phi}$ of size $j$, and $\lambda'_{\phi,j}$ is the number of parts of $\lambda_{\phi}$ of size at least $j$. In order that $|g^G \cap H|$ is nonzero, it is necessary that $g$ has its conjugacy data satisfying $m_1(\lambda_{z-1}(g)) \geq 1$. Then $g^G \cap H$ is a single conjugacy class of $H$, with conjugacy data the same as for $g$ except that a part of size 1 is removed from the partition corresponding to the polynomial $z-1$. Thus one sees that

$$\frac{|g^G \cap H|}{|g^G|} = \frac{|\text{GL}(n-1, q)|}{|\text{GL}(n, q)|} (1 - 1/q^{m_1(\lambda_{z-1}(g))}) q^{2\lambda'_{z-1,1}(g)-1}.$$  

Thus to find $\beta$, it is necessary study the maximum of the function

$$(1 - 1/q^{m_1(\lambda)}) q^{2\lambda'},$$  

among partitions $\lambda$ of size at most $n$ having at least 1 part equal to 1, but excluding the partition of size $n$ that consists of all 1’s. Here $m_1(\lambda)$ denotes the number of parts of $\lambda$ of size 1, and $\lambda'_{z}$ denotes the number of parts of $\lambda$. It is straightforward to see that if $|\lambda| < n$, this function is maximized when $|\lambda| = n-1$ and $\lambda$ consists of $n-1$ 1’s. For $|\lambda| = n$ it is straightforward that the function is maximized for the partition consisting of 1 part of size 2 and $n-2$ parts of size 1. Comparing these two cases one sees that the maximum occurs for the first case. The first case occurs for $q > 2$ but can not occur for $q = 2$ (since $z-1$ is the only polynomial of degree 1 with nonzero constant term), and for $q = 2$ it is straightforward to see that the second case is the maximum.

□

3. Symmetric groups

This section considers the Markov chain $J$ in the case of the symmetric group and develops connections with card shuffling. We assume throughout that the reader is familiar with the Robinson–Schensted–Knuth (RSK) correspondence. See [Sa] for background on this topic.

Consider the symmetric group $S_n$. Let $\Pi = \{\epsilon_1 - \epsilon_2, \ldots, \epsilon_{n-1} - \epsilon_n\}$ be a set of simple roots for the root system consisting of the $n(n-1)$ vectors $\epsilon_i - \epsilon_j$, where $1 \leq i \neq j \leq n$. The positive roots are $\epsilon_i - \epsilon_j$ where $i < j$ and the negative roots are those with $i > j$. The descent set of a permutation $g$ consists of the elements in $\Pi$ that $g$ maps to negative roots. For $L \subseteq \Pi$, let $X_L$ denote the set of permutations whose descent set is disjoint from $L$. It is well-known [H] that $|X_L| = n!/|S_L|$, where $|S_L|$ is the parabolic
subgroup generated by adjacent transpositions corresponding to the roots in \( L \). Consequently if the \( p_L \geq 0 \) satisfy the equality \( \sum_{L \subseteq \Pi} p_L = 1 \), the element \( \sum_{L \subseteq \Pi} \frac{p_L|S_L|}{n!} X_L \) defines a probability measure on the symmetric group.

Given an element \( \sum_{g \in S_n} c_g g \) of the group algebra of the symmetric group, by the inverse element we mean \( \sum_{g \in S_n} c_g g^{-1} \). It is known that the RSK correspondence associates the same partition to \( g \) and to \( g^{-1} \), so when discussing the RSK correspondence one need not be concerned with whether we are considering an element in the group algebra or its inverse. The inverse of the element \( \sum_{L \subseteq \Pi} \frac{p_L|S_L|}{n!} X_L \) can be thought of as a shuffle. For instance if \( p_{\Pi-\{\epsilon_1-\epsilon_2\}} = 1 \), this shuffle is simply the top to random shuffle. One reason these shuffles are important is a result of Solomon \([So]\) that states that \( x_L x_K = \sum_{N \subseteq \Pi} a_{LKN} x_N \) for certain constants \( a_{LKN} \). Thus one can at least in principle compute powers \( (\sum_{L \subseteq \Pi} \frac{p_L|S_L|}{n!} X_L)^r \), which corresponds to understanding iterates of shuffles.

Now the main theorem of this section can be stated. Recall that the irreducible representations of the symmetric group \( S_n \) are parameterized by partitions \( \lambda \) of \( n \).

**Theorem 3.1.** Suppose that \( p_L \geq 0 \) satisfy \( \sum_{L \subseteq \Pi} p_L = 1 \). For \( L \subseteq \Pi \), let \( J[L] \) denote the Markov chain associated to the pair \( G = S_n \) and \( H = S_L \), and let \( J[\vec{p}] = \sum_{L \subseteq \Pi} p_L J[L] \) denote the mixture of the Markov chains \( J[L] \). Then \( J[\vec{p}]^r(\lambda) \) (the chance that the mixed chain started at the trivial representation is at the representation parameterized by \( \lambda \) after \( r \) steps) is equal to the chance that an element of the symmetric group distributed as \( (\sum_{L \subseteq \Pi} \frac{p_L|S_L|}{n!} X_L)^r \) has RSK shape \( \lambda \).

**Proof.** From Proposition 2.3, the functions \( \psi_C(\lambda) \) are a common orthonormal basis of eigenfunctions for the chains \( J[L] \). Hence they are an orthonormal basis of eigenfunctions for the mixed chain \( J[\vec{p}] \). This allows one to compute \( J[\vec{p}]^r(\lambda) \) by the same method used in the proof of Theorem 1.1, and one concludes that it is equal to

\[
\dim(\lambda) \left\langle \chi^\lambda, \left( \sum_L \frac{p_L|S_L|}{n!} \text{Ind}_{S_L} S_n(1) \right)^r \right\rangle.
\]

As explained in the preliminary remarks of Section 4 of \([BBHT]\), the coefficients \( a_{LKN} \) are related to tensor products of representations:

\[
\text{Ind}_{S_L} S_n(1) \times \text{Ind}_{S_K} S_n(1) = \sum_{N \subseteq \Pi} a_{LKN} \text{Ind}_{S_N} S_n(1).
\]
Letting $c_{N,r,\vec{p}}$ denote the coefficient of $X_N$ in
\[
\left( \sum_{L \subseteq \Pi} \frac{p_L|S_L|}{n!} X_L \right)^r,
\]
it follows that $J[\vec{p}]_1^r(\lambda)$ is equal to
\[
\dim(\lambda) \sum_{N \subseteq \Pi} c_{N,r,\vec{p}}(\chi^\lambda, \text{Ind}_{S_N} S_n(1)).
\]

Letting $\mu$ denote the type of $N$ (that is $S_N$ is the direct product of symmetric groups whose sizes are the parts of the partition $\mu$), the multiplicity of $\lambda$ in $\text{Ind}_{S_n} S_N(1)$ is by definition the Kostka–Foulkes number $K_{\lambda\mu}$ discussed in [Sa]. Thus $J[\vec{p}]_1^r(\lambda)$ is equal to
\[
\dim(\lambda) \sum_{\mu} K_{\lambda\mu} \sum_{N: \text{type}(N) = \mu} c_{N,r,\vec{p}}
\]
where the sum is over all partitions $\mu$ of $n$.

Next it is necessary to show this is equal to the chance that an element of the symmetric group distributed as $(\sum_{L \subseteq \Pi} \frac{p_L|S_L|}{n!} X_L)^r$ has RSK shape $\lambda$. By the definition of $c_{N,r,\vec{p}}$, we know that
\[
\left( \sum_{L \subseteq \Pi} \frac{p_L|S_L|}{n!} X_L \right)^r = \sum_{N \subseteq \Pi} c_{N,r,\vec{p}} X_N.
\]

So it suffices to show that the number of summands of the element $X_N$ (or equivalently the inverse of $X_N$) that the RSK correspondence maps to $\lambda$ is $\dim(\lambda)K_{\lambda,\text{type}(N)}$. But writing $S_N = S_{a_1} \times S_{a_2} \cdots \times S_{a_r}$ the summands of the inverse of $x_N$ correspond (in an RSK shape preserving way) to words on the letters $\{1, \ldots, r\}$ in which the letter $l$ appears $a_l$ times. But such words with RSK shape $\lambda$ correspond to pairs $(P,Q)$ of Young tableau with $Q$ standard of shape $\lambda$ and $P$ semistandard of shape $\lambda$ and content $\text{type}(N)$. Since the number of these is $\dim(\lambda)K_{\lambda,\text{type}(N)}$, the theorem is proved. \qed

Corollary 3.2 is an important consequence of Theorem 3.1.

**Corollary 3.2.** Let $tv(r,\vec{p})$ denote the total variation distance between the probability measure $(\sum_{L \subseteq \Pi} \frac{p_L|S_L|}{n!} X_L)^r$ on the symmetric group and the uniform distribution on the symmetric group. Let $\pi$ be the Plancherel measure of $S_n$. Then
\[
\frac{1}{2} \sum_{\lambda \in \text{Irr}(S_n)} \left| \dim(\lambda) \left( \chi^\lambda, \left( \sum_{L \subseteq \Pi} \frac{p_L|S_L|}{n!} \text{Ind}_{S_n} S_n(1) \right)^r \right) - \pi(\lambda) \right| \leq tv(r,\vec{p}).
\]
Proof. From the proof of Theorem 3.1, we know that
\[
\frac{1}{2} \sum_{\lambda \in \text{Irr}(S_n)} \left| \dim(\lambda) \left\langle \chi_\lambda, \left( \sum_{L \subseteq \Pi} p_L \left| S_L \right| \frac{\text{Ind}_{S_L} S_n(1)}{n!} \right)^r \right\rangle - \pi(\lambda) \right|]
\]
is equal to the total variation distance between the measure \(J[p]^r\) and the Plancherel measure of the symmetric group. Theorem 3.1 gives that this is equal to the total variation distance between the RSK pushforward of the measure \(\left( \sum_{L \subseteq \Pi} p_L \left| S_L \right| X_L \right)^r\) and the Plancherel measure. Since the Plancherel measure is the RSK pushforward of the uniform distribution on the symmetric group, the corollary follows.

The significance of Corollary 3.2 is that it allows one to apply work on convergence rates of shuffles to the study of tensor products. We now give some examples showing that the bound of Corollary 3.2 can be much sharper than that of Theorem 1.1. Note that here we only treat examples with \(p_L = 1\) as these are the most natural from the viewpoint of decomposition of tensor products. The convergence rate of the RSK shape for other shuffles is considered in [F2], [F3].

Example 1 (The defining representation). The first example is when \(p_L = 1\) for \(L = \Pi - \{\epsilon_1 - \epsilon_2\}\). Then \(G = S_n\) and \(H = S_{n-1}\). The representation theory problem in this case is the study of decompositions of the \(r\)-th tensor power of the defining (n-dimensional) representation, and the card shuffling problem is the \(r\) fold iteration of the top to random shuffle.

Consider the bound of Theorem 1.1. Letting \(n_1(g)\) denote the number of fixed-points of \(g\), it is clear that \(|g^G \cap H|/|g^G| = n_1(g)/n\). Thus \(\beta = 1 - \frac{2}{n}\). It follows that
\[
\sum_{\lambda \in \text{Irr}(S_n)} \left| \dim(\lambda) \left\langle \chi_\lambda, \left( \text{Ind}_{H}^G(1) \right)^r \right\rangle - \pi(\lambda) \right| \leq \sqrt{n!} \left(1 - \frac{2}{n}\right)^r .
\]
Using Stirling’s approximation [Fe]
\[
n! \leq \sqrt{2\pi n} e^{-n + \frac{1}{2}} (n + \frac{1}{2})^n ,
\]
one sees that for \(r > \frac{n \log(n) + 2c}{2 \log(\frac{n}{2})}\), this is at most
\[
(2\pi)^{1/4} e^{r \log(\beta) + \frac{n \log(n)}{2}} \leq (2\pi)^{1/4} e^{-c}.
\]
For \(c\) fixed and large \(n\), \(\frac{n \log(n) + 2c}{2 \log(\frac{n}{2})}\) is roughly \(\frac{1}{4} n^2 \log n\).

The bound from Corollary 3.2 is much sharper. Indeed, it is known [AD] that for \(r = n \log(n) + cn\), the total variation distance between \(r\) iterations of the top to random shuffle and the uniform distribution is at most \(e^{-c}\), for \(c \geq 0, n \geq 2\).
Next consider lower bounds for
\[
\frac{1}{2} \sum_{\lambda \in \mathrm{Irr}(S_n)} \left| \frac{\dim(\lambda)}{n^r} \langle \chi^\lambda, (\Ind_{H}^{G}(1))^r \rangle - \pi(\lambda) \right|.
\]
By Theorem 3.1, this is equal to the total variation distance between the RSK pushforward of \( r \) iterations of the top to random shuffle and the Plancherel measure. A result of Chapter 5 of [U] is that for large \( n \) at least \( \frac{1}{2}n \log(n) \) iterations of the top to random shuffle are needed to randomize the length of the longest increasing subsequence (actually he states the result for the random to top shuffle, but this is the inverse of top to random). Since the longest increasing subsequence is a function of the RSK shape, it follows that
\[
\frac{1}{2} \sum_{\lambda \in \mathrm{Irr}(S_n)} \left| \frac{\dim(\lambda)}{n^r} \langle \chi^\lambda, (\Ind_{H}^{G}(1))^r \rangle - \pi(\lambda) \right|
\]
requires \( r \) at least \( \frac{1}{2}n \log(n) \) to be small. Thus the upper bound on \( r \) in the previous paragraph is sharp to within a factor of two.

The next two examples generalize Example 1, but in different directions.

**Example 2** \((S_{n-k} \subset S_n)\). This example is the case that
\[
L = \Pi - \{\epsilon_1 - \epsilon_2, \ldots, \epsilon_k - \epsilon_{k+1}\}
\]
where \( k \leq n - 1 \). Then \( G = S_n \) and \( H = S_{n-k} \). The representation theory problem is to study the decomposition of the \( r \)th tensor power of \( \Ind_{S_{n-k}}^{S_n}(1) \), and the relevant card shuffling is the top \( k \) to random shuffle, which proceeds by removing the top \( k \) cards from the deck and sequentially inserting them into random positions (this is equivalent to thoroughly mixing the top \( k \) cards and then riffling them with the rest of the deck — i.e., choosing a random interleaving).

First consider the bound of Theorem 1.1. Using the fact that two elements in a symmetric group are conjugate if and only if they have the same structure, and that a conjugacy class with \( n_i \) cycles of length \( i \) for all \( i \) has size \( n!/\prod_i i^{n_i} n_i! \), one finds that \( \beta = \frac{(n-k)(n-k-1)}{n(n-1)} \). By the same argument as Example 1, it follows that
\[
\sum_{\lambda \in \mathrm{Irr}(S_n)} \left| \frac{\dim(\lambda)}{(n^2 \cdots (n-k+1))^r} \langle \chi^\lambda, (\Ind_{H}^{G}(1))^r \rangle - \pi(\lambda) \right| \leq (2\pi)^{1/4} e^{-c}
\]
when \( r > \frac{n \log(n) + 2c}{2 \log(\frac{1}{\delta})} \). For fixed \( c, k \) and large \( n \), \( \frac{n \log(n) + 2c}{2 \log(\frac{1}{\delta})} \) is roughly \( \frac{n^2 \log(n)}{4k} \).

The convergence rate of the card shuffling problem was studied in [DFiP], where it was shown that for \( k \) fixed and large \( n \), the total variation distance is small for \( r = \frac{n}{k} (\log(n) + c) \). Thus the bound from Corollary 3.2 is much
sharper. The argument for the lower bound also generalizes, showing that \( r \) must be at least \( \frac{1}{2k} n \log(n) \) for

\[
\frac{1}{2} \sum_{\lambda \in \text{Irr}(S_n)} \left| \frac{\dim(\lambda)}{(n(n-1) \cdots (n-k+1))^{r}} \langle \chi^\lambda, (\text{Ind}_G^H(1))^r \rangle - \pi(\lambda) \right|
\]

to be small.

The case of \( k = \frac{n}{2} \) is also of interest. Then for \( n \) large, \( \beta \) is roughly \( \frac{1}{4} \), and the upper bound of Theorem 1.1 shows that \( r \) roughly \( n \log(n) \) is sufficient. Again the bound of Corollary 3.2 is shaper. To see this note that one wants an upper bound on the total variation distance between \( r \) iterates of the shuffle and the uniform distribution. The shuffle is a special case of the Bidigare–Hanlon–Rockmore walks on chambers of hyperplane arrangements, and a convenient upper bound for total variation distance is in [BD] (this bound is somewhat weaker than the bound in [BHR] but is easier to apply). In the case at hand the bound turns out to be \( \left( \frac{n}{2} \right)^{\beta} \), which shows that \( r \) roughly \( \frac{2 \log(n)}{\log(4)} \) is sufficient.

**Example 3** (Action on \( k \)-sets). The next example is the case that \( p_L = 1 \), where \( L = \Pi - \{ \epsilon_k - \epsilon_{k+1} \} \), and \( 1 \leq k \leq n/2 \). Then \( G = S_n \) and \( H = S_k \times S_{n-k} \). The representation theory problem in this case is the study of decompositions of the \( r \)-th tensor power of the permutation representation on \( k \)-sets, and the card shuffling problem is the \( r \)-fold iteration of the shuffle that proceeds by cutting off exactly \( k \) cards, and then riffling them with the other \( n-k \) cards (i.e., choosing a random interleaving).

First consider the bound of Theorem 1.1. The value of \( \beta \) is calculated in [GM] for \( n \geq 5 \) and shown to occur for the conjugacy class of transpositions, where it is \( \frac{(n-2)+(n-2)}{k} \). For \( k \) fixed and large \( n \), \( \log(\frac{1}{\beta}) \) is roughly \( \frac{2k}{n} \), so that \( r \) slightly more than \( \frac{n \log(n)}{4k} \) will make

\[
\sum_{\lambda \in \text{Irr}(S_n)} \left| \frac{\dim(\lambda)}{(n)^r} \langle \chi^\lambda, (\text{Ind}_G^H(1))^r \rangle - \pi(\lambda) \right|
\]

small.

Now consider the bound from Corollary 3.2. To apply it we require an upper bound on the total variation distance between the uniform distribution and \( r \) iterations of the shuffle that cuts off exactly \( k \) cards and riffling them with the rest of the deck. This shuffle too is a special case of the Bidigare–Hanlon–Rockmore walks on chambers of hyperplane arrangements, and a convenient upper bound for total variation distance is in [BD]. In the case at hand one can check that the total variation bound becomes

\[
\left( \frac{n}{2} \right)^{\frac{(n-2)}{k} + \frac{(n-2)}{k}}
\]
which is better than the bound $\sqrt{2} \left( \frac{(n-2) + (n-2)}{2} \right)^r$ from Theorem 1.1. One concludes that $r$ slightly more than $\frac{n \log(n)}{k}$ makes

$$\frac{1}{2} \sum_{\lambda \in \text{Irr}(S_n)} \left| \dim(\lambda) \frac{\langle \chi^\lambda, (\text{Ind}_{G}^{H}(1)) \rangle^r}{(\frac{n}{k})^r} - \pi(\lambda) \right|$$

small. Moreover, the argument for the lower bound in the other examples generalizes, showing that $r$ must be at least $\frac{n \log(n)}{2k}$.

The case of $k = \frac{n}{2}$ is also of interest. Then for $n$ large, $\beta$ is roughly $\frac{1}{2}$, and the upper bound of Theorem 1.1 shows that $r$ roughly $\frac{n \log(n)}{2 \log(2)}$ is sufficient. Again the bound of Corollary 3.2 is shaper, showing that $r$ roughly $\frac{2 \log(n)}{\log(2)}$ is sufficient.

We remark that the fact that nonreversible Markov chains such as top to random are related to the reversible Markov chain $J$ by means of Theorem 3.1 is quite mysterious. As a further result in this direction, we show that the Markov chains $J[p]$ and $\sum_{L \subseteq \Pi} p_{L} [S_{1}]_{n!} X_{L}$ have the same set of eigenvalues (of course the multiplicities are different).

**Proposition 3.3.** The Markov chain $J[p]$ and the element $\sum_{L \subseteq \Pi} p_{L} [S_{1}]_{n!} X_{L}$ have the same set of eigenvalues.

**Proof.** Since the chains $J[L]$ have a common basis of eigenvectors, the eigenvalues of $J[p]$ are linear functions in the $p$'s. Similarly [BHR] finds a formula for the eigenvalues of the element $\sum_{L \subseteq \Pi} p_{L} [S_{1}]_{n!} X_{L}$ and shows that they are linear in the $p$'s. Hence it is enough to prove the result when $p_{L} = 1$ for some $L$.

From Corollary 2.2 of [BHR], the eigenvalues of the element $[S_{1}]_{n!} X_{L}$ are indexed by permutations $g \in S_{n}$. Let $\mu$ be such that the orbits of $S_{L}$ on $\{1, \ldots, n\}$ are $\{1 \cdots \mu_{1}\}, \{\mu_{1} + 1 \cdots \mu_{1} + \mu_{2}\},$ etc.; hence $\mu$ is a composition of $n$. A block ordered partition of the set $\{1, \ldots, n\}$ is by definition a set partition with an ordering on the blocks of the partition. We say that a block ordered partition has type $\mu$ if the first block has size $\mu_{1}$, the second block has size $\mu_{2}$ and so on. The result of [BHR] is that the eigenvalue corresponding to $g$ is the proportion of block ordered partitions of type $\mu$ that are fixed by $g$ in the sense that each block is sent to itself. This is equivalent to requiring that each block is a union of cycles of $g$. Letting $n_{i}$ denote the number of $i$-cycles of $g$, it follows that this proportion is

$$\frac{\mu_{1}! \mu_{2}! \cdots}{n!} \sum_{\sum_{k} a_{i}^{(k)} = n_{i}} \prod_{i \geq 1} \left( \frac{n_{i}}{a_{i}^{(1)}, a_{i}^{(2)}, \ldots} \right).$$
On the other hand, by Proposition 2.3, we know that the eigenvalues of $J$ are parameterized by conjugacy classes $C$ of $S_n$. Let $n_i$ denote the number of cycles of length $i$ for elements in the class $C$. Using the fact that $|C| = n!/\prod i^{n_i} n_i!$, it follows that

$$\frac{|C \cap SL|}{|C|} = \frac{\prod i^{n_i} n_i!}{n!} \sum \frac{\prod \mu_k!}{\prod i^{a_i(k)} a_i(k)!} \prod \sum a_i(k) = n_i \sum a_i(k) = \mu_k \text{ all } k.$$

This is equal to the expression of the previous paragraph, so the proof is complete. □

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A THETA DIVISOR CONTAINING AN ABELIAN SUBVARIETY

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We construct a Jacobian of dimension three whose theta divisor contains an elliptic curve. We work over an algebraically closed field of characteristic zero.

Let $E$ be an elliptic curve and $F$ a principally polarized abelian variety of dimension 3. Let $L$ and $N$ be their principal polarizations.

Lemma 1. There exist $E'$ and $F'$ such that we have isogenies $\psi_E : E' \to E$ and $\psi_F : F' \to F$ of degree two.

Proof. Let $F' = (F'/\mathbb{Z}/2\mathbb{Z})^\vee$ and the same with $E$. 

Let $L' = \psi_E^*L$ and $N' = \psi_F^*N$. Then $\varphi_{L'} : E' \to (E')^\vee$ and $\varphi_{N'} : F' \to (F')^\vee$ have degree four. Let $H_{L'}$ and $H_{N'}$ be their kernels. Then we have theta groups

$$1 \to \mathbb{G}_m \to G_{L'} \to H_{L'} \to 0$$

and

$$1 \to \mathbb{G}_m \to G_{N'} \to H_{N'} \to 0.$$

By Mumford theory we have two torsion elements $\alpha_L$ and $\beta_L$ of $G_{L'}$ such that $\alpha_L \cdot \beta_L = (-1)^4 \beta_L \cdot \alpha_L$ and the same with $N$. Here the images of $\alpha_L$ and $\beta_L$ generate $H_{L'}$. Consider $M = \pi_{F'}^*L' \otimes \pi_F^*N'$. Then $H_M = H_{L'} \times H_{N'}$.

Lemma 2. We have an inclusion $K = (\mathbb{Z}/2\mathbb{Z})^2 \subset G_M$.

Proof. $\alpha_L \otimes \alpha_N$ and $\beta_L \otimes \beta_N$ generate the group. 

Let $X = E' \otimes F'/\text{Im } K$ and let $R$ be the quotient of $M$ by $(\mathbb{Z}/2\mathbb{Z})^2$. Then $R$ gives a principal polarization on $X$. Let $\gamma$ be a nonzero section of $X$. Let $\theta$ be the zeroes of $\gamma$.

Lemma 3. $\theta$ contains some translate of $\text{Im } E'$.
Proof. \( \gamma \) corresponds to a section of \( \mathcal{M} \) that is invariant under \( K \). Let \( \tau \) and \( \mu \) be nonzero sections of \( \mathcal{L} \) and \( \mathcal{N} \) invariant under \( \alpha_\mathcal{L} \) and \( \alpha_\mathcal{N} \). Let \( \tau' = \beta_\mathcal{L}(\tau) \) and \( \mu' = \beta_\mathcal{N}(\mu) \). Then \( \tau' \) and \( \mu' \) are anti-invariant under \( \alpha_\mathcal{L} \) and \( \alpha_\mathcal{N} \). Consider the section \( \eta = \tau \otimes \mu + \tau' \otimes \mu' \neq 0 \) of \( \mathcal{M} \). Then \( \eta \) is invariant under \( (\mathbb{Z}/2\mathbb{Z})^2 \). Then the inverse image of \( \theta \) is the zeroes \( \eta \supset E \times (\mu = \mu' = 0) \), where the second set is nonempty as \( \mathcal{N}^2 \) is ample. \( \square \)

Assume that \( F \) contains no elliptic curve.

Lemma 4. \((\mu = \mu' = 0)\) is a finite set.

Proof. Let \( D \) be the largest divisor in the intersection. Then \( D \) is invariant under the group \( P \) generated by the image of \( \alpha_\mathcal{N} \) and \( \beta_\mathcal{N} \). Then \( D \) comes from an effective divisor \( D' \) on \( F'/P \) where \( \# P = 4 \). So \( \frac{(D^2)}{4} = 4\frac{(D')^2}{2} \) and \( \frac{(D^2)}{2} \leq \frac{(\mu=0)^2}{2} = 2 \). So by [1], \( D' \) comes from a divisor on a quotient of \( F'/D \) which is a point. So \( D' \) is empty. \( \square \)

Lemma 5. \((X, \theta)\) is a Jacobian.

Proof. We need to see that \( \theta \) is irreducible. If \( \theta \) is reducible, we have \( X = E \oplus R \) by [1], where \( \theta \) is the sum of divisors depending on the factors. Thus \( \theta \supset E \times x \) for a curve \( x \). But this contradicts Lemma 4. \( \square \)

References


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We generalize to hypercohomology the Brumer criterion for the strict cohomological dimension of profinite groups.

1. Introduction

The aim of this paper is to obtain a criterion determining the strict cohomological dimension of a profinite group.

The strict cohomological dimension of a profinite group $G$, written $\text{scd}_p G$, is the smallest integer $n$ such that the $p$-primary component of $H^{n+1}(G,M)$ vanishes for all discrete $G$-modules $M$, where $p$ is a prime number. There is also the notion of cohomological dimension of $G$, denoted by $\text{cd}_p G$. This is the smallest integer $n$ such that $H^{n+1}(G,A) = 0$ for all discrete $p$-primary $G$-modules $A$.

It is well-known that the strict cohomological dimension of $G$ is equal to $\text{cd}_p G$ or $\text{cd}_p G + 1$. This result is often useful. In many cases, in fact, it enables us to obtain sufficient information on the cohomology groups of a given profinite group. It is, nevertheless, quite difficult to determine the strict cohomological dimension of a given profinite group. Brumer gave a useful criterion determining it [1]:

**Theorem 1.1** (Theorem 6.1, [1]). *The following are equivalent for a class formation $(G,A)$:

1. $\text{scd}_p G = 2$.
2. For each integer $q$ and each pair $H \subset K$ of open subgroups of $G$ such that $H$ is normal in $K$, the homomorphism induced by the reciprocity map

$$\hat{H}^q(K/H, A^H) \rightarrow \hat{H}^q(K/H, H^{ab})$$

induces an isomorphism onto on the $p$-primary component of the respective cohomology groups.*

Using this criterion, Brumer showed, for example, that the absolute Galois group of a local field has strict cohomological dimension two.

The author developed in [8] the theory of class formations, including complexes and hypercohomology groups. Class formations can be applied
to higher-dimensional local fields and, of course, to the classical class field theory of usual local and global fields. Then the problem is whether one can generalize the Brumer criterion to the case of such class formations. This is the main motivation of this paper. We prove:

**Theorem 1.2** (cf. Theorem 3.2). *The following statements are equivalent for a class formation $(G, A^*)$:*

1. $\text{scd}_p G \leq n + 1$.
2. For each integer $q$ and each pair $H \subset K$ of open subgroups of $G$ such that $H$ is normal in $K$, the homomorphisms induced by the reciprocity map induce isomorphisms
   $$\widehat{H}^q(K/H, H^{-r}(H, A^*))(p) \simeq \widehat{H}^q(K/H, H_{r+1}(H, \mathbb{Z}))(p)$$
   for each $r = 0, \ldots, n - 1$.

This is a generalization of the Brumer criterion, as can be seen by examining the case $n = 1$ (note that $\text{scd}_p G = 1$ if and only if $G = \{1\}$).

As an application, we determine the strict cohomological dimension of a two-dimensional local field, as Brumer did. Note that the point of this paper is to obtain a necessary and sufficient condition determining the strict cohomological dimension. If one wishes to determine only the strict cohomological dimension of higher local fields, there are other methods (see Section 4).

**Notation 1.3.** We will freely use standard notation about complexes and hypercohomology groups. Unless the contrary is explicitly stated, we employ the following notations and conventions:

1. For an abelian group $A$, we denote the group of $n$-torsion elements of $A$ by $A_n$. In particular, in the case $n$ is a prime number, $A(p)$ stands for $p$-primary torsion part of $A$.
2. For a field $F$, we denote the separable closure of $F$ by $F_s$.

## 2. Generalities

In this section we obtain a criterion determining the strict cohomological dimension of a profinite group $G$ in rather general contexts.

Unless the contrary is explicitly stated, let $G$ be a profinite group, and let $H \subset K$ be open subgroups of $G$ such that $H$ is normal in $K$.

First of all, we fix notation used frequently in this paper.

Let $G$ be as above. We may assume that

$$G = \varprojlim U G / U,$$

where $U$ runs through open normal subgroups of $G$. We define the complete group algebra $\mathbb{Z}_p[G]$ of $G$ over the ring of $p$-adic integers to be the inverse
limit of the ordinary group ring of the finite quotients \( G/U \) of \( G \) over \( \mathbb{Z}_p \):

\[
\mathbb{Z}_p[G] = \lim_{\leftarrow} \mathbb{Z}_p[G/U].
\]

For an arbitrary \( \mathbb{Z}_p[[G]] \)-module \( A \), we define the functor \( F_G \) by

\[
F_G(A) = A/I(G),
\]

where \( I(G) \) is the closed ideal of \( \mathbb{Z}_p[[G]] \) generated by \( \{1 - g \mid g \in G\} \).

Next, consider an exact sequence

\[
\cdots \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow \mathbb{Z}_p \rightarrow 0
\]

where, for each \( i \), \( P_i \) is a projective \( \mathbb{Z}_p[[G]] \)-module. We let \( Q_n \) denote \( \text{Ker}(P_n \rightarrow P_{n-1}) \), and formally define \( Q_{-1} \) to be \( \mathbb{Z}_p \). Then the sequence

\[
0 \rightarrow H_{n+1}(H, \mathbb{Z}_p) \rightarrow F_H(Q_n) \rightarrow F_H(P_n) \rightarrow F_H(Q_{n-1}) \rightarrow 0
\]

is exact. From this short exact sequence we obtain an induced homomorphism

\[
\omega^q : \hat{H}^{q-2}(K/H, F_H(Q_{n-2})) \rightarrow \hat{H}^q(K/H, H_n(H, \mathbb{Z}_p))
\]

defined as the composition of two boundary maps.

**Lemma 2.1.** Let \( H \subset K \) be as above and let \( q \) be an arbitrary integer. The following statements are equivalent:

1. \( \hat{H}^{q-2}(K/H, F_H(Q_{n-2})) \simeq \hat{H}^q(K/H, H_n(H, \mathbb{Z}_p)) \).
2. \( \hat{H}^q(K/H, F_H(Q_{n-1})) = 0 \).

This is an immediate consequence of the lemma below:

**Lemma 2.2.** Let \( T \) be a finite group and let

\[
0 \rightarrow A \rightarrow B \twoheadrightarrow C \rightarrow D \rightarrow 0
\]

be an exact sequence of \( T \)-modules; we have an induced homomorphism

\[
d_q : \hat{H}^{q-2}(T, D) \rightarrow \hat{H}^q(T, A)
\]

defined as the composition of two boundary maps. The following statements are equivalent:

1. For each \( q \), \( d_q \) is an isomorphism on the \( p \)-primary components.
2. For each \( q \), \( \omega^* : \hat{H}^q(T, B) \rightarrow \hat{H}^q(T, C) \) is an isomorphism on the \( p \)-primary components.

Lemma 2.1 enables us to prove the following:

**Proposition 2.3.** Assume that

\[
\omega^q : \hat{H}^{q-2}(K/H, F_H(Q_{n-2})) \simeq \hat{H}^q(K/H, H_n(H, \mathbb{Z}_p))
\]

for each integer \( q \). Then \( \text{scd}_p G \leq n + 1 \).
Proof. Form the assumptions and Lemma 2.1 we can conclude that
\[ \widehat{H}^q(K/H, F_H(Q_{n-1})) = 0. \]
Taking into account Lemma 6.7 of [1], we deduce that \( \mathcal{H}d_{Z_p[\mathbb{G}]} Q_{n-1} \leq 1 \). Therefore, we can find projective \( Z_p[G] \)-modules \( P'_{n+1} \) and \( P'_n \) such that
\[ 0 \to P'_{n+1} \to P'_n \to Q_{n-1} \to 0 \]
is exact. Thus, we can take a projective resolution of \( Z_p \) of the form
\[ 0 \to P'_{n+1} \to P'_n \to P_{n-1} \to \cdots \to P_0 \to Z_p \to 0, \]
and therefore \( H_{n+2}(H, \mathbb{Z}_p) = 0 \). Since \( H^{n+2}(H, Q_p/\mathbb{Z}_p) \simeq (H_{n+2}(H, \mathbb{Z}_p))^* \), we obtain \( \text{cd}_p G \leq n + 1 \) (cf. Corollary 4 to Proposition 14, Chap. I, [13]).

The following lemma gives us the converse of Proposition 2.3.

Lemma 2.4. Suppose that we have \( \text{scd}_p G \leq n + 1 \). Then we can conclude
\[ \omega^q : \widehat{H}^{q-2}(K/H, F_H(Q_{n-2})) \simeq \widehat{H}^q(K/H, H_n(H, \mathbb{Z}_p)) \]
for an arbitrary integer \( q \).

Proof. Let
\[ \cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to Z_p \to 0 \]
be a projective resolution of \( Z_p \). Added to this, we denote the complex
\[ \cdots \to F_H(P_n) \to F_H(P_{n-1}) \to F_H(P_{n-2}) \to 0 \to 0 \to \cdots \]
by \( C^\bullet \). Since the sequence
\[ \to P_n \to P_{n-1} \to P_{n-2} \to Q_{n-3} \to 0 \]
is exact,
\[ \tau_{\leq-1} C^\bullet \to C^\bullet \to F_H(Q_{n-3}) \to \tau_{\leq-1} C^\bullet[-1] \]
is a distinguished triangle. For sufficiently large \( q \), we obtain
\[ H_{q-1}(K, Q_{n-3}) \simeq \widehat{H}^{-q}(K/H, C^\bullet). \]
Since \( \text{scd}_p G \leq n + 1 \), we have \( \widehat{H}^{-q}(K/H, C^\bullet) = 0 \) for sufficiently large \( q \), and therefore, for all \( q \), we have \( \widehat{H}^{-q}(K/H, C^\bullet) = 0 \). Thus, we can conclude
\[ \widehat{H}^{q-1}(K/H, F_H(Q_{n-3})) \simeq \widehat{H}^q(K/H, \tau_{\leq-1} C^\bullet), \]
for each integer $q$.

However, the assumption $scd_p G \leq n + 1$ gives $H_{n+1}(H, \mathbb{Z}_p) = 0$, so we know $H^q(\tau_{\leq -1} C^\bullet) = 0$ for $q < -2$. That is, we have a distinguished triangle

$$H_n(H, \mathbb{Z}_p)[2] \to \tau_{\leq -1} C^\bullet \to H_{n-1}(H, \mathbb{Z}_p)[-1] \to H_n(H, \mathbb{Z}_p)[1].$$

This triangle induces a long exact sequence of hypercohomology groups:

$$\cdots \to \widehat{H}^{q+2}(K/H, H_n(H, \mathbb{Z}_p)) \to \widehat{H}^q(K/H, \tau_{\leq -1} C^\bullet) \to \widehat{H}^{q+1}(K/H, H_{n-1}(H, \mathbb{Z}_p)) \to \cdots.$$  \hspace{1cm} (2)

Now consider the commutative diagram

$$\begin{array}{ccc}
B & \xrightarrow{\sim} & B \\
\downarrow & & \downarrow \\
A & \longrightarrow & F_H(P_{n-1}) \longrightarrow F_H(P_{n-2}) \longrightarrow F_H(Q_{n-3}) \\
\downarrow & & \downarrow \approx \downarrow \approx \\
0 \to H_{n-1}(H, \mathbb{Z}_p) & \longrightarrow & F_H(Q_{n-2}) \longrightarrow F_H(P_{n-2}) \longrightarrow F_H(Q_{n-3}) \to 0,
\end{array}$$

where we have put

$A = \text{Ker}(F_H(P_{n-1}) \to F_H(P_{n-2}))$, $B = \text{Ker}(F_H(P_{n-1}) \to F_H(Q_{n-2}))$.

All horizontal sequences are exact. Applying the snake lemma, we have an exact sequence

$$\cdots \to \widehat{H}^{q+1}(K/H, B) \to \widehat{H}^{q+1}(K/H, A) \to \widehat{H}^{q+1}(K/H, H_{n-1}(H, \mathbb{Z}_p)) \to \cdots.$$  \hspace{1cm} (3)

By observing the middle horizontal sequence in the preceding diagram, we have

$$\widehat{H}^{q-1}(K/H, F_H(Q_{n-3})) \cong \widehat{H}^{q+1}(K/H, A),$$

and therefore, from (1), we also have

$$\widehat{H}^{q+1}(K/H, A) \cong \widehat{H}^q(K/H, \tau_{\leq -1} C^\bullet).$$

On the other hand, from the definition of $B$, we know that

$$\widehat{H}^q(K/H, F_H(Q_{n-2})) \cong \widehat{H}^{q+1}(K/H, B).$$

Thus, we have an exact sequence

$$\cdots \to \widehat{H}^q(K/H, F_H(Q_{n-2})) \to \widehat{H}^q(K/H, \tau_{\leq -1} C^\bullet) \to \widehat{H}^{q+1}(K/H, H_{n-1}(H, \mathbb{Z}_p)) \to \cdots.$$  \hspace{1cm} (4)

By comparing this exact sequence with (2), we have

$$\widehat{H}^{q-2}(K/H, F_H(Q_{n-2})) \cong \widehat{H}^q(K/H, H_n(H, \mathbb{Z}_p)).$$

Thus, we obtain the following criterion determining the strict cohomological dimension of $G$:
Theorem 2.5. The following conditions are equivalent:

1. \( \text{scd}_p G \leq n + 1 \).
2. For each \( q \), \( \hat{H}^{q-2}(K/H, F_H(Q_{n-2})) \simeq \hat{H}^q(K/H, H_n(H, Z_p)) \).

3. Relation to class formations

Let \( G \) be a profinite group and \( A^\bullet \) a complex of \( G \)-modules acyclic outside \([−n, 0] \). Assume that the pair \((G, A^\bullet)\) forms a class formation. See [8] about such class formation including complexes. In this case, as in [1], we can make more explicit the conditions to determine \( \text{scd}_p G \).

Lemma 3.1. Assume that the pair \((G, A^\bullet)\) forms a class formation, where \( A^\bullet \) is acyclic outside \([−n, 0] \). The following conditions are equivalent:

1. For each \( r = 0, \ldots, n-1 \),
   \( \hat{H}^q(K/H, H^{-r}(H, A^\bullet)) \simeq \hat{H}^q(K/H, H_{r+1}(H, Z)) \).
2. For each \( q \), \( \hat{H}^{q-2}(K/H, F_H(Q_{n-2})) \simeq \hat{H}^q(K/H, H_n(H, Z_p)) \).

Proof. From the Tate–Nakayama theorem for hypercohomology [8], we have

\[ \hat{H}^q(K/H, \tau \leq -1 R\Gamma(H, A^\bullet)) \simeq \hat{H}^{q-2}(K/H, Z_p) \]

From the exact sequence

\[ 0 \to H_1(H, Z_p) \to F_H(Q_0) \to F_H(P_0) \to Z_p \to 0, \]

we obtain the following long exact sequence of cohomology groups:

\[ \cdots \to \hat{H}^{q-2}(K/H, Z_p) \to \hat{H}^{q}(K/H, H_1(H, Z_p)) \to \hat{H}^q(K/H, F_H(Q_0)) \to \hat{H}^{q-1}(K/H, Z_p) \to \cdots. \]

Noting the results of the Tate–Nakayama theorem and the assumptions, we derive the long exact sequence

\[ \cdots \to \hat{H}^{q}(K/H, \tau_{\leq 0} R\Gamma(H, A^\bullet)) \to \hat{H}^{q}(K/H, H^0(H, A^\bullet)) \to \hat{H}^{q+1}(K/H, \tau_{\leq 0} R\Gamma(H, A^\bullet)) \to \cdots. \]

By comparing the long exact sequence obtained from the distinguished triangle

\[ \tau_{\leq -1} R\Gamma(H, A^\bullet) \to \tau_{\leq 0} R\Gamma(H, A^\bullet) \to H^0(H, A^\bullet) \to \tau_{\leq -1} R\Gamma(H, A^\bullet)[-1], \]

we can conclude that

\[ \hat{H}^{q-2}(K/H, F_H(Q_0)) \simeq \hat{H}^q(K/H, \tau_{\leq -1} R\Gamma(H, A^\bullet)). \]

Similarly we have

\[ \hat{H}^{q-2}(K/H, F_H(Q_r)) \simeq \hat{H}^q(K/H, \tau_{\leq -r-1} R\Gamma(H, A^\bullet)). \]
Therefore,
\[
\hat{H}^{q-2}(K/H, F_H(Q_{n-2})) \simeq \hat{H}^q(K/H, \tau_{\leq -n+1} R\Gamma(H, A^*)) \\
\simeq \hat{H}^q(K/H, H_{n+1}(H, A^*)) \\
\simeq \hat{H}^q(K/H, H_n(H, Z_p)).
\]

The converse can be proven in a similar manner. □

From the above lemma, we have:

**Theorem 3.2.** The following are equivalent:

1. For each \( r = 0, \ldots, n - 1 \),
   \[
   \hat{H}^q(K/H, H^{-r}(H, A^*))(p) \simeq \hat{H}^q(K/H, H_{r+1}(H, Z))(p).
   \]
2. \( \text{scd}_p G \leq n + 1 \).

### 4. An example

We now apply the result of the previous section to higher-dimensional local fields.

Let \( K \) be a two-dimensional local field (a complete discrete valuation field whose residue field is a usual local field). Let \( Z(2) \) be the Lichtenbaum complex for \( K \). (For basic properties and proofs, see [10], [11] and [12].) We know that the pair \((\text{Gal}(K_s/K), Z(2)[2])\) forms a class formation [8].

We will use the criterion established in the previous section to prove the following:

**Theorem 4.1.** Let \( K \) be an \( 2 \)-dimensional local field and \( p \) a prime number different from the characteristic of \( K \). Then \( \text{scd}_p \text{Gal}(K_s/K) = 3 \).

**Remark 4.2.** Tate [14] proved this for the case that \( K \) is a usual local field. Brumer gave a new proof in [1, Theorem 6.1].

For general higher-dimensional local fields, we gave a proof in [9], showing the following:

**Theorem 4.3 ([9]).** Let \( K \) be an \( N \)-dimensional local field and let \( p \) be a prime number distinct from the characteristic of \( K \). Then \( \text{scd}_p \text{Gal}(K_s/K) = N + 1 \).

In [9], we showed a kind of generalization of the Tate duality of Galois cohomology of higher-dimensional local fields with a finite coefficient, and determined the strict cohomological dimension of them as its application.

We have learned from K. Kato that there is a proof based on the duality of \( p \)-adic étale cohomology, which seems to be easier to understand and straightforward (The method given by Kato is rather different from ours). We do not know who first gave the proof mentioned by Kato.
In order to show Theorem 4.1, we may prove the following:

**Proposition 4.4.** Let $L/K$ be an arbitrary finite extension of $K$. The kernel and cokernel of the homomorphisms

$$\omega_q^*: H^{3-q}(L, \mathbb{Z}(2)) \rightarrow H_q(L, \mathbb{Z})$$

are uniquely divisible for $q = 1, 2$. Here the maps $\omega_q^*$ are those induced from the reciprocity map of two-dimensional local class field theory.

Then, from Theorem 3.2, we can conclude $\text{scd}_p \text{Gal}(K_s/K) \leq 3$. On the other hand, from $[4]$ and $[5]$, we already know $\text{cd}_p \text{Gal}(K_s/K) = 3$.

Therefore, we can deduce that the strict cohomological dimension of a two-dimensional local field is 3.

The rest of the section is devoted to the proof of the proposition. We need the following lemma, an easy consequence of $[7, \text{Chap. I, §2, Theorem 2}].$

**Lemma 4.5.** Let $K$ be a 2-dimensional local filed and $p$ a prime number. Suppose that $K$ contains a primitive $p$-th root of unity. Then the canonical homomorphism

$$H^r(K, \mu_p^2) \rightarrow H_{3-r}(K, \mathbb{Z}/p\mathbb{Z})$$

is bijective for $r = 0, 1, 2$.

By using the standard arguments on the restriction and the corestriction maps for the Galois cohomology, we may assume that the field $L$ contains a primitive $p$-th root of unity.

First we prove $\text{Ker} \omega_2^* = \text{Coker} \omega_2^* = 0$.

Consider the commutative diagram

$$
\begin{array}{cccccc}
0 & \rightarrow & H^0(L, \mu_p^2) & \rightarrow & H^1(L, \mathbb{Z}(2)) & \rightarrow & H^1(L, \mathbb{Z}) \\
& & \downarrow \simeq & & \downarrow \omega_2^* & & \downarrow \omega_2^* \\
& & H_3(L, \mathbb{Z}/p\mathbb{Z}) & \rightarrow & H_2(L, \mathbb{Z}) & \rightarrow & H_2(L, \mathbb{Z}),
\end{array}
$$

where the upper and lower horizontal sequences are both exact. From this diagram, $\text{Ker} \omega_2^*$ is $p$-torsion-free. Since the group $\text{Ker} \omega_2^*$ is a torsion group, $\text{Ker} \omega_2^* = 0$. The homomorphism $H_3(L, \mathbb{Z}/p\mathbb{Z}) \rightarrow H_2(L, \mathbb{Z})$ in the diagram is also injective. Thus, we may conclude $\text{Coker} \omega_2^*$ is $p$-torsion-free, and therefore, we have $\text{Coker} \omega_2^* = 0$. (The group $H_2(L, \mathbb{Z})$ is also a torsion group.)

Next we prove that the group $\text{Ker} \omega_2^*$ is uniquely $p$-divisible. Consider the commutative diagram

$$
\begin{array}{cccccc}
0 & \rightarrow & H^1(L, \mathbb{Z}(2))/p & \rightarrow & H^1(L, \mu_p^2) & \rightarrow & K_2K_p & \rightarrow & 0 \\
& & \downarrow \simeq & & \downarrow \simeq & & \downarrow & & \\
0 & \rightarrow & H_2(L, \mathbb{Z})/p & \rightarrow & H_2(L, \mathbb{Z}/p\mathbb{Z}) & \rightarrow & H_1(L, \mathbb{Z})_p & \rightarrow & 0.
\end{array}
$$
Here the fact that the left vertical arrow is an isomorphism can be deduced from the vanishing of \( \text{Ker} \omega_2^* \) and \( \text{Coker} \omega_2^* \). From an investigation of the preceding diagram we see that the map \( K_2 K_p \rightarrow H_1(L, \mathbb{Z})_p \) is an isomorphism. On the other hand, by considering the commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & K_2 K_p \\
\downarrow \cong & & \downarrow \omega_1^* \\
0 & \longrightarrow & H_1(L, \mathbb{Z})_p \\
\end{array}
\begin{array}{ccc}
0 & \longrightarrow & K_2 K \\
\downarrow \omega_1^* & & \downarrow \omega_1^* \\
0 & \longrightarrow & H_1(L, \mathbb{Z}) \\
\end{array}
\begin{array}{ccc}
0 & \longrightarrow & H_1(L, \mathbb{Z})_p \\
\downarrow \omega_1^* & & \downarrow \omega_1^* \\
0 & \longrightarrow & H_1(L, \mathbb{Z}) \\
\end{array}
\]

we can conclude that the groups \( \text{Ker} \omega_1^* \) and \( \text{Coker} \omega_1^* \) are \( p \)-torsion-free.

The divisibility of \( \text{Ker} \omega_1^* \) can be deduced from [2, §2.6, Corollary].

Finally, the divisibility of the group \( \text{Coker} \omega_1^* \) can be deduced from Lemma 4.5.

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ON CERTAIN CUNTZ–PIMSNER ALGEBRAS

ALEX KUMJIAN

In memory of Gert K. Pedersen.

Let $A$ be a separable unital C*-algebra. Let $\pi : A \to \mathcal{L}(\mathcal{H})$ be a faithful representation of $A$ on a separable Hilbert space $\mathcal{H}$ such that $\pi(A) \cap K(\mathcal{H}) = \{0\}$. We show that $\mathcal{O}_E$, the Cuntz–Pimsner algebra associated to the Hilbert $A$-bimodule $E = \mathcal{H} \otimes_C A$, is simple and purely infinite. If $A$ is nuclear and belongs to the bootstrap class to which the UCT applies, the same applies to $\mathcal{O}_E$. Hence by the Kirchberg–Phillips Theorem the isomorphism class of $\mathcal{O}_E$ only depends on the $K$-theory of $A$ and the class of the unit.

In his seminal paper [Pm], Pimsner constructed a C*-algebra $\mathcal{O}_E$ from a Hilbert bimodule over a C*-algebra $A$ as a quotient of a concrete C*-algebra $\mathcal{T}_E$, an analogue of the Toeplitz algebra, acting on the Fock space associated to $E$. There has recently been much interest in these Cuntz–Pimsner algebras (or Cuntz–Krieger–Pimsner algebras), which generalize both crossed products by $\mathbb{Z}$ and Cuntz–Krieger algebras, as well as the associated Toeplitz algebras. The structure of these C*-algebras is not yet fully understood, though considerable progress has been made. For example, Pimsner found a six-term exact sequence for the $K$-theory of $\mathcal{O}_E$ that generalizes the Pimsner–Voiculescu exact sequence (see [Pm, Theorem 4.8]); conditions for simplicity were found in [Sc2, MS, KPW1, DPZ] and for pure infiniteness in [Z].

The purpose of the present note is to analyze the structure of Cuntz–Pimsner algebras associated to a certain class of Hilbert bimodules. Let $A$ be a separable unital C*-algebra and let $\pi : A \to \mathcal{L}(\mathcal{H})$ be a faithful nondegenerate representation of $A$ on a separable Hilbert space $\mathcal{H}$ such that $\pi(A) \cap K(\mathcal{H}) = \{0\}$. Then $E = \mathcal{H} \otimes_C A$ is a Hilbert bimodule over $A$ in a natural way. We show that $\mathcal{O}_E$ is separable, simple and purely infinite. If $A$ is nuclear and in the bootstrap class, then the same holds for $\mathcal{O}_E$ and thus by the Kirchberg–Phillips theorem the isomorphism class of $\mathcal{O}_E$ is completely determined by the $K$-theory of $A$ together with the class of the unit (since $\mathcal{O}_E$ is $KK$-equivalent to $A$).

Many examples of Cuntz–Pimsner algebras found in the literature arise from Hilbert bimodules that are finitely generated and projective; in such
cases the left action must consist entirely of compact operators. Our examples do not fall in this class; in fact, the left action has trivial intersection with the compacts. And this has some interesting consequences: \( \mathcal{O}_E \cong T_E \) (see \[Pm, Corollary 3.14\]) and the natural embedding \( A \hookrightarrow \mathcal{O}_E \) induces a \( KK \)-equivalence (see \[Pm, Corollary 4.5\]).

In §1 we review some basic facts concerning the construction of \( T_E \) as operators on the Fock space of \( E \) and the gauge action \( \lambda : \mathbb{T} \to \text{Aut}(T_E) \). We assume that the left action of \( A \) does not meet the compacts \( \mathcal{K}(E) \) and identify \( \mathcal{O}_E \) with \( T_E \). The fixed point algebra \( \mathcal{F}_E \), the analogue of the AF-core of a Cuntz–Krieger algebra, contains a canonical descending sequence of essential ideals indexed by \( \mathbb{N} \) with trivial intersection. The crossed product \( \mathcal{O}_E \rtimes_\lambda \mathbb{T} \) has a similar collection of essential ideals indexed by \( \mathbb{Z} \) on which the dual group of automorphisms acts in a natural way. By Takesaki–Takai duality,
\[
\mathcal{O}_E \otimes \mathcal{K}(L^2(\mathbb{T})) \cong (\mathcal{O}_E \rtimes_\lambda \mathbb{T}) \rtimes_\lambda \mathbb{Z};
\]
hence, much of the structure of \( \mathcal{O}_E \) is revealed through an analysis of the double crossed product.

In §2 we show that if \( E \) is the Hilbert bimodule over \( A \) associated to a representation as described above, then for every nonzero positive element \( d \in \mathcal{O}_E \) there is a \( z \in \mathcal{O}_E \) such that \( z^*dz = 1 \); it follows that \( \mathcal{O}_E \) is simple and purely infinite (see Theorem 2.8). The proof of this proceeds through a sequence of lemmas and is patterned on the proof of \[R\o, Theorem 2.1\], which is in turn based on a key lemma of Kishimoto (see \[Ks, Lemma 3.2\]). Our argument uses the version of this lemma found in \[OP3, Lemma 7.1\] and this requires that we show that the Connes spectrum of the dual action is full (this is also an ingredient in the proof of simplicity found in \[DPZ\]). We invoke a version of a key lemma of Rørdam for crossed products by \( \mathbb{Z} \) that arise from automorphisms with full Connes spectrum. The fact that \( \mathcal{O}_E \) embeds equivariantly into \( (\mathcal{O}_E \rtimes_\lambda \mathbb{T}) \rtimes_\lambda \mathbb{Z} \) allows us to apply this lemma to \( \mathcal{O}_E \).

In §3 we use the Kirchberg–Phillips theorem to collect some consequences of this theorem as indicated above and discuss certain connections with reduced (amalgamated) free products.

We fix some notation and terminology. Given a \( C^* \)-algebra \( B \) we let \( \hat{B} \) denote its spectrum, that is, the collection of irreducible representations modulo unitary equivalence endowed with the Jacobson topology (see \[Pd, §4.1\]). If \( I \) is an ideal in a \( C^* \)-algebra \( B \), every irreducible representation of \( I \) extends uniquely to an irreducible representation of \( B \). This allows one to identify \( \hat{I} \) with an open subset of \( \hat{B} \), the complement of which consists of the classes of irreducible representations that vanish on \( I \). Given a *-automorphism \( \beta \) of a \( C^* \)-algebra \( B \), let \( \Gamma(\beta) \) denote the Connes spectrum
of $\beta$ (see [O, Co] or [Pd, §8.8]); recall that
\[
\Gamma(\beta) = \bigcap_H \text{Sp} (\beta|_H)
\]
where the intersection is taken over all nonzero $\beta$-invariant hereditary subalgebras $H$. A $C^*$-algebra is said to be purely infinite if every nonzero hereditary subalgebra contains an infinite projection.

1. Preliminaries

We review some basic facts concerning Cuntz–Pimsner algebras; we shall be mainly interested in those that arise from bimodules for which the left action has trivial intersection with the compacts (see Remark 1.3). Let $A$ be a $C^*$-algebra.

**Definition 1.1** (see [L, pp. 2–4], [Ka, pp. 134, 135] and [Ri1, Def. 2.1]).

Let $E$ be a right $A$-module. Then $E$ is said to be a (right) pre-Hilbert $A$-module if it is equipped with an $A$-valued inner product $\langle \cdot, \cdot \rangle_A$ satisfying the following conditions for all $\xi, \eta, \zeta \in E$, $s, t \in \mathbb{C}$, and $a \in A$:

(i) $\langle \xi, s\eta + t\zeta \rangle_A = s\langle \xi, \eta \rangle_A + t\langle \xi, \zeta \rangle_A$.

(ii) $\langle \xi, \eta a \rangle_A = \langle \xi, \eta \rangle_A a$.

(iii) $\langle \eta, \xi \rangle_A = \langle \xi, \eta \rangle_A^*$.

(iv) $\langle \xi, \xi \rangle_A \geq 0$ and $\langle \xi, \xi \rangle_A = 0$ only if $\xi = 0$.

$E$ is said to be a (right) Hilbert $A$-module if it is complete in the norm $\|\xi\| = \|\langle \xi, \xi \rangle_A\|^{1/2}$.

A Hilbert $A$-module $E$ is said to be full if the span of the values of the inner product is dense. The collection of bounded adjointable operators on $E$, $\mathcal{L}(E)$, is a $C^*$-algebra. The closure of the span of operators of the form $\theta_{\xi, \eta}$ for $\xi, \eta \in E$ (where $\theta_{\xi, \eta}(\zeta) = \langle \eta, \zeta \rangle_A$ for $\zeta \in E$) forms an essential ideal in $\mathcal{L}(E)$, denoted by $\mathcal{K}(E)$. A Hilbert space is a Hilbert module over $\mathbb{C}$.

**Definition 1.2.** Let $E$ be a Hilbert $A$-module and let $\varphi : A \to \mathcal{L}(E)$ be an injective $*$-homomorphism. The pair $(E, \varphi)$ is said to be a Hilbert bimodule over $A$ (or a Hilbert $A$-bimodule).

Pimsner defines the Cuntz–Pimsner algebra $\mathcal{O}_E$ as a quotient of the analogue of the Toeplitz algebra, $\mathcal{T}_E$, generated by creation operators on the Fock space of $E$ (see [Pm]). The injectivity of $\varphi$ is not really necessary (see [Pm, Remark 1.2(1)]). We will henceforth assume that $E$ is full (see [Pm, Remark 1.2(3)])..

The Fock space of $E$ is the Hilbert $A$-module
\[
\mathcal{E}_+ = \bigoplus_{n=0}^{\infty} E^\otimes n
\]
where $E^\otimes 0 = A$, $E^\otimes 1 = E$ and for $n > 1$, $E^\otimes n$ is the $n$-fold tensor product: 

$$E^\otimes n = E \otimes_A \cdots \otimes_A E.$$ 

The tensor product used here is called the inner tensor product by Lance (see [L, p. 41], but note Lance uses different notation; see also Theorem 5.9 of [Ri1]). Observe that $\mathcal{E}_+$ is also a Hilbert $A$-bimodule with left action defined by $\varphi_+(a)b = ab$ for $a, b \in A = E^\otimes 0$ and

$$\varphi_+(a)(\xi_1 \otimes \cdots \otimes \xi_n) = \varphi(a) \xi_1 \otimes \cdots \otimes \xi_n$$

for $a \in A$ and $\xi_1 \otimes \cdots \otimes \xi_n \in E^\otimes n$.

Then $T_E \subseteq \mathcal{L}(\mathcal{E}_+)$ is the C*-algebra generated by the creation operators $T_\xi$ for $\xi \in E$ where $T_\xi(a) = \xi a$ and 

$$T_\xi(\xi_1 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n.$$ 

Note that $T_\xi^* T_\eta = \varphi_+(\langle \xi, \eta \rangle_A)$ for $\xi, \eta \in E$. Since $E$ is full, $\varphi_+(A) \subseteq T_E$; let $\iota : A \hookrightarrow T_E$ denote the embedding. One may also define $T_\xi$ for $\xi \in E^\otimes n$ in an analogous manner and we have $T_\xi^* T_\eta = \iota(\langle \xi, \eta \rangle_A)$ for $\xi, \eta \in E^\otimes n$.

There is an embedding $\iota_n : \mathcal{K}(E^\otimes n) \hookrightarrow T_E$ (identify $\mathcal{K}(E^\otimes 0)$ with $A$), given for $n > 0$ by $\iota_n(\theta_{\xi,\eta}) = T_\xi T_\eta^*$ for $\xi, \eta \in E^\otimes n$. Such operators preserve the grading of $\mathcal{E}_+$ and there is an embedding $\mathcal{K}(E^\otimes n) \hookrightarrow \mathcal{L}(E^\otimes m)$ for $m \geq n$. Let $C_n$ denote the C*-subalgebra of $T_E$ generated by operators of the form $T_\xi T_\eta^*$ for $\xi, \eta \in E^\otimes k$ with $k \leq n$ (by convention $C_0 = \iota(A)$). Then the $C_n$ form an ascending family of C*-subalgebras.

**Remark 1.3.** With notation as above the natural map $C_n \rightarrow \mathcal{L}(E^\otimes m)$ is an embedding for $m \geq n$. Suppose $\varphi(A) \cap \mathcal{K}(E) = \{0\}$; then by [Pm, Corollary 3.14] $T_E \cong O_E$ and the inclusion $A \hookrightarrow O_E$ induces a $KK$-equivalence (see [Pm, Corollary 4.5]). Under the isomorphism of $T_E$ with $O_E$, $\bigcup_n C_n$ is mapped to $\mathcal{F}_E$, the analog of the AF core of a Cuntz–Krieger algebra.

For the remainder of this section we shall assume that $\varphi(A) \cap \mathcal{K}(E) = \{0\}$ and identify $T_E$ with $O_E$.

**Proposition 1.4.** For each $n \in \mathbb{N}$ the C*-subalgebra $J_n$ generated by the $\iota_k(\mathcal{K}(E^\otimes k))$ for $k \geq n$ is an essential ideal in $\mathcal{F}_E$. We obtain a descending sequence of ideals

$$J_0 \supset J_1 \supset J_2 \supset \cdots$$

with $J_0 = \mathcal{F}_E$ and $\bigcap_n J_n = \{0\}$. Furthermore, $J_n/J_{n+1} \cong \mathcal{K}(E^\otimes n)$ (thus $J_n/J_{n+1}$ is strongly Morita equivalent to $A$) and the restriction of the quotient map yields an isomorphism $C_n \cong \mathcal{F}_E/J_{n+1}$.

**Proof.** Given $n \in \mathbb{N}$ it is clear that $J_n$ is an ideal (see [Pm, Definition 2.1]). To see that $J_n$ is essential it suffices to show that for every $m$ and nonzero element $c \in C_m$ there is an element $d \in \mathcal{K}(E^\otimes k)$ for some $k \geq n$ such that $c_\iota_k(d) \neq 0$. Let $k$ be an integer with $k \geq \max(m, n)$; since the map from
$C_m$ to $\mathcal{L}(E^\otimes k)$ is an embedding for $k \geq m$, $c\xi \neq 0$ for some $\xi \in E^\otimes k$. Then $cT_\xi T_\xi^* \neq 0$ and we take $d = \theta_{\xi, \xi}$.

The $J_n$ form a descending sequence of ideals by construction. Since $\varphi(A)$ and $K(E)$ have trivial intersection and $K(E) \hookrightarrow \mathcal{L}(E^\otimes k)$ is nondegenerate for $k \geq 1$, the image of $A$ in $\mathcal{L}(E^\otimes k)$ has trivial intersection with $K(E^\otimes k)$ for $k \geq 1$; it follows that

$$\iota_m(K(E^\otimes m)) \cap \iota_n(K(E^\otimes n)) = \{0\}$$

and, hence, $C_m \cap J_n = \{0\}$ for $m < n$. Thus, $\bigcap_n J_n = \{0\}$, for $F_E$ is the inductive limit of the $C_m$. Further, for each $n$ we have

$$J_n = \iota_n(K(E^\otimes n)) + J_{n+1} \quad \text{and} \quad \iota_n(K(E^\otimes n)) \cap J_{n+1} = \{0\};$$

it follows that $J_n/J_{n+1} \cong K(E^\otimes n)$. Finally, since

$$F_E = C_n + J_{n+1} \quad \text{and} \quad C_n \cap J_{n+1} = \{0\},$$

we have $C_n \cong F_E/J_{n+1}$.

There is a strongly continuous action

$$\lambda : \mathbb{T} \to \text{Aut}(O_E)$$

such that $\lambda_t(T_\xi) = tT_\xi$. The fixed point algebra under this action is $F_E$ and we have a faithful conditional expectation $P_E : O_E \to F_E$ given by

$$P_E(x) = \int_\mathbb{T} \lambda_t(x) dt.$$

Consider the spectral subspaces of $O_E$ under this action: for $n \in \mathbb{Z}$

$$(O_E)_n = \{x \in O_E : \lambda_t(x) = t^n x \ \text{for all} \ t \in \mathbb{T}\}.$$

**Remark 1.5.** Note that $(O_E)_n$ is the closure of the span of elements of the form $T_\xi T_\eta^*$, where $\xi \in E^\otimes k$ and $\eta \in E^\otimes l$ with $n = k - l$. For $n \geq 0$ and $x \in (O_E)_n$ we have $x^* x \in F_E$ and $xx^* \in J_n$. We may regard $(O_E)_n$ as a $J_n$-$F_E$-equivalence bimodule (or $J_n$-$F_E$-imprimitivity bimodule; see [Ri1, Def. 6.10]). Hence, $J_n$ is strongly Morita equivalent to $F_E$ for each $n \geq 0$ (see [Ri2, Def. 1.1], [L, p. 74]). If we regard $(O_E)_1$ as a Hilbert $F_E$-bimodule, we have

$$E \otimes_A F_E \cong (O_E)_1,$$

where the isomorphism is implemented by the map $\xi \otimes a \mapsto T_\xi a$ (the Hilbert $F_E$-module $E \otimes_A F_E$ is denoted $E_\infty$ in [Pm, §2]). The crossed product $O_E \rtimes_A \mathbb{T}$ may be identified with the closure of the subalgebra of $O_E \otimes K(\ell^2(\mathbb{Z}))$ consisting of finite sums of the form

$$\sum x_{ij} \otimes e_{ij},$$

where $e_{ij}$ are the standard rank-one partial isometries in $K(\ell^2(\mathbb{Z}))$ and $x_{ij} \in (O_E)_{j-i}$. \hfill \square
Let $\hat{\lambda}: \mathbb{Z} \to \text{Aut}(\mathcal{O}_E \rtimes \chi \mathbb{T})$ denote the dual automorphism group.

**Proposition 1.6.** There is an embedding $\epsilon: \mathcal{F}_E \hookrightarrow \mathcal{O}_E \rtimes \chi \mathbb{T}$ onto a corner and a collection of essential ideals $\{I_n\}_{n \in \mathbb{Z}}$ in $\mathcal{O}_E \rtimes \chi \mathbb{T}$ satisfying the following conditions:

(i) For all $n \in \mathbb{Z}$, $\mathcal{F}_E$ is strongly Morita equivalent to $I_n$ and $A$ is strongly Morita equivalent to $I_n/I_{n+1}$.

(ii) For all $n \geq 0$, $\epsilon(J_n) = \epsilon(1)I_n\epsilon(1)$.

(iii) $I_n \subset I_m$ if $m \leq n$.

(iv) $\bigcap_n I_n = \{0\}$.

(v) $\bigcup_n I_n = \mathcal{O}_E \rtimes \chi \mathbb{T}$.

(vi) $\hat{\lambda}_k(I_n) = I_{n+k}$.

*Proof.* We use the identification, given in Remark 1.5, between $\mathcal{O}_E \rtimes \chi \mathbb{T}$ with a $C^*$-subalgebra of $\mathcal{O}_E \otimes \mathcal{K}(\ell^2(\mathbb{Z}))$. For each $n$ let $I_n$ be the ideal generated by $p_n = 1 \otimes e_{mn}$. Since $\mathcal{F}_E = (\mathcal{O}_E)_0$, it follows that $\mathcal{F}_E$ is isomorphic to the corner determined by $p_n$ and thus is strongly Morita equivalent to $I_n$. The desired embedding $\epsilon: \mathcal{F}_E \hookrightarrow \mathcal{O}_E \rtimes \chi \mathbb{T}$ is given by $\epsilon(a) = a \otimes e_{00}$.

Given an element of the form $a_{mn} = x_{mn} \otimes e_{mn}$ in $\mathcal{O}_E \rtimes \chi \mathbb{T}$ with $m \leq n$, we have

$$a_{mn}^*a_{mn} = x_{mn}^*x_{mn} \otimes e_{nn} \quad \text{and} \quad a_{mn}a_{mn}^* = x_{mn}x_{mn}^* \otimes e_{mm},$$

with $x_{mn}x_{mn}^* \in J_{n-m}$; since $p_n$ may be expressed as a finite sum of elements of the form $a_{mn}^*a_{mn}$, it follows that $I_n \subset I_m$ and that

$$p_m I_n p_m = J_{n-m} \otimes e_{mm}.$$

Moreover, $I_n$ is essential in $I_m$, since $J_{n-m}$ is an essential ideal in $\mathcal{F}_E$ (by Proposition 1.4). Since $q_n = \sum_{i=-n}^n p_i \in I_n$ and $\{q_n\}_n$ forms an approximate identity, we have $\bigcup_n I_n = \mathcal{O}_E \rtimes \chi \mathbb{T}$. Thus $I_n$ is an essential ideal in $\mathcal{O}_E \rtimes \chi \mathbb{T}$ for all $n \in \mathbb{Z}$. Assertion (ii) follows immediately from (i). Assertion (vi) follows from the fact that $\hat{\lambda}_k(p_n) = 1 \otimes p_{n+k}$. The remaining assertions follow from Proposition 1.4. \hfill $\square$

### 2. $\mathcal{O}_E$ is simple and purely infinite

Let $A$ be a separable unital $C^*$-algebra and let $\pi: A \to \mathcal{L}(\mathfrak{H})$ be a faithful nondegenerate representation of $A$ on a separable nontrivial Hilbert space $\mathfrak{H}$; since $\pi$ is nondegenerate we have $\pi(1) = 1$.

**Proposition 2.1.** With $A$ and $\pi: A \to \mathcal{L}(\mathfrak{H})$ as above,

$$E = \mathfrak{H} \otimes_{\mathbb{C}} A$$

is a full Hilbert bimodule over $A$ under the operations

$$\langle \xi \otimes a, \eta \otimes b \rangle_A = \langle \xi, \eta \rangle a^* b, \quad \varphi(a)(\xi \otimes b) = \pi(a)\xi \otimes b$$
for all $\xi, \eta \in \mathfrak{H}$ and $a, b \in A$. Moreover, if $\pi(A) \cap \mathcal{K}(\mathfrak{H}) = \{0\}$, then $\varphi(A) \cap \mathcal{K}(E) = \{0\}$ and $\mathcal{O}_E \cong \mathcal{T}_E$.

**Proof.** $E = \mathfrak{H} \otimes_A A$ is the tensor product of the Hilbert $A$-$C$-bimodule $\mathfrak{H}$ and the Hilbert $C$-$A$-bimodule $A$ as defined by Rieffel in [Ri1, Theorem 5.9] (see also [L, p. 41]). The natural map from $\mathcal{L}(\mathfrak{H})$ to $\mathcal{L}(E) = \mathcal{L}(\mathfrak{H} \otimes_A A)$ induces an embedding $\mathcal{L}(\mathfrak{H})/\mathcal{K}(\mathfrak{H}) \hookrightarrow \mathcal{L}(E)/\mathcal{K}(E)$ (since $\mathcal{K}(\mathfrak{H})$ is mapped into $\mathcal{K}(E)$ and the Calkin algebra $\mathcal{L}(\mathfrak{H})/\mathcal{K}(\mathfrak{H})$ is simple). Hence, if $\pi(A) \cap \mathcal{K}(\mathfrak{H}) = \{0\}$, then $\varphi(A) \cap \mathcal{K}(E) = \{0\}$. The last assertion, $\mathcal{O}_E \cong \mathcal{T}_E$, follows by [Pm, Corollary 3.14].

Henceforth, we assume that $\pi(A) \cap \mathcal{K}(\mathfrak{H}) = \{0\}$ and identify $\mathcal{O}_E$ with $\mathcal{T}_E$. The aim of this section is to show that the Cuntz–Pimsner algebra $O$ is unital and $E$ is simple and purely infinite. Simplicity may be proven directly by invoking [Sc2, Theorem 3.9]: if $A$ is unital and $E$ is full, then $\mathcal{O}_E$ is simple if and only if $E$ is minimal and nonperiodic. Lemma 2.3 would then be a consequence of [OP1, Theorem 6.5]. We follow a more indirect route patterned on the proof of [Ro, Theorem 2.1]; this will also show that $\mathcal{O}_E$ is purely infinite.

**Remark 2.2.** With $E = \mathfrak{H} \otimes_A A$ as above, we have $E^{\otimes n} \cong \mathfrak{H}^{\otimes n} \otimes_A A$ via the map

$$(\xi_1 \otimes a_1) \otimes (\xi_2 \otimes a_2) \otimes \cdots \otimes (\xi_n \otimes a_n) \mapsto (\xi_1 \otimes \pi(a_1) \xi_2 \otimes \cdots \otimes \pi(a_{n-1}) \xi_n) \otimes a_n.$$ 

If $\sigma : A \rightarrow \mathcal{L}(\mathfrak{H})$ is a nondegenerate representation of $A$ on a Hilbert space $\mathfrak{H}$, then

$$E \otimes_A \mathfrak{H} \cong \mathfrak{H} \otimes_A A \otimes_A \mathfrak{H} \cong \mathfrak{H} \otimes_A \mathfrak{H}$$

and, hence,

$$E^{\otimes n} \otimes_A \mathfrak{H} \cong E^{\otimes n-1} \otimes_A E \otimes_A \mathfrak{H} \cong E^{\otimes n-1} \otimes_A \mathfrak{H} \otimes_A \mathfrak{H}.$$

Recall that the action of $\mathcal{F}_E$ on Fock space preserves the natural grading. Let $\tilde{\sigma}_n$ denote the representation of $\mathcal{F}_E$ on $E^{\otimes n} \otimes_A \mathfrak{H}$ given by left action on $E^{\otimes n}$. Then the restriction of $\tilde{\sigma}_n$ to $C_n$ is faithful: indeed, this follows from the facts that the natural map

$$\mathcal{L}(E^{\otimes n-1}) \rightarrow \mathcal{L}(E^{\otimes n-1} \otimes_A \mathfrak{H} \otimes \mathfrak{H}) \cong \mathcal{L}(E^{\otimes n} \otimes_A \mathfrak{H})$$

is an embedding (since $\pi$ is faithful) and that $\tilde{\sigma}_n|_{\mathcal{K}(E^{\otimes n-1})}$ factors through $\mathcal{L}(E^{\otimes n-1})$. Note that $\tilde{\sigma}_n$ is equivalent to the representation of $\mathcal{F}_E$ obtained from $\sigma$ as follows: use the strong Morita equivalence between $A$ and $J_n/J_{n+1}$ to obtain a representation of $J_n/J_{n+1}$ and extend this to a representation of $\mathcal{F}_E$. Since the restriction of $\tilde{\sigma}_n$ to $C_{n-1}$ is faithful, $\ker \tilde{\sigma}_n \subset J_n$ (see Proposition 1.4). It follows that the closure of a point in $\hat{I}_n - \hat{I}_{n+1}$ contains the complement of $J_n$. A similar assertion holds for $\mathcal{O}_E \rtimes_\lambda \mathbb{T}$: for any $n \in \mathbb{Z}$ the closure of a point in $\hat{I}_n - \hat{I}_{n+1}$ contains the complement of $\hat{I}_n$. 
Lemma 2.3. With $A$ and $E$ as above, $\Gamma(\hat{\lambda}) = \mathbb{T}$, where $\hat{\lambda}$ is the dual action of $\mathbb{Z}$ on $\mathcal{O}_E \rtimes_\lambda \mathbb{T}$.

Proof. By [OP2, Theorem 4.6] it suffices to find a dense invariant subset of $(\mathcal{O}_E \rtimes_\lambda \mathbb{T})^\sim$ on which $\hat{\lambda}^*$ acts freely. That is, we must find an irreducible representation $\sigma$ of $\mathcal{O}_E \rtimes_\lambda \mathbb{T}$ such that $\{[\sigma \circ \hat{\lambda}_n] : n \in \mathbb{Z}\}$, the orbit of the unitary equivalence class of $\sigma$ under $\hat{\lambda}^*$, is dense in $(\mathcal{O}_E \rtimes_\lambda \mathbb{T})^\sim$ and satisfies $[\sigma \circ \hat{\lambda}_m] \neq [\sigma \circ \hat{\lambda}_n]$ if $m \neq n$. Let $\sigma_0$ be an irreducible representation of $A$ and use the strong Morita equivalence between $A$ and $I_0/I_1$ to obtain an irreducible representation $\sigma'$ of $I_0/I_1$. Then $\sigma$, the extension of $\sigma'$ to $\mathcal{O}_E \rtimes_\lambda \mathbb{T}$, is also irreducible. The classes $[\sigma \circ \hat{\lambda}_n]$ are distinct, for if $m < n$, $\sigma \circ \hat{\lambda}_m$ vanishes on $I_n$. Moreover, for each $n \in \mathbb{Z}$ the closure of $[\sigma \circ \hat{\lambda}_n]$ in $(\mathcal{O}_E \rtimes_\lambda \mathbb{T})^\sim$ includes the classes of all irreducible representations that vanish on $I_n$ (since $[\sigma \circ \hat{\lambda}_n] \in \widehat{I}_n - \widehat{I}_{n+1}$; see Remark 2.2). Hence, $\{[\sigma \circ \hat{\lambda}_n] : n \in \mathbb{Z}\}$ is dense in $(\mathcal{O}_E \rtimes_\lambda \mathbb{T})^\sim$. \hfill $\square$

Using Takesaki–Takai duality we show below that a C$^*$-algebra $D$ equipped with an action $\alpha$ of $\mathbb{T}$ may be embedded equivariantly as a corner in $(D \rtimes_\alpha \mathbb{T}) \rtimes_\rho \mathbb{Z}$. This fact is related to Rosenberg’s observation that the fixed point algebra under a compact group action embeds as a corner in the crossed product (see [Ro]).

Proposition 2.4. Given a unital C$^*$-algebra $D$ and a strongly continuous action $\alpha : \mathbb{T} \to \text{Aut} (D)$, there is an isomorphism $\psi$ of $D$ onto a full corner of $(D \rtimes_\alpha \mathbb{T}) \rtimes_\rho \mathbb{Z}$ which is equivariant in the sense that $\hat{\alpha} \circ \psi = \psi \circ \alpha_t$ for all $t \in \mathbb{T}$. Moreover, $\psi(1) \in D \rtimes_\alpha \mathbb{T}$.

Proof. By Takesaki–Takai duality [Pd, 7.9.3] there is an isomorphism
\[
\gamma : D \otimes \mathcal{K}(L^2(\mathbb{T})) \cong (D \rtimes_\alpha \mathbb{T}) \rtimes_\rho \mathbb{Z},
\]
which is equivariant with respect to $\alpha \otimes \text{Ad} \rho$ and $\hat{\alpha}$. (where $\rho$ is the right regular representation of $\mathbb{T}$ on $L^2(\mathbb{T})$). The desired embedding is obtained by finding an $\text{Ad} \rho$ invariant minimal projection $p$ in $\mathcal{K}(L^2(\mathbb{T}))$ (cf. [Ro]): set $\psi(d) = \gamma(d \otimes p)$ for $d \in D$. Since $\psi$ is equivariant, $\psi(1)$ is in the fixed point algebra of $\hat{\alpha}$; hence, $\psi(1) \in D \rtimes_\alpha \mathbb{T}$. \hfill $\square$

The following lemma is adapted from [Ro, Lemma 2.4]; the proof follows Rørdam’s but we substitute [OP3, Lemma 7.1] for [KS, Lemma 3.2].

Lemma 2.5. Let $B$ be a C$^*$-algebra, let $\beta$ be an automorphism of $B$ such that $\Gamma(\beta) = \mathbb{T}$, and let $P$ denote the canonical conditional expectation from $B \rtimes_\beta \mathbb{Z}$ to $B$. For every positive element $y \in B \rtimes_\beta \mathbb{Z}$ and $\varepsilon > 0$ there are positive elements $x, b \in B$ such that
\[
\|b\| > \|P(y)\| - \varepsilon, \quad \|x\| \leq 1 \quad \text{and} \quad \|xy - b\| < \varepsilon.
\]
If \( y \) is in the corner determined by a projection \( p \in B \), then \( x, b \) may also be chosen to be in the corner.

Proof. As in the proof of [Ro, Lemma 2.4] we may assume (by perturbing \( y \) if necessary) that \( y \) is of the form

\[
y = y_n u^{-n} + \cdots + y_1 u^{-1} + y_0 + y_1 u + \cdots + y_n u^n
\]

for some \( n \), where \( y_j \in B \) and \( u \) is the canonical unitary in \( B \rtimes_\beta \mathbb{Z} \) implementing the automorphism \( \beta \); note that \( y_0 = P(y) \) is positive.

By [OP3, Theorem 10.4] \( \beta^k \) is properly outer for all \( k \neq 0 \). Hence, by [OP3, Lemma 7.1] there is a positive element \( x \) with \( \|x\| = 1 \) such that

\[
\|xy_0 x\| > \|y_0\| - \varepsilon \quad \text{and} \quad \|xy_k u^k x\| = \|xy_k \beta^k(x)\| < \varepsilon/2n \quad \text{for } 0 < |k| \leq n.
\]

Set \( b = xy_0 x \); then a straightforward calculation yields \( \|xy - b\| < \varepsilon \). We now verify the last assertion. Suppose that \( y \) is in the corner determined by a projection \( p \in B \); we may again assume that \( y \) is of the above form. Since \( P \) is a conditional expectation onto \( B \), \( y_0 = P(y) \) is also in the corner determined by \( p \). In the proof of [OP3, Lemma 7.1] the positive element \( x \) is constructed in the hereditary subalgebra determined by \( y_0 \); hence we may assume that \( x \) and therefore also \( b = xy_0 x \) lies in the same corner. \( \square \)

Recall that \( C_n \) is the \( C^* \)-subalgebra of \( \mathcal{F}_E \) generated by operators of the form \( T_\xi T_\eta^* \) for \( \xi, \eta \in E^{\otimes k} \) with \( k \leq n \) and that they form an ascending family of \( C^* \)-subalgebras with dense union. The subspace \( E^{\otimes n} \) is left invariant by \( C_n \) and there is an embedding \( C_n \hookrightarrow \mathcal{L}(E^{\otimes n}) \).

Lemma 2.6. Given a positive element \( c \in C_n \) and \( \varepsilon > 0 \), there is \( \xi \in E^{\otimes n} \) with \( \|\xi\| = 1 \) such that \( T_\xi c T_\xi^* \in C_0 \) and \( \|T_\xi c T_\xi^*\| > \|c\| - \varepsilon \).

Proof. The first assertion follows from a straightforward calculation: given \( c \in C_n \) and \( \xi \in E^{\otimes n} \), we have \( c\xi \in E^{\otimes n} \) and

\[
T_\xi c T_\xi^* = T_\xi^* T_\xi = \iota(\langle \xi, c\xi \rangle) \in C_0.
\]

The second assertion follows from the embedding \( C_n \hookrightarrow \mathcal{L}(E^{\otimes n}) \) and the fact that

\[
\|d\| = \sup \{ \|\langle \xi, d\xi \rangle_A\| : \xi \in E^{\otimes n}, \|\xi\| = 1 \}
\]

for \( d \in \mathcal{L}(E^{\otimes n}) \) positive. \( \square \)

Lemma 2.7. Given a positive element \( a \in A \) and \( \varepsilon > 0 \) with \( \|a\| > \varepsilon \), there is \( \eta \in E \) with \( \|\eta\| \leq (\|a\| - \varepsilon)^{-1/2} \) such that \( T_\eta^* \iota(a) T_\eta = 1 \).

Proof. Let \( f \) be a continuous nonzero real-valued function supported on the interval \( [\|a\| - \varepsilon, \|a\|] \) and choose a vector \( \zeta \in \pi(f(a)) \mathcal{F} \) such that

\[
\langle \zeta, \pi(a) \zeta \rangle = 1; \quad \text{we have}
\]

\[
(\|a\| - \varepsilon)\|\zeta\|^2 \leq \|\langle \zeta, \pi(a) \zeta \rangle\| = 1.
\]

Then \( \eta = \zeta \otimes 1 \in E \) satisfies the desired conditions. \( \square \)
It will now follow that \( \mathcal{O}_E \) is simple and purely infinite (compare the proof of [Ro, Theorem 2.1]).

**Theorem 2.8.** For every nonzero positive element \( d \in \mathcal{O}_E \) there is a \( z \in \mathcal{O}_E \) such that \( z^* dz = 1 \). Hence, \( \mathcal{O}_E \) is simple and purely infinite.

**Proof.** Let \( d \in \mathcal{O}_E \) be a nonzero positive element and choose \( \varepsilon \) such that \( 0 < \varepsilon < \frac{1}{4} \| P(d) \| \). By Proposition 2.4 there is a \( \mathbb{T} \)-equivariant isomorphism \( \psi \) from \( \mathcal{O}_E \) onto a corner of \( (\mathcal{O}_E \rtimes_{\lambda} \mathbb{T}) \rtimes_{\lambda} \mathbb{Z} \) determined by a projection \( p \in \mathcal{O}_E \rtimes_{\lambda} \mathbb{T} \). We now apply Lemma 2.5 to the element \( y = \psi(d) \) and the automorphism \( \beta = \lambda_1 \) (note that \( \Gamma(\lambda_1) = \mathbb{T} \) by Lemma 2.3). We identify \( \mathcal{O}_E \) with the corner determined by \( p \); under this identification \( \mathcal{F}_E \) is identified with \( p(\mathcal{O}_E \rtimes_{\lambda} \mathbb{T})p \). There are then positive elements \( x, b \in \mathcal{F}_E \) such that
\[
\| b \| > \| P(d) \| - \varepsilon, \quad \| x \| \leq 1 \quad \text{and} \quad \| xdx - b \| < \varepsilon.
\]

Since \( \bigcup_n C_n \) is dense in \( \mathcal{F}_E \) we may assume that \( b \in C_n \) for some \( n \). Hence, by Lemma 2.6 there is \( \xi \in E^\otimes n \) with \( \| \xi \| = 1 \) such that
\[
T_\xi^* b T_\xi \in C_0 \quad \text{and} \quad \| T_\xi^* b T_\xi \| > \| b \| - \varepsilon.
\]

Let \( a \) denote the unique element of \( A \) such that \( \iota(a) = T_\xi^* b T_\xi \); then \( \| a \| > \| P(d) \| - 2\varepsilon \) and
\[
\| T_\xi^* xdx T_\xi - \iota(a) \| = \| T_\xi^* (xdx - b) T_\xi \| < \varepsilon.
\]

By Lemma 2.7 there is \( \eta \in E \) such that \( T_\eta^* \iota(a) T_\eta = 1 \) and
\[
\| \eta \| \leq (\| a \| - \varepsilon)^{-1/2} < (\| P(d) \| - 3\varepsilon)^{-1/2} < \varepsilon^{-1/2}.
\]

It follows that
\[
\| T_\eta^* T_\xi^* xdx T_\xi T_\eta - 1 \| = \| T_\eta^* (T_\xi^* xdx T_\xi - \iota(a)) T_\eta \|
\leq \| T_\xi^* xdx T_\xi - \iota(a) \| (\varepsilon^{-1/2})^2 < 1.
\]

Therefore, \( c = T_\eta^* T_\xi^* xdx T_\xi T_\eta \) is an invertible positive element and we take \( z = xT_\xi T_\eta e^{-1/2} \). \( \square \)

### 3. Applications and concluding remarks

We collect some applications of the theorem above and consider certain connections with the theory of reduced (amalgamated) free product \( C^* \)-algebras. First we consider criteria under which the Kirchberg–Phillips Theorem applies (see [Kr, Theorem C], [Ph, Corollary 4.2.2]).

**Theorem 3.1.** Let \( A \) be a separable nuclear unital \( C^* \)-algebra belonging to the bootstrap class to which the UCT applies (see [RS]); let \( \pi : A \to \mathcal{L}(\mathcal{F}) \) be a faithful nondegenerate representation of \( A \) on a nontrivial separable Hilbert space \( \mathcal{F} \) such that \( \pi(A) \cap \mathcal{K}(\mathcal{F}) = \{0\} \) and let \( E \) denote the Hilbert \( A \)-bimodule \( \mathcal{F} \otimes_{\mathbb{C}} A \). Then \( \mathcal{O}_E \) is a unital Kirchberg algebra (simple, purely
infinite, separable and nuclear) belonging to the bootstrap class. Hence, the Kirchberg–Phillips Theorem applies and the isomorphism class of $O_E$ only depends on $(K_*(A), [1_A])$ and not on the choice of representation $\pi$.

**Proof.** By Theorem 2.8, $O_E$ is simple and purely infinite. If $A$ is nuclear, the argument given in the proof of [DS, Theorem 2.1] shows that $O_E$ must also be nuclear (alternatively, the nuclearity of $O_E$ follows from the structural results discussed in §1). Hence, $O_E$ is a unital Kirchberg algebra. Recall that the inclusion $A \hookrightarrow O_E$ defines a $KK$-equivalence (see [Pm, Corollary 4.5]) that induces a unit-preserving isomorphism $K^*(A) \cong K^*(O_E)$. Hence, if $A$ is in the bootstrap class, so is $O_E$. Therefore, the Kirchberg–Phillips Theorem applies and the isomorphism class of $O_E$ only depends on $(K^*(A), [1_A])$. \[\square\]

Let $X$ be a second countable compact Hausdorff space, let $\mu$ be a nonatomic Borel measure with full support and let $\pi : C(X) \to L(L^2(X, \mu))$ be the representation given by multiplication of functions. Then $\pi$ is faithful and

$$\pi(C(X)) \cap K(L^2(X, \mu)) = \{0\}.$$

Hence, we may apply Theorem 3.1 with $A = C(X)$ and $\mathcal{H} = L^2(X, \mu)$.

**Corollary 3.2.** Let $X$ and $\mu$ be as above. Then

$$E = L^2(X, \mu) \otimes_C C(X)$$

is a Hilbert bimodule over $C(X)$ and $O_E$ is a unital Kirchberg algebra. The embedding $C(X) \hookrightarrow O_E$ induces a (unit preserving) $KK$-equivalence. Hence, the isomorphism class of $O_E$ only depends on $(K_*(C(X)), [1_{C(X)}])$ (and not on $\mu$); moreover, if $X$ is contractible, then $O_E \cong O_\infty$.

The next proposition is Theorem 5.6 of [L] (see also [Ka, Theorem 3]); Lance calls this the Kasparov–Stinespring–Gelfand–Naimark–Segal construction.

**Proposition 3.3.** Let $B$ and $C$ be $C^*$-algebras, let $F$ be a Hilbert $C$-module and let $f : B \to L(F)$ be a completely positive map. Then there is a Hilbert $C$-module $E_f$, a $*$-homomorphism $\varphi_f : B \to L(E_f)$ and an element $v_f \in L(F, E_f)$ such that $f(b) = v_f^* \varphi_f(b)v_f$ and $\varphi_f(B)v_f F$ is dense in $E_f$.

I am grateful to D. Shlyakhtenko for the following observation. Let $T$ be the “usual” Toeplitz algebra ($T_E$, where $E$ is the 1-dimensional Hilbert bimodule over $\mathbb{C}$) and let $g$ denote the vacuum state on $T$.

**Proposition 3.4.** Let $A$ be a separable unital $C^*$-algebra and let $\pi : A \to L(\mathcal{H})$ be a faithful representation of $A$ on a separable Hilbert space $\mathcal{H}$ such that $\pi$ has a cyclic vector $\xi \in \mathcal{H}$. Let $f$ denote the vector state $\langle \xi, \pi(\cdot) \xi \rangle$ and let $\tilde{f}$ denote the corresponding completely positive map from $A$ to $L(A)$.
(given by \( \tilde{f}(a) = f(a) \)). Then \( E = E_{\tilde{f}} \cong \mathcal{H} \otimes A \) and \( T_{E} \) may be realized as a reduced free product (see [A, V]):

\[
(T_{E}, h) \cong (A, f) * (T, g)
\]

for some state \( h \) on \( T_{E} \).

**Proof.** This follows from [Sh, Theorem 2.3, Corollary 2.5]. \( \square \)

As a result of this observation, at least part of Corollary 3.2 follows from the existing literature on reduced free products. The simplicity follows from a theorem of Dykema [Dy, Theorem 2]. Criteria for when reduced free products are purely infinite have been found by Choda, Dykema and Rørdam in a series of papers [DR1, DR2, DC]; but none seem to apply generally to the case considered in the corollary.

A theorem of Speicher (see [Sp]) on reduced amalgamated free products (see [V, §5]) and Toeplitz algebras associated to Hilbert bimodules yields a curious stability property of the algebras we have been considering. The following is the version given in [BDS, Theorem 2.4].

**Proposition 3.5.** Suppose that \( E_{1} \) and \( E_{2} \) are full Hilbert bimodules over the \( C^{*} \)-algebra \( A \). Then

\[
T_{E_{1} \oplus E_{2}} = T_{E_{1}} * A T_{E_{2}}.
\]

**Corollary 3.6.** Let \( A \) be a separable nuclear unital \( C^{*} \)-algebra belonging to the bootstrap class to which the UCT applies (see [RS]) and let \( \pi : A \rightarrow \mathcal{L}(\mathcal{H}) \) be a faithful representation of \( A \) on a separable Hilbert space \( \mathcal{H} \) such that \( \pi(A) \cap \mathcal{K}(\mathcal{H}) = \{0\} \). Let \( E \) be the Hilbert bimodule \( \mathcal{H} \otimes \mathbb{C} A \). Then

\[
O_{E} \cong O_{E} * A O_{E}.
\]

**Proof.** Observe that \( E \oplus E = (\mathcal{H} \oplus \mathcal{H}) \otimes \mathbb{C} A \). Since \( \pi \oplus \pi : A \rightarrow \mathcal{L}(\mathcal{H} \oplus \mathcal{H}) \) is a faithful representation and \( (\pi \oplus \pi)(A) \cap \mathcal{K}(\mathcal{H} \oplus \mathcal{H}) = \{0\} \), the result follows from Theorem 3.1 and the above proposition. \( \square \)

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**References**


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RELATIVE TOPOLOGY OF REAL ALGEBRAIC VARIETIES IN THEIR COMPLEXIFICATIONS

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We investigate, for a given smooth closed manifold $M$, the existence of an algebraic model $X$ for $M$ (i.e., a nonsingular real algebraic variety diffeomorphic to $M$) such that some nonsingular projective complexification $i: X \to X_\C$ of $X$ admits a retraction $r: X_\C \to X$. If such an $X$ exists, we show that $M$ must be formal in the sense of Sullivan’s minimal models, and that all rational Massey products on $M$ are trivial.

We also study the homomorphism on cohomology induced by $i$ for algebraic models $X$ of $M$. Using étale cohomology, we see that mod $p$ Steenrod powers give an obstruction for the induced map on cohomology, $i^*: H^k(X_\C, \Z_p) \to H^k(X, \Z_p)$, to be onto, if we require that $X$ is defined over rational numbers.

1. Introduction

Let $M$ be a closed smooth manifold. In [T] Tognoli, generalizing the results of Seifert [S] and Nash [N], proved that there is a nonsingular real algebraic variety $X$ diffeomorphic to $M$, which we call an algebraic model of $M$. Later Akbulut and King improved Tognoli’s result, proving that if $M \subseteq \R^N$ then $M$ can be isotoped to a nonsingular real algebraic subvariety of $\R^N \times \R$ [AK3, AK4].

In 1978 Kulkarni considered the following problem in [Ku]: Is there an algebraic model $X$ of $M$ such that the inclusion $i: X \to X_\C^0$ of $X$ into some quasiprojective complexification $X_\C^0$ is a homotopy equivalence? He calls such $X_\C^0$ a minimal complexification of $M$. Using mixed Hodge structures he showed that if $M$ has a minimal complexification the Euler characteristic of $M$ is nonnegative. He also showed that any homogeneous space $G/H$, where $G$ is a compact Lie group, admits a canonical minimal complexification.

Since any nonsingular complex projective variety has a fundamental class, by dimension reasons the inclusion of $X$ into any nonsingular projective complexification, $i: X \to X_\C$, is never a homotopy equivalence. Instead one can consider the following problem: Is there an algebraic model $X$ of $M$ with a smooth projective complexification, $i: X \to X_\C$, that admits a retraction $r: X_\C \to X$?
If $M$ is $S^1$ or a closed orientable surface, $M$ has an algebraic model $X$ with a smooth projective complexification that retracts onto $X$. In dimensions larger than two it turns out that, as in the case of Kulkarni’s result, there are topological obstructions to the existence of such $X$ in terms of Massey products and Sullivan’s rational minimal models:

**Theorem 1.1.** Let $M$ be a closed smooth manifold admitting an algebraic model $X$ with a smooth projective complexification that retracts onto $X$. Then, $M$ is formal. In particular, all Massey products in $M$ are trivial.

This is an immediate consequence of Theorems 2.1 and 2.3. The following corollary of Theorem 1.1 will be proved in Section 2:

**Corollary 1.2.** Let $N$ be the total space of an $S^1$-bundle over an orientable closed surface $F$ of positive genus. Then $N$ has an algebraic model $X$ with a projective complexification that retracts onto $X$ if and only if the $S^1$-bundle is trivial; that is, $N = F \times S^1$.

In Section 3, we will see that the manifold $N$ in this corollary has an algebraic model $X$ such that $i^* : H^*(X_{\mathbb{C}}, \mathbb{Z}) \to H^*(X, \mathbb{Z})$ is surjective for any projective complexification $X_{\mathbb{C}}$ (Corollary 3.3). However, by the result above, none of these complexifications retract onto $X$.

On the positive side we have the following result:

**Proposition 1.3.** Let $Z$ be a smooth complex projective variety. Regarded as smooth manifolds, both $Z$ and $Z \times S^1$ admit algebraic models with smooth projective complexifications that retract onto them.

**Remark 1.4.** Let $F_g$ be a Riemann surface of genus $g$. The algebraic surface $(F_g \times \mathbb{CP}^1) \# n \mathbb{CP}^2$, the blowup of $F_g \times \mathbb{CP}^1$ at $n$ points, has Euler characteristic $4(1 - g) + n$, and this can be any integer if $n$ and $g$ are chosen appropriately. Hence, unlike Kulkarni’s result, in Theorem 1.1 there is no restriction on the Euler characteristic.

In the next section we review Sullivan’s theory of minimal models, formality and Massey products. At the end of that section we construct manifolds with nonnegative Euler characteristic that do not admit minimal complexifications in the sense of Kulkarni’s result. Hence, the condition in Kulkarni’s theorem that $\chi(M) \geq 0$ is necessary but not sufficient.

In Section 3, we first review some basic material in real algebraic geometry, needed to prove Proposition 1.3. Then we study the homomorphism on cohomology, induced by $i : X \to X_{\mathbb{C}}$, for algebraic models $X$ of a smooth closed manifold $M$. In particular, using étale cohomology we see that mod $p$ Steenrod powers give an obstruction for the induced map on cohomology, $i^* : H^k(X_{\mathbb{C}}, \mathbb{Z}_p) \to H^k(X, \mathbb{Z}_p)$, to be onto, whenever $X$ is defined over $\mathbb{Q}$. 
2. Minimal models, formality and Massey products

In this section we mainly follow [GM] and [DGMS]. A differential graded algebra is an algebra \( A = \bigoplus_{k \geq 0} A^k \) with a differential \( d : A \to A \) of degree +1, such that \( A \) is graded commutative (i.e., \( xy = (-1)^{\deg x \deg y}yx \)) and \( d \) is a derivation with \( d^2 = 0 \). We will denote its cohomology by \( H^*(A) \). Note that \( H^*(A) \) together with the zero differential, \( d = 0 \), is again a differential graded algebra. In this work, \( A^0 \) denotes the ground field, which will be either \( \mathbb{Q} \) or \( \mathbb{R} \). Each \( H^k(A) \) is assumed to be finite-dimensional.

If \( H^0(A) = A^0 \) then \( A \) is called connected, and if also \( H^1(A) = 0 \) then \( A \) is called 1-connected.

A minimal differential graded algebra \( M \) is a differential graded algebra such that:

1. \( M \) is free as a graded commutative algebra; i.e., it is a tensor product of polynomial algebras on generators of even degrees and exterior algebras on generators of odd degrees.
2. \( d \) is decomposable; i.e., \( d(M) \subseteq M^+ \wedge M^+ \), where \( M^+ = \bigcup_{i>0} M_i \).

A minimal model for a differential graded algebra \( A \) is a degree-zero homomorphism \( \phi : A \to M \) to a minimal algebra \( M \), inducing an isomorphism on cohomology.

It is well-known that any connected differential algebra \( A \) having finite-dimensional cohomology in each degree has a minimal model \( \phi : A \to M \). Moreover, \( M \) is unique up to isomorphism.

Let \( K \) be a simplicial complex and \( E^*(K) \) the rational de Rham complex of \( K \), that is, the complex of \( \mathbb{Q} \)-polynomial forms on \( K \). For a smooth manifold \( M \), let \( E^*(M) \) denote the differential algebra of smooth differential forms on \( M \). Minimal models of these two algebras are related as follows: if \( K \) is a \( C^1 \)-triangulation of \( M \), the minimal model of \( E^*(M) \) is isomorphic to the minimal model of \( E^*(K) \) tensored with \( \mathbb{R} \).

For a simply connected finite \( CW \)-complex \( K \), the rational homotopy of \( K \) can be read from its rational minimal model. Namely, \( \pi_i(K) \otimes \mathbb{Q} \cong I^i(K) \), where \( I^i(K) \) denotes the vector space of irreducibles of degree \( i \) in \( \mathcal{M}(E^*(K)) \).

A differential graded algebra \( A \) is called formal if there is a map \( \psi : A \to H^*(A) \) of degree zero and inducing isomorphism on cohomology, the latter being endowed with the zero differential map. This is equivalent to saying that \( A \) and \( H^*(A) \) have isomorphic minimal models.

A smooth manifold \( M \) is called formal if \( E^*(M) \) is formal. Some examples of formal manifolds are wedges of spheres, compact connected Lie groups, Eilenberg–Mac Lane spaces \( K(\pi, n) \), \( n > 1 \), Riemannian symmetric spaces and compact Kähler manifolds.

Next we give an obstruction against \( M \) being formal, in terms of Massey products. Let \( X \) be a space and let \( a, b, c \in H^*(X, \mathbb{R}) \) classes of degree \( p, q, r \).
respectively, such that \(a \cup b = 0 = b \cup c\), where \(R\) is any commutative ring with unity. Choose cochain representatives \(\alpha, \beta, \gamma\) for \(a, b, c\) respectively. Also let \(\mu\) and \(\tau\) be cochains such that \(d\mu = \alpha \wedge \beta\) and \(d\tau = \beta \wedge \gamma\). Then

\[
\mu \wedge \gamma - (-1)^p \alpha \wedge \tau
\]

is a closed form, called a triple Massey product of \(a, b, c\). It passes to a well-defined class in the quotient group

\[
H^{p+q+r-1}(X, R)/(a \cup H^{q+r-1}(X, R) + c \cup H^{p+q-1}(X, R))
\]

called the Massey triple product of the classes \(a, b, c\) and denoted by \(\langle a, b, c \rangle\).

We will also use the same notation \(\langle a, b, c \rangle\) to denote the coset

\[
\mu \wedge \gamma - (-1)^p \alpha \wedge \tau + (a \cup H^{q+r-1}(X, R) + c \cup H^{p+q-1}(X, R))
\]

i.e., the set of all Massey triple products of the classes \(a, b, c\). There are also higher Massey products, \(\langle a_1, a_2, \ldots, a_k \rangle\), which are defined if all the lower Massey products formed from the elements \(a_1, a_2, \ldots, a_k\) are zero.

Massey products are functorial in the sense that if \(f : X \to Y\) is a continuous map then

\[
f^*(\langle a_1, a_2, \ldots, a_k \rangle) \subseteq \langle f^*(a_1), f^*(a_2), \ldots, f^*(a_k) \rangle.
\]

In particular, if \(f^*\) is an isomorphism on cohomology it preserves Massey products.

Indeed, Massey products can be defined in any differential graded algebra \((A, d)\); all higher Massey products are zero if \(d = 0\). Hence, on a smooth formal manifold \(M\) all higher rational Massey products are 0. In particular:

**Theorem 2.1 ([DGMS]).** Any compact Kähler manifold \(M\) is formal. In particular, all higher rational Massey products in \(M\) are zero.

Any minimal model \(\mathcal{M}\) is isomorphic to

\[
P[V_2 \oplus V_4 \oplus \cdots] \otimes [V_1 \oplus V_3 \oplus \cdots],
\]

where each \(V_i\) is a vector space that contains elements of degree \(i\) only and \(P[V_2 \oplus V_4 \oplus \cdots]\) is the polynomial algebra part and \([V_1 \oplus V_3 \oplus \cdots]\) the exterior algebra part of \(\mathcal{M}\).

Let \(C_i\) denote the subspace of closed elements in \(V_i\). The theorem below shows that \(\mathcal{M}\) is formal if and only if all Massey products are zero in a uniform way:

**Theorem 2.2 ([DGMS]).** \(\mathcal{M}\) is formal if and only if there is in each \(V_i\) a complement \(N_i\) to \(C_i\), such that any closed form \(a\) in the ideal \(I(\oplus N_i)\) is exact. Choosing such an \(N_i\) is equivalent to choosing a \(\psi : \mathcal{M} \to H^*(\mathcal{M})\) that induces an isomorphism on cohomology.

The key observation of this section is:
Theorem 2.3. Let $M$ be closed smooth manifold such that there exists a retraction $r : N \to M$, where $N$ is a closed smooth formal manifold. Then $M$ is formal. In particular, all higher rational Massey products in $M$ are zero.

Remark 2.4.

(1) Theorem 1.1 follows from Theorems 2.1 and 2.3.

(2) It follows from the preceding theorem that the product $M_1 \times M_2$ of two closed manifolds is formal if and only if $M_1$ and $M_2$ are formal.

(3) As stated before, any compact Lie group is formal. On the other hand, we know that any compact Lie group has the structure of a unique real linear algebraic group $[DM]$ that and $G$, equipped with this real algebraic structure, satisfies the following: for any smooth projective complexification $i : G \to G^C$, the induced map on cohomology $i^* : H^i(G^C, \mathbb{Q}) \to H^i(G, \mathbb{Q})$ is trivial for $i > 0$ (see $[O1]$). Hence, for this canonical algebraic structure there is no smooth projective complexification of $i : G \to G^C$ that retracts onto $G$. On the contrary, in Kulkarni’s result mentioned in the introduction the affine complexification of $G$ is homotopy equivalent to $G$ via the inclusion map.

Proof of Theorem 2.3. Let $\mathcal{M}$ and $\mathcal{N}$ denote the minimal models of $M$ and $N$ respectively. We know that $\mathcal{N} \simeq P[V_2 \oplus V_4 \oplus \cdots] \otimes \bigwedge[V_1 \oplus V_3 \oplus \cdots]$ and $\mathcal{M} \simeq P[V'_2 \oplus V'_4 \oplus \cdots] \otimes \bigwedge[V'_1 \oplus V'_3 \oplus \cdots]$ where $V_i$ and $V'_i$ are the vector subspaces that contain the elements of degree $i$ only. The maps $i : M \to N$ and $r : N \to M$ induce homomorphisms $i^* : \mathcal{M} \to \mathcal{N}$ and $r^* : \mathcal{N} \to \mathcal{M}$ such that $i^* \circ r^* = \text{id}|_{\mathcal{M}}$.

Let $C_i$ and $C'_i$ denote the subspaces of closed elements in $V_i$ and $V'_i$. Since $N$ is formal, by Theorem 2.2 there is in each $V_i$ a complement $N_i$ to $C_i$ such that any closed form $a$ in the ideal $I(\oplus N_i)$ is exact. Then $V'_i = (V'_i \cap C'_i) \oplus (V'_i \cap N_i)$, where $(V'_i \cap C'_i)$ is clearly $C'_i$. Hence, again by Theorem 2.2, it suffices to show that any closed element $a'$ in the ideal $I(\oplus N'_i)$ is exact, where $N'_i = V'_i \cap N_i$.

Let $a'$ be a closed element in the ideal $I(\oplus N'_i)$. Then $a = r^*(a')$ is a closed element in $I(\oplus N_i)$ and thus $a = d(b)$ for some $b \in \mathcal{N}$. Now, $a' = i^*(d(b)) = d(i^*(b))$ finishes the proof.

Although the proof below is well-known to experts, we will reproduce it here for the sake of completeness.

Proof of Corollary 1.2. Let $N$ be the total space of an $S^1$-bundle over an orientable closed surface $F$ of positive genus. Suppose that the first Chern
The class of the $S^1$-bundle is nonzero. The Gysin sequence associated to the $S^1$ bundle implies that $H^1(F, \mathbb{R})$ maps isomorphically onto $H^1(N, \mathbb{R})$ and $H^2(F, \mathbb{R})$ maps to zero in $H^2(N, \mathbb{R})$. In particular, we may identify $H^1(F, \mathbb{R})$ with $H^1(N, \mathbb{R})$. Let $a, b \in H^1(F, \mathbb{R})$ such that $c_1(N) = a \cup b$. Then by Chern–Weil theory there is a connection on the $S^1$-bundle whose associated 1-form $\eta$ satisfies $d(\eta) = c_1(N) = a \cup b$. Moreover,

$$(\eta \cup a \cup b)([N]) = (\eta \cup d(\eta))(|N|) = c_1(N)(|F|) \neq 0.$$ 

Hence, the Massey product $\langle a, a, b \rangle$, which is represented by $\eta \cup a$, is not zero. In particular, $N$ is not formal.

The ‘if’ part of the Corollary follows from Proposition 1.3.

Example 2.5. One can find simply connected smooth manifolds containing nonzero Massey products as follows: let $K$ be a simply connected finite CW-complex with nontrivial Massey products. For example, let $K$ be a finite CW-approximation of the free loop space $\Lambda S^2$, $k > 1$, which supports a nontrivial Massey product (see [SVP]). Embed $K$ into some Euclidean space, $\mathbb{R}^N$, and let $M$ be the double of a tubular neighborhood $\nu(K)$ in $\mathbb{R}^N$. Since $M$ retracts onto $K$, by Theorem 2.3 $M$ contains nonzero Massey products.

Example 2.6. For finitely presentable groups there is another notion of formality, called 1-formality. The fundamental group of any formal space is 1-formal. The group

$$\Gamma = \langle x, y, z, t \mid [x, y][z, t], [[[y, x], x], x], y \rangle$$

is not 1-formal; see [ABCKT, Proposition 3.20 and Example 3.22, pp. 32–38]. Therefore, if $M$ is any smooth manifold with fundamental group $\Gamma$ — such an $M$ exists in dimensions at least 4 (see the remark below) — $M$ is not formal but the Massey triple products of elements in $H^1(M, \mathbb{Q})$ are all zero. In particular:

Corollary 2.7. If $M$ is a smooth manifold with fundamental group $\Gamma$ as above, $M$ does not admit an algebraic model $X$ with a smooth projective complexification that retracts onto $X$.

Remark 2.8. By a result of Morgan, the group $\Gamma$ of Example 2.6 cannot be the fundamental group of a smooth quasiprojective complex variety (see Corollary 3.53 and the subsequent paragraph in [ABCKT, p. 463]). Construct a 4-manifold $N$ by taking the connected sum of four copies of $S^3 \times S^1$ and gluing two 2-handles along smooth embedded loops representing the relations of $\Gamma$. Then $N$ has $\Gamma$ as its fundamental group and its Euler characteristic is $-2$. So, if $n \geq 2$, the connected sum $M = N \# n \mathbb{CP}^2$ has nonnegative Euler characteristic $n - 2$ and fundamental group $\Gamma$. Hence, the condition in Kulkarni’s theorem that $\chi(M) \geq 0$ is necessary but not sufficient.
3. Algebraic models

All real algebraic varieties considered here are compact and nonsingular. It is well-known that real projective varieties are affine (Proposition 2.4.1 of [AK2] or Theorem 3.4.4 of [BCR]). Compact affine real algebraic varieties are projective (Corollary 2.5.14 of [AK2]), so we will not distinguish between real compact affine varieties and real projective varieties.

For real algebraic varieties \( X \subseteq \mathbb{R}^r \) and \( Y \subseteq \mathbb{R}^s \), a map \( F : X \to Y \) is said to be entire rational if there exist \( f_i, g_i \in \mathbb{R}[x_1, \ldots, x_r] \), \( i = 1, \ldots, s \), such that each \( g_i \) vanishes nowhere on \( X \) and \( F = (f_1/g_1, \ldots, f_s/g_s) \). We say \( X \) and \( Y \) are isomorphic if there are entire rational maps \( F : X \to Y \), \( G : Y \to X \) such that \( F \circ G = \text{id}_Y \) and \( G \circ F = \text{id}_X \). We regard isomorphic algebraic varieties as being the same. A complexification \( X_C \subseteq \mathbb{CP}^N \) of \( X \) will mean that \( X \) is embedded into some projective space \( \mathbb{RP}^N \) and \( X_C \subseteq \mathbb{CP}^N \) is the complexification of the pair \( X \subseteq \mathbb{RP}^N \). We also require the complexification to be nonsingular (blow up \( X_C \) along smooth centers away from \( X \) defined over reals if necessary, [Hi, BM]). We refer the reader for the basic definitions and facts about real algebraic geometry to [AK2, BCR].

Suppose that \( R \) is a commutative ring with unity and \( X \) is \( R \)-orientable. Let \( KH_k(X, R) \) denote the kernel of the induced map

\[
i_* : H_k(X, R) \to H_k(X_C, R)
\]
on homology. Then \( KH_k(X, R) \) is independent of the complexification \( X \subseteq X_C \) and thus an (entire rational) isomorphism invariant of \( X \). Similarly, the image of the homomorphism

\[
i^* : H^k(X_C, R) \to H^k(X, R),
\]
denoted by \( \text{Im} H^k(X, R) \), is also an isomorphism invariant of \( X \), [O2]. In [BK2], Bochnak and Kucharz also studied \( KH_k(X, R) \) independently.

**Example 3.1.**

a) If \( Z \) is a compact nonsingular complex algebraic variety, we can view \( Z \) as a real algebraic variety, which we denote by \( Z_\mathbb{R} \). This is just the fixed-point set of the antiholomorphic involution \( \sigma : Z \times \bar{Z} \to Z \times \bar{Z} \) given by \( \sigma(x, y) = (\bar{y}, \bar{x}) \), where \( \bar{Z} \) is the complex conjugate of \( Z \). It is well-known that there is a complex algebraic subvariety \( W \) of some projective space \( \mathbb{CP}^N \) defined by real polynomials which is biregularly isomorphic to \( Z \times \bar{Z} \). Moreover, the real part \( W \cap \mathbb{RP}^N \) is isomorphic to \( Z_\mathbb{R} \). However, any projective real algebraic variety is affine (Proposition 3.4.4 in [BCR]) and hence \( Z_\mathbb{R} \) can be viewed as an affine real algebraic variety (see Sections 1 and 2 of [Hu]). Now clearly, there is a retraction \( W = (Z_\mathbb{R})_C \to Z_\mathbb{R} \), and therefore \( \text{Im} H^*(Z_\mathbb{R}, R) = H^*(Z_\mathbb{R}, R) \). In particular, for \( \mathbb{CP}^n \), regarded as a real algebraic variety, we have \( \text{Im} H^*(\mathbb{CP}^n, R) = H^*(\mathbb{CP}^n, R) \); see [O2].
b) The quaternion projective space $\mathbb{HP}^n$ has also a canonical real algebraic structure. The canonical quaternion line bundle $\xi$ over $\mathbb{HP}^n$ is strongly algebraic [BBK] and thus $p_1(\xi) \in \text{Im } H^4(\mathbb{HP}^n, R)$ [O2, AK5]. In particular, $\text{Im } H^*(\mathbb{HP}^n, R) = H^*(\mathbb{HP}^n, R)$.

c) Let $X$ be any nonsingular real algebraic curve diffeomorphic to $S^1$ such that $X_\mathbb{C} - X$ is connected. So, topologically, $X_\mathbb{C}$ is an orientable closed surface and $X$ is a nonseparating simple closed curve on it. Again there is a retraction of $X_\mathbb{C}$ onto $X$ (and therefore $\text{Im } H^*(X, R) = H^*(X, R)$).

Since $X$ is a nonseparating simple closed curve on $X_\mathbb{C}$, its homotopy class, say $\alpha$, is an element of some generating set for the fundamental group $\pi_1(X_\mathbb{C})$. Also, up to homotopy, a retraction $X_\mathbb{C} \to X$ is completely determined by the induced surjective homomorphism $\pi_1(X_\mathbb{C}) \to \pi_1(X) \simeq \langle \alpha \rangle$ sending $\alpha$ to itself. In particular, there are many homotopically different retractions.

**Proof of Proposition 1.3.** The proof follows from parts (a) and (c) of the preceding example. \hfill \square

**Theorem 3.2.** Let $M$ be closed smooth manifold. Then $M$ has an algebraic model $X$ such that $\text{Im } H^*(X, \mathbb{Z})$ contains $H^k(X, \mathbb{Z})$ for $k = 1, 2, 4$. In particular, if the cohomology ring $H^*(X, \mathbb{Z})$ is generated by elements of degree 1, 2 and 4 then $\text{Im } H^*(X, \mathbb{Z}) = H^*(X, \mathbb{Z})$.

In case of rational coefficients, $M$ has an algebraic model $X$ such that $\text{Im } H^*(X, \mathbb{Q})$ contains $H^\text{even}(X, \mathbb{Q})$ and $H^1(X, \mathbb{Q})$.

As an immediate consequence we get:

**Corollary 3.3.** Any closed smooth three-manifold with positive first Betti number has an algebraic model $X$ such that $\text{Im } H^*(X, \mathbb{Z}) = H^*(X, \mathbb{Z})$.

**Proof of Theorem 3.2.** Choose generating sets $a_1, \ldots, a_{n_1}$, $b_1, \ldots, b_{n_2}$ and $c_1, \ldots, c_{n_4}$ for the $\mathbb{Z}$-modules $H^1(M, \mathbb{Z})$, $H^2(M, \mathbb{Z})$ and $H^4(M, \mathbb{Z})$. We can regard each $a_i$, $b_i$ and $c_i$ as a smooth map $a_i : M \to K(\mathbb{Z}, 1) = S^1$, $b_i : M \to K(\mathbb{Z}, 2) = \mathbb{CP}^N$ and $c_i : M \to K(\mathbb{Z}, 4) = \mathbb{HP}^N$ for $N$ large enough ($2N > \dim M$ would suffice).

It is well-known that the Grassmann varieties, in particular $\mathbb{CP}^N$ and $\mathbb{HP}^N$, have canonical real algebraic structures that have totally algebraic homology [AK1]. Let $E$ be a connected nonsingular real elliptic curve diffeomorphic to $S^1$ and regard $a_i$ as a smooth map $a_i : M \to E$. Note that $E_\mathbb{C} - E$ is connected and $E$ has also totally algebraic homology. By a result of Akbulut and King (Lemma 2.7.1 and Theorem 2.8.4 of [AK2]) there exist an algebraic model $X$ of the smooth manifold $M$ and an entire rational map

$$\Phi : X \longrightarrow E^{n_1} \times (\mathbb{CP}^N)^{n_2} \times (\mathbb{HP}^N)^{n_4}$$

homotopic to the product map

$$(a_1, \ldots, a_{n_1}, b_1, \ldots, b_{n_2}, c_1, \ldots, c_{n_4}) : X \longrightarrow E^{n_1} \times (\mathbb{CP}^N)^{n_2} \times (\mathbb{HP}^N)^{n_4}.$$
Now Example 3.1 finishes the proof of the first part.

For the second part, note that the reduced $K$-group $\widetilde{K}_0(M)$ of complex vector bundles of any closed manifold $M$ is finitely generated. So, as above, we choose generators for $\widetilde{K}_0(M)$ and represent all of them by a smooth map from $M$ into a product of Grassmann varieties. Consider all these maps together with the maps $a_i : M \to E$, representing a generating set for the first integer cohomology group. As above, $M$ has an algebraic model $X$, so all these maps are represented by entire rational maps. In particular, any smooth complex vector bundle over $X$ is strongly algebraic. It follows that $\text{Im} H^1(X, \mathbb{Q}) = H^1(X, \mathbb{Q})$ and $\widetilde{K}_0(X) \simeq \widetilde{K}_0(\mathcal{R}(X, \mathbb{C}))$ and thus

$$\widetilde{K}_0(X) \otimes \mathbb{Q} \simeq \widetilde{K}_0(\mathcal{R}(X, \mathbb{C})) \otimes \mathbb{Q},$$

the latter being the reduced $K$-group of strongly algebraic complex vector bundles over $X$ tensored with $\mathbb{Q}$, which is known to be isomorphic to $H^\text{even}_{\text{alg}}(X, \mathbb{Q}) \subset \text{Im} H^\text{even}(X, \mathbb{Q})$ (see [BBK]). Finally, it is well-known that the Chern character gives an isomorphism $\text{Ch} : \widetilde{K}_0(\mathcal{R}(X)) \otimes \mathbb{Q} \to H^\text{even}(X, \mathbb{Q})$ (Theorem 3.27, p. 283 of [Ka]). Hence,

$$H^\text{even}(X, \mathbb{Q}) = H^\text{even}_{\text{alg}}(X, \mathbb{Q}) \subseteq \text{Im} H^\text{even}(X, \mathbb{Q}) \subseteq H^\text{even}(X, \mathbb{Q})$$

and the proof is finished. \hfill $\Box$

Theorem 3.2 does not hold for arbitrary coefficients, in particular for the cyclic group $\mathbb{Z}_p$, if we further require that the algebraic model $X$ is defined over rationals: Let $M$ be a closed manifold having a class $a \in H^3(M, \mathbb{Z}_p)$ with $\beta \circ P^1(a) \neq 0$, where $P^1 : H^3(M, \mathbb{Z}_p) \to H^{2p+1}(M, \mathbb{Z}_p)$ is the first Steenrod power for the prime $p$ and $\beta : H^{2p+1}(M, \mathbb{Z}_p) \to H^{2p+2}(M, \mathbb{Z}_p)$ is the Bockstein homomorphism corresponding to the exact sequence $0 \to \mathbb{Z}_p \to \mathbb{Z}_p^2 \to \mathbb{Z}_p \to 0$. One can take $p = 3$ and $M = S^3/\mathbb{Z}_3 \times S^7/\mathbb{Z}_3$, a product of lens spaces [BHKK]. In this case the triple Massey product $\langle a, a, a \rangle$ is defined and $\beta \circ P^1(a) \notin \langle a, a, a \rangle$ (see [Mc, p. 293] or [Kr]). So the triple Massey product $\langle a, a, a \rangle$ is not trivial. Now suppose that $M$ has an algebraic model $X$ defined over $\mathbb{Q}$ such that

$$\text{Im} H^3(X, \mathbb{Z}_3) = H^3(X, \mathbb{Z}_3).$$

It follows from the naturality of Steenrod operations and Bockstein homomorphisms that for any cohomology class $b \in H^3(X_\mathbb{C}, \mathbb{Z}_3)$ with $i^*(b) = a$, $\beta \circ P^1(b) \neq 0$. As above the triple Massey product $\langle b, b, b \rangle$ is defined [Kr] and hence is nonzero, where $i : X \to X_\mathbb{C}$ is any smooth projective complexification. It is well-known that for any finite abelian group $G$ the singular cohomology of $X_\mathbb{C}$ is isomorphic to the étale cohomology of $X$; i.e., $H^i(X_\mathbb{C}, G) \cong H^i((X_\mathbb{C})_{\text{et}}, G)$ (Theorem 3.12, p. 117 in [Mi]). Since $X$ is defined over rationals, $X$ is also defined over integers. Choose a big prime $q$ such that the mod $q$ reduction $X_q$ of $X$ is a nonsingular variety in some algebraically closed field of characteristic $q$. Now by [Mi, Corollary 4.2, p. 230]
we have $H^i((X_C)_\text{et}, \mathbb{Z}_d) \cong H^i((X_q)_\text{et}, \mathbb{Z}_d)$, for any positive integer $d$ prime to $q$. We may take $d = 3$, in which case the cohomology algebra $H^i((X_q)_\text{et}, \mathbb{Z}_3)$ supports a nontrivial Massey triple product. However, Deligne’s proof of the Weil conjectures implies that any Massey higher product in $H^i((X_q)_\text{et}, \mathbb{Z}_3)$ is trivial, as mentioned in the introduction of [DGMS]; thus we obtain a contradiction.

Indeed the same works for higher mod $p$ Steenrod powers. Let $M$ be a closed smooth manifold with a cohomology class $a \in H^{2i+1}(M, \mathbb{Z}_p)$ satisfying $\beta \circ P^i(a) \neq 0$, where $P^i : H^{2i+1}(M, \mathbb{Z}_p) \to H^{2i+1}(M, \mathbb{Z}_p)$ is the $i$-th mod $p$ Steenrod power and $\beta : H^{2p+1}(M, \mathbb{Z}_p) \to H^{2p+2}(M, \mathbb{Z}_p)$ is the Bockstein map. Then as above $\beta \circ P^i(a) \in \langle a, \ldots, a \rangle$. In particular, we have proved:

**Theorem 3.4.** Let $X$ be an algebraic model for the smooth manifold $M$ in the above paragraph. If $X$ is defined over $\mathbb{Q}$, then $\text{Im} H^{2i+1}(X, \mathbb{Z}_p) \neq H^{2i+1}(X, \mathbb{Z}_p)$.

**Remark 3.5.** Studying the example provided by [BHK] one can see that the class $a \in H^3(M, \mathbb{Z}_3)$ is obtained from the generators of $H^1(S^3/\mathbb{Z}_3, \mathbb{Z}_3)$ and $H^1(S^7/\mathbb{Z}_3, \mathbb{Z}_3)$ via some cohomology operations. Hence, by the results above, at least one of these lens spaces do not admit an algebraic model $X$ defined over rationals with $\text{Im} H^1(X, \mathbb{Z}_3) = H^1(X, \mathbb{Z}_3)$.

We now give a contrasting result: Fix an inclusion-preserving smooth odd order group, $\mathbb{Z}_r$, on the telescoping sequence of odd-dimensional spheres $S^3 \subseteq S^5 \subseteq \cdots \subseteq S^{2k+1} \subseteq \cdots$. Consider the corresponding telescoping sequence lens spaces $S^3/\mathbb{Z}_r \subseteq S^5/\mathbb{Z}_r \subseteq \cdots \subseteq S^{2k+1}/\mathbb{Z}_r \subseteq \cdots$.

**Proposition 3.6.** For any algebraic model $X$ of first lens space $S^3/\mathbb{Z}_r$ in the above sequence and each $k > 1$, the lens space $S^{2k+1}/\mathbb{Z}_r$ has an algebraic model $Y$ that contains $X$ as a subvariety. If $\text{Im} H^1(Y, \mathbb{Z}_r) = H^1(Y, \mathbb{Z}_r)$ then $\text{Im} H^1(X, \mathbb{Z}_r) = H^1(X, \mathbb{Z}_r)$.

**Proof.** Since $r$ is odd, $H^i(S^3/\mathbb{Z}_r, \mathbb{Z}_2) = 0$ for $i = 1, 2$. Now Lemma 1.2 of [BK1] implies that any continuous vector bundle over $X$ is stably trivial. Hence, any continuous vector bundle over $X$ is strongly algebraic (see [BBK]). Consider $X$ as a submanifold of $S^{2k+1}/\mathbb{Z}_r$ in the obvious way. Using a theorem of Akbulut and King (Theorem 2.8.4 of [AK2]), we can find an algebraic model $Y$ for $S^{2k+1}/\mathbb{Z}_r$ having $X$ as a subvariety. In other words, the pair $X \subseteq Y$ is an algebraic model for the pair $S^3/\mathbb{Z}_r \subseteq S^{2k+1}/\mathbb{Z}_r$. The rest follows easily since the $\text{Im} H^*$ is functorial.

We believe that $S^{2k+1}/\mathbb{Z}_r$, for any $k \geq 1$, has no algebraic model $X$ with $\text{Im} H^1(X, \mathbb{Z}_r) = H^1(X, \mathbb{Z}_r)$. 

□
References


R-COVERED BRANCHED SURFACES

Sandra Shields

We give sufficient conditions for an R-covered codimension one foliation $F$ of a closed 3-manifold to be carried by an R-covered branched surface; that is, a branched surface carrying only foliations with the R-covered property. These conditions can be readily verified for many examples. In cases where the branched surface is generated by disks, the R-covered property is stable for $F$ in the sense that all nearby foliations are also R-covered.

Introduction

In this paper, we study codimension one $C^1$ foliations of Riemannian 3-manifolds. In particular, we examine foliations of a closed manifold $M$ that are covered by the canonical foliation of $\mathbb{R}^3$ by parallel hyperplanes, which we refer to as R-covered. These foliations are particularly nice in the sense that they are completely determined by the induced action of $\pi_1(M)$ on the real line, which is the leaf space of the universal cover $[So]$.

The R-covered property has been important in the study of foliations, especially those arising from Anosov flows. For example, if an Anosov foliation is R-covered, then the other Anosov foliation associated with the flow also has this property and the Anosov flow can be shown to be transitive $[So, Ba1, Ba2]$. Certain restrictions on the manifold in combination with the R-covered property have been used to show the Anosov flow is conjugate to a standard model, that is, a geodesic flow or a suspension of an Anosov diffeomorphism $[Pl1, Pl2, Gh]$. More recently, Fenley has used the R-covered hypothesis to uncover the rich structure of metric and homotopy properties of flow lines in many Anosov flows $[Fe1, Fe2]$. He has also shown that each leaf in the lift of an incompressible R-covered foliation (not necessarily Anosov) of a hyperbolic 3-manifold to the universal cover limits on all of $S^2_\infty [Fe3]$.

Foliations of a closed orientable manifold $M \neq S^2 \times S^1$ that are R-covered constitute a subset of the well-studied taut foliations. Tautness is a key to Roussarie’s $[Ro]$ and Thurston’s $[Th]$ results on isotoping incompressible tori and to Thurston’s study of norm minimizing leaves. Gabai $[Ga1, Ga2, Ga3]$ later used these results, by tautly foliating knot complements, to find the minimal genus spanning surface for a large class of knots and links.
While tautness indicates the absence of Reeb components, it does not imply the \(R\)-covered property. For example, there are many Anosov foliations that are not \(R\)-covered. (See [Fr-Wi], [Bo-La].) Unlike the property of being taut, the \(R\)-covered property is remarkably complex. In general, it is quite difficult to determine whether a foliation is \(R\)-covered or not. It is known that foliations with a compact leaf \(L\) are never \(R\)-covered except when the ambient manifold fibers over \(S^1\) with \(L\) as fiber [Go-Sh]. However, when the foliation has no compact leaves, very little is known. One of the few things that is known is that foliations transverse to a non-\(R\)-covered pseudo Anosov flow \(\phi\) are \(R\)-covered precisely when \(\phi\) is regulating; there are, however, many examples of \(R\)-covered Anosov flows transverse to \(R\)-covered foliations that are not regulating [Fe5].

It is well-known that the property of being taut is stable in the sense that all foliations sufficiently close to a taut foliation (in the \(C^1\) metric defined by Hirsch [Hi]) are also taut [Su]. In contrast, the \(R\)-covered property is much more delicate. There are simple examples demonstrating the \(R\)-covered property is not stable in this sense. In fact, Calegari has described a way in which one could use shearing to perturb a foliation of a hyperbolic surface bundle over a circle to possibly obtain an arbitrarily close non-\(R\)-covered foliation [Ca]. While Calegari does not prove whether a non-\(R\)-covered foliation can result from such a shearing, his observations indicate that one might, and his example seems similar to one of Fenley’s examples [Fe5] proving nonstability of the \(R\)-covered property.

In this article, we find conditions that guarantee an \(R\)-covered foliation \(F\) of a closed orientable manifold \(M \neq S^2 \times S^1\) is carried by an \(R\)-covered branched surface \(W\) (i.e., a branched surface carrying only \(R\)-covered foliations). If in addition to being \(R\)-covered, \(W\) is generated by disks, then we can apply a result in [Sh2] to obtain stability of the \(R\)-covered property for all foliations carried by \(W\). As a consequence, we say an \(R\)-covered \(W\) generated by disks is stably \(R\)-covered.

We shall expand on results in [Go-Sh] in which a class of \(R\)-covered branched surfaces was defined. (See Theorem 2.1.) Intuitively, these are branched surfaces that carry only taut foliations and do not contain the local behavior one would expect if there were a pair of leaves in the universal cover corresponding to a pair of nonseparable points in the leaf space. More specifically, a nonseparable pair of leaves in the universal cover gives rise to a smoothly immersed arc in the branched surface \(W\) called a branching arc whose ends branch into smooth local subsets of \(W\) from the same side. These arcs are discussed in Section 2 where we prove:

**Theorem 2.2.** Given a stably taut branched surface \(W\) constructed from a foliation of a closed orientable 3-manifold \(M \neq S^2 \times S^1\), let \(\hat{W}\) be the lift of \(W\) to the universal cover. Suppose there exists a finite set \(\Gamma\) of branching
arcs in $W$ such that the lift of any non-$R$-covered foliation carried by $W$ to a foliation carried by $\tilde{W}$ contains a pair of nonseparable leaves linked by a lift of some arc in $\Gamma$. For any $R$-covered foliation $F$ carried by $W$, there exists a stably $R$-covered branched surface $W'$ also carrying $F$; in particular, the $R$-covered property is stable for $F$.

The branched surface $W'$ does not necessarily satisfy the hypothesis of Theorem 2.1. In particular, the class of stably $R$-covered branched surfaces obtained here properly contains that which was described in [Go-Sh]. It is also worth noting that every $R$-covered foliation is carried by a stably taut branched surface (defined in Section 2) [Go-Sh]. So the key hypothesis of Theorem 2.2 is the existence of the set $\Gamma$.

**Corollary 2.3.** Given a stably taut branched surface $W$ with disk sectors constructed from an $R$-covered foliation $F$ of a closed orientable 3-manifold $M \neq S^2 \times S^1$, let $\tilde{W}$ be the lift of $W$ to the universal cover $\tilde{M}$. If for every non-$R$-covered foliation carried by $W$ there is a branching arc contained in the boundary of some component of $\tilde{M} - \tilde{W}$ linking nonseparable leaves in the universal cover, then there exists a stably $R$-covered branched surface $W'$ carrying $F$ and the $R$-covered property is stable for $F$.

In Section 3, we find verifiable conditions on the branch set $\mu$ of a branched surface $W$ constructed from an $R$-covered foliation $F$ that guarantee the existence of an $R$-covered branched surface carrying $F$. (See Theorem 3.2.) For many explicit examples, the $W$ we construct satisfies these conditions. In fact, it seems likely that all $R$-covered foliations of orientable 3-manifolds are carried by an $R$-covered branched surface (not necessarily generated by disks). So while it is known that the $R$-covered property is not always stable, there is often (perhaps always) a large class of $R$-covered foliations near any $R$-covered foliation $F$ of a closed manifold; specifically, there exists a transverse flow $\phi$ and a finite set $\Delta$ of compact integral surfaces of $F$ such that any foliation transverse to $\phi$ whose leaves contain the elements of $\Delta$ is $R$-covered.

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1. Branched surfaces constructed from foliations

In this section we examine branched surfaces constructed from codimension one, transversely orientable foliations of an orientable 3-manifold $M$. These branched surfaces are in the class of regular branched surfaces introduced by R. Williams [Wi]. Since the construction, first suggested to S. Goodman by C. Danthony, is in an unpublished paper of Christy and Goodman [Ch-Go], we describe it here, including all details necessary for this article.
Branched surface construction. We begin with a foliation $F$, a nonsingular flow $\phi$ transverse to $F$, and a generating set $\Delta = \{D_i\}$ of disjoint embedded compact surfaces with boundary (which is finite if the ambient manifold $M$ is closed), satisfying the following general position requirements:

(i) Each $D_i$ is embedded in a leaf of $F$ (hence is transverse to $\phi$).

(ii) Every orbit of $\phi$ meets the interior of some element of $\Delta$ in forward and backward time.

(iii) For every $i_0$, the set of points in $\text{Bdy}(D_{i_0})$ whose orbit under $\phi$ (or $\phi^{-1}$) meets $\cup \text{Bdy}(D_i)$ before meeting $\cup \text{Int}(D_i)$ is finite.

(iv) Any orbit of $\phi$ meets $\cup \text{Bdy}(D_i)$ at most twice.

It is worth noting that we can always choose a generating set $\Delta$ consisting of embedded disks. For example, cover $M$ with foliation boxes for $F$ and let $\Delta$ contain a slice from each box. Then modify each slice slightly so that $\Delta$ satisfies the general position requirements above.

We cut $M$ open along the interior of each element of $\Delta$ to obtain a submanifold $M^*$ which can be embedded in $M$ so that its boundary contains $\cup \text{Bdy}(D_i)$. This can be thought of as blowing air into the leaves of $F$ to create an air pocket at each element of the generating set. By requirement (ii) above, the restriction of $\phi$ to $M^*$ is a flow $\phi^*$ with the property that each orbit is homeomorphic to the unit interval $[0, 1]$. We next form a quotient space by identifying points that lie on the same orbit of $\phi^*$. That is, we take the quotient $M^*/\sim$, where $x \sim y$ if $x$ and $y$ lie on the same interval orbit of $\phi^*$. We may think of this as enlarging the components of $M - M^*$ until each interval orbit of $\phi^*$ is contracted to a point in $M$. The embedded copy of the resulting quotient space is the branched surface $W$; it is called the branched surface corresponding to $(F, \phi, \Delta)$. By its construction, $W$ can be made transverse to $\phi$. The general position requirements for $\Delta$ imply that $W$ is a connected 2-dimensional complex with a set of charts defining local orientation preserving diffeomorphisms onto one of the models in Figure 1.1, such that the transition maps are smooth and preserve transverse orientation indicated by the arrows. Each local model projects horizontally onto a vertical model of $\mathbb{R}^2$. So $W$ has a smooth structure induced by $T\mathbb{R}^2$ when we pull back each local projection.

A branched surface constructed from a foliation shall be a branched surface embedded in the ambient manifold and obtained in the above fashion using some transverse flow $\phi$. Each is a connected 2-manifold except on a dimension one subset $\mu$ called the branch set. The elements of $\Delta$ can be assumed to be chosen large enough to ensure that $\mu$ is connected. The set $\mu$ is a 1-manifold except at isolated points called crossings where it intersects itself transversely. (There are only finitely many of these points when $M$ is closed.) Each component of $W - \mu$ is called a sector of $W$. 

Curves in the branched surface. Formally, a curve in $M$ is a continuous map from a connected subset of $\mathbb{R}$ into $M$. However, we consider a curve to be the image of such a map, where the map parameterizes the curve. We say the curve is immersed (embedded) if this map is an immersion (embedding respectively.) For example, a curve immersed in a branched surface $W \subseteq M$ is the image of an immersion from a connected subset of $\mathbb{R}$ into $W$. The beginning and end of a curve refer to the negative and positive boundary, respectively, induced by the parameterization. For the sake of simplicity, we consider two curves $\gamma$ and $\gamma'$ to be the same if $\gamma(t) = \gamma'(t)$ after some orientation preserving reparameterization of each. We say a curve $\gamma'$ is contained in a second curve $\gamma$ (or more simply $\gamma'$ is in $\gamma$) only if $\gamma'$ is a subarc of $\gamma$. That is, there exist curves $\alpha_1$ and $\alpha_2$ such that $\gamma$ is equal to the composition $\alpha_2 \ast \gamma' \ast \alpha_1$. A loop in $\gamma$ will be a subarc of $\gamma$ with nonempty interior that begins and ends at the same point. A smooth arc in $W$ is an arc contained in $W$ that is smooth under the structure inherited from $W$. Throughout, we shall not consider curves in $W$ for which some subarc is homotopic, through a sector of $W$, to an arc in the branch set containing no crossings.
Given a point \( x \) contained in the branch set \( \mu \) of \( W \) that is not a crossing, there are locally 3 sectors, \( S_1, S_2, \) and \( S_3 \) that are adjacent at \( x \) such that \( \text{cl}(S_1) \cup S_3 \) and \( \text{cl}(S_2) \cup S_3 \) are smooth submanifolds of \( W \) (i.e., the set of charts locally defines a smooth immersion into a planar subset of \( \mathbb{R}^3 \)). Assume that in a local neighborhood of \( x \), forward orbits under \( \phi \) flow into \( S_1 \) flow into \( S_2 \). For any arc \( \alpha \) immersed in \( W \) that is transverse to \( \mu \) at \( x \), we can choose a subarc \( \beta \) of \( \alpha \) that has nonempty interior and meets \( W \) in forward orbits under \( \phi \). Assume that in a local neighborhood of \( x \), forward orbits under \( \phi \) flow into \( S_1 \) flow into \( S_2 \). For any arc \( \alpha \) immersed in \( W \) that is transverse to \( \mu \) at \( x \), we can choose a subarc \( \beta \) of \( \alpha \) that has nonempty interior and meets \( W \) in forward orbits under \( \phi \). Assume that in a local neighborhood of \( x \), forward orbits under \( \phi \) flow into \( S_1 \) flow into \( S_2 \).

We can extend the notion of incoming and outgoing sectors to a curve \( \alpha \) immersed in the branch set \( \mu \) of \( W \). For each crossing \( x \) of \( \mu \) contained in \( \alpha \), we can choose a subarc \( \beta \) of \( \alpha \) that is bounded by \( x \) and contains no other crossings of \( \mu \). (Each time \( \text{int}(\alpha) \) meets \( x \), \( \beta \) can be chosen to begin or end at \( x \).) There is precisely one sector \( S \) whose boundary is transverse to \( \beta \) at \( x \). If the subarc \( \beta \) can be chosen to begin (end) at \( x \) and the union of \( \text{cl}(S) \) with those sectors whose boundaries both contain \( x \) and do not intersect \( \text{int}(\beta) \) is smooth, then \( S \) is an outgoing (incoming, respectively) sector branching from \( \alpha \). For example, in both Figures 1.2(a) and 1.2(b), a curve \( \beta \) is indicated by the directed arcs. The corresponding sector \( S \) for each is shaded. In Figure 1.2(a), \( S \) is a lower outgoing sector branching from any \( \alpha \) containing \( \beta \), whereas in Figure 1.2(b) the sector \( S \) is not a lower outgoing sector along any such \( \alpha \). In the case where \( \alpha \) ends (begins) at \( x \), the sector \( S \) is an outgoing (incoming, respectively) sector branching from \( \alpha \) if for some subarc \( \beta \) chosen as above, the sector \( S \) is an outgoing (incoming) sector branching from a smooth curve in \( \mu \) that contains \( \beta \) and meets no crossings other than \( x \). For example, the sector \( S \) in Figure 1.2(b) is a lower incoming sector along any \( \alpha \) beginning with \( \beta \).

**Foliations carried by a branched surface.** Note that if we thicken the branched surface \( W \) in the transverse direction to recover the interval orbits of \( \phi^* \), we retrieve \( M^* \) which, for that reason, we shall henceforth call \( N(W), \text{the neighborhood of } W \). Also note that the boundary of \( N(W) \) is an embedded closed surface which has a smooth structure except along curves corresponding to the boundaries of the generating surfaces.

Throughout, \( \pi : N(W) \to W \) will denote the quotient map which identifies points in the same orbit of \( \phi^* \). We say the image \( x \) of a point under
this map is the projection of that point. Accordingly, we say points in the preimage of $x$ lie over $x$. In particular, the interval orbit of $\phi^*$ that projects onto $x$ will be referred to as the fiber of $N(W)$ over $x$. (See Figure 1.3.)

A foliation $F$ transverse to $\phi$ clearly gives rise to a foliation of $N(W)$, which we shall also denote by $F$, with leaves transverse to the fibers of $N(W)$. Each boundary component of $N(W)$ is contained in a leaf of this foliation. These leaves containing the boundary components of $N(W)$ are precisely the (cut-open) leaves of the original foliation containing the elements of $\Delta$. (They can be thought of as leaves of the original foliation with air blown into them.) Figure 1.4 shows a local picture of the foliation of $N(W)$.

There are, of course, many foliations like $F$ that are transverse to the fibers of $N(W)$ with the property that each boundary component of $N(W)$ is contained in a leaf. When we collapse the components of $M - N(W)$ (i.e., the air pockets), each of these foliations of $N(W)$ yields a foliation of $M$,
also transverse to $\phi$, which we say is *carried by* $W$. Indeed, $W$ carries a foliation $G$ if and only if $W$ corresponds to $(G,\phi,\Delta)$.

In what follows, $W$ will be a branched surface constructed from a foliation $F$ of a closed manifold $M$ as described above, and $\widehat{W}$ will be the lift of $W$ to the universal cover $\widehat{M}$. In other words, if $W$ corresponds to $(F,\Delta,\phi)$, then $\widehat{W}$ is the branched surface corresponding to $(\widehat{F},\widehat{\Delta},\widehat{\phi})$, where $\widehat{F}$, $\widehat{\Delta}$ and $\widehat{\phi}$ represent the lifts of $F$, $\Delta$, and $\phi$, respectively, to the universal cover. The covering map $\rho_M$ of $M$ by $\widehat{M}$ acts on $N(\widehat{W})$ and induces a covering map $p : \widehat{W} \to W$ such that $\pi \circ \rho_M = \rho \circ \widehat{\pi}$, where $\widehat{\pi}$ is the quotient map from $N(\widehat{W})$ onto $\widehat{W}$. (Details are given in [Sh1].)

### 2. Stability of the R-covered property

Throughout, $F$ will be a codimension one $C^1$ foliation of a closed orientable 3-manifold $M \neq S^2 \times S^1$. Passing to a double cover of $M$ if necessary, we may assume that $F$ is transversely orientable.

If $W$ is a branched surface constructed from a foliation $F$ whose generating set $\Delta$ consists of embedded disks, then all foliations sufficiently close to $F$ are also carried by $W$ [Sh2]. (Here we are using the $C^1$ metric defined in [Hi], where a nearby foliation is obtained by perturbing the tangent bundle to the leaves to another integrable plane field.) In particular, if a branched surface $W$ is generated by disks and every foliation carried by $W$ has a certain topological property, then we know that property is stable for all foliations carried by $W$.

Here we give conditions that guarantee a branched surface $W$ constructed from a foliation is **R-covered**; that is, $W$ has the property that every foliation carried by it is **R-covered**. If a branched surface $W$ is **R-covered** and generated by disks, then we say it is **stably R-covered** since any foliation $F$ carried by that $W$ is **R-covered** and this property is stable for each such $F$ (in the sense that all foliations sufficiently close to $F$ are also **R-covered**). We realize that this latter terminology is slightly misleading. That is, a branched
surface could carry only foliations for which the $\mathbb{R}$-covered property is stable without being generated by disks. However, our stability results rely on the branched surface being generated by disks. So for simplicity in our discussion, we define stably $\mathbb{R}$-covered as above.

Every $\mathbb{R}$-covered foliation of a closed orientable manifold $M \neq S^2 \times S^1$ is taut; i.e., there exists a positive transverse arc from any leaf to any other leaf. (For details, see Lemma B [Go-Sh].) So we first review the characterization of branched surfaces carrying only taut foliations.

Clearly any leaf in a foliation carried by a branched surface $W$ projects onto several sectors of $W$ and each sector has a transverse orientation, which it inherits from the leaves. So to ensure tautness, it suffices that there exists a positive transversal to $W$ between any two sectors. When we regard the branched surface as a simplicial complex, this translates to a condition on the dual graph. More precisely, we define the dual graph to the branched surface $W$ as follows: each component of $M - W$ will contain one vertex. There will be an edge through each sector of $W$, oriented according to the transverse orientation of $W$, joining two vertices. In particular, the dual graph is a directed graph in $M$ that is transverse to $W$. If for any ordered pair of vertices $(v, w)$ of the dual graph there is a positively oriented path from $v$ to $w$, then the dual graph is said to be transitive. When a foliation is carried by a branched surface with a transitive dual, then there is a positively oriented transverse arc from any leaf to any other leaf; specifically, there exists a positively oriented transverse loop through each leaf, so the foliation is taut.

The converse is also true. Indeed, any branched surface constructed from a taut foliation using a volume-preserving transverse flow has a transitive dual. So the transitive dual property characterizes branched surfaces carrying only taut foliations [Go-Sh]. Consequently, if $W$ has a transitive dual we say it is a taut branched surface. If, in addition, $W$ is generated by disks, we say $W$ is stably taut.

Now suppose $W$ is a taut branched surface corresponding to $(F, \Delta, \phi)$, where the surfaces in the generating set $\Delta$ are not necessarily embedded disks. In this case, there is a transverse loop through every leaf of $F$, which ensures the absence of Reeb components. This, in turn, precludes vanishing cycles. So when we lift $F$ to the universal cover, we obtain a foliation $\bar{F}$ of $\mathbb{R}^3$ where all the leaves are topologically closed planes (since $M$ is assumed to be orientable and not homeomorphic to $S^2 \times S^1$) and the leaf space is a 1-manifold, possibly non-Hausdorff [Ha]. At this stage, the only obstruction to $F$ being $\mathbb{R}$-covered is the existence of two leaves in the universal cover which are nonseparable in the leaf space; in other words, a pair of leaves $\bar{A}$ and $\bar{B}$ in $\bar{F}$ which do not have disjoint saturated neighborhoods. In particular, there is a 1-parameter family $\{\bar{K}_n\}$ of leaves, parameterized by $n \in I$ (where $I$ is the real interval $(0, 1)$), which lie on the same side of
Figure 2.1.

\( \widehat{A} \) and \( \widehat{B} \) and monotonically approach both \( \widehat{A} \) and \( \widehat{B} \) in the leaf space as \( n \to 1 \). (We assume points in the leaf space are ordered according to the transverse orientation of \( \widehat{F} \) induced by the transverse flow \( \widehat{\phi} \)). This sort of local behavior occurs, for example, in the lifts of certain Anosov foliations, like the one constructed in [Fr-Wi]. We can choose points \( \widehat{x}_n, \widehat{y}_n \) in \( \widehat{K}_n \) for each \( n \), such that \( \{ \widehat{x}_n \} \) converges to a point \( \widehat{x} \) in \( \widehat{A} \) along an orbit of \( \widehat{\phi} \) (or \( \widehat{\phi}^{-1} \)) and \( \{ \widehat{y}_n \} \) converges to a point \( \widehat{y} \) in \( \widehat{B} \) along an orbit of \( \widehat{\phi} \) (or \( \widehat{\phi}^{-1} \)), respectively. Now for every \( N \in (0, 1) \) the orbit from \( \widehat{x}_N \) to \( \widehat{x} \) is finite, so there are at most finitely many points along this orbit that are contained in a generating surface for \( \widehat{\mathcal{W}} \). (If not, such points would accumulate along this orbit on some point \( \widehat{z} \). However the generating set for \( \mathcal{W} \) is finite and each element is compact. So if we consider the point \( z \) covered by \( \widehat{z} \), the intersection of all generating surfaces for \( \mathcal{W} \) with any evenly covered neighborhood \( U \) of \( z \) has finitely many components. Furthermore, we can choose \( U \) small enough so that any orbit of \( \phi|_U \) intersects each of these components in at most one point, a contradiction.) So if each \( \widehat{K}_n \) lies on the negative (positive) side of \( \widehat{A} \) and \( \widehat{B} \), then we can choose \( N \) large enough so that orbits of \( \widehat{\phi} \) (or \( \widehat{\phi}^{-1} \), respectively) from \( \widehat{x}_N \) to \( \widehat{A} \) and from \( \widehat{x}_N \) to \( \widehat{B} \) do not meet any generating surface for \( \widehat{\mathcal{W}} \). These orbits are, up to orientation, contained in fibers of \( N(\widehat{\mathcal{W}}) \), so the local picture in \( \widehat{\mathcal{W}} \) is as shown in Figure 2.1.

More precisely, if \( \widehat{F} \) contains a pair of leaves \( \widehat{A} \) and \( \widehat{B} \) that are nonseparable on their negative (positive) sides, then there is a smoothly embedded arc \( \widehat{\beta} \) in \( \widehat{\mathcal{W}} \) transverse to \( \widehat{\mu} \) and containing no crossings of \( \widehat{\mu} \) with the property that the ends of \( \widehat{\beta} \) branch into the negative (positive) side of two smooth local subsets of \( \widehat{\mathcal{W}} \). In particular, the arc \( \widehat{\beta} \) has an upper (lower) outgoing sector branching from its initial point and an upper (lower) incoming sector branching into its terminal point. Any arc embedded in \( \widehat{\mathcal{W}} \) with these properties is called a negatively (positively) branching arc in \( \widehat{\mathcal{W}} \). Each branching
arc $\tilde{\beta}$ in $\tilde{W}$ covers a branching arc $\beta$ in $W$. A branching arc corresponding to a nonseparable pair of leaves $\tilde{A}$ and $\tilde{B}$ as above can be chosen so that its ends lie in $\tilde{\pi}(\tilde{A})$ and $\tilde{\pi}(\tilde{B})$, respectively, and some integral curve $\tilde{\kappa}$ of $\tilde{\pi}$ lies over it in $N(\tilde{W})$. In this case, we say $\tilde{\beta}$ is a branching arc linking $\tilde{A}$ and $\tilde{B}$.

If the branched surface $\tilde{W}(W)$ has only disk sectors, then certain branching arcs in $\tilde{W}$ ($W$, respectively) can be regarded as essentially the same. In particular, the boundary of each disk sector $S$ is contained in the branch set and can be partitioned by crossings into disjoint open arcs. Any two curves whose interiors lie in $S$ are equivalent if their initial points both lie in the interior of $S$ or in the same interval of the partition, and if the same condition holds for their terminal points. Now, any finite topologically closed curve immersed in $\tilde{W}(W)$ that is transverse to the branch set is a composition of finitely many curves whose interiors are contained in the disk sectors of $\tilde{W}$ ($W$, respectively). Two such curves are equivalent if they contain no crossings of the branch set and are piecewise equivalent. It is worth emphasizing that this definition applies only to curves contained in branched surfaces with disk sectors. Also, note that if $\tilde{\beta}$ is a branching arc in $\tilde{W}$ linking nonseparable leaves $\tilde{A}$ and $\tilde{B}$, and $\tilde{\beta}'$ is equivalent to $\tilde{\beta}$, then $\tilde{\beta}'$ is also a branching arc linking $\tilde{A}$ and $\tilde{B}$.

Now let $W$ be a taut branched surface carrying a foliation $F$ that has sectors of any type and is not necessarily generated by disks. Although nonseparable leaves in the lift $\tilde{F}$ of $F$ to the universal cover give rise to branching arcs in a branched surface $\tilde{W}$ carrying $\tilde{F}$, not every branching arc in $\tilde{W}$ indicates a pair of nonseparable leaves. In fact, any smoothly embedded integral curve of $\tilde{F}$ whose interior is contained in an element of $\tilde{\Delta}$ (generating $\tilde{W}$) and whose endpoints are in the boundary of that generating surface has two images in $\tilde{W}$, both of which are branching arcs. So branching arcs also occur in branched surfaces carrying only $R$-covered foliations. Certain curves embedded in $\tilde{W}$, called bypasses, offer a topological means for identifying those branching arcs that cannot link nonseparable leaves. Specifically, a smoothly embedded arc $\tilde{\delta}$ in $\tilde{W}$ transverse to $\tilde{\mu}$ and fixed point homotopic to a negatively branching arc $\tilde{\beta}$ is a bypass for $\tilde{\beta}$ if it contains no crossings of $\tilde{\mu}$ or negatively branching arcs. (Figure 2.2 shows a bypass for a branching arc $\tilde{\beta}$.) A bypass for a positively branching arc is defined in an analogous manner. Clearly a bypass $\tilde{\delta}$ for a branching arc $\tilde{\beta}$ in $\tilde{W}$ covers an immersed arc $\delta$ homotopic to $\beta$ (the branching arc covered by $\tilde{\beta}$) with the same properties. Accordingly, such an arc $\delta$ is a bypass for $\beta$.

We say that a branching arc $\beta$ in a taut branched surface $W$ (as above) is critical if it lifts to an arc linking nonseparable leaves in some foliation carried by $\tilde{W}$. In particular, critical branching arcs have no bypasses. For
Figure 2.2.

suppose, to the contrary, that a branching arc \( \hat{\beta} \) has a bypass \( \hat{\delta} \) yet links leaves \( \hat{A} \) and \( \hat{B} \) which are nonseparable on, say, their negative sides. If the projection of \( \hat{A} \) branches away from \( \hat{\delta} \) along an upper outgoing sector at some point \( \hat{x} \) in \( \hat{\delta} \), then there are no upper incoming sectors branching into \( \hat{\delta} \) after \( \hat{x} \). In this case, there exists a positive transversal from \( \hat{B} \) to \( \hat{A} \), a contradiction. So \( \hat{\pi}(\hat{A}) \) branches away from \( \hat{\delta} \) along a lower outgoing sector at some point \( \hat{x} \) in \( \hat{\delta} \). Similarly, we find \( \hat{\pi}(\hat{B}) \) branches into \( \hat{\delta} \) along a lower incoming sector at some point after \( \hat{x} \) in \( \hat{\delta} \); more precisely, \( \hat{\delta} \) contains a positively branching arc with \( \hat{\pi}(\hat{A}) \) and \( \hat{\pi}(\hat{B}) \) branching from its respective ends. However, the uppermost leaf whose projection meets this arc is cut by positive transversals (not necessarily fibers) from \( \hat{A} \) and \( \hat{B} \), contradicting our assumption that \( \hat{A} \) and \( \hat{B} \) are nonseparable on their negative sides. So if a foliation \( F \) is carried by a taut branched surface \( W \) and every branching arc has a bypass, then \( F \) is \( \mathbb{R} \)-covered. If, in addition, \( W \) is stably taut (hence, is generated by disks), then the \( \mathbb{R} \)-covered property is stable for \( F \).

These observations lead to the following:

**Theorem 2.1 ([Go-Sh])**. Given a stably taut branched surface \( W \) constructed from a foliation of a closed orientable 3-manifold \( M \neq S^2 \times S^1 \), if every branching arc has a bypass, then \( W \) is \( \mathbb{R} \)-covered and the \( \mathbb{R} \)-covered property is stable for each foliation \( F \) carried by \( W \); in particular, \( F \) and all foliations sufficiently close to \( F \) are \( \mathbb{R} \)-covered.

If a foliation \( F \) is \( \mathbb{R} \)-covered, then we can construct a branched surface \( W \) from it using a generating set consisting of embedded disks and a transverse flow that is volume-preserving. As noted earlier, \( W \) will be stably taut. However, such a \( W \) will often contain branching arcs without bypasses.
In [Go-Sh], a technique was given for modifying $W$ to produce bypasses for certain branching arcs. Unfortunately, this can create annuli in the generating set. So while $W$ will still be taut after these modifications, it might not be stably taut. Moreover, application of this technique frequently does not produce a branched surface where every branching arc has a bypass. One obstruction is that it can only be applied to branching arcs that are contained in the projection of an integral curve of $F$. Another is that most branched surfaces contain infinitely many branching arcs. For example, an infinite set of branching arcs may share a nontrivial loop and differ only in the number of times this loop is traversed. (Here a loop in a branching arc is a subarc with nonempty interior that begins and ends at the same point.)

Consequently, we shall weaken the hypotheses of Theorem 2.1 significantly. To prove stability of the $R$-covered property under these new conditions, we modify $W$ using a different technique that will not create annuli in the generating set.

The main result is as follows:

**Theorem 2.2.** Given a stably taut branched surface $W$ constructed from a foliation of a closed orientable 3-manifold $M \neq S^2 \times S^1$, let $\hat{W}$ be the lift of $W$ to the universal cover. Suppose there exists a finite set $\Gamma$ of branching arcs in $W$ such that the lift of any non-$R$-covered foliation carried by $W$ to a foliation carried by $\hat{W}$ contains a pair of nonseparable leaves linked by a lift of some arc in $\Gamma$. For any $R$-covered foliation $F$ carried by $W$, there exists a stably $R$-covered branched surface $W'$ also carrying $F$; in particular, the $R$-covered property is stable for $F$.

**Proof.** Suppose $W$ and $\Gamma$ are as in the hypotheses. In particular, $W$ is stably taut, so it is generated by disks and all foliations carried by $W$ are taut. Let $F$ be an $R$-covered foliation carried by $W$ and let $\hat{\beta}$ be any critical branching arc in $\Gamma$. Without loss of generality, we may assume $\hat{\beta}$ is negatively branching. Substituting a homotopic branching arc if necessary, we may also assume that $\hat{\beta}$ does not begin and end at the same point; that is, $\hat{\beta}(0) \neq \hat{\beta}(1)$. We first consider the case where $\hat{\beta}$ and its lift $\hat{\beta}$ to $\hat{W}$ are projections of integral curves of $F$ and $\hat{F}$ respectively. In particular, $\hat{\beta}$ is contained in $\hat{\pi}(\hat{L})$ for some leaf $\hat{L}$ of $\hat{F}$.

Since $\hat{\beta}$ has no bypass, the positive ends of the fibers over $\hat{\beta}(0)$ and $\hat{\beta}(1)$ lie in distinct boundary components of $N(\hat{W})$ which correspond respectively to elements $\hat{D}_0$ and $\hat{D}_1$ of $\hat{\Delta}$. Let $\hat{L}_0$ and $\hat{L}_1$ be the leaves of $\hat{F}$ containing $\hat{D}_0$ and $\hat{D}_1$ respectively. In the natural ordering of the leaves of $\hat{F}$ (induced by a copy of the leaf space $R$ which, in turn, is oriented according to the direction of the transverse flow $\hat{\phi}$), we have either:

(i) $\hat{L}_0 < \hat{L}_1$,
(ii) $\hat{L}_0 > \hat{L}_1$, or
(iii) $\hat{L}_0 = \hat{L}_1$. 

Figure 2.3.

In case (i), \( \hat{L} < \hat{L}_0 < \hat{L}_1 \), and since the fiber \( \hat{t}_1 \) over \( \hat{\beta}(1) \) meets \( \hat{L} \) and \( \hat{L}_1 \), it meets all leaves between \( \hat{L} \) and \( \hat{L}_1 \). It follows that \( \hat{t}_1 \) also meets \( \hat{L}_0 \), so we can find a curve \( \hat{\gamma}_F \) smoothly embedded in \( \hat{L}_0 \) that begins in \( \partial \hat{D}_0 \) and cuts the fiber \( \hat{t}_1 \). When choosing \( \hat{\gamma}_F \), we shall want to ensure that its quotient \( \gamma_F \) does not cut \( t_0 \) or \( t_1 \) more than once: i.e., we choose \( \hat{\gamma}_F \) so it descends to a smoothly immersed integral curve \( \gamma_F \) of \( F \) whose interior does not meet \( t_0 \) or \( t_1 \). We can then find a disk \( \hat{D} \) embedded in a leaf of \( \hat{F} \) that does not intersect any generating disks for \( \hat{W} \) and lies sufficiently close to \( \hat{D}_0 \) on the positive side so that all orbits from \( \hat{\gamma}_F \) meet \( \hat{D} \) before meeting a generating disk. Adding \( \hat{D} \) to the generating set \( \hat{\Delta} \) (and its quotient \( D \) to \( \Delta \)) changes the branched surface by splitting it along an embedded disk containing \( \hat{\pi}(\hat{\gamma}_F) \) (\( \pi(\gamma_F) \) respectively). By the way we chose \( \hat{\gamma}_F \), we can ensure no covering translate of \( \hat{D} \) cuts \( \hat{t}_0 \) or the new fiber \( \hat{t}_1 \) over \( \hat{\beta}(1) \). So after this modification, there is a smoothly embedded arc \( \hat{\delta} \) in \( \hat{W} \) containing \( \hat{\pi}(\hat{\gamma}_F) \) which is fixed point homotopic to \( \hat{\beta} \) and has only one upper branch along its interior at the initial point of \( \hat{\pi}(\hat{\gamma}_F) \). (See Figure 2.3.) In this way, we create bypasses \( \hat{\delta} \) and \( \delta \) for \( \hat{\beta} \) and \( \beta \) respectively.

Cases (ii) and (iii) are analogous. (In case (iii), the curve \( \hat{\delta} \) that we create has an upper incoming sector branching into it at \( \hat{\pi}(\hat{\gamma}_F(0)) \) and an upper outgoing sector branching from it at \( \hat{\pi}(\hat{\gamma}_F(1)) \), but contains no negatively branching arcs.)

So whenever a critical branching arc \( \beta \) is the projection of an integral curve of \( F \), we can modify \( W \) to create a bypass \( \delta \) for \( \beta \). If, on the contrary, \( \beta \) is not the projection of an integral curve of \( F \), then it can be destroyed. Reversing the orientation of \( \beta \) if necessary, there exists an integral curve \( \gamma_F(t)_{0 \leq t \leq 1} \) of \( F \) beginning in the intersection of \( \partial N(W) \) with the interior of a fiber over \( \text{int}(\beta) \) such that for some \( \varepsilon > 0 \), \( \pi(\gamma_F(t))_{0 \leq t \leq 1-\varepsilon} \) is contained in \( \beta \), \( \beta \cap \pi(\gamma_F(t))_{1-\varepsilon < t < 1} = \varnothing \) and \( \gamma_F(1-\varepsilon) \notin \partial N(W) \). See Figure 2.4. So there exist incoming and outgoing branches along \( \beta \) at \( \pi(\gamma_F(0)) \) and \( \pi(\gamma_F(1-\varepsilon)) \)
respectively. Moreover, we can choose $\gamma_F$ so that if the sector branching into $\beta$ at $\pi(\gamma_F(0))$ is an upper (lower) incoming sector, then the sector branching from $\beta$ at $\pi(\gamma_F(1-\varepsilon))$ is a lower (upper, respectively) outgoing sector containing $\pi(\gamma_F(t))_{1-\varepsilon < t \leq 1}$. (For details, see [Sh1].) Clearly, the initial point of $\gamma_F$ lies in the boundary of some generating disk $D_0$ as before.

If there is an upper (lower) sector branching into $\beta$ at $\pi(\gamma_F(0))$, we may add an embedded disk $D$ to the generating set which lies sufficiently close to $D_0$ on the positive (negative) side so that all forward (backward, respectively) orbits from $\gamma_F$ meet $D$ before meeting any other generating disk. This change in $\Delta$ corresponds to splitting $W$ along a disk containing $\pi(\gamma_F)$ to destroy $\beta$. Specifically, no foliation carried by the original branched surface $W$ with an integral curve over $\beta$ is carried by the new branched surface $W'$. So given any critical branching arc $\beta \in \Gamma$, we may modify $W$ to obtain a new branched surface $W'$, also generated by disks and carrying $F$, such that any foliation carried by $W$ and containing nonseparable leaves linked by a lift $\tilde{\beta}$ of $\beta$ is not carried by $\tilde{W}'$. By applying Lemma A [Go-Sh], we see that $W'$ also has a transitive dual and so is stably taut. Furthermore, every foliation carried by $W'$ is carried by $W$, since $N(W')$ can be obtained from $N(W)$ by pinching together a component of $\partial N(W')$ (contained in a branched leaf of the foliation). This amounts to removing a disk from the generating set for $W'$. By hypothesis, the set $\Gamma = \{\beta_1, \ldots, \beta_n\}$ is finite and the lift of any non-$\mathbb{R}$-covered foliation carried by $W$ to a foliation carried by $\tilde{W}$ contains a pair of nonseparable leaves linked by a lift of some arc in $\Gamma$. Moreover, we may assume each arc in $\Gamma$ is critical. In particular, we can modify $W$ as above to either destroy $\beta_1$ or create a bypass for it. We do so in a way that splits $N(W)$ along a curve meeting each $\beta_i$, $i \neq 1$, either transversely at interior points or not at all. In this sense, the new branched surface $W_1$ contains a branching arc corresponding to $\beta_i$ for every $i > 1$. Furthermore, the lift of
any non-$R$-covered foliation carried by $W_1$ contains a pair of leaves linked by a lift of one such arc.

We next apply the same technique to either destroy the branching arc in $W_1$ corresponding to $\beta_2$ or create a bypass for it; this yields a new branched surface $W_2$. Continuing in this manner, we obtain a finite string of branched surfaces $W = W_0, W_1, \ldots, W_n$, each carrying $F$. Let $W' = W_n$.

Now suppose there is a foliation $G$ carried by $W'$ whose lift $\hat{G}$ to the universal cover has a non-Hausdorff leaf space. The foliation $G$ is also carried by $W_i$, so for some $1 \leq i \leq n$, a lift $\hat{\beta}_i$ of $\beta_i$ links nonseparable leaves $\hat{A}$ and $\hat{B}$ in $\hat{G}$. However, $W_i$ was obtained by modifying $W_{i-1}$ to either destroy the branching arc (in $W_{i-1}$) corresponding to $\beta_i$ or create a bypass for it. Since some lift of this branching arc also links $\hat{A}$ and $\hat{B}$, the foliation $\hat{G}$ is not carried by $\hat{W}_i$, a contradiction.

It follows that every foliation carried by $W'$ is $R$-covered. □

As noted above, any transversely orientable $R$-covered foliation of a closed orientable 3-manifold $M \neq S^2 \times S^1$ is carried by a stably taut branched surface. So the essential hypothesis of Theorem 2.2 is the existence of the set $\Gamma$. It is also worth mentioning that if we omit the assumption that disks generate $W$, the proof above can still be used to show $F$ is carried by an $R$-covered branched surface $W'$. The only thing that is lost is the guarantee that $W'$ is stably $R$-covered.

A negatively (positively) branching arc $\beta$ with no upper (lower respectively) branches along its interior is contained in the boundary of some component of $M - W$. In particular, $\beta$ is the projection of a smoothly immersed curve whose interior is contained in some element of the generating set $\Delta$ and whose endpoints are in the boundary of that generating surface. If $W$ is generated by disks and has all disk sectors, then there are at most finitely many such branching arcs in $W$ up to equivalence. In particular, we have the following:

**Corollary 2.3.** Given a stably taut branched surface $W$ with disk sectors constructed from an $R$-covered foliation $F$ of a closed orientable 3-manifold $M$, let $\hat{W}$ be the lift of $W$ to the universal cover $\hat{M}$. If for every non-$R$-covered foliation carried by $W$ there is a branching arc contained in the boundary of some component of $\hat{M} - \hat{W}$ linking nonseparable leaves in the universal cover, then there exists a stably $R$-covered branched surface $W'$ carrying $F$ and the $R$-covered property is stable for $F$.

### 3. $R$-covered branched surfaces

In this section, we find a condition on the branch set of a taut branched surface $W$ constructed from $R$-covered foliation $F$ of a closed orientable manifold $M \neq S^2 \times S^1$ that guarantees the existence of an $R$-covered branched
to $\gamma$ can be written as a composition \[ \hat{\gamma} \] of a closed subarc $F$(surface carrying $K$ in $W$) not assuming that disks generate $W$ unless this is explicitly stated. Since this means the map of $W(W)$ into the ambient manifold $M$ ($\hat{M}$ respectively) by inclusion is not always $\pi_1$-injective, we assume the range of all homotopy maps is the branched surface unless otherwise stated.

An arc embedded in the branch set $\hat{\mu}$ of $\hat{W}$ with an upper (lower) outgoing sector branching from its initial point and an upper (lower respectively) incoming sector branching into its terminal point is called a *branching arc in $\hat{\mu}$.* (Each branching arc in $\hat{\mu}$ covers a branching arc in $\mu$.)

For example, if $W$ is a taut branched surface with disk sectors, then any branching arc $\hat{\beta}$ in $\hat{W}$ transverse to $\hat{\mu}$ is homotopic to a curve containing such an arc, provided that it links nonseparable leaves $\hat{A}$ and $\hat{B}$ in the lift of some foliation $G$ carried by $W$. Specifically, there is, by definition, a smoothly embedded curve $\hat{\kappa}$ in some leaf $\hat{K}$ of $\hat{G}$ that projects injectively onto $\hat{\beta}$. It follows that $\hat{\beta}$ is smoothly embedded in $\hat{\pi}(\hat{K})$ and there exists an $\hat{\alpha}$ immersed in $\hat{\mu}$ that is piecewise homotopic to $\hat{\beta}$ through disk sectors of $\hat{W}$ contained in $\hat{\pi}(\hat{K})$. By Reeb stability [Re], each of these disk sectors is covered injectively by a disk embedded in $\hat{\kappa}$. So there is a curve $\hat{\gamma}$ immersed in $\hat{\kappa}$ and homotopic to $\hat{\kappa}$ whose projection $\hat{\pi}(\hat{\gamma})$ is a curve in $\hat{\mu}$ homotopic to $\hat{\beta}$. If $\hat{\gamma}$ contains a loop, i.e., if there exist subarcs $\hat{\gamma}_1$ and $\hat{\gamma}_2$ of $\hat{\gamma}$ and a closed subarc $\hat{\lambda}$ of $\hat{\gamma}$ that begins and ends at the same point such that $\hat{\gamma}$ can be written as a composition $\hat{\gamma}_1 * \hat{\lambda} * \hat{\gamma}_2$, then $\hat{\lambda}$ is null homotopic in $\hat{\kappa}$. In this case, the curve $\hat{\gamma}_1 * \hat{\gamma}_2$, obtained by removing $\hat{\lambda}$ from $\hat{\gamma}$ and reparameterizing the remaining arcs, $\hat{\gamma}_1$ and $\hat{\gamma}_2$, is also homotopic to $\hat{\kappa}$ in $\hat{\kappa}$. So we may assume $\hat{\gamma}$ contains no loops. It follows that $\hat{\pi}(\hat{\gamma})$ contains no loops, since otherwise there are two distinct points $\hat{x}$ and $\hat{y}$ in $\hat{\gamma}$ that project onto the same point of $\hat{W}$. However, if this occurs, we could construct a closed curve by taking the composition of a curve in $\hat{\kappa}$ joining $\hat{x}$ and $\hat{y}$ with a curve contained in a fiber of $N(\hat{\pi}(\gamma))$ up to orientation. Modifying this curve slightly, we could then produce a closed loop transverse to $\hat{\pi}(\hat{\gamma})$ that is null homotopic in $\hat{\mu}$, contradicting our assumption that every foliation carried by $W$ is taut. Now since $\hat{\pi}(\hat{\gamma})$ begins and ends in $\hat{\pi}(\hat{A})$ and $\hat{\pi}(\hat{B})$ respectively, it has a subarc $\hat{\beta}_\mu$ which is also embedded in $\hat{\mu}$ and whose ends both branch into the negative (or positive) side of two smooth local subsets of $\hat{\pi}(\hat{A})$ and $\hat{\pi}(\hat{B})$ respectively. Furthermore, there is a curve embedded in $\hat{\kappa}$ that lies over $\hat{\beta}_\mu$ in $N(\hat{\pi}(\gamma))$; namely the corresponding subcurve of $\hat{\gamma}$. An arc $\hat{\beta}_\mu$ with these properties is called a *critical branching arc in $\hat{\mu}$ linking $\hat{A}$ and $\hat{B}$.* Its quotient $\beta_\mu$ is a *critical branching arc in $\mu$.* Since $\hat{\beta}_\mu$ contains...
no loops that are null homotopic in $\hat{W}$, the curve $\beta_\mu$ contains no loops that are null homotopic in $W$.

It is worth noting that branching arcs in the branch set are not necessarily smooth. Moreover, a branching arc in $\hat{\mu}$ with $\hat{\pi}(\hat{A})$ and $\hat{\pi}(\hat{B})$ branching from its respective ends is not necessarily critical, since to link $\hat{A}$ and $\hat{B}$ the arc must lie in a smoothly embedded plane in $\hat{W}$. It is also worth noting that there seems to be no natural way to extend the notion of equivalence to branching arcs in $\hat{\mu}$.

A bypass for a negatively (positively) branching arc in $\hat{\mu}$ will be a homotopic curve, either embedded in $\hat{\mu}$ or smoothly embedded in $\hat{W}$ and transverse to $\hat{\mu}$, that contains no negatively (positively) branching arcs. Its quotient in $W$ is a bypass for the corresponding branching arc in $\mu$. We say a finite arc $\tau$ embedded in $\mu$ is inescapable if every upper (lower) outgoing sector along $\tau$ branches away from some negatively (positively) branching arc that is both contained in $\tau$ and has a bypass with no upper (lower, respectively) outgoing sectors. A finite arc $\hat{\tau}$ embedded in $\hat{\mu}$ is inescapable if it is the lift of an inescapable arc in $\mu$.

**Lemma 3.1.** Given a branched surface $W$ constructed from a foliation $F$ of a closed orientable 3-manifold $M$, let $\hat{F}$ and $\hat{W}$ be the lifts of $F$ and $W$, respectively, to the universal cover. If the projection $\hat{\pi}(\hat{L})$ of some leaf $\hat{L}$ of $\hat{F}$ meets the initial point of an inescapable curve $\hat{\tau}$ in the branch set $\hat{\mu}$ of $\hat{W}$, then $\hat{\pi}(\hat{L})$ also meets the terminal point of $\hat{\tau}$.

**Proof.** Suppose $\hat{\tau}(t)_{0 \leq t \leq 1}$ is an inescapable curve embedded in the branch set $\hat{\mu}$ of $\hat{W}$. Let $\hat{L}$ be a leaf in a foliation $\hat{F}$ carried by $\hat{W}$ such that its projection $\hat{\pi}(\hat{L})$ meets $\hat{\tau}(0)$. Any time $\hat{\pi}(\hat{L})$ branches away from $\hat{\tau}$, it does so along an outgoing sector at a crossing of $\hat{\mu}$ contained in $\hat{\tau}$. By our construction of $W$ and $\hat{W}$, crossings of $\hat{\mu}$ cannot accumulate in the lift of an evenly covered neighborhood of $W$. So since $\hat{\tau}$ is compact, it contains only finitely many crossings. As a consequence, if $\hat{\pi}(\hat{L})$ does not meet $\hat{\tau}(1)$, there is a last point $\hat{\tau}(t')$ along $\hat{\tau}$ that is contained in $\hat{\pi}(\hat{L})$.

In this case, $\hat{\pi}(\hat{L})$ branches away from $\hat{\tau}$ along, say, an upper outgoing sector at $\hat{\tau}(t')$, $0 \leq t' < 1$. By hypothesis, there exists $t_0 \leq t' \leq t_1$, $t_0 \neq t_1$, such that $\hat{\tau}(t)_{t_0 \leq t \leq t_1}$ is a negatively branching arc. Furthermore, this branching arc has a bypass $\delta$ with no upper outgoing sectors branching from it. So there is an embedded curve $\tilde{\eta}$ over $\tilde{\delta}$, beginning and ending at the uppermost points over $\hat{\tau}(t_0)$ and $\hat{\tau}(t_1)$ respectively, which is a composition of integral curves of $\hat{F}$ and positive transversals contained in fibers of $N(\hat{W})$. Furthermore, we can find an embedded curve $\hat{\gamma}(t)_{t_0 \leq t \leq t_1}$ where $\hat{\gamma}(t)_{t_0 \leq t \leq t'}$ lies over $\hat{\tau}(t)_{t_0 \leq t \leq t'}$ and is contained in a leaf below $\hat{L}$, and $\hat{\gamma}(t')_{t' \leq t \leq t_1}$ lies over $\hat{\tau}(t)_{t' \leq t \leq t_1}$ and consists of integral curves of $\hat{F}$ and transversals to $\hat{F}$.
Indicates \( \hat{\tau} \) and indicates \( \hat{\pi}(\hat{L}) \).

Indicates leaves of \( \hat{F} \) and indicates \( \hat{\eta} \) and \( \hat{\gamma} \).

Figure 3.1.

contained in fibers of \( N(\hat{W}) \) up to orientation. See Figure 3.1. We then have a disk in \( \hat{M} \) bounded by \( \hat{\gamma}, \hat{\eta} \), and transversals over \( \hat{\tau}(t_0) \) and \( \hat{\tau}(t_1) \) respectively which can be put in general position with respect to the foliation \( \hat{F} \). The leaf \( \hat{L} \) meets the boundary of this disk transversely at some point over \( \hat{\tau}(t_0) \). Furthermore, \( \hat{L} \) cannot cross any of the positive transversals in \( \hat{\eta} \) since \( \hat{F} \) is transversely orientable. So \( \hat{L} \) must cross either \( \hat{\gamma}(t)_t \leq t \leq t_1 \) or the transversal over \( \hat{\tau}(t_1) \). In both cases, we have a contradiction to the way we chose \( t' \). So \( \hat{\pi}(\hat{L}) \) contains \( \hat{\tau}(1) \).

In what follows, a parameterized embedded copy of \( S^1 \) in \( \mu \) that does not bound an embedded disk in \( W \) will be called an essential loop. Furthermore, we shall say a branching arc in \( \mu \) is elementary if it contains no loops.

The following theorem offers a verifiable condition on the branch set of a branched surface carrying an \( R \)-covered foliation \( F \) that guarantees the existence of an \( R \)-covered branched surface carrying \( F \).

**Theorem 3.2.** Given a taut branched surface \( W \) with disk sectors constructed from a foliation of a closed orientable 3-manifold \( M \neq S^2 \times S^1 \), let \( \alpha(t)_{0 \leq t \leq 1} \) be an arc embedded in the branch set \( \mu \) that has an outgoing
sector branching from its initial point and contains no loops. Suppose that
for every such $\alpha$ and every essential loop $\lambda$ in $\mu$ based at $\alpha(1)$, there exist
homotopic curves $\eta \subseteq \alpha * \lambda$ and $\tau \subseteq \mu$ beginning at $\alpha(0)$ such that $\eta$ has
nonempty interior and $\tau$ is inescapable. For any $\mathbb{R}$-covered foliation $F$
carried by $W$, there exists an $\mathbb{R}$-covered branched surface $W'$ also carrying $F$; in
particular, when $W$ is generated by disks, the $\mathbb{R}$-covered property is stable
for $F$.

Proof. Suppose $G$ is a non-$\mathbb{R}$-covered foliation carried by $W$ and let $\hat{\beta}$ be a
critical branching arc in the branch set $\hat{\mu}$ of $\hat{W}$ linking nonseparable leaves
$\hat{A}$ and $\hat{B}$ of $\hat{G}$. We may choose $\hat{\beta}$ so that it contains no other branching arc
linking $\hat{A}$ and $\hat{B}$. We may also assume, without loss of generality, that $\hat{A}$
and $\hat{B}$ are nonseparable on their negative sides, so $\hat{\beta}$ is negatively branching.
By definition, $\hat{\beta}$ is the projection of an integral curve of $\hat{G}$ onto $\hat{W}$. As noted
above, it descends to a branching arc $\beta$ in $\mu$ that contains no loops which
are null homotopic in $W$ (i.e., there is no subarc of $\beta$ with nonempty interior
that begins and ends at the same point and is null homotopic).

So if $\beta$ contains a loop, then it contains an essential loop. In this case,
let $\lambda$ be the first essential loop in $\beta$ and let $\alpha$ be the arc in $\beta$ from its initial
point $\beta(0)$ to the initial point of $\lambda$. We may now choose homotopic curves
$\eta$ in $\alpha * \lambda$ and $\tau$ in $\mu$ as in the hypotheses. Since $\tau$ is inescapable it is, by
definition, embedded in $\mu$. So $\tau$ lifts to a curve $\tilde{\tau}$ embedded in $\hat{\mu}$ with the
same initial point as $\hat{\beta}$. In particular, $\tilde{\tau}$ begins in $\hat{\pi}(\hat{A})$ and, by Lemma 3.1,$\tilde{\tau}$ ends in $\hat{\pi}(\hat{A}) \cap \text{int}(\hat{\beta})$. By assumption, $\hat{B}$ meets the fiber over $\hat{\beta}(1)$, so
$\tilde{\pi}(\hat{A})$ cannot meet this fiber. Therefore, $\hat{\pi}(\hat{A})$ must branch away from $\text{int}(\hat{\beta})$
along an upper sector, a contradiction since we chose $\hat{\beta}$ to contain no other
critical branching arcs linking $\hat{A}$ and $\hat{B}$.

It follows that any pair of nonseparable leaves $\hat{A}$ and $\hat{B}$ in $\hat{G}$ is linked by a
critical branching arc $\hat{\beta}$ in $\mu$ which descends to an elementary branching arc $\beta$
in $\mu$. Furthermore, $\beta$ can be perturbed slightly onto adjacent sectors to
obtain a homotopic branching arc that is nowhere tangent to $\mu$, contains no
crossings of $\mu$ and also lifts to a branching arc linking $\hat{A}$ and $\hat{B}$. Now for every
elementary branching arc $\beta$ in $\mu$, there are at most finitely many homotopic
branching arcs (up to equivalence) that can be obtained by perturbing $\beta$ in
this manner. Since there are at most finitely many elementary branching
arcs in $\mu$, the result now follows from the remark after Theorem 2.2. □

As above, let $W$ be a taut branched surface with disk sectors and let $F$
be an $\mathbb{R}$-covered foliation carried by $W$. We say a negatively (positively)
branching arc in the branch set $\mu$ of $W$ is simple if it is elementary and
has no upper (lower, respectively) branches along its interior. Clearly there
are at most finitely many simple branching arcs in $\mu$ and each is contained
in the boundary of some component of $M - W$. In particular, each simple
branching arc $\beta$ is the projection of an integral curve of $F$. So we can split $N(W)$ along disks embedded in leaves of $F$, as in the proof of Theorem 2.2, to create bypasses for all the simple branching arcs in $\mu$. (For each simple negatively (positively) branching arc $\beta$, the disk we use is met by forward (backward) orbits of an integral curve $\gamma_F$ of $F$ which meets the uppermost (lowermost) point over one end of $\beta$ and also meets the fiber over the other end.) The branched surface we obtain, $V$, carries only foliations carried by $W$, including $F$. (If the lift of each non-$R$-covered foliation carried by $W$ contains a pair of nonseparable leaves linked by a simple branching arc in $\hat{\mu}$, then the branched surface $V$ is $R$-covered.)

Now, we can choose an orientation for each simple negatively (positively) branching arc $\beta$ in $\mu$ so that for any foliation carried by $\hat{V}$, the leaves meeting the fiber of $N(\hat{W})$ over $\hat{\beta}(0)$ either meet the fiber over $\hat{\beta}(1)$ or branch away from $\hat{\beta}$ along a lower (upper respectively) outgoing sector. (For example, if $\beta$ is negatively branching, the orientation is chosen so that $\hat{\gamma}_F$ is contained in the uppermost leaf of $\hat{F}$ over $\hat{\beta}(0)$; if some leaf of $\hat{F}$ meets the uppermost point over both ends of $\hat{\beta}$, then either orientation will suffice.) Then any curve $\tau$ in the branch set of $W$ yields an inescapable curve in $V$ if every upper (lower) outgoing sector branches from a negatively (positively) branching arc contained in $\tau$ that either has a bypass with no upper (lower respectively) outgoing sectors or is simple and oriented as above. So the $\tau$ in Theorem 3.2 need not be inescapable; rather, it need only satisfy this weaker condition. In other words, it is not necessary to explicitly construct $V$ from $W$ to check whether it satisfies the hypotheses of Theorem 3.2.

For example, let $G_0$ be a foliation of $T^2$ with two unstable Reeb components, and consider a foliation $G$ of $T^2 \times S^1$ where each leaf is the product of a leaf in $G_0$ with an $S^1$ fiber. Clearly an $R$-covered foliation $F$ of $T^2 \times S^1$ can be chosen arbitrarily close to $G$. We construct a branched surface carrying both $F$ and $G$ as follows:

Begin by choosing two annuli $A_1$ and $A_2$, contained in leaves of both $F$ and $G$, as shown in Figure 3.2. (We assume the figure is contained in a copy of $[0,1] \times [0,1] \times [0,1]$ where opposite horizontal and vertical sides are identified in the natural way to obtain $T^2 \times S^1$.) Orbits of a flow $\phi$ transverse to both $F$ and $G$ are indicated by the oriented dashed curves. The generating set $\{A_1, A_2\}$ together with the flow $\phi$ can be used to construct the branched surface $W$ shown in Figure 3.3. Note that this branched surface does not have disk sectors and its branch set is not connected. However, we modify it so that it has these desired properties by splitting along two disks $D_1$ and $D_2$ embedded in $W$ as shown in Figure 3.4(a), and enlarging the two components in the complement of $W$. The latter type of modification involves splitting $W$ along strips $V_1$ and $V_2$, like those shown in Figure 3.4(b), that do not intersect the disks $D_1$ and $D_2$. (We can think this as “sticking our finger”
into the branch set so that it meets the boundary of one of these strips \( V_i \) and pushing it into the one-sheeted side at this branching so that it separates the branched surface along \( V_i \).) Each of these splittings corresponds to a modification of the original generating set. The splittings along \( D_1 \) and \( D_2 \) are the result of adding two disks to the original generating set, and the splittings along \( V_1 \) and \( V_2 \) are the result of enlarging \( A_1 \) and \( A_2 \) respectively. See \([\text{Sh}1]\) for details.

Figure 3.5 shows the branch set for the resulting \( W \). The 3-sheeted side of each crossing of \( \mu \) has been labeled \( R \) or \( L \) to distinguish between the two possible local neighborhoods in Figures 1.1(c) and 1.1(d), respectively.
We have labeled the two arcs bounding the 3-sheeted side of each crossing with + or − to indicate an upper outgoing sector or a lower outgoing sector, respectively, as we leave the crossing along that arc. The labels make it easy to identify the negative and positive branching arcs in μ. However, determining whether a branching arc has a bypass and finding such a bypass if one does exist, requires that we visualize how the splittings described above change the original branched surface in Figure 3.3. For example, the six elementary branching arcs indicated by the dotted lines in Figure 3.5 are each contained in the image of either V₁ or V₂ after the splittings. Figure 3.6 shows three copies of the image of V₂ in W. In each, a curve corresponding to one of the three branching arcs in Figure 3.5 labeled λ₁, λ₂ and λ₃ is indicated. Clearly each of these branching arcs has a bypass with no branches along its interior.

Next consider the eight remaining simple branching arcs in μ. Each lies in the boundary of either D₁ or D₂. Precisely four of these branching arcs, indicated by the dashed lines, have no bypass. (The other four each have a bypass whose interior lies in a sector of W.) However, we can split N(W) along disks in leaves of F, as described above, to create bypasses for these branching arcs. For each of the two positively branching arcs, the disk we use is met by the backward orbits of an integral curve γ₊ which begins and ends with the lowermost points of the fibers over the respective ends of that branching arc. Likewise, the disk we use for each of the two negatively branching arcs is met by forward orbits of an integral curve γ₆ which begins and ends with the uppermost points of the fibers over both ends of the branching arc.

It is now straightforward to verify that for every α and λ as in the hypotheses of Theorem 3.2, a subarc η of α * λ can be found homotopic to some curve τ embedded in μ with the property that every upper (lower)
outgoing sector along $\tau$ branches from a negatively (positively) branching arc in $\tau$ that either has a bypass with no upper (lower respectively) outgoing sectors or is simple. Consequently, $W$ can be modified to obtain an $\mathbf{R}$-covered branched surface.

There are, of course, branched surfaces carrying $\mathbf{R}$-covered foliations that do not satisfy the hypotheses of Theorem 3.2. For example, let $M = \Sigma_2 \times S^1$, where $\Sigma_2$ is the compact surface of genus two. Figure 3.7 shows a projection $p(\mu)$ of the branch set $\mu$ for a stably taut branched surface $W$, constructed
from the trivial foliation $F$ of $M$ by compact surfaces transverse to the $S^1$ fibers, onto the base space $\Sigma_2$. (This projection $p(\mu)$ is shown embedded in a planar model of $\Sigma_2$, where we are assuming opposite sides of the octagon are identified.) By the way we chose the generating disks, $p(\mu)$ is symmetric under both horizontal and vertical reflection of Figure 3.7 and under forty five degree clockwise and counterclockwise rotation of this planar model about its center.

In particular, the generating set for $W$ consists of six disks chosen in three different leaves (so $W$ carries all foliations sufficiently close to $F$). The transverse flow $\phi$ used for the construction is orthogonal to the foliation $F$, so its orbits are closed and contained in the $S^1$ fibers. Consequently, the boundary of each generating disk yields a smooth embedded copy of $S^1$ in the branch set. For example, Figure 3.8(a) shows projected parts of four generating disks $D_1$, $D_2$, $D_3$ and $D_4$. Each boundary point marked with an “X” yields a crossing of $W$. The remaining portions of the disk boundaries have been labeled 1, 2, 3 or 4 to indicate whether the points in this portion flow forward under $\phi$ into $D_1$, $D_2$, $D_3$ or $D_4$, respectively, before meeting another disk. Figure 3.8(b) shows the corresponding piece of $p(\mu)$. We
see, for example, that the two loops indicated by the dashed and bold lines respectively correspond to essential loops in $W$ which, up to orientation, are homotopic to $S^1$ fibers. (There are many other essential loops in $\mu$.)

Now, one can construct non-$\mathbf{R}$-covered foliations arbitrarily close to the foliation $F$ ([Ca], [Fe5]) so, as discussed in Section 2, $W$ cannot be stably $\mathbf{R}$-covered. Indeed, if we choose $\lambda$ to be the essential loop indicated by the dashed curve in Figure 3.7 and let $\alpha$ be the embedded arc indicated by the dotted line, it is straightforward to check that there are no corresponding curves $\eta$ and $\tau$ as in the hypotheses of Theorem 3.2. (As in Figure 3.5, we have labeled the two arcs bounding the 3-sheeted side of each crossing with $+$ or $-$ to indicate an upper outgoing sector or a lower outgoing sector, respectively, as we leave the crossing along that arc.)

Finally, we note that for Theorem 3.2 we need only consider $\alpha$ and $\lambda$ for which $\alpha \ast \lambda$ is contained in the projection of a surface smoothly embedded in $N(W)$ and transverse to the fibers. (This is necessary for a branching arc $\beta$ containing $\alpha \ast \lambda$ to be critical.) In the previous example, the indicated curve $\alpha \ast \lambda$ satisfies this additional restriction. So, as we should expect, the hypotheses of Theorem 3.2 still fail to hold under these weaker conditions. It is, however, possible that an $\mathbf{R}$-covered branched surface carrying $F$ could be constructed using generating surfaces other than disks.

In fact, for many explicit examples of $\mathbf{R}$-covered foliations it is possible to construct a branched surface that satisfies the hypotheses of either Theorem 2.2 or Theorem 3.2 if we allow the generating set to contain surfaces other than disks. At this stage, we cannot claim that any $\mathbf{R}$-covered foliation is carried by some $\mathbf{R}$-covered branched surface, although we conjecture that it is so. On the other hand, not every $\mathbf{R}$-covered foliation is carried by a stably $\mathbf{R}$-covered branched surface, since there are examples of non-$\mathbf{R}$-covered foliations arbitrarily near $\mathbf{R}$-covered foliations. However, it is possible that every $\mathbf{R}$-covered foliation for which the $\mathbf{R}$-covered property is stable is carried by a stably $\mathbf{R}$-covered branched surface.

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OPERATORS AND DIVERGENT SERIES

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We give a natural extension of the classical definition of Césaro convergence of a divergent sequence/function. This involves understanding the spectrum of eigenvalues and eigenvectors of a certain Césaro operator on a suitable space of functions or sequences. The essential idea is applicable in identical fashion to other summation methods such as Borel’s. As an example we show how to obtain the analytic continuation of the Riemann zeta function \( \zeta(z) \) for \( \text{Re}\, z \leq 1 \) directly from generalised Césaro summation of its divergent defining series. We discuss a variety of analytic and symmetry properties of these generalised methods and some possible further applications.

1. Introduction

Methods for assigning generalised limits (sums) to divergent sequences (series) have been studied for centuries. A large variety of definitions exist, due to Césaro, Borel and others, each applicable to a different class of sequences/series.

In this paper we describe an approach which permits generalisation of many of these definitions, expanding the class of divergent sequences/series to which generalised limits/sums can be attached by each method. In describing this approach we will principally consider just one method, that of Césaro. In the final two sections, however, we will describe how this approach naturally generalises to other methods such as Borel’s.

In more detail, in §2 we give our generalisation of the definition of Césaro convergence, both in a discrete (i.e., sequences and series) and continuous (i.e., functions and what we call pictures) setting. The key is to recast the existing definition in terms of a Césaro operator and use its spectrum of eigenvalues and eigenvectors. We state an alternative practical formulation of this definition and investigate its precise relation to our original version. We also discuss a notion of Césaro asymptotics and certain functorial properties which we will require.

In §3 we turn to the example of analytically continuing the Riemann zeta function \( \zeta(z) \equiv \sum_{n=1}^{\infty} n^{-z} \) outside its half-plane of convergence \( \text{Re}\, z > 1 \). We show how, using our continuous generalised Césaro scheme, we obtain
this analytic continuation directly by analysing the divergent defining series for \( \text{Re } z \leq 1 \). Moreover we show how to obtain the location and residue of the simple pole of \( \zeta \) at \( z = 1 \) in this framework.

In \( \S 4 \) we reconsider this example in the context of the discrete Césaro scheme where certain anomalous errors arise in trying to perform the analogous analytic continuation of \( \zeta \). We show how these relate to the singularity arguments of \( \S 3 \) and how to adapt those arguments to rectify the errors.

In \( \S 5 \) we discuss basic analyticity properties of our Césaro schemes. The main result clarifies why the extensions of \( \zeta \) obtained in \( \S 3 \) and \( \S 4 \) must both \textit{a priori} be the unique analytic continuation.

In \( \S 6 \) we then discuss, at varying levels of rigor, some possible implications of our Césaro analysis for Dirichlet series more generally. These include observations regarding scaling, dilation and translation invariances of our schemes, possible criteria for detecting poles and zeros of ordinary Dirichlet series, and example applications to other Dirichlet series arising from analysis of self-adjoint elliptic operators on manifolds.

In \( \S 7 \) we turn back to describing how our approach to extending Césaro’s definition of generalised convergence can be applied more broadly to a whole class of definitions. We illustrate by introducing a new notion of Borel summation and comparing it with the existing definition.

Finally in \( \S 8 \) we outline briefly some possible extensions of this work in a variety of directions. We discuss higher-dimensional schemes for series or functions of several variables, the concept of “ratio eigenfunctions”, schemes associated to arbitrary measures, and some speculative relations with recent dynamical-systems treatments of the zeta function.

2. Generalised Césaro convergence

2.1. Existing definitions. Let \( S \) be the space of all sequences \( a = \{a_n\}_{n=1}^{\infty} \). The usual definition of the Césaro\(^1\) limit of \( a \) can be phrased as follows: let \( P_D : S \to S \) be the linear (discrete) Césaro operator given by \( P_D[a]_n \equiv \frac{1}{n} \sum_{j=1}^{n} a_j \). We define \( a \) to have Césaro limit \( L \), and write \( \text{C}_D\lim_{n \to \infty} a_n = L \), if for some positive integer \( r \) the sequence \( P_D^r[a] \) converges classically to \( L \). Since \( P_D \) is a \textit{regular} operator (meaning that if \( a \) is classically convergent then so is \( P_D[a] \) with the same limit) Césaro convergence is a well-defined generalisation of the notion of classical convergence.

The discrete Césaro sum of a divergent series is then the Césaro limit of its sequence of partial sums: thus, for example, \( \sum_{n=1}^{\infty} (-1)^{n-1} \) has discrete Césaro sum \( \frac{1}{2} \) while \( \sum_{n=1}^{\infty} (-1)^{n-1} n \) has discrete Césaro sum \( \frac{1}{4} \).

We can also define corresponding notions of continuous Césaro convergence and Césaro integration of functions. Precisely, let \( F \) be the space of

\(^1\)Technically this is Hölder’s formulation (\([2]\), \( \S 5 \)) but the origin of the idea is Césaro’s.
complex-valued functions on $[0, \infty)$ given by

$$\mathcal{F} = \left\{ f : \int_0^x |f(t)(\ln(t))^m|dt < \infty \text{ for all } x \geq 0 \text{ and for all } m \in \mathbb{Z}_{\geq 0} \right\}. $$

Then we define the (continuous) Cesaro operator $P : \mathcal{F} \to \mathcal{F}$ by $P[f](x) \equiv \frac{1}{x} \int_0^x f(t)dt$ and say that $f$ has Cesaro limit $L$, written $\text{Clim}_{x \to \infty} f(x) = L$, if for some positive integer $r$, $P^r[f]$ converges classically to $L$. The function space, $\mathcal{F}$, is defined to ensure that $P$ sends $\mathcal{F}$ back into itself and hence that $P^r$ is well-defined. This can be verified by integration by parts and an elementary estimate. Once again $P$ is a regular operator and can be used to define values for certain divergent improper integrals, $\int_0^\infty f(t)dt$, by application to the associated partial-integral function $F(x) \equiv \int_0^x f(t)dt$.

$P$ can in fact also be used as an alternative to $P_D$ in analysing series $\sum_{n=1}^\infty a_n$, by considering not the associated sequence of partial sums $\{s_k\}_{k=1}^\infty$ but instead the partial sum function $s(x) \equiv \sum_{n \leq x} a_n$ and its continuous Cesaro limit. This corresponds geometrically to viewing the terms in the series as being added in at the integer points along the positive real axis.

Although these definitions of discrete and continuous Cesaro convergence enlarge the class of series/integrals to which we can attach values to include, for instance, the alternating series $\sum_{n=1}^\infty (-1)^{n-1}$ and $\sum_{n=1}^\infty (-1)^{n-1}n$ above, it is readily checked that they do not allow evaluation of nonalternating series like $\sum_{n=1}^\infty 1$ and $\sum_{n=1}^\infty n$, which arise as the formal defining series for $\zeta(0)$ and $\zeta(-1)$. We thus turn now to extending these definitions in order to handle these examples and obtain the correct values of $\zeta(0) = \frac{-1}{2}$ and $\zeta(-1) = \frac{-1}{12}$ as their generalised Cesaro sums.

2.2. Generalised definitions. Consider first the definition of continuous Cesaro convergence. Its key feature was the regularity of the operators $P$ and hence $P^r$. In operator terms, however, the restriction to regular operators which are pure powers of $P$ is clearly unnecessary. In particular it is natural to consider arbitrary regular polynomials in $P$, $q(P)$. Any such $q(P)$ is immediately well-defined as an operator (unlike a power series or more general function of $P$), and the condition of regularity is clearly equivalent to simply requiring $q(1) = 1$. We thus generalise the definition of continuous Cesaro convergence as follows:

**Definition 1.** We say that $f \in \mathcal{F}$ has generalised Cesaro limit $L$ if there exists $q(P)$ a regular polynomial in $P$ ($q(1) = 1$) such that $q(P)[f](x) \to L$ classically as $x \to \infty$. We continue to write $\text{Clim}_{x \to \infty} f(x) = L$ in this case.

Note that $L$ is uniquely determined in this definition: if $q_1(P)[f](x) \to L_1$ and $q_2(P)[f](x) \to L_2$, then by the regularity of each $q_i(P)$ we see that $L_2 = \lim_{x \to \infty} q_1(P)q_2(P)[f](x) = \lim_{x \to \infty} q_2(P)q_1(P)[f](x) = L_1$.

The generalisation of the definition of discrete Cesaro convergence follows identical lines. For the remainder of this section and §3, however, we
now restrict attention solely to the setting of functions and our continuous Césaro definitions. We shall refer simply to the Césaro operator (meaning $P$ not $P_0$) and Césaro convergence and summation (meaning their continuous versions).

### 2.3. Interpretation of definition

To identify what the generalisation in Definition 1 achieves, note that $P$ is a linear operator and consider its spectrum of eigenvalues and eigenfunctions. If $f \in F$ is an eigenfunction of $P$ with eigenvalue $\lambda \in \mathbb{C}$ then by definition $(P - \lambda)f = 0$. Although $(P - \lambda)$ is not a regular operator, for $\lambda \neq 1$ the constant multiple $\frac{1}{1 - \lambda}(P - \lambda)$ is.

Taking $q(P) = \frac{1}{1 - \lambda}(P - \lambda)$ in our definition, we thus obtain the following:

**Lemma 1.** If $f \in F$ is any eigenfunction of $P$ with eigenvalue $\lambda \neq 1$ then we have $\text{Clim}_{x \to \infty} f(x) = 0$.

Note that the exclusion of the case $\lambda = 1$ is to be expected; constant functions, which are eigenfunctions of $P$ with eigenvalue 1, should have their limits preserved by regular polynomials $q(P)$ instead of having generalised Césaro limit 0.

Lemma 1 does, however, extend to *generalised* eigenfunctions with eigenvalue $\lambda \neq 1$, that is functions $f \in F$ such that $(P - \lambda)^{n}f = 0$ for some $n \in \mathbb{Z}_{\geq 1}$. In this case, taking $q(P) = (\frac{1}{1 - \lambda})^n(P - \lambda)^n$ we obtain likewise:

**Lemma 2.** If $f \in F$ is any generalised eigenfunction of $P$ with eigenvalue $\lambda \neq 1$ then $\text{Clim}_{x \to \infty} f(x) = 0$.

Linear combinations of eigenfunctions and generalised eigenfunctions of $P$ with eigenvalues all not equal to 1 must also have generalised Césaro limit 0. This follows immediately from the following easy observation:

**Lemma 3.** If $\text{Clim}_{x \to \infty} f_1(x) = L_1$ and $\text{Clim}_{x \to \infty} f_2(x) = L_2$ and $c \in \mathbb{C}$ then $\text{Clim}_{x \to \infty} cf_1(x) = cL_i$ for each $i = 1, 2$ and $\text{Clim}_{x \to \infty} (f_1 + f_2)(x) = L_1 + L_2$.

**Proof.** By definition there exist regular polynomials $q_1(P)$ and $q_2(P)$ such that $q_i(P)[f_i](x) \to L_i$ classically for each $i = 1, 2$. The first result follows trivially by linearity of the $q_i(P)$. The second follows on using the regular polynomial $q(P) = q_1(P)q_2(P)$, since, by commuting the $q_i(P)$ as required, we have

$$q(P)[f_1 + f_2](x) = q_2(P)q_1(P)[f_1](x) + q_1(P)q_2(P)[f_2](x) \to L_1 + L_2.$$  

In light of this last lemma we have now proved at least the following proposition as a consequence of our generalised definition of Césaro convergence.

**Lemma 4.** Suppose $f \in F$ can be written as $f(x) = \sum_{j=1}^{n} c_j f_j(x) + R(x)$ where each $c_j \in \mathbb{C}$, each $f_j$ is an eigenfunction or generalised eigenfunction of $P$ with eigenvalue $\lambda_j \neq 1$, and $R(x)$ is a remainder function satisfying $P^r[R](x) \to L$ classically as $x \to \infty$ for some nonnegative integer $r$. Then $\text{Clim}_{x \to \infty} f(x) = L$. 

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**RICHARD STONE**
Two questions immediately arise. First, whether the converse of Lemma 4 also holds, thus giving a characterisation of generalised Cesàro convergence, or whether Definition 1 is strictly stronger. This is the question of whether the new class of functions to which we can now assign generalised Cesàro limit 0 consists precisely just of the eigenfunctions and generalised eigenfunctions of $P$ with eigenvalue $\lambda \neq 1$, or whether it also includes other types of functions. The second is the more immediate question of identifying explicitly the eigenfunctions and generalised eigenfunctions of $P$. The following lemma answers this question first.

**Lemma 5.**

(i) The functions $x^\rho$, $\rho \in \mathbb{C}$, Re $\rho > -1$ are all eigenfunctions of $P$ in $\mathcal{F}$ with eigenvalue $\frac{1}{\rho+1}$. Each spans a one-dimensional eigenspace of $P$.

(ii) For each eigenvalue $\frac{1}{\rho+1}$ the corresponding generalised eigenfunctions of $P$ are then the functions $x^\rho(\ln(x))^m$, $m = 1, 2, 3, \ldots$.

**Proof.** (i) It is trivial that $P[\tilde{x}^\rho](x) \equiv \frac{1}{\rho+1}x^\rho$ for any Re $\rho > -1$ (where here and throughout we adopt a convention of attaching tildes to dummy variables used in defining functions). Now consider the eigenvalue equation $P[f] = \frac{1}{\rho+1}f$. This means that $\int_0^x f(t)dt \equiv \frac{1}{\rho+1}xf(x)$ as functions on $(0, \infty)$, and differentiating with respect to $x$ then implies $x\frac{df}{dx} = \rho f(x)$. Since this is a homogeneous first-order linear ODE its solution space must be one-dimensional as claimed.

(ii) It is easily verified by an induction argument based on repeated integration by parts that each $x^\rho(\ln(x))^m$ satisfies $(P - \frac{1}{\rho+1})^{m+1}[\tilde{x}^\rho(\ln(\tilde{x}))^m] \equiv 0$. That the generalised eigenspace of solutions of $(P - \frac{1}{\rho+1})^{m+1}[f] \equiv 0$ in $\mathcal{F}$ is precisely of dimension $m + 1$ (hence spanned by the functions $x^\rho, x^\rho \ln(x), \ldots, x^\rho(\ln(x))^m$) is then established inductively along the lines of the argument in (i), by translating to an equivalent first-order linear ODE.

Lemmas 1, 2 and 5, together with the observation that for Re $\rho \leq -1$ the functions $x^\rho(\ln(x))^m$ already converge classically to 0, establish that for any $\rho \neq 0$ and any nonnegative integer $m$, Clim$_{x \to \infty} x^\rho(\ln(x))^m = 0$. Lemma 4 thus translates into the following more explicit proposition:

**Lemma 6.** Suppose $f \in \mathcal{F}$ can be written as $f(x) = \sum_{j=1}^n c_j x^{\rho_j}(\ln(x))^{m_j} + R(x)$ for some collection of constants $c_j \in \mathbb{C}$, $\rho_j \in \mathbb{C} \setminus \{0\}$ and $m_j \in \mathbb{Z}_{\geq 0}$, and some remainder function $R(x)$ satisfying $P^r[R](x) \to L$ classically as $x \to \infty$ for some nonnegative integer $r$. Then Clim$_{x \to \infty} f(x) = L$.

**Note.** The relationship here between the collection of constants $\rho_j$, $m_j$ appearing in the expansion of $f$ and the simplest polynomial $q(P)$ satisfying $q(P)[f](x) \to L$ is as follows: take the list $\rho_1, \ldots, \rho_n$. For each distinct value,
ρ, in this list consider those ρ_j with ρ_j = ρ and let m be the largest of the corresponding values of m_j. Then include in the construction of q(P) a regular factor of the form \((\frac{\rho+1}{\rho})^{m+1}(P - \frac{1}{\rho+1})^{m+1}\). The product of these regular factors over all distinct ρ-values, together with a final factor of \(P^r\), gives a polynomial q(P) with the required property.

Definition 1 thus extends the existing definition of Césaro convergence by allowing us to assign generalised limits not just to functions which become classically convergent upon repeated application of P, but also to ones which have additional power and power-log divergences. For example, we can now correctly evaluate the previously intractable formal defining series for \(\zeta(0)\) and \(\zeta(-1)\) mentioned in §2.1. Let \(x = k + \alpha\) with \(k = \lfloor x \rfloor\) and \(\alpha \in [0, 1)\). Then:

(i) For \(\sum_{n=1}^{\infty} 1\) the partial sum function is \(s(x) = k = x - \alpha\). Since the saw-tooth function \(R(x) = \alpha\) clearly satisfies \(P[R](x) \to \frac{1}{2}\) as \(x \to \infty\) it follows immediately from Lemma 6 that \(\text{Clim}_{x \to \infty}s(x) = -\frac{1}{2}\), i.e., \(\sum_{n=1}^{\infty} 1 = -\frac{1}{2}\) in a generalised Césaro sense, as desired.

(ii) For \(\sum_{n=1}^{\infty} n\) we have

\[
\begin{align*}
  s(k + \alpha) &= \frac{1}{2}(k^2 + k) = \frac{1}{2}(k + \alpha)^2 + \left(\frac{1}{2} - \alpha\right)k - \frac{1}{2}\alpha^2 = \frac{1}{2}x^2 + R(x) \\
  R(k + \alpha) &= \left(\frac{1}{2} - \alpha\right)k - \frac{1}{2}\alpha^2.
\end{align*}
\]

where \(R(k + \alpha) = \left(\frac{1}{2} - \alpha\right)k - \frac{1}{2}\alpha^2\). Now

\[
P[R](k + \alpha) = \frac{1}{k + \alpha}\left(\sum_{j=0}^{k-1} \left(\frac{1}{2} - \int_0^1 \beta \, d\beta\right)j - \left(\frac{1}{2} \int_0^1 \beta^2 \, d\beta\right)\right)k
\]

\[
+ \left(\int_0^\alpha \frac{1}{2} - \beta \, d\beta\right)k - \frac{1}{2} \int_0^\alpha \beta^2 \, d\beta
\]

\[
= \left(-\frac{1}{6} + \frac{\alpha}{2} - \frac{\alpha^2}{2}\right) + O\left(\frac{1}{k}\right)
\]

and so as, \(k \to \infty\), \(P^2[R](k + \alpha) \to -\frac{1}{6} + \frac{1}{4} - \frac{1}{6} = -\frac{1}{12}\). In Lemma 6 we thus obtain that \(\sum_{n=1}^{\infty} n = \frac{1}{12}\) in a generalised Césaro sense, again as desired.

Returning to the first of our earlier two questions now, it turns out that Lemma 6 is in fact slightly weaker than Definition 1. This is due to the existence, for certain eigenvalues, of nontrivial asymptotic eigenfunctions in addition to the exact eigenfunctions calculated in Lemma 5. Here we are using the following:

**Definition 2.** A function \(f \in \mathcal{F}\) is an asymptotic eigenfunction of \(P\) with eigenvalue \(\lambda\) if \((P - \lambda)[f](x) = o(1)\).
A nontrivial asymptotic eigenfunction is one that does not merely differ from an exact eigenfunction by a $o(1)$-function. Since we clearly still have $\text{Clim}_{x \to \infty} f(x) = 0$ for any asymptotic eigenfunction with eigenvalue $\lambda \neq 1$, nontrivial such functions will be ones which can be assigned generalised Césaro limits under Definition 1, but which are not simply of the form described in Lemma 6. The following Tauberian-type lemma and corollary prove (constructively) the existence of such functions, and clarify more precisely the relationship between Lemma 6 and Definition 1.

**Lemma 7.** Let $S_{1,1}^1 \setminus \{0\}$ be the circle in $\mathbb{C}$ with centre $\frac{1}{2}$ and radius $\frac{1}{2}$. Note that $S_{1,1}^1 \setminus \{0\}$ is the image of the imaginary axis under the mapping $\rho \mapsto \frac{1}{\rho + 1}$.

(i) Suppose that $\text{Re} \rho > -1$, $\text{Re} \rho \neq 0$, and that $f \in \mathcal{F}$ satisfies \((\frac{\rho + 1}{\rho}) (P - \frac{1}{\rho + 1})[f](x) \to 0 \text{ classically as } x \to \infty\). Then $f(x) = Cx^\rho + o(1)$ for some constant $C$. The converse of Lemma 6 thus holds at least for $q(P)$ of degree 1 with root not lying on $S_{1,1}^1 \setminus \{0\}$.

(ii) However, for any $\text{Re} \rho = 0$, $\rho \neq 0$, there exist functions $f \in \mathcal{F}$ such that \((\frac{\rho + 1}{\rho}) (P - \frac{1}{\rho + 1})[f](x) \to 0 \text{ classically as } x \to \infty\), but $f$ is not of the form $f(x) = Cx^\rho + o(1)$. Thus the converse of Lemma 6 fails when $q(P)$ has a root lying on $S_{1,1}^1 \setminus \{0, 1\}$.

**Proof.** (i) The proof is principally due to Andrew Stone. The given eigenvalue equation states that $\frac{1}{x} \int_0^x f(t) dt - \frac{1}{\rho + 1} f(x) = r(x)$ where $r(x) = o(1)$.

Writing $F(x) \equiv \int_0^x f(t) dt$ this becomes the asymptotic differential equation $\frac{1}{x} F(x) - \frac{1}{\rho + 1} F'(x) = r(x)$, which can be rewritten as $\frac{d}{dx} (x^{-(\rho + 1)} F(x)) = -(\rho + 1)x^{-(\rho + 1)} r(x)$. Integrating implies $x^{-\rho} F(x) = F(1) - (\rho + 1) \phi(x)$ where $\phi(x) \equiv \int_1^x t^{-(\rho + 1)} r(t) dt$. Consider two cases separately.

Case (a): $\text{Re} \rho > 0$. In this case $\lim_{x \to \infty} \phi(x)$ exists. Denoting it by $\phi_\infty(\rho)$ we obtain

$$F(x) = C_\rho x^{\rho + 1} + (\rho + 1)x^{\rho + 1} \int_x^\infty t^{-(\rho + 1)} r(t) dt$$

where $C_\rho = F(1) - (\rho + 1) \phi_\infty(\rho)$ is a constant, and differentiating then yields

$$f(x) = C_\rho (\rho + 1)x^{\rho} + (\rho + 1)^2 x^{\rho} \int_x^\infty t^{-(\rho + 1)} r(t) dt - (\rho + 1)r(x).$$

Since $r(x) = o(1)$ the result thus follows immediately if we can show also that $x^\rho \int_x^\infty t^{-(\rho + 1)} r(t) dt = o(1)$. But to see this let $X_\epsilon$, for any $\epsilon > 0$, be such that $|r(x)| \leq \epsilon$ whenever $x > X_\epsilon$. Then, for any $x > X_\epsilon$ we have

$$\left| x^\rho \int_x^\infty t^{-(\rho + 1)} r(t) dt \right| \leq x^{\text{Re} \rho} \int_x^\infty t^{-(\text{Re} \rho + 1)} \epsilon dt \leq \frac{\epsilon}{\text{Re} \rho}$$

and the result follows.
Case (b): \(-1 < \Re \rho < 0\). In this case we need to show that \(f(x)\) itself is \(o(1)\) so we directly consider

\[
f(x) = F'(x) = (\rho + 1)F(1)x^\rho - (\rho + 1)^2x^\rho \phi(x) - (\rho + 1)r(x).
\]

But here the first and third terms are immediately \(o(1)\) and the term

\[
x^\rho \phi(x) = x^\rho \int_1^x t^{-(\rho+1)}r(t)dt
\]

is also seen to be \(o(1)\) by a similar argument to the one just given, on writing the integral over \([1,x]\) as a sum of integrals over \([1,X]\) and \([X, x]\). Hence the result follows in this case also.

(ii) For any \(\rho = i\beta\) with \(\beta \neq 0\) real, it is easy to check that, for example,

\[
f(x) \equiv \begin{cases} 0, & 0 \leq x \leq e \\ x^{i\beta} \ln(\ln x), & x > e \end{cases}
\]

is a function with the required properties and thus a nontrivial asymptotic eigenfunction of \(P\) with eigenvalue \(\frac{1}{1+i\beta} \in S_{\frac{1}{2}, \frac{1}{2}} \setminus \{0, 1\}\). \(\square\)

**Note.** In the cases of eigenvalue 0 or 1 (\(\rho = \infty\) or 0) excluded in (ii), \(P\) certainly does still have nontrivial asymptotic eigenfunctions. For eigenvalue 0 this is of course the reason the original definition of Césaro convergence is stronger than simply classical convergence. In the case of eigenvalue 1 the example in the proof still yields a suitable eigenfunction, on taking \(\beta = 0\). We omitted this case only because factors of \((P - 1)\) cannot arise in a regular polynomial \(q(P)\), so that it need not be considered initially in analysing Definition 1.

The following corollary, which follows easily from part (i) of Lemma 7, then extends the discussion there to polynomials of arbitrary degree.

**Corollary 1.** Suppose \(q(P)\) is any regular polynomial in \(P\) none of whose roots lie on \(S_{\frac{1}{2}, \frac{1}{2}} \setminus \{0\}\), and that \(f \in \mathcal{F}\) satisfies \(q(P)f(x) \to L\) classically as \(x \to \infty\). Then \(f(x)\) must be of the form \(f(x) = \sum_{j=1}^n c_j x^{\rho_j} (\ln(x))^{m_j} + R(x)\) where the relationship between the exponents \(\rho_j, m_j\) and the roots of \(q(P)\) is precisely as outlined in the note subsequent to Lemma 6, and where \(P^r[R](x) \to L\) with \(r\) being the multiplicity of 0 as a root of \(q(P)\).

Thus the converse of Lemma 6 holds subject to these root conditions on \(q(P)\).

We now consider finally in §2.3 the important case of eigenvalue \(\lambda = 1\) (\(\rho = 0\)) omitted throughout Lemmas 1–4 and Lemma 6. By Lemma 5, the exact eigenfunctions with eigenvalue 1 are the constant functions and the \(\lambda = 1\) generalised eigenspace is spanned by the functions \((\ln x)^m, m = 1, 2, \ldots\). As constant functions, the eigenfunctions with eigenvalue 1 of course have classical limits, not just generalised Césaro ones. This is not so, however, for the generalised eigenfunctions with eigenvalue 1. For example, since \(P[\ln(x)] = \ln x - 1\) and \(P\) is regular it follows that no generalised Césaro limit, \(L\), can be assigned to \(\ln x\), since \(L\) would have to satisfy \(L = L - 1\).
Working inductively, the same conclusion holds for all \((\ln x)^m, m \in \mathbb{Z}_{>0}\). We thus arrive at the following observation which we shall use frequently.

**Lemma 8.** For any integer \(m \geq 1\) the generalised eigenfunction, \((\ln x)^m\), of \(P\) with eigenvalue 1 cannot be assigned a generalised Césaro limit.

### 2.4. Césaro asymptotics.

We conclude §2 by defining a notion which will prove useful in §3 and elsewhere.

**Definition 3.** We say that two functions \(f\) and \(g\) in \(F\) are Césaro asymptotic, and write \(f \sim C g\), if \(\lim_{x \to \infty} (f - g)(x) = 0\).

This definition satisfies the following basic functorial properties:

**Lemma 9.** For any functions in \(F\) we have:

1. If \(f \sim C g\) and \(\lim_{x \to \infty} f(x) = L\) then \(\lim_{x \to \infty} g(x) = L\).
2. \(\sim C\) is an equivalence relation.
3. If \(f_1 \sim C f_2\) and \(g_1 \sim C g_2\) then \(f_1 + g_1 \sim C f_2 + g_2\).

The proofs of these properties are all elementary, involving commutation arguments as in the proof of Lemma 3, the regularity and linearity of \(P\), and the closure of the space of regular polynomials in \(P\) under multiplication.

### 3. The Riemann zeta function

In this section we illustrate the scope of our generalised definition of Césaro convergence by sketching how it yields the analytic continuation of the Riemann zeta function, \(\zeta(z)\), directly from analysis of its divergent defining series.

Let \(\zeta^{\text{ext}}\) be the function on \(\mathbb{C}\) whose value at any \(z\) is the generalised Césaro sum of the series \(\sum_{n=1}^{\infty} n^{-z}\). Clearly \(\zeta^{\text{ext}}(z) = \zeta(z)\) for \(\Re z > 1\). To show that \(\zeta^{\text{ext}}(z) = \zeta(z)\) for all \(\Re z \leq 1\) also (verified explicitly for \(z = 0, -1\) in §2.3) we will need to interpret both \(\zeta\) and \(\zeta^{\text{ext}}\) in terms of the Euler–MacLaurin sum formula.

As a preliminary, however, note that \(\zeta^{\text{ext}}\) does have a singularity at \(z = 1\) as it should. At \(z = 1\) the defining series for \(\zeta^{\text{ext}}\) is \(\sum_{n=1}^{\infty} \frac{1}{n}\) with partial sum function \(s(x) = \ln x + \gamma + O(1)\) where \(\gamma\) is Euler’s constant. But by §2.3, Lemma 8, no generalised Césaro limit can be assigned to the function \(\ln x\) and so \(\zeta^{\text{ext}}(1)\) is undefined. We will show later that \(z = 1\) is the only singularity of \(\zeta^{\text{ext}}\) and is in fact a simple pole with residue 1 as required for agreement with \(\zeta\).

We now turn, however, to considering the general case. Throughout the next sections let \(z \neq 1\) be a fixed complex number with \(\Re z \leq 1\).

#### 3.1. The Euler–MacLaurin sum formula.

The version we use here is essentially the formulation in [2], §13.
Theorem 1 (Euler–MacLaurin Sum Formula). Suppose that \( f \in C^\infty(0, \infty) \cap L^1_{\text{loc}}(0, \infty) \) and that \( f \) and its successive derivatives form an asymptotic scale. Then we have

\[
\sum_{n=1}^{k} f(n) \sim \int_{0}^{k} f(x) \, dx + C_f + \frac{1}{2} f(k) + \sum_{r=1}^{\infty} \frac{(-1)^{r-1} B_r}{(2r)!} f^{(2r-1)}(k).
\]

Here \( C_f \) is a constant, the \( B_r \) are the Bernoulli numbers \( B_1 = \frac{1}{6}, \ B_2 = \frac{1}{30}, \ B_3 = \frac{1}{42}, \ldots \) and the expansion is asymptotic in the usual sense that truncating the infinite sum at any point yields a remainder which can be estimated in little-o terms by the last term retained.

Applying this to the case of \( f(x) = x^{-z} \) we find that the partial sum function for the divergent series defining \( \zeta(z) \), \( \sum_{n=1}^{\infty} n^{-z} \), is given by

\[
s_{\zeta,z}(k + \alpha) = k^{-z+1} + C_{\zeta,z} + \frac{1}{2} k^{-z} + \sum_{r=1}^{\infty} \frac{(-1)^r B_r}{(2r)!} z(z + 1) \cdots (z + 2r - 2) k^{-z-2r+1}.
\]

This expression truncates after some finite number of terms (depending on \( z \)), with a remainder which is \( o(1) \) as \( k \to \infty \) and can therefore be neglected in evaluating \( \lim_{k \to \infty} s_{\zeta,z}(k + \alpha) \). This observation, strengthened by noting the local uniformity of the \( o(1) \)-estimate in \( z \)-neighbourhoods in \( \mathbb{C} \), allows one to deduce easily (see e.g., [2], §13.10) the following simple formula for \( \zeta(z) \) in terms of expansion (2):

Theorem 2. For any \( z \neq 1 \) the value of \( \zeta(z) \) is given by

\[
\zeta(z) = C_{\zeta,z}
\]

It remains to prove that the same formula holds for \( \zeta^{\text{ext}}(z) \). We will do this by re-expressing Equation (2) in simpler form in terms of Césaro asymptotics:

Lemma 10. For any \( \text{Re } z \leq 1, \ z \neq 1, \)

\[
s_{\zeta,z}(k + \alpha) \sim \frac{(k + \alpha)^{-z+1}}{1 - z} + C_{\zeta,z}.
\]

The desired formula for \( \zeta^{\text{ext}}(z) \) will follow at once from this together with Lemmas 6 and 9, completing the proof that \( \zeta^{\text{ext}} = \zeta \).

3.2. The proof of Lemma 10. Proving Lemma 10 involves obtaining a general Césaro asymptotic expression for a term of the form \( (k + \alpha)^\gamma \), \( \text{Re } \gamma \geq 0 \). Indeed taking \( \gamma = 1 - z \) in Equation (2), Lemma 10 is precisely equivalent to verifying the following such expression:
Lemma 11. For any $\text{Re} \gamma \geq 0$ we have

\begin{equation}
(k + \alpha)^\gamma \lesssim k^\gamma + \frac{1}{2} \gamma k^{\gamma - 1} \sum_{r=1}^{\infty} (-1)^{r-1} \frac{B_r}{(2r)!} \gamma(\gamma - 1) \cdots (\gamma - 2r + 1) k^{\gamma - 2r}.
\end{equation}

When $0 \leq \text{Re} \gamma < 1$ this says simply that $(k + \alpha)^\gamma \lesssim k^\gamma$, while for $1 \leq \text{Re} \gamma < 2$ it states that $(k + \alpha)^\gamma \lesssim k^\gamma + \frac{1}{2} \gamma k^{\gamma - 1}$. Both cases are of course easily verified, the first holding classically, the second because $P[(\overline{k + \alpha})^\gamma - \overline{k}^\gamma - \frac{1}{2} \gamma \overline{k}^{\gamma - 1}](k + \alpha) = o(1)$. For $\text{Re} \gamma$ arbitrarily large, however, we need to work indirectly, starting with the Taylor series expansion

\begin{equation}
(k + \alpha)^\gamma = k^\gamma + \sum_{l=1}^{[\text{Re} \gamma]} \frac{\gamma(\gamma - 1) \cdots (\gamma - l + 1)}{l!} k^{\gamma - l} \alpha^l + o(1).
\end{equation}

Our strategy involves first obtaining a Césaro asymptotic formula for expressions of the form $k^\delta \alpha^r$, for any $\text{Re} \delta \geq 0$ and $r \in \mathbb{Z}_{>0}$. We shall then apply this to each term in Equation (6) to obtain Lemma 11.

Lemma 12. For any $\text{Re} \delta \geq 0$ and any nonnegative integer $r$ we have

\begin{equation}
k^\delta \alpha^r \lesssim \frac{1}{r + 1} k^\delta + \sum_{j=1}^{[\text{Re} \delta]} c_j(\delta, r) k^{\delta - j}
\end{equation}

where

\begin{equation}
c_j(\delta, r) = \delta(\delta - 1) \cdots (\delta - j + 1) d_j(r)
\end{equation}

and

\begin{equation}
d_j(r) = \frac{(-1)^{[j/2] - 1}}{(r + 1) \cdots (r + j + 1)} \sum_{l=0}^{[j/2] - 1} (-1)^l \binom{r + j + 1}{2([j/2] - l)} B_{[j/2] - l} + \frac{(-1)^{[j/2]}}{2 (r + j - 1)}.
\end{equation}

Proof. We argue by induction on $\text{Re} \delta$, verifying (7) for $0 \leq \text{Re} \delta < 1$ (and arbitrary $r$) and then proceeding to $1 \leq \text{Re} \delta < 2$, . . . .

In the base case $0 \leq \text{Re} \delta < 1$, (7) reduces to the formula

\begin{equation}
k^\delta \alpha^r \lesssim \frac{1}{r + 1} k^\delta
\end{equation}

which is immediate, as for any $r$, $P[(\overline{\alpha^r - \frac{1}{r + 1}})\overline{k}^{\delta}](k + \alpha) = O(k^{\delta - 1}) = o(1)$. For the inductive step, suppose Equation (7) holds for all $\text{Re} \delta < l$ (and arbitrary $r$) for some positive integer $l$. To show that it continues to hold for $l \leq \text{Re} \delta < l + 1$ (and arbitrary $r$ again), we work in a sequence of steps.
Step (i). First we observe that

\[ P \left( \left( \tilde{\alpha} r - \frac{1}{r+1} \right) \tilde{k}^{\delta} \right)(k + \alpha) = \frac{1}{k + \alpha} \left\{ k^{\delta} \left( \frac{\alpha r + 1}{r + 1} - \frac{\alpha}{r + 1} \right) \right\} \]

\[ = k^{\delta - 1} \left( \frac{\alpha r + 1}{r + 1} - \frac{\alpha}{r + 1} \right) \left( 1 - \frac{\alpha}{k} + \frac{\alpha^2}{k^2} - \cdots \right) \]

\[ = k^{\delta - 1} \left( \frac{\alpha r + 1}{r + 1} - \frac{\alpha}{r + 1} \right) - k^{\delta - 2} \left( \frac{\alpha^r + 2}{r + 1} - \frac{\alpha^2}{r + 1} \right) \]

\[ + k^{\delta - 3} \left( \frac{\alpha^r + 3}{r + 1} - \frac{\alpha^3}{r + 1} \right) + \cdots . \]

Step (ii). Next we apply the inductive hypothesis to each term on the right-hand side in this expression. Using Lemma 9 we thus rewrite it as a Césaro asymptotic equation involving a linear combination of terms \( k^{\delta - j} \) with certain real constant coefficients,

\[ P \left( \left( \tilde{\alpha} r - \frac{1}{r+1} \right) \tilde{k}^{\delta} \right)(k + \alpha) \sim a_1^{(0)} k^{\delta - 1} + a_2^{(0)} k^{\delta - 2} + a_3^{(0)} k^{\delta - 3} + \cdots . \]

We now need to write the entire expression on the right here as the image of some other expression under \( P \), at least Césaro asymptotically. We do this by iteratively “inverting \( P \) at top order”.

Step (iii). At top order we have

\[ a_1^{(0)} k^{\delta - 1} = P \left[ a_1^{(0)} \tilde{k}^{\delta - 1} \right](k + \alpha) + \pi_2^{(0)}(\alpha) k^{\delta - 2} + \pi_3^{(0)}(\alpha) k^{\delta - 3} + \cdots + o(1) \]

where each \( \pi_j^{(0)}(\alpha) \) is some constant-coefficient polynomial in \( \alpha \) arising from the sub-leading terms in \( P \left[ a_1^{(0)} \tilde{k}^{\delta - 1} \right](k + \alpha) \) (whose evaluation entails using the Euler–MacLaurin sum formula).

Step (iv). We thus obtain

\[ P \left( \left( \tilde{\alpha} r - \frac{1}{r+1} \right) \tilde{k}^{\delta} \right)(k + \alpha) \sim P \left[ a_1^{(0)} \tilde{k}^{\delta - 1} \right](k + \alpha) \]

\[ + \left( a_2^{(0)} + \pi_2^{(0)}(\alpha) \right) k^{\delta - 2} + \cdots \]

and invoking the inductive hypothesis again, this can in turn be rewritten as

\[ P \left( \left( \tilde{\alpha} r - \frac{1}{r+1} \right) \tilde{k}^{\delta} \right)(k + \alpha) \sim P \left[ a_1^{(0)} \tilde{k}^{\delta - 1} \right](k + \alpha) \]

\[ + a_2^{(1)} k^{\delta - 2} + a_3^{(1)} k^{\delta - 3} + \cdots \]

for some new collection of constant coefficients \( a_j^{(1)} \), \( j = 2, 3, \ldots \).
Step (v). Iterating Steps (iii) and (iv), starting next at order \( k^{\delta-2} \) and dropping successively by one order in \( k \) at each iteration, we ultimately obtain a complete Césaro asymptotic expression for the right-hand side as the image of some expression under \( P \):

\[
P \left[ \left( \tilde{\alpha}r - \frac{1}{r+1} \right) \tilde{k}^\delta \right] (k + \alpha) \sim \frac{1}{r+1} k^\delta + a_1^{(0)} \delta k^{\delta-1} + a_2^{(1)} (\delta - 1) k^{\delta-2} + \cdots.
\]

Step (vi) This then finally yields the desired Césaro asymptotic expression for \( k^\delta \alpha^r \),

\[
k^\delta \alpha^r \sim \frac{1}{r+1} k^\delta + a_1^{(0)} \delta k^{\delta-1} + a_2^{(1)} (\delta - 1) k^{\delta-2} + \cdots
\]

and it remains only to verify that the coefficients in this expansion continue to be given by formulae (7)–(9) to complete the inductive step and hence the proof of Lemma 12.

Unfortunately detailed verification of this appears to be combinatorially messy, owing to the need to iteratively invoke the inductive hypothesis and keep track of the lower-order correction terms in each top-order inversion of \( P \). We will only show how the strategy proceeds in the two simplest situations, going from the base case to the case \( 1 \leq \text{Re} \delta < 2 \), and then from this to the case \( 2 \leq \text{Re} \delta < 3 \). We note, however, that such computations as far as \( \text{Re} \delta < 7 \) have been performed and were essential in guessing the correct form of Lemma 12 in the first place.

For \( 1 \leq \text{Re} \delta < 2 \), in Step (i) we have simply

\[
P \left[ \left( \tilde{\alpha}r - \frac{1}{r+1} \right) \tilde{k}^\delta \right] (k + \alpha) = k^{\delta-1} \left( \frac{\alpha^{r+1}}{r+1} - \frac{\alpha}{r+1} \right) + o(1)
\]

and, by the base case Equation (10), this yields in Step (ii) that

\[
P \left[ \left( \tilde{\alpha}r - \frac{1}{r+1} \right) \tilde{k}^\delta \right] (k + \alpha) \sim \frac{-r}{2(r+1)(r+2)} k^{\delta-1}.
\]

Inverting at top order, and noting all sub-leading terms are \( o(1) \) in this case, we thus obtain in Step (iii) that

\[
P \left[ \left( \tilde{\alpha}r - \frac{1}{r+1} \right) \tilde{k}^\delta \right] (k + \alpha) \sim P \left[ \frac{-r \delta}{2(r+1)(r+2)} \tilde{k}^{\delta-1} \right] (k + \alpha)
\]

and this means immediately in Step (vi) that

\[
k^\delta \alpha^r \sim \frac{1}{r+1} k^\delta - \frac{r \delta}{2(r+1)(r+2)} k^{\delta-1}.
\]

This verifies Lemma 12 for this case.
Now suppose $2 \leq \text{Re} \delta < 3$. Then in Step (i) we have that
\[
P \left[ \left( \bar{\alpha}^r - \frac{1}{r + 1} \right) \bar{k}^\delta \right] (k + \alpha)
= k^{\delta - 1} \left( \frac{\alpha^{r+1}}{r+1} - \frac{\alpha}{r+1} \right) - k^{\delta - 2} \left( \frac{\alpha^{r+2}}{r+1} - \frac{\alpha^2}{r+1} \right) + o(1)
\]
and by Equations (7)–(9) this becomes, in Step (ii), the equation
\[
P \left[ \left( \bar{\alpha}^r - \frac{1}{r + 1} \right) \bar{k}^\delta \right] (k + \alpha) \sim C \frac{-r}{2(r + 1)(r + 2)} k^{\delta - 1}
+ \left\{ \frac{(r^2 - r)\delta + (3r^2 + 9r)}{12(r + 1)(r + 2)(r + 3)} \right\} k^{\delta - 2}.
\]
Inverting at top order in Step (iii), note that
\[
P \left[ \frac{r\delta}{2(r + 1)(r + 2)} \bar{k}^{\delta - 1} \right] (k + \alpha)
= \left( \frac{r\delta}{2(r + 1)(r + 2)} \right) \frac{1}{k + \alpha} \left( \sum_{j=0}^{k-1} j^{\delta - 1} + k^{\delta - 1} \alpha \right)
= \left( \frac{r\delta}{2(r + 1)(r + 2)} \right) \left( \frac{(k - 1)^\delta}{\delta} + \frac{1}{2} (k - 1)^{\delta - 1} \right.
+ k^{\delta - 1} \alpha + O(k^{\delta - 2}) \left) \frac{1}{k} \left( 1 - \frac{\alpha}{k} + \ldots \right)
= \left( \frac{r}{2(r + 1)(r + 2)} \right) k^{\delta - 1} + \left( \frac{-r\delta}{4(r + 1)(r + 2)} + \frac{r\delta\alpha}{2(r + 1)(r + 2)} \right.
- \frac{r\alpha}{2(r + 1)(r + 2)} \right) k^{\delta - 2} + o(1).
\]
We thus obtain initially in Step (iv) that
\[
P \left[ \left( \bar{\alpha}^r - \frac{1}{r + 1} \right) \bar{k}^\delta \right] (k + \alpha)
\sim P \left[ \frac{-r\delta}{2(r + 1)(r + 2)} \bar{k}^{\delta - 1} \right] (k + \alpha) + \left( \frac{(r^2 - r)\delta + (3r^2 + 9r)}{12(r + 1)(r + 2)(r + 3)} \right.
- \frac{r\delta}{4(r + 1)(r + 2)} + \frac{r\delta\alpha}{2(r + 1)(r + 2)} - \frac{r\alpha}{2(r + 1)(r + 2)} \right) k^{\delta - 2}.
and, on invoking Equations (7)–(9) again, this reduces to simply
\[ P \left[ \left( \frac{\tilde{\alpha}r - \frac{1}{r+1}}{\delta} \right) k^\delta \right] (k + \alpha) \lesssim P \left[ \frac{-r \delta}{2(r+1)(r+2)} k^{\delta-1} \right] (k + \alpha) \]
\[ + \left( \frac{r^2 - r \delta}{12(r+1)(r+2)(r+3)} \right) k^{\delta-2}. \]

But then iterating Step (iii) by inverting now at order \( k^{\delta-2} \) we deduce that
\[ \left( \frac{(r^2 - r) \delta}{12(r+1)(r+2)(r+3)} \right) k^{\delta-2} \]
\[ = P \left[ \frac{r(r-1) \delta(\delta-1)}{12(r+1)(r+2)(r+3)} k^{\delta-2} \right] (k + \alpha) + o(1) \]
and hence overall (Step (v)) that
\[ P \left[ \left( \frac{\tilde{\alpha}r - \frac{1}{r+1}}{\delta} \right) k^\delta \right] (k + \alpha) \lesssim P \left[ \frac{-r \delta}{2(r+1)(r+2)} k^{\delta-1} \right] (k + \alpha) \]
\[ + P \left[ \frac{r(r-1) \delta(\delta-1)}{12(r+1)(r+2)(r+3)} k^{\delta-2} \right] (k + \alpha). \]

It follows immediately in Step (vi) that
\[ k^\delta \alpha^r \lesssim \frac{1}{r+1} k^\delta - \frac{r \delta}{2(r+1)(r+2)} k^{\delta-1} + \frac{r(r-1) \delta(\delta-1)}{12(r+1)(r+2)(r+3)} k^{\delta-2} \]
and this again verifies Lemma 12, for the case \( 2 \leq \text{Re} \delta < 3. \)

Having sketched a proof of Lemma 12 it now remains to verify that it does in turn yield Lemma 11. We turn to this now.

Invoking Lemma 12 term by term in the Taylor series expansion (6), we obtain the Cesàro asymptotic equation
\[ (k + \alpha)^\gamma \lesssim k^\gamma + \sum_{l=1}^{\lfloor \gamma \rfloor} \sum_{j=0}^{\lfloor \gamma \rfloor - l} \frac{\gamma(\gamma-1)\cdots(\gamma-l+1)}{l!} c_j(\gamma - l, l) k^{\gamma-l-j} \]
where here we have extended the definition of \( c_j(\delta, r) \) in Equations (8) and (9) by setting \( c_0(\delta, r) \equiv \frac{1}{r+1} \) so Equation (7) becomes simply
\[ k^\delta \alpha^r \lesssim \sum_{j=0}^{\lfloor \delta \rfloor} c_j(\delta, r) k^{\delta-j}. \]

Letting \( p = j + l \) and swapping the order of summation this becomes
\[ (k + \alpha)^\gamma \lesssim k^\gamma + \sum_{p=1}^{\lfloor \gamma \rfloor} \beta_p(\gamma) k^{\gamma-p} \]
where

\[(12) \quad \beta_p(\gamma) = \sum_{l=1}^{p} \frac{\gamma(\gamma - 1) \cdots (\gamma - l + 1)}{l!} c_{p-l}(\gamma - l, l).\]

Comparing Equations (11) and (5) we see that proving Lemma 11 reduces to showing on the one hand that

\[(13) \quad \beta_1(\gamma) = \frac{1}{2} \gamma \quad \text{and} \quad \beta_p(\gamma) = 0 \quad \text{for all} \quad p \quad \text{odd}, \quad p \geq 3 \]

and on the other that

\[(14) \quad \beta_p(\gamma) = (-1)^{\frac{p-2}{2}} \frac{p-2}{p!} B_{\frac{p-2}{2}} (\gamma - 1) \cdots (\gamma - p + 1) \quad \text{for} \quad p \quad \text{even}, \quad p \geq 2.\]

Consider Equation (13) first. It is easy to see that \(\beta_1(\gamma) = \frac{1}{2} \gamma\) so it remains to show that \(\beta_{2m+1}(\gamma) = 0\) for any positive integer \(m\).

Now in Equation (12) for \(\beta_{2m+1}(\gamma)\) consider first just the \(l = 1\) term. This is simply \(\gamma c_{2m}(\gamma - 1, 1)\) and so is a multiple of \(d_{2m}(1)\) where, by Equation (9),

\[
d_{2m}(1) = \frac{(-1)^{m-1}}{(2m + 2)!} \sum_{i=0}^{m-1} (-1)^i \left( \frac{2m + 2}{2(m - i)} \right) B_{m-i} + (-1)^m m
\]

\[
= \frac{-1}{(2m + 2)!} \left( \sum_{q=1}^{m} (-1)^q \left( \frac{2m + 2}{2q} \right) B_q + m \right).
\]

But this expression is in fact identically zero for any \(m \geq 1\). To see this we use the Bernoulli polynomials which, for even index, are given by

\[(15) \quad B_{2n}(x) \equiv x^{2n} - nx^{2n-1} + \sum_{q=1}^{n} (-1)^{q-1} \left( \frac{2n}{2q} \right) B_q x^{2n-2q}
\]

([4], §9.6, adjusting for a different convention regarding the indexing of the Bernoulli numbers). Letting \(n = m + 1\), splitting off the \(q = m + 1\) term from the sum and rearranging, this becomes the equation

\[
\sum_{q=1}^{m} (-1)^{q-1} \left( \frac{2m + 2}{2q} \right) B_q x^{2m+2-2q} = B_{2m+2}(x) - x^{2m+2} + (m + 1)x^{2m+1} + (-1)^{m-1} B_{m+1}.
\]

But now recall ([4], §9.6]) that \(B_{2m+2}(1) = (-1)^m B_{m+1}\). Substituting \(x = 1\) into our expression it follows that

\[(16) \quad \sum_{q=1}^{m} (-1)^{q-1} \left( \frac{2m + 2}{2q} \right) B_q = m \]

and hence we obtain at once that \(d_{2m}(1) = 0\) for all \(m \geq 1\) as claimed.
It follows that for any \( m \geq 1 \) Equation (12) for \( \beta_{2m+1}(\gamma) \) becomes simply

\[
\beta_{2m+1}(\gamma) = \sum_{l=2}^{2m+1} \frac{\gamma(\gamma-1) \cdots (\gamma-l+1)}{l!} c_{2m+1-l}(\gamma-l,l).
\]

Splitting the terms in this sum into pairs, we can next rewrite this as

\[
\beta_{2m+1}(\gamma) = \sum_{q=1}^{m} A_q(\gamma)
\]

where

\[
A_q(\gamma) = \frac{\gamma(\gamma-1) \cdots (\gamma-2q+1)}{(2q)!} c_{2m-2q+1}(\gamma-2q,2q)
\]

\[
+ \frac{\gamma(\gamma-1) \cdots (\gamma-2q)}{(2q+1)!} c_{2m-2q}(\gamma-2q-1,2q+1).
\]

But using Equations (8) and (9) we find that, for any \( 1 \leq q \leq m-1 \), both terms in this expression for \( A_q(\gamma) \) in fact collapse to the same quantity, giving

\[
A_q(\gamma) = \frac{\gamma(\gamma-1) \cdots (\gamma-2m)}{(2m+2)!} \left( \sum_{s=1}^{m-q} (-1)^{s-1} \left( \begin{array}{c} 2m+2 \\ 2s \end{array} \right) B_s - m \right)
\]

while for the case \( q = m \), recalling that \( c_0(\delta,r) \equiv \frac{1}{r+1} \), we obtain easily that

\[
A_m(\gamma) = \frac{\gamma(\gamma-1) \cdots (\gamma-2m)}{(2m+2)!} (1-m).
\]

Substituting these expressions for \( A_q(\gamma) \) we thus obtain overall that

\[
\beta_{2m+1}(\gamma) = 2 \left( \frac{\gamma(\gamma-1) \cdots (\gamma-2m)}{(2m+2)!} \right)
\]

\[
\cdot \left( \frac{(1-m)}{2} + \sum_{q=1}^{m-1} \left( \sum_{s=1}^{m-q} (-1)^{s-1} \left( \begin{array}{c} 2m+2 \\ 2s \end{array} \right) B_s - m \right) \right)
\]

\[
= 2 \left( \frac{\gamma(\gamma-1) \cdots (\gamma-2m)}{(2m+2)!} \right)
\]

\[
\cdot \left( -\frac{(2m+1)(m-1)}{2} + \sum_{s=1}^{m-1} (-1)^{s-1}(m-s) \left( \begin{array}{c} 2m+2 \\ 2s \end{array} \right) B_s \right)
\]

where, in the last step, we have combined all terms not involving Bernoulli numbers and reduced the double sum by reversing the order of summation and noting that the summand is independent of \( q \).
But now consider again the Bernoulli polynomials defined by Equation (15). Differentiating with respect to \( x \) and setting \( x = 1 \) yields that

\[
2 \sum_{q=1}^{n} (-1)^{q-1} (n - q) \left( \frac{2n}{2q} \right) B_q = B'_{2n}(1) + 2n^2 - 3n
\]

and since \( B'_{2n}(1) = 0 \) for any \( n \) ([4], §9.6 again) this in turn reduces to the equation

\[
(17) \quad \sum_{q=1}^{n-1} (-1)^{q-1} (n - q) \left( \frac{2n}{2q} \right) B_q = \frac{2n^2 - 3n}{2}.
\]

But setting \( n = m+1 \) it then follows directly from our earlier calculation (16) that

\[
(18) \quad \sum_{q=1}^{m} (-1)^{q-1} (m - q) \left( \frac{2m + 2}{2q} \right) B_q = \frac{(2m + 1)(m - 1)}{2}.
\]

Substituting into our last formula for \( \beta_{2m+1}(\gamma) \) this immediately implies \( \beta_{2m+1}(\gamma) = 0 \) for any \( m \geq 1 \), as claimed, and this completes the proof of identity (13).

The proof of the second identity (14) follows in similar fashion.

But this then completes our sketch of the proof of Lemma 11 using Lemma 12; hence of Lemma 10; and hence ultimately, as noted, of our central result in §3, namely that \( \zeta_{\text{ext}} = \zeta \) on all of \( \mathbb{C} \setminus \{1\} \).

3.3. The singularity. We conclude §3 by showing how we can also determine the nature of the singularity of \( \zeta_{\text{ext}} = \zeta \) at \( z = 1 \) within our generalised Césaro framework.

The key is to observe that in the proof of Lemma 11, the Césaro asymptotic formula (5) is in fact obtained by applying a pure power of \( P \), namely \( P^{\lfloor \text{Re} \gamma \rfloor} \). Recalling our identification \( \gamma = 1 - z \) it thus follows from Lemmas 10 and 6 that, for any \( z \neq 1 \), the regular polynomial \( q(z, P) \) needed to evaluate \( \zeta_{\text{ext}}(z) \) as the generalised Césaro limit of \( s_{\zeta,z}(k + \alpha) \) (i.e., to obtain \( q(z, P)[s_{\zeta,z}](k + \alpha) \to C_{\zeta,z} \) as \( k \to \infty \)) is given explicitly by

\[
(19) \quad q(z, P) = \left( \frac{2 - z}{1 - z} \right) \left( P - \frac{1}{2 - z} \right) P^{\lfloor -\text{Re} z \rfloor + 1}.
\]

This observation immediately leads to a deeper explanation for the presence of the singularity of \( \zeta_{\text{ext}} \) at \( z = 1 \): it occurs because of the breakdown of regularity and analyticity of \( q(z, P) \) at \( z = 1 \), arising from the presence of the factor \( \left( \frac{2 - z}{1 - z} \right) \).
To see the precise form of this singularity, consider \( \lim_{z \to 1} (z - 1) \zeta_{\text{ext}}(z) \) within our Césaro framework. We have

\[
\lim_{z \to 1} (z - 1) \zeta_{\text{ext}}(z) = \lim_{z \to 1} (z - 1) \lim_{k \to \infty} q(z, P)[s_{\zeta, z}](k + \alpha) \\
= -\lim_{z \to 1} \lim_{k \to \infty} ((2 - z)P - 1)[s_{\zeta, z}](k + \alpha) \\
= -\lim_{k \to \infty} (P - 1)[s_{\zeta, z}](k + \alpha) \\
= -\lim_{k \to \infty} (P - 1)\ln x + \gamma + o(1) \quad (k + \alpha) = 1
\]

on swapping limits in the third line and recalling that \( P[\ln(x)] = \ln(x) - 1 \). It follows that, for \( z \) near 1, \( \zeta_{\text{ext}}(z) = \frac{1}{z - 1} + \text{analytic} \). Thus \( z = 1 \) is a simple pole of \( \zeta_{\text{ext}} = \zeta \) with residue 1.

4. Discrete Césaro summation and \( \zeta \)

It is interesting to briefly reconsider the example of \( \zeta \) in the context of our original discrete Césaro scheme for sequences/series.

4.1. Basic spectrum of \( P_D \). We first need to mimic our analysis of \( P \) and identify the spectrum of eigenvalues and eigensequences/generalised eigensequences of \( P_D \).

Clearly the unique eigensequences of \( P_D \) with eigenvalue \( \lambda = 1 \) are the constant eigensequences, and by analogy with \( \sec x \) it seems natural, for \( \lambda \neq 1 \), to look at sequences of the form \( \{j^\rho\}_{j=1}^\infty \) for arbitrary \( \text{Re} \rho \geq 0, \rho \neq 0 \) (we again ignore \( \text{Re} \rho < 0 \) since then \( \{j^\rho\}_{j=1}^\infty \) is already classically convergent to 0). Here, however, we have to split into two cases:

Case (i). For \( \rho \in \mathbb{Z}_{>0} \) a simple induction argument shows that the exact eigensequence, \( \{a_j\}_{j=1}^\infty \), of \( P_D \) with eigenvalue \( \lambda = \frac{1}{\rho + 1} \) is given not simply by \( a_j = j^\rho \) but rather by

\[
a_j = \prod_{i=1}^{\rho} (j - i) = (j - 1)(j - 2) \cdots (j - \rho).
\]

Case (ii). For \( \rho \notin \mathbb{Z}_{>0} \) we need to work in steps. Consider first sequences \( \{j^\rho\}_{j=1}^\infty \) with \( 0 \leq \text{Re} \rho < 1, \rho \neq 0 \). By the Euler–MacLaurin sum formula

\[
P_D[\{j^\rho\}]_k = \frac{1}{k} \left( \frac{k^\rho + 1}{\rho + 1} + O(k^\rho) \right) = \frac{k^\rho}{\rho + 1} + o(1)
\]

and it follows that in this case \( \{j^\rho\}_{j=1}^\infty \) is an asymptotic eigensequence of \( P_D \) (in the obvious sense analogous to Definition 2) with eigenvalue \( \frac{1}{\rho + 1} \). This means at once (cf. \( \sec \) analogous to Definition 2) that

\[
C_D \lim_{k \to \infty} \{k^\rho\} = 0 \quad \text{for any} \quad 0 \leq \text{Re} \rho < 1, \rho \neq 0.
\]
Next consider the case $1 \leq \Re \rho < 2$, $\rho \neq 1$. Here we obtain

$$P_D[\{j^\rho\}]_k = \frac{1}{\rho + 1} k^\rho + \frac{1}{2} k^{\rho-1} + o(1)$$

so $\{j^\rho\}_{j=1}^\infty$ is no longer an asymptotic eigensequence of $P_D$. It is easy to turn it into one, however, simply by adding a suitable lower-order correction term. A short computation (similar to our top-order inversions of $P$ in §3) yields that in fact in this case the desired asymptotic eigensequence with eigenvalue $\frac{1}{\rho + 1}$ is $\{j^\rho - \frac{\rho(\rho + 1)}{2} j^{\rho-1}\}_{j=1}^\infty$ and we deduce that

$$C_D \lim_{k \to \infty} \left\{ k^\rho - \frac{\rho(\rho + 1)}{2} k^{\rho-1} \right\} = 0 \quad \text{for any} \quad 1 \leq \Re \rho < 2, \, \rho \neq 1.$$ 

In light of Equation (21) (and the discrete analogue of §2, Lemma 9), however, this still implies simply

$$C_D \lim_{k \to \infty} \{k^\rho\} = 0 \quad \text{for any} \quad 1 \leq \Re \rho < 2, \, \rho \neq 1. \tag{22}$$

In the same way, for $2 \leq \Re \rho < 3$, $\rho \neq 2$ we find that the asymptotic eigensequence of $P_D$ is now $\{j^\rho - \frac{\rho(\rho + 1)}{2} j^{\rho-1} + \frac{\rho(\rho + 1)(\rho - 1)}{24} j^{\rho-2}\}_{j=1}^\infty$, but, in light of Equations (21) and (22), this clearly still implies

$$C_D \lim_{k \to \infty} \{k^\rho\} = 0 \quad \text{for any} \quad 2 \leq \Re \rho < 3, \, \rho \neq 2 \tag{23}$$

and continuing in this fashion we see that in general

$$C_D \lim_{k \to \infty} \{k^\rho\} = 0 \quad \text{for any} \quad \Re \rho \geq 0, \, \rho \notin \mathbb{Z}. \tag{24}$$

We shall limit our analysis of Case (ii) to this observation. Results (20) and (24) can then be combined, after an elementary computation, into a single lemma summarising our cursory spectral analysis of $P_D$ so far:

**Lemma 13.** For any $\Re \rho \geq 0$ we have

$$C_D \lim_{k \to \infty} \{k^\rho\} = \begin{cases} 1 & \text{if } \rho \in \mathbb{Z}_{\geq 0} \\ 0 & \text{otherwise.} \end{cases} \tag{25}$$

(Cf. the continuous Césaro scheme for which $\text{Clim}_{k \to \infty} \{k^\rho\} = 0$ for $\rho \notin \mathbb{Z}_{\geq 0}$ but $\text{Clim}_{k \to \infty} \{k^\rho\} = (-1)^\rho \frac{1}{\rho + 1}$ for $\rho \in \mathbb{Z}_{\geq 0}$.)

**4.2. $\zeta(z)$ and the discrete Césaro scheme.** Defining $\zeta^{\text{ext,}D}(z)$ as the generalised discrete Césaro sum of $\sum_{n=1}^\infty n^{-z}$, we now investigate, as in §3, whether $\zeta^{\text{ext,}D} = \zeta$ for all $\Re z \leq 1$.

By working almost identical to that in §3, we may first easily verify that $z = 1$ is a singularity of $\zeta^{\text{ext,}D}$ also, as we want. Turning to $z \neq 1$, the results of Lemma 13 suggest that we should divide our analysis into two cases:
Case (i) \( z \notin \mathbb{Z}_{\leq 0} \). In this case the partial sum sequence for \( \sum_{n=1}^{\infty} n^{-z} \), which is given (cf. Equation (2)) by
\[
(s_{\zeta,z})_k = \frac{k^{-z+1}}{1-z} + C_{\zeta,z} + \frac{1}{2} k^{-z} - \frac{1}{12} z k^{-z-1} + \cdots + o(1)
\]
has only noninteger powers of \( k \) in the terms other than \( C_{\zeta,z} \). It follows at once from Lemma 13 that \( C_D \lim_{k \to \infty} \{(s_{\zeta,z})_k\} = C_{\zeta,z} \) and, in light of Theorem 2, this means immediately that we do indeed have
\[
\zeta^{\text{ext,}D}(z) = \zeta(z) \quad \text{for all } \Re z \leq 1, \; z \notin \mathbb{Z}_{\leq 0}.
\] (26)

Case (ii) \( z \in \mathbb{Z}_{\leq 0} \). In this case, however, we obtain the following result:

Lemma 14. For any \( z \in \mathbb{Z}_{\leq 0} \)
\[
\zeta^{\text{ext,}D}(z) = 1.
\] (27)

Proof. Writing \( z = -r \) we need to show that \( C_D \lim_{k \to \infty} \{s^{(r)}_k\} = 1 \) for all nonnegative integers \( r \), where \( s^{(r)}_k \equiv \sum_{j=1}^{k} j^r \). But \( \sum_{j=1}^{k} j^r \) can be expanded as a linear combination of nonnegative integer powers of \( k \), whose generalised discrete Césaro limits are all 1 by Lemma 13. It follows that \( C_D \lim_{k \to \infty} \{s^{(r)}_k\} \) can be computed simply by setting \( k = 1 \) in the expression \( \sum_{j=1}^{k} j^r \) for \( s^{(r)}_k \). This trivially implies Equation (27) since \( \sum_{j=1}^{1} j^r = 1 \). □

Lemma 14 means of course that
\[
\zeta^{\text{ext,}D}(z) \neq \zeta(z) \quad \text{for all } \quad z \in \mathbb{Z}_{\leq 0}.
\] (28)

Thus in the discrete setting \( \zeta^{\text{ext,}D} \) yields a countable collection of anomalous evaluations of the zeta function at \( z = 0, -1, \ldots \). Closer analysis of our working, however, explains this problem.

4.3. Correcting the anomalies. We would like a diagnostic way of identifying in advance that the values of \( \zeta^{\text{ext,}D}(z) \) are potentially anomalous at \( z = 0, -1, \ldots \) and, in turn, a way of computing the true values of \( \zeta \) at these points.

The key, as in §3.3, is to consider explicitly the polynomials \( q(z, P_D) \) used to evaluate \( \zeta^{\text{ext,}D}(z) \) as \( \lim_{k \to \infty} (q(z, P_D)[(s_{\zeta,z})_k])_k \). We examine these in successive vertical strips moving leftwards in the complex plane.

Consider first the strip \(-1 < \Re z < 1\) containing 0. Here \( (s_{\zeta,z})_k = \frac{k^{-z+1}}{1-z} + C_{\zeta,z} + \frac{1}{2} k^{-z} + o(1) \), and writing this in terms of asymptotic eigensequences of \( P_D \) we have
\[
(s_{\zeta,z})_k = \frac{1}{1-z} \left( k^{-z} - \frac{(1-z)(2-z)}{2} k^{-z} \right) + \frac{3-z}{2} k^{-z} + C_{\zeta,z} + o(1).
\] (29)
It follows that in this strip the regular polynomial we need to use is
\begin{equation}
q(z, P_D) = -\left(\frac{2-z}{z}\right) \left( P_D - \frac{1}{2-z} \right) \left( P_D - \frac{1}{1-z} \right),
\end{equation}
the first factor here annihilating the eigensequence \(\{k^{1-z} - \frac{1-z}{2^{1-z}} k^{-z}\}_{k=1}^{\infty}\) and the second the other eigensequence \(\{k^{-z}\}_{k=1}^{\infty}\) in the expression (29).

But it is now clear why the calculation of \(\zeta^{\text{ext},D}(0)\) was anomalous: the analyticity of \(q(z, P_D)\) breaks down at \(z = 0\) due to the regularising factor \((\frac{2-z}{z})\). As in §3.3, this signals the presence of a singularity in \(\zeta^{\text{ext},D}\) at \(z = 0\). In this case, however, \(z = 0\) is not a pole but a removable singularity and we can obtain the correct value of \(\zeta(0)\) simply by applying L'Hôpital's law within our discrete Césaro scheme:

\[
\zeta(0) = \lim_{z \to 0} e^{\text{ext},D}(z) = \lim_{z \to 0} \lim_{k \to \infty} q(z, P_D)[\{(s_{\zeta,z})_k\}]_k
\]
\[
= -\lim_{k \to \infty} \lim_{z \to 0} \left(\frac{(2-z)P_D - 1(P_D - \frac{1}{1-z})}{z}\right)[\{(s_{\zeta,z})_k\}]_k
\]
\[
= -\lim_{k \to \infty} \lim_{z \to 0} \left\{ (2-z)P_D - 1 \left( P_D - \frac{1}{1-z} \right) \left\{ \frac{d}{dz}(s_{\zeta,z})_k \right\}_k \right.
\]
\[
+ \left( \frac{1}{1-z} \right) (2-z)P_D - 1 \left\{ (s_{\zeta,z})_k \right\}_k
\]
\[
- P_D \left( P_D - \frac{1}{1-z} \right) \left\{ (s_{\zeta,z})_k \right\}_k
\]
\[
= -\lim_{k \to \infty} \left\{ (2P_D - 1)(P_D - 1) \left[ \left\{ -\ln \tilde{k} + \tilde{k} + \frac{d}{dz} C_{\zeta,z} \right|_{z=0} - \frac{1}{2} \ln \tilde{k} \right\}_k \right.
\]
\[
- \left( (2P_D - 1)[(\tilde{k})]_k - P_D(P_D - 1)[(\tilde{k})]_k \right)
\]
\[
= -\lim_{k \to \infty} \left\{ \left( \frac{5}{4} - \frac{1}{4} k + \frac{1}{2} \right) - 1 + \frac{1}{4} (k - 1) \right\} = -\frac{1}{2}.
\]

In the final steps here we have used that \((2P_D - 1)[(\ln \tilde{k})]_k = \ln k - \frac{1}{2} k + o(1)\) and \((P_D - 1)[(\ln \tilde{k})]_k = -1 + o(1)\).

A more careful analysis within our discrete Césaro scheme thus explains the anomalous discrepancy between \(\zeta^{\text{ext},D}\) and \(\zeta\) at \(z = 0\) and how to correct it. The same explanatory framework is easily seen to apply also at \(z = -1, -2, \ldots\).

Indeed, if we work in an open neighbourhood of any \(z_0 \in \mathbb{Z}_{<0}\) and express \((s_{\zeta,z})_k\) as a linear combination of asymptotic eigensequences of \(P_D\) as in Equation (29), we find that the analyticity of the regular polynomial \(q(z, P_D)\) we need to use in this neighbourhood breaks down as \(z \to z_0\). This occurs
because the lowest-order eigensequence in this expansion becomes a constant sequence at \( z = z_0 \). Since constant sequences are eigensequences of \( P_D \) with eigenvalue 1, one of the factors in \( q(z, P_D) \) thus degenerates into \((P - 1)\) as \( z \to z_0 \), and since \( q(z, P_D) \) must be regular, this leads immediately to a breakdown of analyticity in \( z \). However since, at \( z = z_0 \), \( \{(s_{\zeta,k})_k\}_{k=1}^{\infty} \) is acquiring just an eigensequence of \( P_D \) with eigenvalue 1 rather than a generalised eigensequence of the form \( \{(\ln k)^m\}_{k=1}^{\infty} \) (as occurs at \( z = 1 \)), the singularity at \( z_0 \) is a removable singularity rather than a pole. A L’Hôpital’s calculation can thus again be used to correct the anomaly in \( \zeta^{\text{ext},D}(z_0) \) and yield the correct value of \( \zeta(z_0) \).

In this fashion then it is possible to obtain the correct analytic continuation of the zeta function to all of \( \mathbb{C} \backslash \{1\} \) within our discrete Césaro scheme also.

4.4. Final remarks. Our definitions of generalised Césaro convergence give new notions of convergence for sequences and functions and hence, in some sense, new topologies on \( \mathbb{C} \). However, as the presence of the anomalies in our calculations in §4 shows, when applied to a sequence/family of functions of a complex variable \( z \), evaluation of generalised limits pointwise does not guarantee analyticity of the limit function. As with Weierstrass’ classical theorem, we will see in the next section that to guarantee analyticity of generalised limits one needs to work in open neighbourhoods, using families of polynomials \( q(z, P_D) \) or \( q(z, P) \) which are analytic in \( z \) and regular throughout these entire neighbourhoods, and obtaining convergence of the transformed sequences of functions which is appropriately uniform. We begin concretely by reconsidering the cases of \( \zeta^{\text{ext}} \) and \( \zeta^{\text{ext},D} \).

5. Analyticity and generalised convergence

In §3 and §4 we deduced the analyticity of \( \zeta^{\text{ext}} \) on \( \mathbb{C} \backslash \{1\} \) and of \( \zeta^{\text{ext},D} \) on \( \mathbb{C} \backslash \mathbb{Z} \leq 1 \) only “after the fact” by deriving formulas for them in agreement with the known formula (3) for the analytic continuation of \( \zeta \). If we could prove these extensions were \textit{a priori} analytic, however, their coincidence with \( \zeta \) for \( \text{Re } z > 1 \) would imply that they are in fact both the unique analytic continuation of \( \zeta \) and formula (3) would follow as a corollary. We consider \( \zeta^{\text{ext},D} \) first.

5.1. A priori analyticity of \( \zeta^{\text{ext},D} \). We shall need a pair of lemmas, the first of which is as follows:

\textbf{Lemma 15.} Suppose \( \{r_k(z)\}_{k=1}^{\infty} \) is a sequence of analytic functions on \( U \subseteq \mathbb{C} \), converging uniformly to zero on compact subsets of \( U \), and suppose \( q(z, P_D) \) is a family of polynomials in \( P_D \) analytic in \( z \) and regular throughout \( U \). Then the transformed sequence \( \{q(z, P_D)[\{r_k(z)\}_{k=1}^{\infty}]\} \) of analytic functions also converges uniformly to zero on compact subsets of \( U \).
This follows by reducing to the case \( q(z, P_D) = P_D \) and elementary estimates.

The second lemma we need concerns local uniformity of convergence in our eigenvalue equations for \( P_D \). For any \( \Re \rho \geq 0 \), let \( \{a_k(\rho)\}_{k=1}^\infty \) be the eigensequence (asymptotic for \( \rho \notin \mathbb{Z}_{\geq 0} \)) of \( P_D \) with eigenvalue \( \frac{1}{\rho+1} \) described in §4.1; i.e., \( a_k(\rho) = k^\rho \) for \( 0 \leq \Re \rho < 1 \), \( a_k(\rho) = k^\rho - \frac{\rho(\rho+1)}{2} k^{\rho-1} \) for \( 1 \leq \Re \rho < 2 \), and so on.

Now consider first any \( \rho_0 \) in the interior of one of these strips \( l \leq \Re \rho < l + 1 \). If we take \( \delta_0 = \min\left\{ \frac{\Re \rho_0 - l}{2}, \frac{l + 1 - \Re \rho_0}{2} \right\} \), the open ball \( B(\rho_0, \delta_0) \) lies within this strip and at strictly positive distance at least \( \frac{l + 1 - \Re \rho_0}{2} \) from its right-hand side. By standard Euler–MacLaurin remainder analysis like that in [2], §13.10, it is then easy to see that the remainder sequences \( \{r_k(\rho)\}_{k=1}^\infty \) in the eigenvalue equation

\[
(31) \quad \left( P_D - \frac{1}{\rho+1} \right) \{a_k(\rho)\}_k \equiv r_k(\rho) = o(1)
\]

in fact converge uniformly to zero for all \( \rho \in B(\rho_0, \delta_0) \).

The same is not true, however, for \( \rho_0 \) on the edge of a strip, say \( \Re \rho_0 = l \), \( l \in \mathbb{Z}_{\geq 0} \). For such \( \rho_0 \) it is readily verified that there is no local neighbourhood on which the remainders \( \{r_k(\rho)\}_{k=1}^\infty \) in Equation (31) all converge uniformly to zero.

To rectify this we change to an alternative family of asymptotic eigensequences. Specifically, for any \( l \in \mathbb{Z}_{\geq 0} \) let \( \{a_k^{(l)}(\rho)\}_{k=1}^\infty \) be the family of sequences defined, for \( -1 < \Re \rho < l + 1 \), by simply applying the formula for \( \{a_k(\rho)\}_{k=1}^\infty \) in the strip \( l \leq \Re \rho < l + 1 \) throughout the region \( -1 < \Re \rho < l \) also. This yields an equally valid family of eigensequences for \( -1 < \Re \rho < l + 1 \) since the leading terms in the formula for \( \{a_k(\rho)\}_{k=1}^\infty \) in any strip always coincide with the formulae for the \( \{a_k(\rho)\}_{k=1}^\infty \) in strips further left, so that for \( -1 < \Re \rho < l \) each \( \{a_k^{(l)}(\rho)\}_{k=1}^\infty \) differs from the known asymptotic eigensequence \( \{a_k(\rho)\}_{k=1}^\infty \) only by a sequence which is classically \( o(1) \).

Using these adapted families of eigensequences instead in Equation (31) the problems above with obtaining locally uniform convergence of remainders at integer values of \( \Re \rho_0 \) disappear and we obtain the following simple uniformity result:

**Lemma 16.** For any \( \Re \rho_0 > -1 \), set \( \delta_0 = \min\left\{ \frac{\Re \rho_0 + 1}{2}, \frac{l + 1 - \Re \rho_0}{2} \right\} \) where \( [\Re \rho_0] = l \). Then the remainder sequences \( \{r_k^{(l)}(\rho)\}_{k=1}^\infty \) in the alternative eigenvalue equation

\[
(32) \quad \left( P_D - \frac{1}{\rho+1} \right) \left\{a_k^{(l)}(\rho)\right\}_k \equiv r_k^{(l)}(\rho) = o(1)
\]

converge uniformly to zero for all \( \rho \in B(\rho_0, \delta_0) \).
With Lemmas 15 and 16 it finally becomes easy to show the desired a priori analyticity of $\zeta^{ext,D}$ on $\mathbb{C} \setminus \mathbb{Z}_{\leq 1}$. Let $z_0$ be any point in $\mathbb{C}$ with $\text{Re } z_0 \leq 1$, $z_0 \notin \mathbb{Z}_{\leq 1}$. Let $l = \lfloor 1 - \text{Re } z_0 \rfloor$ and take $\delta_0 = \min\{\text{Re } z_0 + l, \frac{|1 - z_0|}{2}\}$. Here the first possibility for $\delta_0$ arises from setting $\rho_0 = 1 - z_0$ in the formula for $\delta_0$ in Lemma 16, while the second guarantees that $B(z_0, \delta_0)$ stays a strictly positive distance from the set $\mathbb{Z}_{\leq 1}$ where anomalies/regularity breakdowns occur.

Using expression (2) for $(s_{\zeta,z})_k$, it is clear that, for $z \in B(z_0, \delta_0)$, we can write $(s_{\zeta,z})_k$ as a linear combination of adapted asymptotic eigensequences of $P_D$ (cf. §4.2 Equation (29))

\[
(s_{\zeta,z})_k = \sum_{i=0}^{l} \lambda_i(z)a_k^{(l-i)}(1 - z - i) + C_{\zeta,z} + (R_{\zeta,z})_k
\]

where $\lambda_0(z) = \frac{1}{1 - z}$, the other coefficients $\lambda_i(z)$ are all simply polynomials in $z$, and the remainders $(R_{\zeta,z})_k$ are uniformly $o(1)$ on $B(z_0, \delta_0)$.

But now consider the transformed family of sequences

\[
\{q(z, P_D)[\{(s_{\zeta,z})_k\}_k]\}_{k=1}^{\infty}
\]

on $B(z_0, \delta_0)$ where, in light of expression (33) (cf. §4.3), we take

\[
q(z, P_D) = \prod_{j=0}^{l} \left( \frac{2 - z - j}{1 - z - j} \right) \left( P_D - \frac{1}{2} \frac{1}{2 - z - j} \right).
\]

By our choice of $\delta_0$, $q(z, P_D)$ is clearly both analytic and regular throughout $B(z_0, \delta_0)$, so the transformed remainder sequences

\[
\{q(z, P_D)[\{(R_{\zeta,z})_k\}_k]\}_{k=1}^{\infty}
\]

remain uniformly $o(1)$ on compact subsets of $B(z_0, \delta_0)$ by Lemma 15.

Take next the terms $\lambda_i(z)a_k^{(l-i)}(1 - z - i)$, $i = 0, \ldots, l$ in expression (33). For $i = 0$ it follows directly from the way we chose $\delta_0$ that Lemma 16 applies, and we obtain that $(\frac{2 - z}{1 - z})(P_D - \frac{1}{2})[\{(\lambda_0(z)a_k^{(0)}(1 - z))\}_k]$ is uniformly $o(1)$ on $B(z_0, \delta_0)$. By Lemma 15 this uniformity is then preserved on compact subsets of $B(z_0, \delta_0)$ by the other factors in the product for $q(z, P_D)$, and so $q(z, P_D)[\{(\lambda_0(z)a_k^{(0)}(1 - z))\}_k]$ is uniformly $o(1)$ on compact subsets of $B(z_0, \delta_0)$. As for $i > 0$, in this case the variables $1 - z - i$ in the eigensequences $a_k^{(l-i)}(1 - z - i)$ lie in integer translates of our ball, but since we are using the $(l-i)$-family of eigensequences we can still immediately invoke Lemma 16. It follows that $(\frac{2 - z}{1 - z})(P_D - \frac{1}{2})[\{(\lambda_i(z)a_k^{(l-i)}(1 - z - i))\}_k]$ is likewise uniformly $o(1)$ on $B(z_0, \delta_0)$ for all $i = 1, \ldots, l$, and this is again preserved on compact subsets of $B(z_0, \delta_0)$ by the other factors in $q(z, P_D)$.

The transformed sequence of analytic functions $\{(q(z, P_D)[\{(s_{\zeta,z})_k\}_k]\}_{k=1}^{\infty}$ thus in fact converges uniformly to $C_{\zeta,z} = \zeta^{ext,D}(z)$ on compact subsets of
$B(z_0, \delta_0)$ and, by the classical Weierstrass theorem, this implies the analyticity of $\zeta^{\text{ext},D}$ on $B(z_0, \delta_0)$. The desired \textit{a priori} analyticity of $\zeta^{\text{ext},D}$ on all of $\mathbb{C} \setminus \mathbb{Z}_{\leq 1}$ follows at once.

5.2. A priori analyticity of $\zeta^{\text{ext}}$. For $\zeta^{\text{ext}}$ the approach to proving a priori analyticity is broadly similar, but with certain nontrivial variations. We will provide only a schematic overview.

The first step remains obtaining a continuous analogue of Lemma 15 and is straightforward. The only variant needed is an extra condition on the 1-parameter family of functions $r(x,z)$, that for any compact $K \subseteq U$ and fixed $x \geq 0$ there exists $M > 0$ with \[ \int_0^x |r(\tilde{x}, z)| \, d\tilde{x} \leq M \text{ for all } z \in K. \]
If the $r(x,z)$ are remainders in partial sum functions, as in §3, this holds trivially.

The other ingredient now, however, is not an analogue of Lemma 16 (our eigenfunctions of $P$ are already exact rather than asymptotic). Instead observe that, while $q(z, P_D)$ never involved pure powers of $P_D$ but acquired progressively more nontrivial factors $(P_D - \lambda)$ leading to anomalies and necessitating the analysis of Lemma 16, in the continuous case $q(z, P)$ only ever has one nontrivial factor $(2 - z)(P - \frac{1}{2 - z})$ but requires successively higher pure powers of $P$ to obtain the necessary Césaro asymptotic behaviour in Lemma 10. The second result we need is thus a proof that the pointwise Césaro asymptotic relationship in Lemma 10 can in fact be obtained locally uniformly.

It turns out this can be achieved simply by including one extra factor of $P$ beyond what was used in the pointwise arguments of §3. More precisely, for any $z_0 \in \mathbb{C} \setminus \{1\}$ with, say, $\text{Re } z_0 < \frac{5}{3}$, there exists an open neighbourhood of $z_0$ such that

\[ P[\text{Re } (z_0) + 2] \left[ s_{\zeta, z}(k + \alpha) - \frac{(k + \alpha)^{-z+1}}{1 - z} - C_{\zeta, z} \right] (k + \alpha) = o(1) \]

uniformly on compact subsets of this neighbourhood. We call this Lemma 10'.

To prove this we need first to prove a corresponding locally uniform version, Lemma 12', of Lemma 12 by including an extra power of $P$, then work backwards via Lemma 11. We shall only sketch the argument.

Consider first the case of $k^\delta \alpha^r$ for $-\frac{2}{3} < \text{Re } \delta < \frac{2}{3}$. We have

\[ P \left[ -k^\delta \left( \frac{\alpha^r - 1}{r + 1} \right) \right] (k + \alpha) = k^{\delta-1} \left( \frac{\alpha^{r+1} - \alpha}{r + 1} \right) + \ldots \]

and the desired locally uniform version Lemma 12' follows immediately in this strip by Euler–MacLaurin remainder analysis.

Next suppose $\frac{1}{3} < \text{Re } \delta < \frac{5}{3}$. Our working from the case of $1 \leq \text{Re } \delta < 2$ in the proof of Lemma 12 implies, on keeping careful track of powers of $P$
in Steps (i) and (ii), that we likewise have

\[ P^2 \left[ \tilde{k}^\delta \left( \tilde{\alpha} - \frac{1}{r+1} \right) \right] (k + \alpha) = P \left[ \frac{-r}{2(r + 1)(r + 2)} \tilde{k}^{\delta - 1} \right] (k + \alpha) + o(1) \]

uniformly throughout this strip. Moreover, the top-order inversion (Step (iii)) giving

\[ P \left[ \frac{-r\delta}{2(r + 1)(r + 2)} \tilde{k}^{\delta - 1} \right] (k + \alpha) = \frac{-r}{2(r + 1)(r + 2)} k^{\delta - 1} + o(1) \]

also holds uniformly for \( \frac{1}{3} < \text{Re} \delta < \frac{5}{3} \). It follows at once that we have

\[ P^2 \left[ \tilde{k}^\delta \tilde{\alpha} - \frac{1}{r + 1} \tilde{k}^\delta + \frac{r\delta}{2(r + 1)(r + 2)} \tilde{k}^{\delta - 1} \right] (k + \alpha) = o(1) \]

uniformly for \( \frac{1}{3} < \text{Re} \delta < \frac{5}{3} \) and this is the desired locally uniform version Lemma 12’ for this strip as well.

Continuing in this way, working in successive overlapping open strips, we obtain the requisite locally uniform version Lemma 12’ for any \( \delta \) with \( \text{Re} \delta > -\frac{2}{3} \).

The transition back via Lemma 11 to Lemma 10’ is then elementary except for one observation. When we apply Lemma 12’ term by term in the Taylor series expansion (6) to obtain a uniform version of Lemma 11, the different terms come with different powers of \( P \) associated with them from Lemma 12’. We need, however, to apply the single power, \( P^{[\text{Re} \gamma + 2]} \), to all terms in expression (5). We thus need to apply extra powers of \( P \) to each of the terms in expression (6). That uniformity is preserved in doing this is due to our continuous analogue of Lemma 15 and this explains the restriction to compact subsets of our local neighbourhoods in Lemma 10’.

With Lemma 10’ it is finally trivial, on invoking the analogue of Lemma 15 once more in relation to the factor \( (\frac{2-z}{1-z})(P - \frac{1}{\tilde{k}^{\delta - 1}}) \), to deduce the desired local uniform convergence of

\[ \left( \frac{2-z}{1-z} \right) \left( P - \frac{1}{2-z} \right) P^{[\text{Re} (z_0) + 2]} [s_{\zeta, z}(k + \alpha)] (k + \alpha) \]

to \( C_{\zeta, z} = \zeta^{\text{ext}}(z) \) for all \( z \) in some sufficiently small neighbourhood of any \( z_0 \neq 1 \). This yields at once the claimed \( a \text{ priori} \) analyticity of \( \zeta^{\text{ext}} \) throughout \( \mathbb{C} \setminus \{1\} \).

5.3. Commuting differentiation and generalised convergence. In §5.1 and §5.2 we proved the \( a \text{ priori} \) analyticity of the generalised limit, \( f \), of a sequence of analytic functions \( \{f_k(z)\}_{k=1}^\infty \) (or family \( f(x, z) \)) by finding local regular analytic families of polynomials \( q(z, P_D) \) (or \( q(z, P) \)) for which the transformed sequence/family of functions converges to \( f \) locally uniformly, and then applying the classical Weierstrass theorem. The following result shows that under such circumstances the derivative \( f' \) is then
also the generalised limit of the sequence \( \{f'_k(z)\}_{k=1}^{\infty} \) (or family \( \frac{\partial}{\partial z} f(x, z) \)) and hence that, in general, differentiation with respect to \( z \) commutes with taking generalised limits. We only state the result in the discrete Césaro setting, but translation to the continuous setting is trivial, modulo minor details regarding differentiation under integrals.

**Lemma 17.** Suppose \( U \subseteq \mathbb{C} \) is open, \( \{f_k(z)\}_{k=1}^{\infty} \) is a sequence of analytic functions on \( U \), and \( q(z, P_D) \) is a family of polynomials in \( P_D \), analytic in \( z \) and regular throughout \( U \), such that \( q(z, P_D)\{\{f'_k(z)\}\}_{k} \) converges to \( f(z) \) uniformly on compact subsets of \( U \). Then \( f \) is analytic in \( U \) and \( (q(z, P_D))^{2}\{\{f'_k(z)\}\}_{k} \) converges to \( f' \) uniformly on compact subsets of \( U \).

**Proof.** Fix \( K \subseteq U \) compact. We have that \( q(z, P_D)\{\{f'_k(z)\}\}_{k} = f(z) + R_k(z) \) with \( R_k(z) \) uniformly convergent to zero on \( K \). Since all terms in this equation are analytic in \( U \) we may differentiate throughout with respect to \( z \). We obtain

\[
q(z, P_D)\{\{f'_k(z)\}\}_{k} + q'(z, P_D)\{\{f'_k(z)\}\}_{k} = f'(z) + R'_k(z)
\]

and applying \( q(z, P_D) \) again yields that

\[
(q(z, P_D))^{2}\{\{f'_k(z)\}\}_{k} + q'(z, P_D)q(z, P_D)\{\{f'_k(z)\}\}_{k} = f'(z) + q(z, P_D)\{\{R'_k(z)\}\}_{k}
\]

on noting trivially that \( q(z, P_D) \) is regular and \( f'(z) \) is independent of \( k \). Now since \( \{R_k(z)\}_{k=1}^{\infty} \) is uniformly convergent to zero on compact subsets of \( U \), the classical Weierstrass theorem implies that the same is true for \( \{R'_k(z)\}_{k=1}^{\infty} \). By Lemma 15 this remains true for \( \{q(z, P_D)\{\{R'_k(z)\}\}_{k}\}_{k=1}^{\infty} \) and it remains only to prove that

\[
q'(z, P_D)q(z, P_D)\{\{f'_k(z)\}\}_{k} = q'(z, P_D)\{\{f(z) + R_k(z)\}\}_{k}
\]

also converges uniformly to zero on \( K \). But note that since \( q(z, 1) = 1 \) for all \( z \in U \), by regularity, we have \( q'(z, 1) = 0 \) for all \( z \in U \). Thus \( q'(z, P_D) \) is a polynomial in \( P_D \) whose coefficients are analytic functions of \( z \) with sum zero throughout \( U \). It follows that \( q'(z, P_D)\{\{f(z)\}\}_{k} \) is actually identically zero on \( U \), while the uniform convergence of \( q'(z, P_D)\{\{R_k(z)\}\}_{k} \) to zero on \( K \) follows by invoking Lemma 15 repeatedly for each term \( P^n \) in \( q'(z, P_D) \) in turn, and noting the boundedness of its analytic coefficient on the compact set \( K \). \( \square \)

Applied to a sequence of partial sums, this lemma of course represents an analogue, for generalised convergence, of Weierstrass’ theorem on term-by-term differentiation of power series inside their circles of convergence.

Note that if \( q(z, P_D) \) produces generalised convergence of \( \{f_k(z)\}_{k=1}^{\infty} \) to \( f \), then it is \( (q(z, P_D))^{2} \) that yields generalised convergence of \( \{f'_k(z)\}_{k=1}^{\infty} \) to \( f' \) and, iterating the arguments in the proof, \( (q(z, P_D))^{n+1} \) that gives
generalised convergence of \( \{ f_k^{(n)}(z) \}_{k=1}^{\infty} \) to \( f^{(n)} \). To understand more concretely why this is so, consider again \( \zeta^{\text{ext,}D}(z) \) as the generalised limit of \( \{(s_{\zeta,z})_k\}_{k=1}^{\infty} \) for \( z \notin \mathbb{Z} \leq 1 \).

By Equation (2), the sequence \( \{(s_{\zeta,z})'_k\}_{k=1}^{\infty} \) (which is the partial sum sequence for the derivative defining series \(- \sum_{n=1}^{\infty} n^{-z} \ln n \)) is given by

\[
(s_{\zeta,z})'_k = \left( -\frac{1}{1-z} k^{1-z} \ln k + \frac{1}{(1-z)^2} k^{1-z} \right) + C'_{\zeta,z} - \frac{1}{2} k^{-z} \ln k + \ldots
\]

where the terms omitted are all products of analytic functions of \( z \) with terms of the form either \( k^{-z-2r+1} \) or \( k^{-z-2r+1} \ln k \). The terms involving only powers of \( k \) can be grouped and rewritten as a linear combination in the fashion of Equation (33), and hence can be uniformly locally asymptotically annihilated by \( q(z,P_D) \) alone from Equation (34).

To handle the terms of the form \( k^{-z-2r+1} \ln k \), however, note that these arise not in eigensequences of \( P_D \) but in generalised eigensequences of \( P_D \). We omitted discussion of this in §4.1 but it is readily verified that multiplying the formulae from §4.1 for our eigensequences \( \{a_k(\rho)\}_{k=1}^{\infty} \) by factors \((\ln k)^m, m \in \mathbb{Z}_{\geq 1}\), yields generalised asymptotic eigensequences of \( P_D \) with the same eigenvalue, at least after including further lower-order correction terms (e.g., for \(-1 < \text{Re} \rho < 1\), \( \{k^\rho \ln k\}_{k=1}^{\infty} \) is at once a generalised eigensequence of \( P_D \), while for \( 1 \leq \text{Re} \rho < 2 \) we need to take \( \{(k^\rho - \frac{\rho(\rho+1)}{2} k^{\rho-1}) \ln k - \frac{(2\rho+1)k^{\rho-1}}{2} \}_{k=1}^{\infty} \), and so on).

It follows that when we group these terms in Equation (35) and write them as a corresponding linear combination of generalised asymptotic eigensequences, their uniform local asymptotic annihilation requires using \( q(z,P_D) \)\(^2 \) rather than just \( q(z,P_D) \), so that each factor \( (\frac{k^\rho - \frac{\rho(\rho+1)}{2} k^{\rho-1}}{2} \ln k - \frac{(2\rho+1)k^{\rho-1}}{2} \) occurs with exponent one higher.

Each further differentiation of Equation (2) in turn leads to terms with one more factor of \( \ln k \), whose annihilation requires one higher power on each factor of \( q(z,P_D) \), and this explains, at least for this example, the need to take one higher power each time of the polynomial \( q(z,P_D) \).

In fact such behaviour occurs much more generally. In §7 we shall define arbitrary (non-Césaro) convergence schemes and consider examples using them, but in these cases too we shall find that where eigenvectors of the operator in the scheme must be annihilated in order to extend a series outside its domain of classical convergence, it is generalised eigenvectors that need to be annihilated in order to treat the derivative series and ensure that the resulting extension is analytic. An analogue of Lemma 17 will thus hold for convergence schemes in general.

We conclude §5 now, however, by returning briefly to the example of \( \zeta \) and showing how the ideas of this subsection lead to a new explanation of why
anomalies/removable singularities arise at the nonpositive integer points for \( \zeta^{ext,D} \) but not for \( \zeta^{ext} \).

Consider the partial sum sequence/function for the derivative defining series \(-\sum_{n=1}^{\infty} n^{-z} \ln n \) at, for example, \( z = 0 \): 

\[
(s_{\zeta,z})'(k_{z=0}) = (s_{\zeta,z})'(k + \alpha)|_{z=0} = -(k + \frac{1}{2}) \ln k + k + C'_{\zeta,z}|_{z=0}.
\]

Within the continuous scheme it is readily verified that we have 

\[
(s_{\zeta,z})'(k + \alpha)|_{z=0} \sim -(k + \alpha) \ln(k + \alpha) + (k + \alpha) + C'_{\zeta,z}|_{z=0}
\]

and hence that \((s_{\zeta,z})(k + \alpha)|_{z=0}\) converges in a generalised continuous Césaro sense (via the polynomial \((2P - 1)^2\) as per Lemma 17) to the correct value \( C'_{\zeta,z}|_{z=0} \) (which may be computed explicitly as \(-\frac{1}{2} \ln(2\pi)\) using Sterling’s theorem).

Within the discrete scheme, however, no generalised Césaro limit can be attached to \( \{(s_{\zeta,z})'_k|_{z=0}\}_{k=1}^{\infty} \), because any attempt to write expression (36) in terms of eigensequences and generalised eigensequences of \( P_D \) leaves pure factors of \( \ln k \) left over. This gives a new indication that our original pointwise evaluation of \( \zeta^{ext,D}(0) \) must represent an anomaly/removable singularity, since it shows that although \( \zeta^{ext,D} \) can be successfully evaluated at \( z = 0 \) as a pointwise generalised limit of the defining series for \( \zeta \), the derivative series cannot be handled there by the same pointwise approach.

Of course in §4.3 we saw how to identify the location of such anomalies, and correct them, by considering explicitly the polynomial we use, \( q(z, P_D) \). It is noteworthy, however, that they can be detected even within a naive pointwise approach and without ever needing to identify \( q(z, P_D) \), simply by considering not just the original defining series, but also its derivative series whose behaviour tells us about the analyticity of our extension.

6. Conjectures, results for Dirichlet series

We now consider new directions that can be explored using our Césaro schemes. We begin with some notions still related to \( \zeta \) and its number-theoretic role.

6.1. Pictures, dilation and scaling. An additional basic question arises in using the continuous Césaro scheme. In §2.1 we supplied a necessary procedure for going from series to partial sum functions of the continuous variable \( x \) by adding in the terms at the integer points along the positive axis. This choice, however, was arbitrary; we could have chosen to add in the terms at points \( \lambda_n \) for any monotonically increasing, unbounded sequence \( \{\lambda_n\}_{n=1}^{\infty} \).

It follows that our analysis of \( \zeta \) using \( P \) in §3 really involved analysing not just the defining series \( \sum_{n=1}^{\infty} n^{-z} \), but rather a defining “picture” for
ζ, namely this series together with a geometric prescription of the points λ_n = n at which the terms of the series were added in.

This raises an obvious question: had we chosen a different defining picture for ζ (i.e., different \{λ_n\}_{n=1}^{∞}), would we still have obtained the correct analytic continuation of ζ from the corresponding \( \tilde{\zeta}^{\text{ext}} \)?

The initial answer is no. For example if we take λ_n = e^{n-1} or λ_n = \ln n then we are unable even to evaluate \( \tilde{\zeta}^{\text{ext}}(z) \) within our continuous Cesàro scheme for any \Re z \leq 1, and so obtain no extension at all.

However, for choices of λ_n given just by linear combinations of powers of n, i.e., λ_n = \sum_{i=1}^{r} c_i n^{\rho_i} with each \rho_i \in \mathbb{R}_{\geq 0} and \max\{\rho_i\} > 0, the answer turns out to be effectively yes, reflecting the fact that the eigenfunctions of \( P \) are themselves powers of x. We defer a proof of this claim to another paper ([7]) but we make here two remarks.

First, for such pictures the extension \( \tilde{\zeta}^{\text{ext}} \) generically involves a countable family of anomalies/removable singularities like those which arose in our discrete analysis in §4, and these must of course be corrected for in order to obtain the full analytic continuation of ζ. Thus, although we have a broad class of pictures for which we can still correctly perform continuous Cesàro extension of ζ, the standard picture λ_n = n in §3 remains special in having no anomalies/removable singularities and thus requiring only pointwise analysis. Within the continuous Cesàro scheme this picture is thus, in some sense, especially well adapted to analysing ζ. We conjecture that this is related to the basic role of the integers in the definition of ζ itself, and with this in mind, it would seem interesting to see to what degree this conjecture, and indeed our whole Cesàro approach, can be carried over to the setting of more general algebraic zeta functions.

Our second remark then concerns the following corollary question: within this class of pictures with \lambda_n = \sum_{i=1}^{r} c_i n^{\rho_i}, is the case λ_n = n the only one where no anomalies arise, or is there some subclass for which this is true?

In fact there is such a subclass, arising from consideration of the one-parameter dilation and scaling groups, \{D_r\}_{r>0} and \{S_r\}_{r>0}, given by

\[ D_r[f](x) \equiv f(rx) \] and \[ S_r[f](x) \equiv f(x^r) \] respectively. Again we defer proof until [7], but we will show there that the generalised continuous Cesàro limit of any function in \( F \) is preserved under the actions of \( D_r \) and \( S_r \) for any \( r > 0 \). It follows that for any picture related to the standard one by either a simple dilation (\( \lambda_n = rn \)) or scaling (\( \lambda_n = n^r \)) we also obtain the analytic continuation of ζ directly from pointwise calculations without anomalies.

It is interesting to consider whether such invariances could possibly be exploited to understand ζ (and its zeros) better, by viewing it as a \( \tilde{\zeta}^{\text{ext}} \) with the associated picture dilated or rescaled in a way depending on z.
6.2. Detecting poles and zeros. The simple pole of $\zeta$ at $z = 1$ coincides with the presence of a pure log divergence in the partial sum function $s_{\zeta,1}(x) = \ln x + \gamma + o(1)$. Similarly the $m^{th}$ derivative of $\zeta$ has its pole of order $m + 1$ at $z = 1$ signaled by a pure log divergence $(\ln x)^{m+1}$ in $s_{\zeta^{(m)},1}(x)$. As generalised eigenfunctions of $P$ with eigenvalue 1, the functions $(\ln x)^{m+1}$ arise only at $z$-values where the regularity/analyticity of $q(z, P)$ breaks down and, moreover, cannot be ascribed generalised limits within our continuous Césaro scheme.

Based on these examples we might conjecture that for any function, $f(z)$, obtained by using the continuous (or discrete) Césaro scheme to analytically continue a defining picture outside its domain of classical convergence, its poles should coincide with $z$-values where the associated partial sum function $s_{f,z}(x)$ contains a pure logarithmic divergence, the order of the pole coinciding with the power of the log divergence.

This, however, is false. For example the series $\sum_{n=1}^{\infty} \frac{z n^{-1-z^2}}{n^{2} + 2n^{-z}}$, classically convergent for Re $z > \frac{3}{2}$. In a neighbourhood of $z = 1$ this has expansion

$$\sum_{n=1}^{\infty} \{ n^{2-2z} \ln n + 2(1-z)n^{1-2z} \ln n + (2z^2 - 2z + 1)n^{-2z} \ln n + \ldots \}.$$  

Near $z = 1$ it thus represents $-2(1-z)\zeta'(2z - 1) + f(z)$, $f$ analytic, and our extension should have a simple pole with residue $\frac{1}{2}$ at $z = 1$. But the partial sum function at $z = 1$ again has no pure log divergence at all, owing to the factor of $(1-z)$ in front of the $n^{1-2z} \ln n$ term (the $(k+\frac{1}{2}) \ln k$ arising from the leading order term is not a pure log divergence — as in §5.3, it comes from a generalised eigenfunction of $P$ with eigenvalue $\frac{1}{2}$ rather than 1 in the continuous Césaro scheme).
In this case, however, note that if we take the derivative series, its partial sum function at $z = 1$ does have a pure $(\ln x)^2$ divergence. This suggests amending our conjecture for this class of Dirichlet series/pictures to the following claim: poles occur precisely at those $z$-values where pure log divergences arise in the partial sum function for either the original series or its derivative series, with the order of the pole at such $z$ given by the highest power of pure log-divergence appearing minus the order of the derivative with respect to $z$ taken.

This form of the conjecture is finally true and is actually relatively easy to prove by combining the sorts of Taylor expansion arguments used above with our basic results from the case of $\zeta$. In fact the proof shows that the result may be extended slightly to include both removable singularities as poles of order 0 (e.g., the case of $\zeta^{ext,D}$ at $z = 0$ in §5.3), and zeros as poles of negative order (e.g., $\sum_{n=1}^{\infty} (z - 1)^2 n^{-z}$ yields an $(\ln x)^1$ divergence at $z = 1$ after taking two derivatives, reflecting a “pole of order $-1$” or order 1 zero of $(z - 1)^2 \zeta(z)$ at $z = 1$). In addition, the proof also yields a simple relationship between the coefficient of the highest pure log divergence in the partial sum function at any given $z$-value and the residue of the associated pole there.

Rather than pursue these issues here, however, we instead turn briefly now to consider a different class of number-theoretic Dirichlet series which arise in the study of the Riemann zeta function. These are not generally of the type we have just considered. For instance, the well-known Dirichlet series for $-\zeta'$ is given by $\sum_{n=1}^{\infty} \Lambda(n)n^{-z}$ where $\Lambda(n)$ is the von Mangoldt function

$$\Lambda(n) = \begin{cases} \ln p, & n = p^r \text{ and } p \text{ prime,} \\ 0, & \text{otherwise} \end{cases}$$

and clearly this series does not lie within the class just discussed.

For series like this the irregular behaviour of the coefficients makes it unclear whether they even can be extended outside their domains of classical convergence by Cesaro methods. Let us assume for a moment, however, that they can, and moreover that, as in the result we have just discussed, poles of their analytic continuations coincide with $z$-values where the associated partial sum functions for either the original series or their derivatives contain pure log divergences.

Then since the nontrivial zeros of $\zeta$ correspond precisely to simple poles of $\frac{\zeta'}{\zeta}$ in the critical strip $0 < \text{Re}(z) < 1$, it would follow that these zeros should occur at $z$-values where the particular partial sum functions

$$s\left(\left.\frac{\zeta'}{\zeta}\right|^{(m)}_{k+z}(k + \alpha) = (-1)^m \sum_{n=1}^{k} \Lambda(n)(\ln n)^m n^{-z}$$


have a pure log divergence for some \( m \in \mathbb{Z}_{\geq 0} \). We would thus obtain an explicit pointwise criterion for the locations of the nontrivial zeros of \( \zeta \), purely in terms of the asymptotic behaviour of the partial sum functions for the single Dirichlet series \( \sum_{n=1}^{\infty} \Lambda(n)n^{-z} \) and its derivatives.

Now of course this is all highly speculative. Nevertheless it is interesting to ask to what degree Césaro methods can be applied to such number-theoretic Dirichlet series and conjectures regarding their poles confirmed, falsified or refined. The central difficulty is in obtaining expressions for the associated partial sum functions in the first place, but at an anecdotal level we may at least note a few intriguing “experimental” observations.

First, for example, \( -\frac{\zeta'(0)}{\zeta(0)} \) certainly has a simple pole at \( z = 1 \), consistent with the fact that \( \sum_{n=1}^{k} \Lambda(n)n^{-1} \sim \ln k \) ([5], pp12). Next, consider the so-called “explicit formula” for \( \sum_{n \leq X} \epsilon_{X}(n)\Lambda(n) = X - \sum_{\rho \in Z} X^{\rho} - \ln 2\pi - \frac{1}{2} \ln(1 - X^{-2}) \)

where \( \epsilon_{X}(n) \) is 1 if \( n < X \), \( \frac{1}{2} \) if \( n = X \), and 0 if \( n > X \), and where \( Z \) is the set of critical zeros of \( \zeta \). At a qualitative level this too seems to fit our Césaro framework, at least loosely, in that if we assign the generalised Césaro limit 0 to all the powers of \( X \) (and ignore for the moment that there are infinitely many of them) we obtain the correct generalised limit of \( -\ln 2\pi \) for the partial sum function.

Observations like this (see also the discussion in §8.1 of product series and multi-dimensional schemes) suggest that Césaro methods may in fact be applicable to the analysis of number-theoretic Dirichlet series.

6.3. Zeta functions for elliptic operators. We conclude §6 by considering zeta functions of positive self-adjoint elliptic differential operators on compact manifolds. The zeta function of such an operator, \( A \), of order \( m \) on a compact manifold of dimension \( n \), is defined by \( \zeta_{A}(z) = \sum_{j=1}^{\infty} M_{j}\lambda_{j}^{-z} \) where \( \lambda_{j} \) is the \( j \)th distinct eigenvalue of \( A \) and \( M_{j} \) its multiplicity, the series being classically convergent for \( \text{Re} \, z > \frac{n}{m} \) (see e.g., [6], §13).

Such zeta functions can often, like the Riemann zeta function, be analytically continued by our Césaro schemes. For example, for the Laplacian \( \Delta \) on any sphere \( S^{n} \), both \( M_{j} \) and \( \lambda_{j} \) are real polynomials, of degrees \( n - 1 \) and 2 respectively, and so \( \zeta_{\Delta_{S^{n}}} \) lies within the Césaro-amenable class discussed in §6.2. Indeed in this case these facts alone are sufficient to deduce from the Césaro approach certain universal qualitative features ([6], §13) of such zeta functions.

To begin with, it follows that \( \zeta_{\Delta_{S^{n}}} \) extends to a meromorphic function on \( \mathbb{C} \) whose only poles lie among the points \( z_{i} = \frac{n-1}{2}, i \in \mathbb{Z}_{\geq 0} \) and certainly not at the points 0, −1, −2, …. This follows from the result of §6.2 since,
on performing Taylor expansion in descending powers of \( j \), such \( z_i \) are the only points where terms of the form \( \frac{a}{j} \) can arise in the summand \( M_j \lambda_j^{-z} \), and hence the only points where the partial sum sequence/function can have a pure log divergence. This log-divergence test can, moreover, be applied explicitly in any given case to decide whether a prospective point \( z_i \) is, in fact, a pole. For example, for \( \Delta \) on \( S^3 \) we have \( \lambda_j = (j^2 + 2j) \) and \( M_j = (j + 1)^2 \). Thus at \( z = 1 \) we have \( (s_{\zeta_{\Delta S^3}})_k = k + \frac{3}{4} + o(1) \) and there is actually no pole at \( z = 1 \).

Further, the fact that the \( M_j \) have no \( \ln j \) factors (which we have actually just used implicitly) implies immediately that any poles in \( \zeta_{\Delta S^3} \) must be simple. For any such simple pole, finally, the result outlined in §6.2 implies simply that its residue is half the coefficient of the associated log-divergence, and this yields the explicit formula for the residue of the simple pole at \( z_0 = \frac{n}{2} \) given in [6], §13.

Unfortunately, for self-adjoint elliptic differential operators \( A \) more generally, things are not as straightforward. For example, even for the Laplacian on the 2-torus, \( T \), its eigenvalues \( \{p^2 + q^2\} \) form a 2-parameter family which is hard to order into a 1-parameter sequence \( \{\lambda_j\} \), making Césaro analysis problematic (in this case it may turn out to be better to use a two-dimensional Césaro scheme of the type we will discuss briefly in §8). We nonetheless remain hopeful that Césaro methods should still be applicable to such zeta functions. In conjecturing this we conclude simply by noting that, at least for any such continuous Césaro analysis, our computations suggest one should use a picture adapted to \( A \) in the sense that the \( j^{th} \) term \( M_j \lambda_j^{-z} \) should be added in not at the point \( x = j \) but rather at \( x = \lambda_j \). This appears to prevent the occurrence of anomalous removable singularities in the continuous Césaro extensions \( \zeta_A^{\text{ext}} \).

7. General schemes, the Borel scheme

So far we have considered only generalised notions of convergence based on the continuous or discrete Césaro operator. Clearly, however, we may define similar generalised convergence schemes using any regular, linear operator, \( A \), defined either on the space, \( S \), of arbitrary sequences or some suitable function space, \( F_A \), analogous to \( F \). Given any sequence \( s \in S \) (resp. \( f \in F_A \)) define \( s \) (resp. \( f \)) to have generalised \( A \)-limit \( L \), and write \( A \lim_{k \to \infty} s_k = L \) (resp. \( A \lim_{x \to \infty} f(x) = L \)), if there exists a regular polynomial, \( q(A) \), such that \( \lim_{k \to \infty} (q(A)[\{s\}]_k = L \) (resp. \( \lim_{x \to \infty} q(A)[f](x) = L \)). Regularity of \( q \) is again equivalent to the condition \( q(1) = 1 \).

Such alternative schemes can be used to analytically continue series not amenable to Césaro methods, such as ones diverging more rapidly than the ordinary (or nearly ordinary) Dirichlet series considered so far.
For example, consider the simple geometric series $g(z) \equiv \sum_{n=1}^{\infty} z^{n-1}$, classically convergent for $|z| < 1$. Its partial sum sequence is given, for $z \neq 1$, by

$$ (s_{g,z})_k = \left( \frac{z}{z-1} \right) z^{k-1} + \frac{1}{1-z} $$

(37)

and for any fixed $|z| > 1$ this diverges exponentially in $k$. It thus cannot be handled by our Césaro schemes, whose eigensequences/eigenfunctions involve only power divergences.

Traditionally (e.g., [2], §4.12) this series has been analysed instead using the classical Borel operator, $B_{cl}$, from sequences, $\{a_n\}_{n=0}^{\infty}$, to functions, $B_{cl}[\{a\}](x)$ given by

$$ B_{cl}[\{a\}](x) = \frac{1}{e^x} \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}. $$

Applied to $\{(s_{g,z})_n\}$ (trivially reindexed to start at $n = 0$) this yields a function which converges to $\frac{1}{1-z}$ for all $\text{Re} \, z < 1$. Classical Borel summation thus yields the correct analytic continuation of $g$ to the half-plane $\text{Re} \, z < 1$, but does not succeed in extending to $\text{Re} \, z \geq 1$.

By instead using a suitably adapted generalised convergence scheme of the type just described, however, we may obtain at once the correct analytic continuation $g(z) = \frac{1}{1-z}$ throughout the whole complex plane.

Define a new Borel operator $B : S \to S$ by

$$ B[\{a\}]_k \equiv \frac{1}{e^{k-1}} \left\{ a_1 + \sum_{j=2}^{k} \left( e^{j-1} - e^{j-2} \right) a_j \right\}. $$

(38)

Clearly $B$ is both linear and regular, so we may consider the (discrete) “Borel scheme” with this as its fundamental operator. For this, we have the following lemma, which may be verified directly by elementary computations.

**Lemma 18.** The asymptotic eigensequences of $B$ are the sequences $\{z^{n-1}\}_{n=1}^{\infty}$, $z \in \mathbb{C} \setminus \{\frac{1}{e}\}$, each with eigenvalue $\frac{(e-1)z}{ez-1}$ (for $z = \frac{1}{e}$, the sequence $\{e^{-n}\}_{n=1}^{\infty}$ is already classically convergent to zero). The sequences $\{n^{\rho}\}_{n=1}^{\infty}$ are asymptotic generalised eigensequences of $B$ with eigenvalue $1$ for any $\rho \in \mathbb{C}$ and, for any $z \in \mathbb{C} \setminus \{\frac{1}{e}\}$, the asymptotic generalised eigensequences of $B$ with eigenvalue $\frac{(e-1)z}{ez-1}$ are the sequences $\{z^{n-1}n^m\}_{n=1}^{\infty}$, $m \in \mathbb{Z}_{\geq 1}$.

Since eigensequences and generalised eigensequences of $B$ with eigenvalue $\lambda \neq 1$ all have generalised $B$-limit 0 in the usual fashion, it follows immediately from this lemma and Equation (37) that, in the example of the geometric series, we have $g^{\text{ext},B}(z) = \frac{1}{1-z}$ for any $z \neq 1$. We thus do obtain the correct analytic continuation of $g$ to the whole complex plane.
via this new Borel scheme as promised. That \( z = 1 \) is \textit{a priori} a simple pole with residue \(-1\) follows, moreover, in similar fashion to our earlier singularity computations in §3.3 — the polynomial we have used is 

\[
q(z, B) = \frac{e^z - 1}{z - 1} (B - \frac{(e^z - 1)}{z - 1})
\]

and its analyticity breaks down at \( z = 1 \) with 

\[
\lim_{z \to 1} \lim_{k \to \infty} (z - 1)q(z, B)[\{ s_{g,z} \}]_k = (e - 1) \lim_{k \to \infty} (B - 1)[\{ \tilde{k} \}]_k = -1.
\]

Although the analytic continuation in this example is, of course, trivial by other means, it nonetheless shows how an effective generalised convergence scheme may be constructed in general using an operator whose choice is adapted to the type of series requiring extension. An infinite variety of such schemes exists (see e.g., §8.3) and they may easily be adapted to other natural settings such as integrals (with complex parameter) having divergences at finite points rather than just as \( x \to \infty \). We conclude §7, however, with two further brief remarks regarding the example of the geometric series and Borel scheme.

First, if we write \( z = e^{-w} \) then \( \sum_{n=1}^{\infty} z^{n-1} \) becomes a Dirichlet series in the variable \( w \), \( \sum_{n=1}^{\infty} a_n e^{-\lambda_n w}, \) with \( a_n = 1 \) and \( \lambda_n = (n - 1) \). Comparing with Equation (38) we conjecture that a Dirichlet series \( \sum_{n=1}^{\infty} a_n e^{-\lambda_n w} \) (with all \( \lambda_n \geq 0 \)) should in general be analysed by a discrete generalised convergence scheme with fundamental operator \( A : \mathcal{S} \to \mathcal{S} \) given by

\[
A[\{a\}]_k \equiv \frac{1}{e^{\lambda_k}} \left\{ e^{\lambda_1}a_1 + \sum_{j=2}^{k} (e^{\lambda_j} - e^{\lambda_{j-1}})a_j \right\}.
\]

The earlier case of \( PD \) for the Dirichlet series \( \sum_{n=1}^{\infty} n^{-z} \) in §4 fits within this framework.

Secondly, this new Borel scheme may be useful more broadly in defining generalised Fourier series to handle periodic functions with nonintegrable singularities and convergence off the real line. We investigate this more fully in a separate paper, in preparation ([8]).

8. Final observations

We conclude by schematically discussing some further topics of potential interest related to the notions of generalised convergence introduced.

8.1. Higher-dimensional schemes. One is often interested (e.g., \( \zeta_{\triangle T} \) in §6.3) in sums of arrays of numbers indexed by two or more parameters. They arise for example when taking a product of two series, \( \sum_i a_i \) and \( \sum_j b_j \), producing a double series \( \sum_{i,j} a_i b_j \). It is thus natural to seek to construct higher-dimensional convergence schemes for defining generalised limits of functions/sequences of several variables. We shall work only in the context of the continuous Cesaro scheme and two dimensions but the treatment of other schemes and higher dimensions is obviously analogous.
The initial difficulties are to decide what it should mean even at the classical level to say that a function, \( f(x, y) \), is convergent to limit \( L \), and then what the function space and operators should be for a 2-d Césaro scheme. Two requirements are natural — first, if \( f_1(x) \) converges classically to \( L_1 \) and \( f_2(y) \) converges classically to \( L_2 \) then \( f_1(x)f_2(y) \) should converge classically to \( L_1L_2 \), and secondly, this should continue to hold for generalised Césaro limits.

Based on the first of these conditions we define \( f(x, y) \) to converge classically to limit \( L \) if and only if for all \( \epsilon > 0 \) there exist \( M, N > 0 \) such that
\[
|f(x, y) - L| < \epsilon
\]
whenever both \( x > M \) and \( y > N \).

Turning to the second, consider operators \( P_1 \) and \( P_2 \) defined by
\[
P_1[f](x, y) \equiv \frac{1}{x^2} \int_0^x f(t, y) \, dt \quad \text{and} \quad P_2[f](x, y) \equiv \frac{1}{y^2} \int_0^y f(x, t) \, dt.
\]
If \( P_1 \) and \( P_2 \) were regular operators permitted within our 2-d Césaro scheme we would obtain at once the desired result on generalised limits of products.

Now in order for each \( P_i \) even to map back into the same function space, \( \mathcal{F}(2) \), for our 2-d scheme we need functions \( f \) in this space to satisfy the condition in our definition of the 1-d Césaro space, \( \mathcal{F} \), on each slice parallel to the \( x \) or \( y \) axes. That is, for any fixed \( y \) we need that
\[
\int_0^x |f(t, y)(\ln t)^m| \, dt < \infty \quad \text{for all} \quad x \geq 0 \quad \text{and} \quad m \in \mathbb{Z}_{\geq 0},
\]
and similarly for any fixed \( x \).

These conditions alone, however, are insufficient to guarantee that each \( P_i \) is regular. For example, if we take
\[
f(x, y) = \begin{cases} 
1, & x \leq 1, \ y \leq 1 \\
y, & x \leq 1, \ y > 1 \\
x, & x > 1, \ y \leq 1 \\
x^2-y^2-x, & x > 1, \ y > 1
\end{cases}
\]
then \( f \) is continuous, satisfies the above slice conditions, and has classical limit 0 under our definition, but neither \( P_1[f] \) nor \( P_2[f] \) is classically convergent under this definition.

To overcome this we restrict \( \mathcal{F}(2) \) further by imposing uniformity requirements in our slice conditions. Our final definition is that \( f \) lies in \( \mathcal{F}(2) \) if:

(a) For all \( y \) we have \( \int_0^x |f(t, y)(\ln t)^m| \, dt < \infty \) for all \( x \geq 0 \) and \( m \in \mathbb{Z}_{\geq 0} \), and for any fixed \( X > 0 \) and \( m \in \mathbb{Z}_{\geq 0} \) there exist \( Y, C > 0 \) such that
\[
\int_0^X |f(t, y)(\ln t)^m| \, dt < C \quad \text{for all} \quad y > Y.
\]

(b) The same holds with the roles of \( x \) and \( y \) reversed.

With this definition it is readily verified that each \( P_i \) still maps \( \mathcal{F}(2) \) back into itself but is now also a regular operator as wanted. We obtain our desired 2-d Césaro scheme finally by defining \( f \in \mathcal{F}(2) \) to have generalised \( C(2) \)-limit \( L \) if there exists a regular polynomial \( q(P_1, P_2) \) (i.e., \( q(1, 1) = 1 \)) such that \( q(P_1, P_2)[f](x, y) \) converges classically to \( L \). Since \( P_1 \) commutes with \( P_2 \), the same argument as in the 1-d case shows that this formulation is
well-defined, and, as remarked earlier, the 2-d generalised Césaro limit of a product of functions is now clearly equal to the product of their generalised 1-d Césaro limits.

With this in mind, consider briefly again our discussion from §6.2 about the amenability of number-theoretic Dirichlet series to Césaro analysis. Take for example the series $\sum_{n=1}^{\infty} d_2(n)n^{-z}$, absolutely convergent for $\Re z > 1$, where $d_2(n)$ is the cardinality of the set of ordered pairs $(a, b)$ such that $ab = n$. This is precisely of the kind discussed in §6.2 where the irregularity of the coefficients makes it unclear whether 1-d Césaro analysis is possible. For $\Re z > 1$, however, this series is simply the product $\zeta(z)^2 = \left(\sum_{n=1}^{\infty} n^{-z}\right)^2$ ([5], §1). It is therefore certainly amenable to successful 2-d Césaro extension by treating the double-series $\sum_{i,j=1}^{\infty} i^{-z} j^{-z}$.

The fact that Césaro methods thus can be employed to analytically continue a Dirichlet series like this, albeit by rewriting it as a product and using a 2-d scheme, suggests perhaps that Césaro analysis may indeed be applicable to number-theoretic Dirichlet series more generally.

8.2. Ratio eigenfunctions. In §5.3 we saw, in the context of the discrete Césaro scheme, that where generalised convergence of $\{s_n(z)\}_{n=1}^{\infty}$ is obtained by annihilating eigensequences of $P_D$, the generalised convergence of $\{d\frac{d}{dz}s_n(z)\}_{n=1}^{\infty}$ requires annihilation of generalised eigensequences of $P_D$, and similarly for further derivatives. The same pattern holds for arbitrary convergence schemes of the type defined in §7 such as the Borel scheme.

Taking antiderivatives is also interesting. Consider, for example, the family of functions $s(x, z) = x^z$. As $x \to \infty$, $s(x, z)$ converges in a generalised continuous Césaro sense to the zero function on $\mathbb{C}\setminus\{0\}$ with a removable singular value 1 at $z = 0$. This occurs via polynomials $q(z, P)$ whose coefficients are uniformly bounded on any set a strictly positive distance from the origin.

Taking such a set $U$ of the form $\mathbb{C}\setminus D_r$, where $D_r$ is a disk of radius $r$, consider now the antiderivative family $\tilde{s}(x, z) = \frac{x^z}{\ln x}$ on $U$ (of course for any $z$ the function $\frac{x^z}{\ln x}$ is not strictly in $\mathcal{F}$, but $\frac{x^{z-1}}{\ln x}$ is, and since $\frac{1}{\ln x}$ converges classically to 0 we may ignore this technicality — alternatively we could simply redefine both families $s$ and $\tilde{s}$ as zero for $x < 2$). We would hope that on $U$ antidifferentiation should commute with generalised convergence just as differentiation did. Since the antiderivative of the zero function is a constant function and $\frac{x^z}{\ln x}$ already converges classically to zero for all $z$ in $U$ with $\Re z < 0$, this means we should have

**Conjecture 1.** $\text{Clim}_{x \to \infty} \frac{x^z}{\ln x} = 0$ for all $z \in U$.

Unfortunately it is easy to see that, for any $\Re z > 0$ in $U$, no polynomial $q(z, P)$ exists which annihilates $\frac{x^z}{\ln x}$ even asymptotically. For if we consider
\( P[\frac{\ln^2 x}{x}] (x) \) we obtain an infinite asymptotic expansion

\[
P \left[ \frac{x^z}{\ln x} \right] (x) \sim \frac{x^z}{(z + 1) \ln x} \cdot \left\{ 1 + \frac{1!}{(z + 1) \ln x} + \frac{2!}{(z + 1)^2 (\ln x)^2} + \frac{3!}{(z + 1)^3 (\ln x)^3} + \ldots \right\}
\]

and the presence of classically divergent terms \( \frac{x^z}{(\ln x)^k} \) for \( k \) arbitrarily large makes it impossible to construct any annihilating polynomial of finite degree.

It follows that our current definition of generalised \( \zeta \)-Cesaro convergence is inadequate to validate Conjecture 1, and the same is true for the families \( x^z (\ln x)^m \), \( m \in \mathbb{Z}_{>1} \), which arise from further antidifferentiation on \( U \). An identical situation arises for convergence schemes in general when we consider antiderivatives leading to ratios of arbitrary eigenfunctions/sequences over powers of generalised eigenfunctions/sequences with eigenvalue 1 (or indeed products of arbitrary eigenfunctions/sequences with \( \rho^{th} \) powers of generalised eigenfunctions/sequences with eigenvalue 1 for \( \rho \notin \mathbb{Z}_{\geq 0} \)).

The functions \( \tilde{s}(x, z) = \frac{x^z}{\ln x} \) do, however, behave in a similar way to eigenfunctions of \( P \) with eigenvalue \( \frac{1}{z+1} \) in one sense. Although they don’t satisfy either the exact eigenfunction equation \( (P - \frac{1}{z+1})[\tilde{s}(\cdot, z)] = 0 \) or its asymptotic counterpart, they do satisfy an infinite descending chain of asymptotic relations involving the operator \( (P - \frac{1}{z+1}) \). Writing \( \tilde{s}(x, z) = \tilde{s}_0(x, z) \) we have

\[
(P - \frac{1}{z+1})[\tilde{s}_0(\tilde{x}, z)](x) = \tilde{s}_1(x, z) \quad \text{with} \quad \tilde{s}_1(x, z) = o(\tilde{s}_0(x, z)),
(P - \frac{1}{z+1})[\tilde{s}_1(\tilde{x}, z)](x) = \tilde{s}_2(x, z) \quad \text{with} \quad \tilde{s}_2(x, z) = o(\tilde{s}_1(x, z)),
\]

\[
\vdots
\]

Since each relation in this chain can be rewritten (at least loosely) as the ratio

\[
\frac{(P - \frac{1}{z+1})[\tilde{s}_i(\tilde{x}, z)](x)}{\tilde{s}_i(x, z)} = o(1),
\]

we shall call the function \( \tilde{s}_0(x, z) = \frac{x^z}{\ln x} \) a “ratio eigenfunction” of \( P \) with eigenvalue \( \frac{1}{z+1} \). The functions \( \frac{x^z}{(\ln x)^2}, \frac{x^z}{(\ln x)^3}, \ldots \), are all also ratio eigenfunctions of \( P \) with eigenvalue \( \frac{1}{z+1} \).

It remains, however, to determine how to extend our definition of generalised \( \zeta \)-Cesaro convergence to obtain the desired generalised limit 0 for such ratio eigenfunctions.

At present we do not have a fully satisfactory answer, but one possibility, again developed jointly with Andrew Stone, consists of extending to permit polynomials constructed not just from the basic operator \( P \), but also from conjugates of the form \( A_m = M_{(\ln x)^m} \circ P \circ M_{(\ln x)^m} \), where \( M_{f(x)} \) is the
operator of multiplication by \( f(x) \) on \( \mathcal{F} \). Since the functions \((\ln x)^m\) are generalised eigenfunctions of \( P \) with eigenvalue 1, these conjugate operators are all regular, and each function \( \frac{x^z}{(\ln x)^m} \) is now an exact eigenfunction of \( A_r \) and generalised eigenfunction of \( A_m, m > r \). It is possible to frame an extended definition of generalised Césaro convergence, using such \( A_m \), which handles linear combinations of such ratio eigenfunctions and satisfies uniqueness of generalised limits, although this now requires some care due to the noncommutativity of \( A_m \) and \( A_n \) for \( m \neq n \). We shall not go into details here, however, and instead close with one remark.

It is that more exotic ratio eigenfunctions than just \( \frac{x^z}{(\ln x)^m} \) also arise naturally in many applications and may necessitate extending further still, to permit conjugation by multiplication operators \( M_f(x) \) where \( f \) is an arbitrary asymptotic generalised eigenfunction of \( P \) with eigenvalue 1. Such a further generalisation has yet to be fully worked out, but we note that an ability to handle ratio eigenfunctions (for arbitrary schemes) is essential for the application to generalised notions of Fourier theory which we mentioned at the end of §7 and shall discuss in [8].

8.3. Schemes and measures. The continuous Césaro convergence scheme is only one example of a class of schemes associated to measures on \([0, \infty)\).

If we take a measure \( \mu(t)dt \) where \( \mu \) is any positive locally integrable function with \( \lim_{x \to \infty} \int_0^x \mu(t)dt = \infty \), we can define an associated convergence scheme with fundamental regular operator \( P_\mu \) given by

\[
P_\mu[f](x) = \frac{1}{\int_0^x \mu(t)dt} \int_0^x f(t)\mu(t)dt.
\]

Continuous Césaro corresponds to ordinary Lebesgue measure with \( \mu(t) \equiv 1 \).

Letting \( \nu(x) \equiv \int_0^x \mu(t)dt \) it is trivial to verify that, for any such scheme, the functions \((\nu(x))^z\) are the eigenfunctions of \( P_\mu \) with eigenvalue \( \frac{1}{1+z} \), with the generalised eigenfunctions being simply \((\nu(x))^z(\ln(\nu(x)))^m, m \in \mathbb{Z}_{>0}\).

If we have some family of divergent series or integrals to which we want to ascribe generalised limits, in order to say to perform an explicit analytic continuation, we now see that one approach is to choose a \( \mu(t) \) with the eigenfunctions/generalised eigenfunctions of the resulting scheme involving divergences of the same type as the ones we have to deal with. This is essentially what we did in using the continuous Césaro scheme (where \( \nu(x) = x \)) to handle the power divergences in the ordinary Dirichlet series for \( \zeta \).

In [8] we shall likewise use a scheme with \( \mu(t) = e^t \) adapted to handling exponential divergences which arise there in treating Fourier transforms.

For the remainder of this section, however, we now briefly consider just one family of schemes arising from taking \( \mu \) as \( \mu_r(t) = t^r, r > -1 \). The schemes in this class are all closely related to the single continuous Césaro case \( r = 0 \); for example, they all have the same eigenfunctions and generalised eigenfunctions, albeit with varying eigenvalues. Indeed the operators
\( P_r \equiv P_t \) are all conjugates of \( P \) by rescaling operators of the kind described in §6.1:

\[
P_r = S^{-1}_{r+1} \circ P \circ S_{1/r+1}
\]

(41)

(if we view all functions as partial sum functions arising from some underlying picture on \([0, \infty)\), this relationship just says that we may equivalently think either of the picture as fixed and the measure varying, or alternatively of the measure as fixed but the underlying picture being suitably rescaled).

Now since, as remarked in §6.1, generalised continuous Cesàro limits are invariant under rescalings, Equation (41) means that the generalised limits of functions are the same for all the schemes in our family. Let us consider, however, the limiting cases \( r = -1 \) and \( r = \infty \).

For \( r = -1 \) a technicality arises since the function \( \frac{1}{t} \) is not integrable at zero, but if instead we take \( \mu_{-1}(t) = 0 \) for \( 0 \leq t < 1 \) and \( \mu_{-1}(t) = \frac{1}{t} \) for \( t \geq 1 \) this problem disappears and we can still consider \( \mu_{-1} \) as a limiting case of a slightly amended family in which we similarly truncate the functions \( \mu_r(t) \), \( r > -1 \). The case \( r = -1 \) then yields the operator \( P_{-1} \) given by

\[
P_{-1}[f](x) = \frac{1}{\ln x} \int_1^x f(t) \frac{1}{t} dt.
\]

This is the continuous analogue of Riesz’ logarithmic mean, as discussed in [3].\(^2\) In terms of rescalings we have \( P_{-1} = S_{\ln} \circ P \circ S_{\exp} \) where \( S_{\exp}[f](x) \equiv f(e^x) \) and \( S_{\ln}[f](x) \equiv f(\ln x) \).

An interesting phenomenon occurs with the eigenfunctions and generalised eigenfunctions of our schemes in this limit. Under \( P_r \) for any \( r > -1 \) the eigenfunction of \( P \) with eigenvalue \( \lambda \) (namely \( x^\frac{1}{\lambda} - 1 \)) remains an eigenfunction but with eigenvalue \( \frac{(r+1)\lambda}{\lambda + 1} \). As \( r \to -1 \) these eigenvalues all flow towards zero, except for \( \lambda = 1 \) which stays fixed. For \( \lambda = 1 \), however, there is still variation, with the associated generalised eigenvalue equation becoming \( r \)-dependent:

\[
(P_r - 1)[\ln \tilde{x}](x) = -\frac{1}{r + 1}.
\]

What happens in the actual limiting case \( r = -1 \)? To begin with, the functions \( (\ln x)^z \), which were formerly all generalised eigenfunctions with eigenvalue 1, now become a full array of eigenfunctions of \( P_{-1} \) with eigenvalues \( \frac{1}{1+z} \). Some nontrivial asymptotic eigenfunctions (not generalised eigenfunctions) with eigenvalue 1, such as \( \ln(\ln x) \), become generalised eigenfunctions of \( P_{-1} \) with eigenvalue 1. Others like \( \sin(\sqrt{\ln x}) \) now fall directly in the asymptotic kernel of \( P_{-1} \). Passing to the limiting case \( r = -1 \) thus separates the asymptotic eigenfunctions and generalised eigenfunctions of

\(^2\) \( P_{-1} \) is also useful in giving an alternative way of obtaining the gamma function, \( \Gamma(z) \), starting from the divergent product \( \prod_{n=1}^{\infty} (1 + \frac{z}{n}) \).
With eigenvalue 1 into different hierarchies according to their behaviour under \( P_{-1} \).

As for the other eigenfunctions and generalised eigenfunctions of \( P \) (and hence of \( P_r \) for any \( r > -1 \)), these no longer generally remain eigenfunctions/generalised eigenfunctions of \( P_{-1} \). But interestingly the action of \( P_{-1} \) intertwines them with the ratio eigenfunctions of \( P \) discussed in the previous subsection. For example we have

\[
P_{-1}[^{\tilde{x}z}](x) = \frac{1}{z} \frac{x^z - 1}{\ln x} = \frac{1}{z} \frac{x^z}{\ln x} + o(1)
\]

for any \( z \in \mathbb{C} \setminus \{0\} \). Only for \( \text{Re} \, z = 0 \) does this still represent a nontrivial asymptotic eigenvalue equation, with eigenvalue 0 for all such \( z \) in line with our earlier eigenvalue flow remarks. Note that this intertwining means that, rather than conjugating \( P \) by multiplication operators \( M_{(\ln x)^m} \) to handle ratio eigenfunctions in \( §8.2 \), we could alternatively conjugate by powers of the operator \( P_{-1} \).

As for the case \( r \to \infty \), it is not immediately obvious how to interpret this limit in terms of a limiting measure. In this case, however, the eigenvalues \( \lambda \neq 1 \) all flow towards 1. This suggests that any limiting operator, \( P_{\infty} \), should have the functions \( x^z \) as eigenfunctions or generalised eigenfunctions with eigenvalue 1 for all \( z \in \mathbb{C} \). Such an operator is easily manufactured by taking \( P_{\infty} \equiv S_{\exp} \circ P \circ S_{\ln} \). More explicitly, \( P_{\infty} \) is then just the operator

\[
P_{\infty}[f](x) = \frac{1}{e^x} \int_{-\infty}^{x} f(t)e^t \, dt
\]

which is very close to being the operator \( P_{\mu} \) with \( \mu(t) = e^t \) mentioned earlier in this subsection. Its eigenfunctions are the exponentials \( e^{zx} \), \( z \in \mathbb{C} \), with generalised eigenfunctions \( e^{zx}x^m \), \( m \in \mathbb{Z}_{>0} \), and they are intertwined with the associated ratio eigenfunctions of \( P_{\infty} \) by the action of \( P \).

We see that we can construct schemes using \( P_{-1} \) and \( P_{\infty} \), corresponding to logarithmic and exponential underlying rescalings of our original continuous Césaro scheme, which are, in some sense, limits of our invariant family \( \{ P_r \}_{r>-1} \). Passage to these limiting cases involves interesting behaviour in the eigenvalues and eigenfunctions/generalised eigenfunctions of the associated operators.

8.4. Dynamical systems, quantisation and symmetries for \( \zeta \). The inverse of the Césaro operator, \( P \), used in \( §3 \) to analyse \( \zeta \), is the differential operator \( P^{-1} = x \frac{d}{dx} + 1 \). There has been tremendous recent interest (e.g., [1] and many others) in trying to understand the zeros of \( \zeta \) in terms of the spectrum of the quantised Hamiltonian operator of some chaotic dynamical system. In this context it is interesting that \( P^{-1} \) is in fact the canonical quantisation, modulo ordering choices and a factor of \(-i\), of precisely the
classical Hamiltonian $H_{cl} = XP$ suggested for this role in [1]\(^3\). In light of
the possible role also of the dilation group $D_r$, discussed in §6.1, in this
dynamical systems approach (e.g., [1], §6), it seems interesting to ask whether
this relationship between $P^{-1}$ and the putative $H_{cl}$ could be of significance.
In particular, it is interesting to speculate whether the scaling group, $S_r$,
which has played a role in this paper, could also be of value in the dynamical
systems approach.

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\[^3\]In [1] it is actually the symmetrised expression $\frac{1}{2}(XP + PX)$ which is quantised to
guarantee hermiticity, but this just corresponds, up to a factor of $-i/2$, to considering the
inverse of the related rescaled operator $P^{-1}_r$, defined in §8.3, rather than $P$.
ON THE CUT LOCUS IN ALEXANDROV SPACES AND APPLICATIONS TO CONVEX SURFACES

Tudor Zamfirescu

Alexandrov spaces are a large class of metric spaces that includes Hilbert spaces, Riemannian manifolds and convex surfaces. In the framework of Alexandrov spaces, we examine the ambiguous locus of analysis and the cut locus of differential geometry, proving a general bisecting property, showing how small the ambiguous locus must be, and proving that typically the ambiguous locus and a fortiori the cut locus are dense.

Introduction

A metric space is called an Alexandrov space if it is an intrinsic metric space with curvature bounded below in the sense of Alexandrov (see definition on page 376). Such spaces were introduced by A.D. Alexandrov in 1955, along with spaces with curvature bounded above.

Hilbert spaces, Riemannian manifolds, convex surfaces, and convex subsets of these are examples of Alexandrov spaces.

This paper considers the ambiguous locus of analysis and the cut locus of differential geometry in the framework of Alexandrov spaces. We prove a general bisecting property, then show how small the ambiguous locus must be, and finally prove that typically the ambiguous locus and a fortiori the cut locus must be dense.

In fact, we discover that the apparently distant notions of cut locus and ambiguous locus share a common soul. Some more (metamathematical) searching brings to light another two independent developments of the same notion.

The ambiguous locus has been investigated by — among others — de Blasi and Myjak [3], [4], [5]; de Blasi, Kenderov and Myjak [6]; Myjak and Rudnicki [21]; Zhivkov [41]; de Blasi and Zamfirescu [7].

The cut locus, introduced by Poincaré in 1905 [25], received its name from Whitehead in 1936; it was studied by Myers in 1935–36 [19], [20], and subsequently by many other authors in the context of Riemannian geometry. Their contributions grew to what is today a large, well-established body of results.
More recently, Otsu and Shioya [22], Shiohama and Tanaka [27], Itoh [15], Zamfirescu [38] and others have investigated the cut locus in Alexandrov spaces. Shiohama and Tanaka considered in [27] the cut locus with respect to a compact set.

Independently, A. Riviè re [26] studied the cut locus (under the name of “nervure”) with respect to closed subsets of a Euclidean space.

A fourth independent appearance of the cut locus (or, more exactly, of the ambiguous locus), with a more applied background, under the name of “medial axis”, appears in papers by Lee [17], Lee and Drysdale [18], Yap [29], Choi, Choi and Moon [10], and others.

Kunze [16] and Zamfirescu [39] have treated problems on the cut locus in the case of closed convex hypersurfaces without any differentiability assumptions.

By making use of the notion of porosity we shall see how small (σ-porous) the ambiguous locus must always be, and also how large (dense) it can sometimes be. These results relate to work in Banach spaces and in Riemannian manifolds.

Definitions. We work in a metric space $(\mathcal{A}, \rho)$.

A segment between two distinct points is a shortest path between them—a minimal segment in Otsu and Shioya’s terminology [22].

A segment between a point $x$ and a closed set $K$ not containing $x$ is an arc between $x$ and a point of $K$, not longer than any other such arc.

A geodesic is a curve which is locally a segment (for a precise definition see, e.g., [38]).

A geodesic triangle is a triangle with segments as sides.

Let $S_k$ denote the 2-dimensional complete simply-connected Riemannian manifold of constant curvature $k < 0$ (a Lobachevskii plane).

If $a, b, c$ belong to the metric space $(\mathcal{A}, \rho)$, let $\angle^* abc$ denote the angle of the geodesic triangle in $S_k$ of side-lengths $\rho(a, b), \rho(b, c), \rho(c, a)$, opposite to the side of length $\rho(c, a)$.

Here, $(\mathcal{A}, \rho)$ is called an intrinsic metric space if any two points admit a midpoint. An Alexandrov space is a complete intrinsic metric space $(\mathcal{A}, \rho)$ such that every point of $\mathcal{A}$ has a neighbourhood in which, for any four distinct points $a, b, c, d$, we have

$$\angle^* bac + \angle^* cad + \angle^* dab \leq 2 \pi.$$

This angle condition says that $(\mathcal{A}, \rho)$ has curvature bounded below by $k$ in the sense of A.D. Alexandrov (note the dependence of $\angle^*$ on $k$).

Berestovskii [2] proved that a complete intrinsic metric space $(\mathcal{A}, \rho)$ is an Alexandrov space if and only if every point of $\mathcal{A}$ has a neighbourhood in which any four points admit an isometric embedding in $S_{k'}$ for some $k' \geq k$. 
Several other characterizations of Alexandrov spaces are given by Burago, Gromov and Perelman in [9]. We use some basic concepts and results developed in [9] and further investigated by Perelman [23, 24]. So, for example, in any Alexandrov space a geodesic starting at \( x \) has a direction at \( x \) (see the definition below) and the angle between two geodesics exists. Moreover, geodesics do not branch. The interested reader should consult the recent book of Burago, Burago and Ivanov [8].

**Prerequisites**

If \( x \) is a point in the Alexandrov space \( \mathcal{A} \), the space \( \Sigma_x \) of directions at \( x \) is defined as the completion of the metric space \( \Sigma'_x \) consisting of classes of geodesics starting at \( x \), all the geodesics of the same class overlapping, and the distance being the angle between representatives (see [9], p. 23).

We shall identify the elements of \( \Sigma'_x \) with the directions of representatives. Every space of directions is known to be itself an Alexandrov space if \( \mathcal{A} \) has finite dimension [9].

Let \( xy \) denote the direction of the segment \( xy \) at \( x \).

We say that a set \( A \in \mathcal{A} \) has a direction \( \tau \) at its accumulation point \( x \) if, for \( y \rightarrow x \) with \( y \in A \setminus \{x\} \), \( xy \) converges in \( \Sigma_x \) to \( \tau \).

If \( pa, pb \) are geodesics, let \( \angle apb \) denote the angle between \( pa \) and \( pb \), i.e., the distance from \( pa \) to \( pb \).

We now recall three basic results from [9].

**Lemma 1** ([9], p. 6). If \( pa, pb, pc \) are geodesics in an Alexandrov space,

\[ \angle apb + \angle bpc + \angle cpa \leq 2\pi. \]

Here is a generalized form of Toponogov’s comparison theorem:

**Lemma 2** ([9], p. 7). For any geodesic triangle \( abc \) in an Alexandrov space,

\[ \angle^* abc \leq \angle abc, \quad \angle^* bca \leq \angle bca, \quad \angle^* cab \leq \angle cab. \]

**Lemma 3** ([9], p. 6). If in an Alexandrov space the segment \( piq_i \) converges to \( pq \) and the segment \( pr_i \) converges to \( pr \), then

\[ \angle qpr \leq \liminf_{i \to \infty} \angle q_i p_i r_i. \]

Let \( \mathcal{A} \) be an Alexandrov space and \( K \subset \mathcal{A} \) a closed set.

The **cut locus** \( C(K) \) of \( K \) is the set of all points \( x \in \mathcal{A} \) such that there is a segment from \( x \) to \( K \) not extendable as a segment beyond \( x \).

The **multijointed locus** of \( K \) is the set \( M(K) \) of all points \( x \in \mathcal{A} \) such that the distance from \( x \) to \( K \) is realized by at least two distinct segments from \( x \) to (not necessarily distinct) points in \( K \). The points of \( M(K) \) are said to be multijointed to \( K \) (see also [37]).
The ambiguous locus of $K$ is the set $A(K)$ of all points $x \in \mathcal{A}$ such that the distance from $x$ to $K$ is realized by at least two segments from $x$ to distinct points in $K$.

If $K$ consists of a single point $x$, we write $C(x)$ and $M(x)$ instead of $C(\{x\})$ and $M(\{x\})$, respectively.

A point in $\mathcal{A}$ is called an endpoint if it is not interior to any segment; let $E$ denote the set of all endpoints of $\mathcal{A}$.

Clearly, $A(K) \subset M(K) \subset C(K)$ and $E \subset C(K)$.

The open ball with centre $x \in \mathcal{A}$ and radius $\alpha$ is denoted by $B(x, \alpha)$.

The boundary of a compact convex set with interior points in $\mathbb{R}^n$ is called a convex hypersurface. The space of all convex hypersurfaces, equipped with the Pompeiu–Hausdorff metric, is a Baire space.

Most (or typical) means “all, except those in a set of first Baire category”.

**Lemma 4 ([31]).** On most convex hypersurfaces, most points are endpoints.

### Equidistant sets

Both the ambiguous locus and the cut locus are known to enjoy a bisecting property with respect to certain pairs of segments.

Let $a$, $b$ be (possibly coinciding) points in $\mathcal{A}$. The set $E(a, b)$ of all points joined by equally long but distinct segments with $a$ and $b$ will be called the equidistant set of $\{a, b\}$. Thus $E(a, b)$ coincides with $A(\{a, b\})$ if $a \neq b$, and with $M(\{a\})$ if $a = b$.

We shall prove a bisecting property of $E(a, b)$ which lies at the core of these phenomena.

Let $x \in E(a, b)$, and consider distinct segments $xa, xb \subset \mathcal{A}$ and $\alpha, \beta \subset \Sigma_x$ such that $\alpha = \overline{xa}$, $\beta = \overline{xb}$. Also, suppose $\alpha \beta$ is a segment in $\Sigma_x$.

A bisector of $xa, xb$ is a nondegenerate arc $\Gamma \subset E(a, b)$ starting at $x$ such that, for any $y \in \Gamma \setminus \{x\}$:

1) There is a segment $xy$ with $\overline{xy} \in \alpha \beta$.

2) There are segments $ya, yb$ such that $ya \rightarrow xa$ and $yb \rightarrow xb$ for $y \rightarrow x$.

This name is explained by the next theorem.

There may exist no bisector of $xa, xb$. This is the case, for example, if $a = b$ and $a, x$ are antipodal on the standard sphere, because then $E(a, a)$ is a single point and cannot include any nondegenerate arc. But we may also have no bisector if $a \neq b$: consider a smooth convex (2-dimensional) surface with precisely two segments from $x$ to $a$, say $\sigma_1, \sigma_2$, and precisely one segment $\sigma_3$ from $x$ to $b$ such that, for $s_i \in \text{int}\sigma_i$,

$$\angle s_1xs_2 + \angle s_2xs_3 = \angle s_1xs_3 < \pi.$$  

Then $\sigma_1$ and $\sigma_3$ admit no bisector, because condition 2 is not satisfied. In this case, however, $\sigma_1, \sigma_2$, as well as $\sigma_2, \sigma_3$, would admit a bisector each.
Theorem 1. Any bisector of \( xa, xb \) has a direction at \( x \), namely the midpoint of \( \alpha \beta \), where \( \alpha = \overline{xa} \) and \( \beta = \overline{xb} \).

Proof. Let \( \Gamma \) be a bisector of \( xa, xb \). If \( y \in \Gamma \setminus \{x\} \) converges to \( x \) then \( \angle^*xay \to 0 \), which implies \( \angle^*axy + \angle^*ayx \to \pi \). Analogously, \( \angle^*bxy + \angle^*byx \to \pi \). Hence

\[
\angle^*axy + \angle^*bxy + \angle^*ayx + \angle^*byx \to 2\pi.
\]

By Lemma 2,

\[
\liminf_{y \to x} (\angle axb + \angle ayx + \angle byx) \geq 2\pi.
\]

Since \( \overline{xy} \in \alpha \beta \),

\[
\angle axb = \angle axy + \angle bxy.
\]

Hence

\[
\liminf_{y \to x} (\angle axb + \angle ayx + \angle byx) \geq 2\pi.
\]

Suppose

\[
\limsup_{y \to x} (\angle axb + \angle ayx + \angle byx) > 2\pi.
\]

By Lemma 3,

\[
\liminf_{y \to x} \angle ayb \geq \angle axb.
\]

Then

\[
\limsup_{y \to x} (\angle ayb + \angle ayx + \angle byx) > 2\pi,
\]

which contradicts the inequality

\[
\angle ayb + \angle ayx + \angle byx \leq 2\pi,
\]

which holds, by Lemma 1, for all \( y \). Hence

\[
\angle axb + \angle ayx + \angle byx \to 2\pi.
\]

Using Lemma 2 again, we get \( \angle axy - \angle^*axy \to 0 \), \( \angle bxy - \angle^*bxy \to 0 \), \( \angle ayx - \angle^*ayx \to 0 \), \( \angle byx - \angle^*byx \to 0 \). But \( \angle^*axy = \angle^*bxy \); therefore

\[
\angle axy - \angle bxy \to 0
\]

and

\[
\lim_{y \to x} \angle axy = \lim_{y \to x} \angle bxy = \frac{1}{2} \angle axb.
\]

Thus, the proof is finished.
Multijointed loci in Alexandrov spaces: the arbitrary case

A set \( A \) in a metric space \((\mathcal{M}, \rho)\) is said to be porous at \( x \in \mathcal{M} \) if there is a number \( \alpha > 0 \) such that every ball centered at \( x \) contains a ball \( B(y, \alpha \rho(x, y)) \) disjoint from \( A \). It is called porous if it is porous at all its points, and \( \sigma \)-porous if it is a countable union of porous sets.

The main result of this section establishes the \( \sigma \)-porosity of the multi-jointed locus of any closed set in an arbitrary Alexandrov space \( \mathcal{A} \). This extends results of Gruber \cite{12, 13} and the author \cite{35}.

Otsu and Shioya \cite{22} have shown that the cut locus \( C(x) \) of a point \( x \) in an \( n \)-dimensional Alexandrov space \( A \) has \( n \)-dimensional Hausdorff measure 0. However, we cannot expect \( C(x) \) to be \( \sigma \)-porous because, by Lemma 4, there are Alexandrov spaces in which \( C(x) \) is residual. About the multijointed locus, Otsu and Shioya proved the stronger assertion \( \dim M(x) \leq n - 1 \) \cite{22}.

Also, Shiohama and Tanaka described in detail, for \( n = 2 \), the structure of \( C(K) \) and \( M(K) \) for any compact set \( K \) \cite{27}. According to them, \( M(K) \) is a countable union of Jordan arcs.

Unfortunately, the Hausdorff dimension being at most \( n - 1 \) does not imply \( \sigma \)-porosity (see Zajícek \cite{30}), nor are Jordan arcs necessarily \( \sigma \)-porous (see Foran \cite{11}).

Lemma 5. Let \( \mathcal{A} \) be an Alexandrov space and let \( A \subset \mathcal{A} \) be such that, for some points \( x, x' \) in \( \mathcal{A} \) and some positive number \( \varepsilon \), all points \( z \) in a neighbourhood of \( x \) that satisfy \( \angle^* zxx' \leq \varepsilon \) do not belong to \( A \). Then \( A \) is porous at \( x \).

Proof. Let \( N \) be the neighbourhood of \( x \) considered in the statement. It suffices to take \( \alpha = \sin \varepsilon \) in the definition of porosity. Then, for any segment \( xx' \) and any point \( y \in xx' \),

\[
z \in B(y, \alpha \rho(x, y)) \cap N \implies \sin \angle^* zxy \leq \alpha \implies \angle^* zxx' \leq \varepsilon.
\]

Hence \( z \in N \setminus A \), and \( A \) is porous at \( x \).

Theorem 2. In an Alexandrov space, the multijointed locus of any closed set is \( \sigma \)-porous.

Proof. Let \( \mathcal{A} \) be an Alexandrov space and let \( K \subset \mathcal{A} \) be closed. For any \( m \in \mathbb{N} \), let \( M_m \subset M(K) \) be the set of all points \( u \) joined with \( K \) by (at least) two segments making an angle at \( x \) not less than \( m^{-1} \). We have

\[
M(K) = \bigcup_{n=1}^{\infty} M_m,
\]

and we only have to prove that \( M_m \) is porous for each \( m \in \mathbb{N} \).

Consider \( u, v \in M_m \) and let \( a, b \) be interior points of the two segments from \( u \) to \( K \) under consideration. If simultaneously

\[
\angle awu > \pi - (2m)^{-1} \quad \text{and} \quad \angle buv > \pi - (2m)^{-1},
\]

then \( M_m \) is \( \sigma \)-porous.
then
\[ \angle auv + \angle buv + \angle aub > 2\pi, \]
which contradicts Lemma 1. Thus, there exists a segment \( ux \) from \( u \) to \( K \)
such that, in the geodesic triangle \( xuv \),
\[ \angle xuv \leq \pi - (2m)^{-1}. \]
By Lemma 2, \( \angle^* xuv \leq \pi - (2m)^{-1} \) too.
Now, keep \( v \) fixed and let \( u \in M_m \) converge to \( v \). Then \( \angle^* uxv \to 0 \).
But this yields \( \angle^* xvu > (3m)^{-1} \) for all \( u \in M_m \) in some neighbourhood
of \( v \). Then, by Lemma 5, \( M_m \) is porous at \( v \). Thus, \( M_m \) is porous, \( M(K) \)
is \( \sigma \)-porous, and the proof is finished.

For the special case of a point \( x \) on a convex surface, the \( \sigma \)-porosity of
\( M(x) \) was proved in [35] (see also Gruber [12], [13]).

Ambiguous and multijoined loci in Alexandrov spaces:
the generic case

In this section we consider the ambiguous locus in typical cases. We investigate
typical compact sets or take the ambient space to be typical. Let \( \mathcal{K}(\mathcal{A}) \)
be the space of all compact sets in the Alexandrov space \((\mathcal{A}, \rho)\), endowed
with its usual Pompeiu–Hausdorff metric \( h \), based on \( \rho \).
The next theorem generalizes Theorem 1 from [34] considerably. This
result of [34] has already been repeatedly strengthened, for example in [3],
[4], [5], [6], [41], [7].

**Theorem 3.** For most compact sets in a separable Alexandrov space of di-
mension at least 2, the ambiguous locus is dense.

**Proof.** Let \( B(x_0, \varepsilon) \) be an open ball in the Alexandrov space \( \mathcal{A} \). We prove
that the compact sets \( K \subset \mathcal{A} \) for which the ambiguous locus \( A(K) \) does not
meet \( B(x_0, \varepsilon) \) form a nowhere dense set.
Indeed, in any open set \( \mathcal{O} \subset \mathcal{K}(\mathcal{A}) \), there exists a compact set \( K \) avoiding
\( x_0 \). Let \( y_0 \in K \) be a point closest to \( x_0 \) and take \( y_1 \) different from \( y_0 \) on a
segment \( \sigma \) from \( x_0 \) to \( y_0 \), so that \( K \cup \{y_1\} \in \mathcal{O} \). There is a whole ball \( B(y_1, \eta) \)
disjoint from \( K \) such that for any finite set \( F \subset B(y_1, \eta) \), \( K \cup F \in \mathcal{O} \).
Since \( \dim \mathcal{A} \geq 2 \), we can choose \( y_2 \in B(y_1, \eta) \setminus \sigma \).
Consider the point \( y_3 \in \sigma \) with \( \rho(x_0, y_2) = \rho(x_0, y_3) \). Then \( y_3 \in B(y_1, \eta) \)
too. Let \( \sigma_2, \sigma_3 \) be segments from \( x_0 \) to \( y_2, y_3 \) respectively. Choose points
\( x_2 \in \sigma_2, x_3 \in \sigma_3 \) such that \( \rho(x, x_2) = \rho(x, x_3) \) \( < \varepsilon \). Clearly, \( \rho(x_2, y_3) >
\rho(x_2, y_2), \rho(x_3, y_2) > \rho(x_3, y_3) \).
Let
\[ \nu < \min \{ \rho(x_2, y_3) - \rho(x_2, y_2), \rho(x_3, y_2) - \rho(x_3, y_3) \}. \]
If \( h(K', K \cup \{y_2, y_3\}) < \nu/2 \) in \( K(A) \), then \( K' \) meets both \( B(y_1, \nu/2) \) and \( B(y_2, \nu/2) \). Therefore, for \( i = 2, 3 \), the point of \( K' \) closest to \( x_i \) lies in \( B(y_i, \nu/2) \). The function

\[
f(x) = d(x, K' \cap B(y_2, \nu/2)) - d(x, K' \cap B(y_3, \nu/2)),
\]

where \( d(x, M) \) means the infimum of \( \rho(x, y) \) for \( y \in M \), is continuous, \( f(x_2) < 0 \) and \( f(x_3) > 0 \). Thus there is a point \( x \in x_2x_0 \cup x_0x_3 \) with \( f(x) = 0 \); that is, \( x \in A(K') \).

Hence the set \( K_{m,n} \subseteq K(A) \) of all compact sets \( K \) for which \( A(K) \cap B(x_m, 1/n) = \emptyset \) is nowhere dense. Since \( A \) is separable, \( \{x_m\}_{m=1}^\infty \) can be chosen to be dense in \( A \), and then the set \( \bigcup_{m=1}^\infty K_{m,n} \) of all compact sets with nondense ambiguous locus is of first Baire category.

For an interesting analog of Theorem 3 concerning convex hypersurfaces instead of compact sets, see [40].

There are Alexandrov spaces in which the multijoined locus is dense for more compact sets than “just” those of a second Baire family. More precisely, in these spaces, the multijoined locus of any element of an open subset of \( K(A) \) is dense in its complement.

The following result strengthens in several directions Theorem 2 in [35] and C. Vălcu’s second assertion in the first theorem of [28].

**Theorem 4.** Let the compact Alexandrov space \( A \) be an \( n \)-dimensional topological manifold (\( n \geq 2 \), finite), and \( A \subseteq A \). Assume that the set of endpoints of \( A \) is dense in \( A \) (with respect to its relative topology) and \( K \) is a closed subset of a union \( B \) of components of \( A \setminus A \). If \( B \) is not dense in \( A \) then the multijoined locus \( M(K) \) is dense in the interior of \( A \setminus B \).

**Proof.** Suppose there is an open set \( O \subseteq A \setminus B \) homeomorphic to \( \mathbb{R}^n \) and disjoint from \( M(K) \). Since every point \( y \in A \setminus M(K) \) is joined by precisely one segment \( \sigma_y \) to the closest point \( \pi_y \) of \( K \), the mapping \( y \mapsto \sigma_y \) is continuous in \( O \). The mapping \( y \mapsto \pi_y \) is continuous too.

If \( 0 \leq r \leq 1 \) and \( \phi(y, r) \) denotes the point of \( \sigma_y \) at distance \( r \rho(y, \pi_y) \) from \( y \), then \( \phi \) is continuous in both variables.

Moreover the function \( \psi_r \) defined by \( \psi_r(y) = \phi(y, r) \) is injective for \( 0 \leq r < 1 \), since geodesics do not branch and therefore \( y \neq y' \) implies \( \sigma_y \subset \sigma_{y'} \) or \( \sigma_y' \subset \sigma_y \) or \( \sigma_y \cap \sigma_y' = \{\pi_y\} \cap \{\pi_{y'}\} \). The inverse of \( \psi_r \), defined on \( \psi_r(O) \), is obviously continuous too.

Let \( y_0 \in O \). For some \( r \) distinct from 1, \( \psi_r(y_0) \) belongs to \( A \) and is different from \( y_0 \). We may consider \( O \) chosen such that \( \psi_r(O) \) lies in a neighbourhood of \( y_0 \) homeomorphic to \( \mathbb{R}^n \). Then \( \psi_r(O) \) is open (see, e.g., [14, Theorem 6-54]). Hence \( \psi_r(O) \cap A \) is nonempty and open in the relative topology of \( A \). Now choose the endpoint \( z \) in this set. Then \( z \) must belong to a segment \( \sigma_y \) with \( y \in O \), and is not an endpoint of \( \sigma_y \). This contradiction completes the proof.
Corollary 1. In a space $A$ as in Theorem 4, assume that the closed set $K$ is strictly included in the open set $O$, and that the set of endpoints of $A$ is dense in $O \setminus K$. Then $M(K)$ is dense in $A \setminus K$.

Example. Consider a smooth torus $T$ embedded in $\mathbb{R}^3$ and a point $p \in T$ of positive Gauss curvature. Near $p$ and outside $T$ take a convex hypersurface $C$ of the type described in Lemma 4.

If $C$ is small enough and close enough to $p$, each line segment $s$ joining a point of $C$ with a point of $T$ and not meeting the interiors (bounded components of the complements) of $C$ and $T$ has an endpoint on $T$ of positive Gauss curvature. The union of all line segments $s$ has a boundary $S$. Then the (topological) closure of
\[ C \cup S \cup T \setminus (C \cap S) \cup (S \cap T) \]
is a torus $T'$, the endpoints of which lie densely in $C \setminus S$. By Corollary 1, for any compact set $K \subset C$ disjoint from $S$, $M(K)$ is dense in $T' \setminus K$.

Theorem 5. On most convex hypersurfaces $S$, for any compact set $K \subset S$, the multijoined locus $M(K)$ is dense in $S \setminus K$.

This follows from Corollary 1 and Lemma 4.

By Theorems 3 and 5, we can say that every compact set on most convex hypersurfaces and most compact sets on every convex hypersurface have multijoined loci dense in their complements.

In a Baire metric space, “nearly all” means “all, except those in a $\sigma$-porous set” [32].

Lemma 6. On any convex hypersurface, nearly all compact sets are porous.

This follows from Theorem 2 in [33] applied to $\mathbb{R}^{n-1}$, together with the observation that the convex hypersurface can be tiled into finitely many pieces admitting bi-Lipschitz bijections to pieces of $\mathbb{R}^{n-1}$, while porosity and $\sigma$-porosity are invariant under bi-Lipschitz transformations.

Lemma 6 and Theorem 5 immediately imply the following result:

Theorem 6. On most convex hypersurfaces $S$, for nearly all compact sets the multijoined locus is dense on $S$.

Suggestion. Show that Theorem 6 is also true for the ambiguous locus instead of the multijoined locus, although Theorem 5 is not.

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ON THE CLASSIFICATION OF FINITE GROUPS ACTING ON HOMOLOGY 3-SPHERES

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In previous work we showed that the only finite nonabelian simple group acting by diffeomorphisms on a homology 3-sphere is the alternating or dodecahedral group $A_5$. Here we characterize finite nonsolvable groups that act on a homology 3-sphere preserving orientation. We find exactly the finite nonsolvable groups that act orthogonally on the 3-sphere, plus two families of groups for which we do not know at present if they really can act on a homology 3-sphere.

1. Introduction

A finite group acting freely on a homology $n$-sphere has periodic cohomology of period $n + 1$, and the groups with periodic cohomology have been characterized by Zassenhaus and Suzuki (see [AM, Theorem 6.15] or [W, Chapter 6.3]). Specializing to dimension three one gets a list of the possible finite groups which may act freely on a homology 3-sphere (see [Mn]). The complete classification of such groups remains open (the difficulty lies in the class of groups $Q(8n, k, l)$ defined below).

Here we are interested in (not necessarily free) actions of finite groups on homology 3-spheres. Such groups no longer have periodic cohomology. It has been shown in [Z2] that the only finite nonabelian simple group acting on a homology 3-sphere is the alternating group $A_5 \cong PSL(2, 5)$. Continuing work begun in [R] (see also [Z1]), the main result of the present paper is a characterization of the finite nonsolvable groups which act on homology 3-spheres. We find exactly the finite nonsolvable groups admitting orthogonal actions on the 3-sphere (subgroups of $SO(4)$), plus two additional classes of groups for which we cannot decide at present if they can really act on a homology 3-sphere. We remark that a similar analysis should be possible also for the case of solvable groups, but the list of such groups will be inevitably much longer and more technical.

In order to state our results we introduce some notation.

By $[Mn]$, any finite group acting freely on a homology $n$-sphere has at most one involution, which consequently belongs to the center of the group; in the following, we denote by $Z \cong \mathbb{Z}_2$ the subgroup generated by such an
involution. If \( G_1 \) and \( G_2 \) are two such groups, we denote by

\[
G_1 \times \mathbb{Z} G_2
\]

the central product of \( G_1 \) and \( G_2 \), which is the product of the two groups with the two central involutions identified (the quotient of the product by the order-two subgroup generated by (involution of \( G_1 \), involution of \( G_2 \)); see [S1, p. 137]). Thus \( G_1 \) and \( G_2 \) commute elementwise and \( G_1 \cap G_2 = \mathbb{Z} \).

We denote by \( \mathbb{D}_{4n}^*, A_4^*, S_4^* \) and \( A_5^* \) the binary dihedral, tetrahedral, octahedral and dodecahedral group (of orders 4, 24, 48 and 120, respectively). These are the preimages under the surjection of Lie groups \( S^3 \rightarrow SO(3) \) of the dihedral group \( \mathbb{D}_{2n} \), the tetrahedral group \( A_4 \), the octahedral group \( S_4 \) and the dodecahedral group \( A_5 \). Together with the cyclic groups \( \mathbb{Z}_n \), these are exactly the finite subgroups of the orthogonal group \( SO(3) \).

Following [Mn], for relatively coprime positive integers \( 8n, k \) and \( l \), let \( Q(8n, k, l) \) denote the group with presentation

\[
\langle x, y, z \mid x^2 = (xy)^2 = y^{2n}, z^{kl} = 1, xzx^{-1} = z^r, yzy^{-1} = z^{-1} \rangle,
\]

where \( r \equiv -1 \mod k \) and \( r \equiv +1 \mod l \). Then \( Q(8n, k, l) \) is an extension with normal subgroup \( \mathbb{Z}_k \times \mathbb{Z}_l \cong \mathbb{Z}_{kl} \) and factor group the binary dihedral group \( \mathbb{D}_{8n}^* \cong Q(8n, 1, 1) \). Among the groups \( Q(8n, k, l) \) are the only candidates of finite groups not admitting free orthogonal actions on the 3-sphere but possibly admitting nonorthogonal free actions. Some of these groups act freely on homology 3-spheres, but it is not known if any of them can act on \( S^3 \) (see Section 2).

**Theorem.** Let \( G \) be a finite nonsolvable group of orientation-preserving diffeomorphisms of a homology 3-sphere. Then \( G \) is isomorphic to one of the following groups:

(i) \( A_5 \) or \( A_5 \times \mathbb{Z}_2 \).
(ii) \( A_5^* \times Z A_5^*, A_5^* \times Z S_4^*, A_5^* \times Z A_4^*, A_5^* \times Z \mathbb{D}_{4n}^* \) or \( A_5^* \times Z \mathbb{Z}_{2n} \).
(iii) \( A_5^* \times Z Q(8n, k, l) \), for relatively coprime integers \( 8n, k \) and \( l \), with \( n \) odd and \( n > k > l \geq 1 \).
(iv) \( A_5^* \times Z (\mathbb{D}_{4n}^* \times \mathbb{Z}_k) \), with \( n \) odd and \( k > 1 \) coprime to \( 4n \).

In case (i), each involution of \( A_5 \) has nonempty connected fixed-point set and in all other cases, each factor of the central products acts freely.

The groups of type (i) and (ii) are exactly the finite nonsolvable groups that admit orientation-preserving orthogonal actions on the 3-sphere.

Considering \( S^3 \) as the group of unit quaternions, we have the surjection of Lie groups \( S^3 \times S^3 \rightarrow SO(4) \) induced by left and right multiplication of \( S^3 \). The kernel of this surjection is the group \( \mathbb{Z} \) of order two generated by \((-1, -1)\), so \( SO(4) \) is isomorphic to the central product \( S^3 \times Z S^3 \). The finite subgroups of \( S^3 \) are exactly the cyclic groups \( \mathbb{Z}_n \) and the binary polyhedral groups \( \mathbb{D}_{4n}, A_4^*, S_4^* \) and \( A_5^* \). Thus all groups of type (ii) in the theorem act...
orthogonally on the 3-sphere. (See [DV] for a list of all finite groups acting orthogonally on $S^3$.)

The group $A^*_5 \times Z A^*_5$ occurs as the symmetry group of the 4-dimensional regular 120-cell whose boundary is the 3-sphere with a regular tessellation by 120 regular spherical dodecahedra with dihedral angles $2\pi/3$. The characteristic cell (simplex) of the regular spherical $2\pi/3$-dodecahedron is the Coxeter tetrahedron of type $[3,3,5]$, so $A^*_5 \times Z A^*_5$ is the orientation-preserving subgroup of index two in the corresponding Coxeter group (generated by the reflections in the faces of the tetrahedron). Similarly, an action of $A_5$ on $S^3$ comes from the orientation-preserving symmetry group of the regular 4-simplex, which induces a regular tessellation of $S^3$ by five regular $2\pi/4$-tetrahedra having the Coxeter tetrahedron of type $[3,3,3]$ as their characteristic cell. There is another action of $A_5$ on $S^3$, obtained by doubling the standard action of $A_5$ on the 3-ball. See [Du] for a geometric description of the quotient orbifolds of the finite subgroups of SO(4) occurring in the theorem.

2. Preliminaries

Any finite group $G$ acting freely on the 3-sphere or on a homology 3-sphere has cohomological period four and at most one involution. By [Mn], the groups of period four and with at most one involution are exactly the following:

2.1. The groups $1, D^*_4, A^*_4, S^*_4$ and $A^*_5$.

2.2. The split metacyclic groups $D_{2^k(2n+1)}$ with presentation

$$\langle x, y \mid x^{2^k} = 1, y^{2n+1} = 1, xyx^{-1} = y^{-1} \rangle,$$

where $k \geq 2$ and $n \geq 1$. The group $D_{4(2n+1)}$ is isomorphic to $D^*_4$.

2.3. The groups $P'_{8,3^k}$ with presentation

$$\langle x, y, z \mid x^2 = (xy)^2 = y^2, zxz^{-1} = y, zyz^{-1} = xy, z^{3^k} = 1 \rangle,$$

where $k \geq 1$; these groups are extensions with normal subgroup the quaternion group $Q_8$ of order eight (generated by $x$ and $y$) and factor group the cyclic group of order $3^k$. The group $P'_{24}$ is isomorphic to $A^*_4$.

2.4. The groups $Q(8n,k,l)$, for relatively coprime integers $8n, k$ and $l$ and either:

(a) $n$ odd and $n > k > l \geq 1$, or
(b) $n \geq 2$ even and $k > l \geq 1$. 
2.5. The groups $P_{4r}^r$, with $r \geq 3$ odd, which are extensions with normal subgroup $\mathbb{Z}_r$ and factor group $S_4^*$ such that the 3-Sylow subgroup is cyclic, and such that the commutator subgroup $A_4^*$ of $S_4^*$ acts trivially on $\mathbb{Z}_r$ and the remaining elements by -identity. See [L, p. 195] for presentations of these groups.

2.6. The product of any of these groups with a cyclic group of relatively prime order.

The groups of types 2.1–2.3 and their products with a cyclic group of relatively prime order are exactly the groups that act orthogonally and freely on $S^3$. It has been shown in [L] that the groups of types 2.4(b) and 2.5 do not act freely on a homology 3-sphere. The situation for the groups of type 2.4(a) is not completely understood: some of them act freely on homology 3-spheres, some do not, and it doesn’t seem to be known at present if any of them can act on the 3-sphere (see [Mg]; see also [K, problem 3.37, p. 173]: in contrast to the claim in the updated version of the problem list the classification of the finite groups acting on the 3-sphere remains open, however).

We collect some other results needed for the proof of the main theorem. For a proof of Proposition 1, see [B, Theorems 7.9 and 8.1], and [RZ, Lemma 3] for part (c).

**Proposition 1.**

(a) The fixed-point set of a diffeomorphism of prime period $p$ of a $\mathbb{Z}_p$-homology 3-sphere is either empty or connected.

(b) For a prime $p$, the group $\mathbb{Z}_p \times \mathbb{Z}_p$ does not act freely on a $\mathbb{Z}_p$-homology 3-sphere.

(c) For a prime $p$, let $A \cong \mathbb{Z}_p \times \mathbb{Z}_p$ be a finite group of orientation-preserving diffeomorphisms of a $\mathbb{Z}_p$-homology 3-sphere. Then there are exactly two subgroups $\mathbb{Z}_p$ of $A$ with nonempty fixed-point sets (two disjoint circles) or, if $p = 2$, all three involutions in $A$ may have nonempty fixed-point sets which are three circles intersecting in exactly two points (so $A$ has two global fixed points).

**Proposition 2.** Let $G$ be a finite group of orientation-preserving diffeomorphisms of a closed orientable 3-manifold. Suppose that $G$ contains an element $h$ with nonempty connected fixed-point set. Then the normalizer $N_{Gh}$ of the subgroup generated by $h$ in $G$ is isomorphic to a subgroup of a semidirect product $\mathbb{Z}_2 \ltimes (\mathbb{Z}_a \times \mathbb{Z}_b)$, for some nonnegative integers $a$ and $b$, where $\mathbb{Z}_2$ acts on the normal subgroup $\mathbb{Z}_a \times \mathbb{Z}_b$ by sending each element to its inverse.

**Proof.** The fixed-point set of $h$ is a simple closed curve $K$ invariant under the action of $N_{Gh}$. By a result of Newman [N] (see also [Dr]), a periodic transformation of a connected manifold that is the identity on an open subset
is the identity. Thus the action of an element of $N_G h$ is determined by its action in a regular neighbourhood of $K$ where it is a standard action on a solid torus. Every element of $N_G h$ restricts to a reflection (strong inversion) or to a (possibly trivial) rotation on $K$. The subgroup of rotations is abelian and has index one or two in $N_G h$. It has a cyclic subgroup (the elements acting trivially on $K$), with cyclic quotient group, so it is abelian of rank at most two.

□

Proposition 3.

(a) The only finite abelian group of rank $\geq 3$ that acts on a homology 3-sphere preserving orientation is the elementary abelian 2-group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

(b) Let $H$ be a metacyclic group, with normal subgroup $\mathbb{Z}_p$ and factor group $\mathbb{Z}_q$, for a prime number $p$ and an integer $q \geq 2$. If $H$ acts by orientation-preserving diffeomorphisms on a homology 3-sphere, any generator of $\mathbb{Z}_q$ acts as $\pm 1$ identity on $\mathbb{Z}_p$.

Proof. (a) An abelian group $A$ of rank three has subgroups $\mathbb{Z}_p \times \mathbb{Z}_p$, for some prime $p \geq 2$. By Proposition 1, some element $h$ of order $p$ has nonempty connected fixed-point set, so $A$ is a semidirect product of the type described in Proposition 2. The only abelian group of rank three of such type is the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

(b) Follows from [Z2, Proof of Proposition 1].

□

3. Proof of the theorem

A finite group $F$ is quasisimple if it is perfect (the abelianized group is trivial) and the factor group $F/Z(F)$ of $F$ by its center $Z(F)$ is a nonabelian simple group. A group $E$ is semisimple if it is perfect and the factor group $E/Z(E)$ is a direct product of nonabelian simple groups. A semisimple group $E$ is a central product of quasisimple groups, which are uniquely determined. Any finite group $G$ has a unique maximal semisimple subgroup $E = E(G)$ (see [S2, Chapter 6.6]).

Let $G$ be a finite group of orientation-preserving diffeomorphisms of a homology 3-sphere, and $E$ its maximal semisimple subgroup. If $E$ is trivial it is shown in [R, Section 4, case (a)] that $G$ is solvable. We will assume in the following that $E$ is nontrivial. It is shown in [Z2] that the only nonabelian simple group acting on a homology 3-sphere is the dodecahedral group $A_5$. Now it follows from [Z1, Proposition 3] that the maximal semisimple subgroup $E$ of $G$ is of one of the following three types:

1. $E \cong A_5$, and every involution in $A_5$ has nonempty connected fixed-point set.
2. $E \cong A_5^r$, and $E$ acts freely.
3. $E \cong A_5^r \times Z A_5^r$, and each factor $A_5^r$ acts freely.
Since $E$ is normal in $G$ its centralizer $C_G E$ is also normal. Let $C$ be the subgroup of $G$ generated by $E$ and $C_G E$. Then $C$ is normal in $G$, and the factor group $G/C$ is a subgroup of the outer automorphism group

$$\text{Out } E = \text{Aut } E / \text{Inn } E$$

of $E$. The intersection of $E$ and $C_G E$ is the center $Z$ of $E$, which implies

$$C \cong E \times Z C_G E.$$}

Because the normalizer in $C$ of any element in $C_G E$ contains $E$ and hence is nonsolvable, by Propositions 1(a) and 2 the group $C_G E$ acts freely.

We now consider separately the three cases above.

**3.1.** Suppose first that $E \cong A_5$, and that every involution in $A_5$ has nonempty connected fixed-point set.

Then $E$ contains a subgroup $A \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ (the Sylow 2-subgroup of $A_5$). Let $h$ be a nontrivial involution in $A$. As all three involutions in $A$ have nonempty fixed-point set, by Proposition 1(c) the two other involutions in $A$ act as reflections (strong inversions) on the fixed-point set of $h$. The centralizer $C_G E$ of $E$ in $G$ is contained in the centralizer $C_A A$ of $A$ in $G$, which in turn is contained in the centralizer $C_G h$ of $h$ in $G$. Now Propositions 2 and 3 imply that $C_G A$, and hence also $C_G E$, is an elementary abelian 2-group of rank at most three. Then $C$ is isomorphic to $A_5$ or to $A_5 \times \mathbb{Z}_2$.

The outer automorphism group of $A_5$ has order two and is generated, modulo inner automorphisms, by conjugation of $A_5$ with any odd permutation in the symmetric group $S_5$. Thus $C$ has index one or two in $G$. In the symmetric group $S_5$ the equation $xyx^{-1} = y^2$ holds for the cycles $x = (4532)$ of order four and $y = (12345)$ of order five. Suppose that $C$ has index two in $G$. Then there is an element in $G$ that induces by conjugation the same automorphism of $A_5$ as the cycle $x$, which contradicts Proposition 3. Thus $G = C$, and $G$ is isomorphic to $A_5$ or $A_5 \times \mathbb{Z}_2$.

**3.2.** Suppose that $E \cong A_5^*$. Since the outer automorphism group of $A_5^*$ has order two, $C \cong A_5^* \times Z C_G E$ has index one or two in $G$. Representing the elements of $A_5^* \cong \text{SL}(2,5)$ by $2 \times 2$-matrices of determinant one, the unique nontrivial outer automorphism is given by conjugation with the matrix of order four $A := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The equation $ABA^{-1} = B^2$ holds for the matrix $B := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ representing an element of order five in $\text{SL}(2,5)$.

Supposing that $C$ has index two in $G$ we get again a contradiction to Proposition 3. Thus $G = C \cong A_5^* \times Z C_G E$.

Recall that $C_G E$ acts freely, hence $C_G E$ is isomorphic to one of the groups in the list 2.1–2.6 (actually, it cannot be isomorphic to $A_5^*$ because otherwise $E$ would not be the maximal semisimple subgroup of $G$; for the following arguments this makes no difference, however).

We denote by $Q_8 = \{ \pm 1, \pm i, \pm j, \pm k \}$ the quaternion group of order eight (isomorphic to $D_8^*$); the center $Z \cong \mathbb{Z}_2$ of $Q_8$ is generated by $-1$. 
Lemma. For a positive integer $m > 1$, the groups $Q_8 \times Z Q_8 \times Z_m$ and $Q_8 \times Z Q_8 \times Z_{2m}$ do not act on a homology 3-sphere preserving orientation.

Proof. The central product $Q_8 \times Z Q_8$ has the subgroup $\{ (\pm 1, 1), (\pm i, i), (\pm j, j), (\pm k, k) \}$ isomorphic to $Z_2 \times Z_2 \times Z_2$. Now the lemma follows from Proposition 3(a).

Using the lemma we shall exclude various types of groups 2.1–2.6.

(i) Suppose that $CGE$ is isomorphic to a group $P_{8,3k}^r$ of type 2.3. This contains a subgroup $Q_8 \times Z_{Q_8} \times Z_{3k-1}$. The Sylow 2-subgroup of $A_5^*$ is the quaternion group $Q_8 \cong D_8^*$, hence $G \cong A_5^* \times Z QGE$ contains a subgroup $Q_8 \times Z Q_8 \times Z_{3k-1}$. By the lemma this is possible only for $k = 1$, so $CGE$ is isomorphic to the binary tetrahedral group $A_4 \cong P_{24}^r$.

(ii) Suppose that $CGE$ is isomorphic to a group $D_{2k(2n+1)}$ of type 2.2. Then $G \cong A_4^* \times Z D_{2k(2n+1)}$ contains a subgroup $Q_8 \times Z_4$ where $Z_4$ is generated by $u := x^{2k-2}$ (see the presentation of $D_{2k(2n+1)}$ in 2.2). Suppose $k > 2$. The subgroup of $G$ containing the elements $(1, 1)$ and $(i, u)$ is isomorphic to $Z_2 \times Z_2$ (note that $u^2$ is the unique central involution in $D_{2k(2n+1)}$ that is identified with $-1$ in $Q_8$). By Proposition 2, the central involution $(-1, 1) = (1, u^2)$ in $G$ has empty fixed-point set, so by Proposition 1 we can assume that the involution $h := (i, u)$ has nonempty connected fixed-point set. The centralizer of $h$ in $G$ contains the generators $x$ and $y$ of $D_{2k(2n+1)}$. Then both $x$ and $y$ act as rotations along the circle of fixed points of $h$ (and not as reflections, see the proof of Proposition 2). This implies that $x$ and $y$ commute, which is a contradiction. Thus $k = 2$ and $D_{2k(2n+1)} = D_{4(2n+1)}$ is isomorphic to the binary dihedral group $D_{4(2n+1)}$.

(iii) Suppose that $CGE$ is isomorphic to a group $P_{48r}^r$ of type 2.5. As $CGE$ acts freely, it follows from [Mn, Lemma 2] that $r = 3^k$, and by [L, Corollary 4.17] the group $P_{48r}^r$ does not act freely on a homology 3-sphere, for $k \geq 1$.

A direct argument that, for $r > 1$, the group $G \cong A_4^* \times Z P_{48r}^r$ does not act on a homology 3-sphere is as follows: the group $P_{48r}^r$ has $S_4^*$ as a quotient; the preimage of $A_4^* \subset S_4^*$ in $P_{48r}^r$ is a group of type 2.3 and contains a subgroup $Q_8 \times Z_r$ (see part (i) of the proof). Then $G$ has a subgroup $Q_8 \times Z Q_8 \times Z_r$ and the lemma implies $r = 1$, so $CGE$ is isomorphic to the binary octahedral group $S_4 \cong P_{48}^r$.

By (i)–(iii), the remaining possibilities for $CGE$ are the groups of type 2.1, 2.4 and the product of any of these groups with a cyclic group of relatively prime order (type 2.6).

Suppose that $CGE$ is a product of one of the groups $D_{4n}^*$ with $n$ even, $A_4^*$, $S_4^*$, $A_5^*$ or $Q(8n, k, l)$ with a cyclic group $Z_m$. Then $CGE$ has a subgroup $Q_8 \times Z m$, so $G \cong A_5^* \times Z CGE$ has a subgroup $Q_8 \times Z Q_8 \times Z_m$ and the lemma implies $m = 1$. 

By [L, Corollary 4.15] the groups $Q(8n, k, l)$ with $n$ even do not act freely on a homology 3-sphere. This leaves for $G$ exactly the possibilities (ii), (iii) and (iv) of the theorem.

3.3. Finally, suppose that $E \cong \mathbb{A}_5^* \times \mathbb{Z} \mathbb{A}_5^*$. Then $E$ has a subgroup $Q_8 \times \mathbb{Z} Q_8$. It follows from the lemma that, for $C \cong E \times \mathbb{Z} C G E$, the group $C G E$ is equal to $\mathbb{Z}$. Thus $C = E \cong \mathbb{A}_5^* \times \mathbb{Z} \mathbb{A}_5^*$.

The outer automorphism group of $E \cong \mathbb{A}_5^* \times \mathbb{Z} \mathbb{A}_5^*$ is the dihedral group of order eight: considering centralizers of elements, each element $(x, 1)$ is mapped to an element of the form $(y, 1)$ or $(1, y)$, so each automorphism of $E$ either preserves each factor (and the subgroup generated by such automorphisms is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$) or exchanges them. It follows that $E$ has index at most eight in $G$.

It follows as in case 3.2 of the proof of the theorem that any nontrivial outer automorphism of $E$ induced by conjugation with an element of $G$ has to interchange the two factors of $E \cong \mathbb{A}_5^* \times \mathbb{Z} \mathbb{A}_5^*$; in particular, $E$ has index at most two in $G$. Up to inner automorphisms, an outer automorphism $\alpha$ of order two of $\mathbb{A}_5^* \times \mathbb{Z} \mathbb{A}_5^*$ and interchanging its two factors is of the form $\alpha(x, y) = (\beta(y), \beta(x))$, for an automorphism $\beta$ of $\mathbb{A}_5^*$.

Suppose that $\alpha$ is induced by conjugation by an element $g$ in $G$. We consider the subgroup $\mathbb{A}_5$ of $\mathbb{A}_5^* \times \mathbb{Z} \mathbb{A}_5^*$ consisting of all elements of the form $(x, x)$. If $\beta$ represents the unique nontrivial outer automorphism of $\mathbb{A}_5^*$ then conjugation by $g$ induces the unique nontrivial outer automorphism of $\mathbb{A}_5$. By the argument in case 3.1 of the proof of the theorem this is not possible, so we can assume $\beta$ is the trivial automorphism of $\mathbb{A}_5^*$. Then $\alpha^2$ is the trivial automorphism of $\mathbb{A}_5^* \times \mathbb{Z} \mathbb{A}_5^*$, and $g$ induces the trivial automorphism of $\mathbb{A}_5$. Proposition 2 implies that $g$ acts without fixed points (because its centralizer contains $\mathbb{A}_5$).

There are the two possibilities $g^2 = 1$ or $g^2 = z$ (where $z$ denotes the nontrivial element of $Z$). If $g^2 = 1$ then the subgroup of $G$ generated by $g$ and $z$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, and Proposition 1 implies that $g$ or $z$ has nonempty fixed-point set, which is not possible by Proposition 2. If $g^2 = z$ then $g$ has order four and $G$ contains a subgroup $\mathbb{A}_5 \times \mathbb{Z}_4$ and hence also a subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$, which is not possible by the lemma.

Thus $G = E$, and $g$ is isomorphic to $\mathbb{A}_5^* \times \mathbb{Z} \mathbb{A}_5^*$. This finishes the proof of the theorem.

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