Pacific Journal of Mathematics

A UNIFIED APPROACH TO UNIVERSAL INEQUALITIES FOR EIGENVALUES OF ELLIPTIC OPERATORS

Mark S. Ashbaugh and Lotfi Hermi

Volume 217 No. 2 December 2004

PACIFIC JOURNAL OF MATHEMATICS
Vol. 217, No. 2, 2004

A UNIFIED APPROACH TO UNIVERSAL INEQUALITIES FOR EIGENVALUES OF ELLIPTIC OPERATORS

Mark S. Ashbaugh and Lotfi Hermi

We present an abstract approach to universal inequalities for the discrete spectrum of a self-adjoint operator, based on commutator algebra, the Rayleigh–Ritz principle, and one set of "auxiliary" operators. The new proof unifies classical inequalities of Payne–Pólya–Weinberger, Hile–Protter, and H.C. Yang and provides a Yang type strengthening of Hook's bounds for various elliptic operators with Dirichlet boundary conditions. The proof avoids the introduction of the "free parameters" of many previous authors and relies on earlier works of Ashbaugh and Benguria, and, especially, Harrell (alone and with Michel), in addition to those of the other authors listed above. The Yang type inequality is proved to be stronger under general conditions on the operator and the auxiliary operators. This approach provides an alternative route to recent results obtained by Harrell and Stubbe.

1. Introduction

There has been much work dedicated to extending and strengthening the classical gap inequality of Payne, Pólya, and Weinberger [**28**], [**29**] (see also [**2**], [**4**], [**5**], [**6**], [**7**], [**34**]). This result states that

$$\lambda_{m+1} - \lambda_m \le \frac{4}{n} \frac{\sum_{i=1}^m \lambda_i}{m}, \tag{1.1}$$

where $0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots$ designate the eigenvalues of the membrane problem (multiplicities included)

$$\begin{aligned} -\Delta u &= \lambda u \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{1.2}$$

The set $\Omega \subset \mathbb{R}^n$ is a bounded domain and the Laplacian Δ is given by $\Delta \equiv \sum_{i=1}^n \partial^2/\partial x_i^2$.

On the strengthening side we find the work of Hile and Protter [**21**], who showed that

$$\frac{mn}{4} \le \sum_{i=1}^m \frac{\lambda_i}{\lambda_{m+1} - \lambda_i}. \tag{1.3}$$

We also find the work of H.C. Yang [**35**] (see also [**2**], [**3**], [**8**]), who showed that

$$\sum_{i=1}^{m}(\lambda_{m+1}-\lambda_i)^2 \leq \frac{4}{n}\sum_{i=1}^{m}\lambda_i(\lambda_{m+1}-\lambda_i). \tag{1.4}$$

Harrell and Stubbe [**20**] extended this inequality further to

$$\sum_{i=1}^{m}(\lambda_{m+1}-\lambda_i)^p \leq \frac{2p}{n}\sum_{i=1}^{m}\lambda_i(\lambda_{m+1}-\lambda_i)^{p-1} \quad \text{for } p \geq 2 \tag{1.5}$$

(inequality (14) in Thm. 9, p. 1805) and

$$\sum_{i=1}^{m}(\lambda_{m+1}-\lambda_i)^p \leq \frac{4}{n}\sum_{i=1}^{m}\lambda_i(\lambda_{m+1}-\lambda_i)^{p-1} \quad \text{for } p \leq 2 \tag{1.6}$$

(inequality (11) in Thm. 5, p. 1801). The Hile–Protter and H.C. Yang inequalities appear as special cases of this latter inequality for $p = 0$ and 2 respectively. It can be shown, however, that (1.6) for $p < 2$ is always weaker than (1.4) (i.e., the $p = 2$ case of (1.6)). In fact, the bounds (1.6) can be shown to improve monotonically with p (see [**10**]).

These inequalities are called *universal* because they do not involve domain-dependencies [**31**] (see also [**2**], [**3**], for example).

As extensions of the classical PPW and HP results we find applications of the methods to various geometric and physical situations. Cheng [**14**] produced the first estimates of this type for a compact hypersurface minimally immersed in $\mathbb{R}^{n+1}$. He also treated the case of an inhomogeneous membrane and subdomains of $\mathbb{S}^2$. P.C. Yang and S.-T. Yau [**36**] produced similar estimates for a hypersurface minimally immersed in $\mathbb{S}^n$. Li [**26**] treated the case of compact homogeneous spaces. Leung [**25**] corrected the constants in Yang and Yau's PPW type bound and produced an HP like version of their work in the spirit of Li's estimate in [**26**] (see also [**18**]).

It was Harrell and Davies (see [**16**]) who first realized that many of the original PPW arguments rely on facts involving operators, their commutators, and traces. In 1988, Harrell [**16**] first published a fully "algebraicized" version of the PPW argument. Then in 1993, using projections [**17**], he produced bounds on the eigenvalue gap for the Dirichlet problem for subdomains of a given Riemannian manifold in terms of their geometry. Harrell and Michel [**18**], [**19**] produced an algebraic inequality based on two sets of auxiliary operators and a set of spectral projections. They applied their formula to produce various geometric inequalities and bounds for partial differential operators. Their work improved earlier bounds of Harrell [**16**], [**17**]. They also significantly improved and simplified the bounds of Cheng [**14**] (see [**27**] where the results are displayed explicitly), Li [**26**], Yang and Yau [**36**], and Leung [**25**]. Their HP type inequalities appear in the "natural" form described above. At the heart of Harrell and Michel's improvement appears the exploitation of certain symmetry and commutation properties of the eigenfunctions and eigenvalues. Parallel considerations in the spirit of [**5**] (see also [**4**]) were used by the authors [**9**] to produce new domain-dependent versions of (1.1), (1.3), and (1.4). Ideas along these lines were also used by Lee [**24**] to produce HP type bounds for the eigenvalues of the Laplace–Beltrami operator on p-forms. These bounds extend and generalize the results of Cheng [**14**] and Yang and Yau [**36**]. They are *extrinsic*, meaning they depend on a curvature operator appearing in the Weitzenböck formula and a mean curvature vector field. The domains in Lee's work are not minimally immersed, but only immersed in $\mathbb{R}^{n+1}$ or $\mathbb{S}^n$ isometrically.

Harrell and Michel's approach is similar in spirit to that of Hook [**23**], though they used a somewhat different method of proof. Hook, for his part, generalized the original argument of Hile and Protter [**21**] in an abstract setting and was able to reproduce their result and improve on results of Hile and Yeh [**22**] in the context of the biharmonic operator with Dirichlet boundary conditions. He applied his abstract framework to various operators of mathematical physics and produced HP type bounds for them (Schrödinger operators with magnetic potential, second-order elliptic operators with constant coefficients, the Sturm–Liouville problem, the Lamé system). We have recently found a new proof [**12**] of his results that avoids Hile and Protter's "free parameters" and allows us to develop his conclusions in a fashion paralleling that of the present work.

In this article, we produce a set of algebraic inequalities from which the classical inequalities of Payne, Pólya, and Weinberger, Hile and Protter, and H.C. Yang described earlier follow.

A self-adjoint semibounded operator is given and we study its eigenvalues. A set of auxiliary symmetric operators is introduced. Various inequalities relating the eigenvalues to commutators and projections are proved.

Here, we use one set of auxiliary operators to produce many of Hook's results, and do without the projections of Harrell *et al.* We use solely the Rayleigh–Ritz principle and properties of commutators. Using this method we are able to improve on and generalize recent work by Harrell and Stubbe [**20**].

The approach we present here has a "unifying" aspect. The extensions of Hile–Protter and H.C. Yang will be proved to follow — save for the addition of a key idea — from the same set of principles as the original PPW inequality. New proofs for the HP and H.C. Yang inequalities, in their abstract setting, are provided. These inequalities can be seen as conditions that certain discriminants be nonpositive [**12**]. We also prove that both of Yang's inequalities (see [**2**], [**3**]) are stronger than both the PPW and HP results, thus supplying an argument left aside in [**35**]. An alternative proof of this latter result appears in [**3**] but our proof here is simpler and more direct.

Our work follows the spirit of [**2**] and [**3**]. In these works one of us (MSA) produced an argument partially based on the work of H.C. Yang [**35**], which does away with the "free parameters" of Hile–Protter [**21**] and Hook [**23**] (see also [**31**], [**32**]). Ashbaugh and Benguria produced the earliest "parameter-free" proof of the HP inequality in [**7**]. We recommend the article [**2**] where a discussion of the history is presented and an explicit version for the Dirichlet Laplacian of some of the arguments we generalize here is laid out (see also [**6**] for more results and conjectures).

In [**9**], we present versions of domain-dependent estimates of a related type. Then in [**10**] we develop the Harrell–Stubbe type inequalities given above in the spirit of this abstract formulation. In particular, we show that (1.5) is weaker than H.C. Yang's (1.4) if p is restricted to integer values $p > 2$. We also show that (1.6) is intermediate between the H. C. Yang and Hile–Protter inequalities and interpolates between them for $0 \leq p \leq 2$. In our work we adopt much of the notation of Hook in [**23**].

Our abstract formulation allows for Yang type and Harrell–Stubbe type improvements for the various physical and geometric problems described above. This is presented in [**11**], where we improve on Harrell and Michel's works [**18**], [**19**], and Hook's work [**23**]. In that article, we adopt the point of view of Bandle [**13**] to produce Yang type inequalities for eigenvalues of domains in $\mathbb{S}^2$ and $\mathbb{H}^2$ to improve results in [**14**], [**17**], [**18**], [**19**] by treating the eigenvalue problem in these space forms as inhomogeneous membrane problems.

2. The classical inequalities with one set of auxiliary operators

Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle\ ,\ \rangle$, $A : \mathcal{D} \subset \mathcal{H} \to \mathcal{H}$ a self-adjoint operator defined on a dense domain $\mathcal{D}$ that is bounded below and has a discrete spectrum $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$,

$$\{B_k : A(\mathcal{D}) \to \mathcal{H}\}_{k=1}^{N}$$

a collection of symmetric operators leaving $\mathcal{D}$ invariant, and $\{u_i\}_{i=1}^{\infty}$ the normalized eigenvectors of A, u_i corresponding to λ_i. We may further assume that $\{u_i\}_{i=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$. $[A, B]$ denotes the commutator of two operators defined by $[A, B] = AB - BA$, and $\|u\| = \sqrt{\langle u, u\rangle}$.

Let

$$\rho_i = \sum_{k=1}^{N} \big\langle [A, B_k]u_i, B_k u_i \big\rangle \tag{2.1}$$

and

$$\Lambda_i = \sum_{k=1}^{N} \big\|[A, B_k]u_i\big\|^2. \tag{2.2}$$

The following is motivated by the classical PPW, HP, and H.C. Yang inequalities. We will show that they spring from these inequalities and the general set-up of this theorem.

Theorem 2.1. *The eigenvalues λ_i of the operator A satisfy the inequalities*

$$\sum_{i=1}^{m} \rho_i \leq \frac{\sum_{i=1}^{m} \Lambda_i}{\lambda_{m+1} - \lambda_m}, \tag{2.3}$$

$$\sum_{i=1}^{m} \rho_i \leq \sum_{i=1}^{m} \frac{\Lambda_i}{\lambda_{m+1} - \lambda_i}, \tag{2.4}$$

and

$$\sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^2 \rho_i \leq \sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i) \Lambda_i. \tag{2.5}$$

The proof of this theorem will be given in Section 3.

Remark. Roughly speaking, the quantities Λ_i are analogues of the kinetic energy term corresponding to the eigenstate u_i. Since $\{u_i\}_{i=1}^{\infty}$ is complete, one can write, using Parseval's decomposition

$$[A, B_k] u_i = \sum_{j=1}^{\infty} b_{ij}^k u_j \tag{2.6}$$

where $b_{ij}^k = \big\langle [A, B_k] u_i, u_j \big\rangle$. This implies that

$$\big\| [A, B_k] u_i \big\|^2 = \sum_{j=1}^{\infty} \big| \langle [A, B_k] u_i, u_j \rangle \big|^2,$$

or yet

$$\Lambda_i = \sum_{k=1}^{N} \sum_{j=1}^{\infty} \big| \langle [A, B_k] u_i, u_j \rangle \big|^2. \tag{2.7}$$

When A is a Dirichlet Laplacian or a Schrödinger operator with magnetic potential, (2.7) reduces to Equation (5) on p. 1798 of [**20**] with a factor of 4, i.e., $\Lambda_i = 4T_i$, the kinetic energy term in Harrell and Stubbe's notation (the quantity $\big|\langle [A, B_k] u_i, u_j \rangle\big|^2$ is similar to T_{kij} in their notation). Identity (2.7) is exploited in [**10**] to produce an alternative route to some of the results of [**20**].

Lemma 2.2. *The quantity ρ_i can be written in the simpler form*

$$\rho_i = \frac{1}{2} \sum_{k=1}^{N} \big\langle [B_k, [A, B_k]] u_i, u_i \big\rangle. \tag{2.8}$$

Proof. For A, B as introduced above and u an eigenvector u_i of A, one has

$$\begin{aligned}\big\langle [B,[A,B]]u,u\big\rangle &= \big\langle B[A,B]u,u\big\rangle - \big\langle [A,B]Bu,u\big\rangle \\ &= \big\langle [A,B]u,Bu\big\rangle + \big\langle Bu,[A,B]u\big\rangle \\ &= 2\mathrm{Re}\,\big\langle [A,B]u,Bu\big\rangle \\ &= 2\,\big\langle [A,B]u,Bu\big\rangle,\end{aligned} \tag{2.9}$$

since

$$\begin{aligned}\big\langle [A,B]u,Bu\big\rangle &= \langle ABu,Bu\rangle - \langle BAu,Bu\rangle \\ &= \big\langle A(Bu),(Bu)\big\rangle - \lambda\langle Bu,Bu\rangle\end{aligned} \tag{2.10}$$

is clearly real by the self-adjointness of A and the fact that u is an eigenvector of A (in the above, we have taken, $Au = \lambda u$, and λ, as an eigenvalue of A, is necessarily real). Lemma 2.2 follows in view of (2.9) and the definition of ρ_i. □

Lemma 2.3. *Let $A = -\sum_{k=1}^N T_k^2$ where the T_k's are skew-symmetric with domains $\mathcal{D}(T_k)$ such that $T_k(\mathcal{D}) \subset \mathcal{D}(T_k)$ and $\mathcal{D}(T_k) \supset \mathcal{D}(A) = \mathcal{D}$. If $[T_\ell, B_k]u = \delta_{\ell k}u$, for a vector u, then $[A, B_k]u = -2T_k u$ and $\big[B_k, [A, B_k]\big]u = 2u$.*

Proof. By the formal commutator identity $[A, BC] = B[A, C] + [A, B]C$, one has

$$\begin{aligned}[A,B_k]u &= -\sum_{\ell=1}^N [T_\ell^2, B_k]u = \sum_{\ell=1}^N [B_k, T_\ell^2]u \\ &= \sum_{\ell=1}^N \big(T_\ell[B_k,T_\ell] + [B_k,T_\ell]T_\ell\big)u = -2\sum_{\ell=1}^N \delta_{\ell k}T_\ell u = -2T_k u.\end{aligned} \tag{2.11}$$

Therefore,

$$\big[B_k,[A,B_k]\big]u = -2\ [B_k,T_k]u = 2u, \tag{2.12}$$

and the desired result follows. □

We now conclude the generalization of the classical results.

Corollary 2.4. *If $A = -\sum_{k=1}^N T_k^2$, where the T_k's are skew-symmetric with domains as delineated above, and if $[T_\ell, B_k]u = \delta_{\ell k}u$ for u an eigenvector of A, then $\rho_i = N$, $\Lambda_i = 4\lambda_i$, and*

$$\lambda_{m+1} - \lambda_m \le \frac{4}{Nm}\sum_{i=1}^m \lambda_i, \tag{2.13}$$

$$\frac{Nm}{4} \le \sum_{i=1}^m \frac{\lambda_i}{\lambda_{m+1}-\lambda_i}, \tag{2.14}$$

and

$$\sum_{i=1}^{m}(\lambda_{m+1}-\lambda_i)^2 \le \frac{4}{N}\sum_{i=1}^{m}(\lambda_{m+1}-\lambda_i)\lambda_i. \tag{2.15}$$

If C is a symmetric operator and $\alpha \in \mathbb{R}$, we write $C \ge \alpha$ if $\langle Cu, u\rangle \ge \alpha\langle u, u\rangle$ for all vectors $u \in \mathcal{D}(C)$. We define $A \ge B$ for symmetric operators A and B which are bounded below if $\mathcal{D}(A) \subset \mathcal{D}(B)$ and $A - B \ge 0$ on $\mathcal{D}(A)$.

Theorem 2.5. *Suppose there exist γ, β such that*

$$0 < \gamma \le \big[B_k, [A, B_k]\big] \tag{2.16}$$

and

$$-\sum_{k=1}^{N}[B_k, A]^2 \le \beta A. \tag{2.17}$$

Then

$$\lambda_{m+1}-\lambda_m \le \frac{2\beta}{mN\gamma}\sum_{i=1}^{m}\lambda_i, \tag{2.18}$$

$$\frac{mN\gamma}{2\beta} \le \sum_{i=1}^{m}\frac{\lambda_i}{\lambda_{m+1}-\lambda_i}, \tag{2.19}$$

and

$$\sum_{i=1}^{m}(\lambda_{m+1}-\lambda_i)^2 \le \frac{2\beta}{N\gamma}\sum_{i=1}^{m}(\lambda_{m+1}-\lambda_i)\lambda_i. \tag{2.20}$$

Proof. By virtue of (2.16) and (2.17), $\rho_i \ge \frac{1}{2}N\gamma$ and $\Lambda_i \le \beta\lambda_i$. □

3. Proof of Theorem 2.1

Let ϕ be a trial function for λ_{m+1} in the Rayleigh–Ritz inequality. Then

$$\lambda_{m+1} \le \frac{\langle A\phi, \phi\rangle}{\langle \phi, \phi\rangle} \tag{3.1}$$

and

$$\langle \phi, u_j\rangle = 0 \tag{3.2}$$

for $j = 1, \dots, m$.

Let ϕ be given by

$$\phi_i = Bu_i - \sum_{j=1}^{m} a_{ij}u_j, \tag{3.3}$$

where B is one of the B_k's, $k = 1, \ldots, N$. Condition (3.2) makes $a_{ij} = \langle Bu_i, u_j \rangle$. Since B is symmetric, we have $a_{ji} = \overline{a_{ij}}$. Moreover,

$$\|\phi_i\|^2 = \langle Bu_i, \phi_i \rangle, \tag{3.4}$$

and

$$\begin{aligned}
\langle A\phi_i, \phi_i \rangle &= \left\langle ABu_i - \sum_{j=1}^{m} a_{ij}\lambda_j u_j, \phi_i \right\rangle = \langle ABu_i, \phi_i \rangle \\
&= \langle BAu_i, \phi_i \rangle + \big\langle [A,B]u_i, \phi_i \big\rangle = \lambda_i \langle Bu_i, \phi_i \rangle + \big\langle [A,B]u_i, \phi_i \big\rangle \\
&= \lambda_i \langle \phi_i, \phi_i \rangle + \langle [A,B]u_i, \phi_i \rangle.
\end{aligned} \tag{3.5}$$

Thus, (3.1) reduces to

$$\lambda_{m+1} - \lambda_i \leq \frac{\langle [A,B]u_i, \phi_i \rangle}{\langle \phi_i, \phi_i \rangle}. \tag{3.6}$$

Now

$$\big\langle [A,B]u_i, \phi_i \big\rangle = \big\langle [A,B]u_i, Bu_i \big\rangle - \sum_{j=1}^{m} \overline{a_{ij}} \big\langle [A,B]u_i, u_j \big\rangle. \tag{3.7}$$

Let $b_{ij} = \big\langle [A,B]u_i, u_j \big\rangle$. Then

$$\big\langle [A,B]u_i, \phi_i \big\rangle = \big\langle [A,B]u_i, Bu_i \big\rangle - \sum_{j=1}^{m} \overline{a_{ij}} b_{ij}. \tag{3.8}$$

We now observe that

$$b_{ij} = -\overline{b_{ji}} = (\lambda_j - \lambda_i) a_{ij}. \tag{3.9}$$

This is evident from

$$\begin{aligned}
b_{ij} &= \big\langle [A,B]u_i, u_j \big\rangle = \langle ABu_i, u_j \rangle - \langle BAu_i, u_j \rangle \\
&= \langle Bu_i, Au_j \rangle - \langle BAu_i, u_j \rangle = (\lambda_j - \lambda_i) \langle Bu_i, u_j \rangle = (\lambda_j - \lambda_i) a_{ij}.
\end{aligned} \tag{3.10}$$

Therefore,

$$\big\langle [A,B]u_i, \phi_i \big\rangle = \big\langle [A,B]u_i, Bu_i \big\rangle - \sum_{j=1}^{m} (\lambda_j - \lambda_i) |a_{ij}|^2. \tag{3.11}$$

Lemma 3.1 ("Optimal" Cauchy–Schwarz)**.** *With the notation and choice of ϕ_i as given above, we have*

$$\frac{\big\langle [A,B]u_i, \phi_i \big\rangle}{\langle \phi_i, \phi_i \rangle} \leq \frac{\big\| [A,B]u_i \big\|^2 - \sum_{j=1}^{m} |b_{ij}|^2}{\big\langle [A,B]u_i, \phi_i \big\rangle}. \tag{3.12}$$

Proof. Since $\langle u_j, \phi_i\rangle = 0$, for $1 \le i, j \le m$ and $\big\langle [A,B]u_i, \phi_i\big\rangle$ is real (by the self-adjointness of A; see, for example, (3.11) or (3.6)) we have

$$\begin{aligned}\big(\langle [A,B]u_i, \phi_i\rangle\big)^2 &= \left(\left\langle [A,B]u_i - \sum_{j=1}^m b_{ij}u_j, \phi_i\right\rangle\right)^2\\ &\le \left\|[A,B]u_i - \sum_{j=1}^m b_{ij}u_j\right\|^2 \|\phi_i\|^2\\ &= \left(\|[A,B]u_i\|^2 - \sum_{j=1}^m |b_{ij}|^2\right)\|\phi_i\|^2.\end{aligned}$$

Hence, noting that $\langle [A,B]u_i, \phi_i\rangle \ge 0$ (by (3.6), for example), we get the result. □

Remark. Written in more explicit terms, (3.10) reads

$$\big\langle [A,B]u_i, u_j\big\rangle = (\lambda_j - \lambda_i)\langle Bu_i, u_j\rangle. \tag{3.13}$$

Domain considerations aside, this gap formula is at the heart of many good estimates for eigenvalues [**16**], [**19**]. Hence the coefficient b_{ij} is "natural" in this context. It is the incorporation of the "counterterms" involving the b_{ij}'s that makes the use of the Cauchy–Schwarz inequality to obtain (3.12) "optimal". If these counterterms are dropped both the PPW and HP results are obtained (see (3.18) and (3.20) below). The orthogonality of the ϕ_i's to the u_j's is what allows us to include these counterterms. As remarked above in (2.6), the b_{ij}'s are precisely the components of $[A,B]u_i$ along the u_j. This choice of components is optimal for minimizing the norm of any expression of the form $[A,B]u_i - \sum_{j=1}^m c_j u_j$. This can be regarded as the key element that allows us to derive Yang's strengthened version of the classical inequalities, explaining our designation "optimal use of the Cauchy–Schwarz inequality" in connection with our method. This approach was first noted in preliminary form by Ashbaugh and Benguria [**8**]. Yang's derivation [**35**] is more circuitous, and in particular does not make explicit use of this crucial element of our proof. A full explanation of the method appeared first in Ashbaugh [**2**] (see also [**3**], [**9**]).

Corollary 3.2. *Under the assumptions of the problem, the following inequality holds:*

$$\begin{aligned}(\lambda_{m+1} - \lambda_i)\left(\langle [A,B]u_i, Bu_i\rangle - \sum_{j=1}^m (\lambda_j - \lambda_i)|a_{ij}|^2\right)&\\ \le \|[A,B]u_i\|^2 - \sum_{j=1}^m (\lambda_j - \lambda_i)^2|a_{ij}|^2&.\end{aligned} \tag{3.14}$$

Proof. Start with (3.6), and use Lemma 3.1 along with (3.9) and (3.11). □

Since B is one of the B_k's, $a_{ij} = a_{ij}^k$. Let

$$A_{ij} \equiv \sum_{k=1}^{N} |a_{ij}^k|^2. \tag{3.15}$$

We have $A_{ij} = A_{ji} \geq 0$. Summing (3.14) over k, for $1 \leq k \leq N$, and incorporating the definitions of ρ_i, Λ_i, and A_{ij}, we get

$$(\lambda_{m+1} - \lambda_i)\left(\rho_i - \sum_{j=1}^{m} (\lambda_j - \lambda_i) A_{ij}\right) \leq \Lambda_i - \sum_{j=1}^{m} (\lambda_j - \lambda_i)^2 A_{ij}. \tag{3.16}$$

By dropping the last term on the right-hand side, one has

$$(\lambda_{m+1} - \lambda_i)\left(\rho_i - \sum_{j=1}^{m} (\lambda_j - \lambda_i) A_{ij}\right) \leq \Lambda_i. \tag{3.17}$$

Remark. The quantity $\rho_i - \sum_{j=1}^{m} (\lambda_j - \lambda_i) A_{ij} \geq 0$ by virtue of (3.6) (see also (3.11)).

The General PPW Bound for $\lambda_{m+1} - \lambda_m$. From (3.17), we pass to

$$(\lambda_{m+1} - \lambda_m)\left(\rho_i - \sum_{j=1}^{m} (\lambda_j - \lambda_i) A_{ij}\right) \leq \Lambda_i. \tag{3.18}$$

Summing over $i = 1, \ldots, m$ yields (2.3). The double sum

$$\sum_{i=1}^{m} \sum_{j=1}^{m} A_{ij} (\lambda_j - \lambda_i) \tag{3.19}$$

vanishes by antisymmetry.

The Hile–Protter Bound. We rewrite (3.17) as

$$\rho_i - \sum_{j=1}^{m} (\lambda_j - \lambda_i) A_{ij} \leq \frac{\Lambda_i}{\lambda_{m+1} - \lambda_i}. \tag{3.20}$$

Summing on i, $1 \leq i \leq m$, yields (2.4).

The H.C. Yang Bound. We rewrite (3.16) as

$$(\lambda_{m+1} - \lambda_i)\rho_i \leq \Lambda_i + \sum_{j=1}^{m} (\lambda_{m+1} - \lambda_j)(\lambda_j - \lambda_i) A_{ij}. \tag{3.21}$$

Multiplying by $(\lambda_{m+1} - \lambda_i)$ and summing on i yields (2.5). The double sum

$$\sum_{i=1}^{m} \sum_{j=1}^{m} (\lambda_{m+1} - \lambda_i)(\lambda_{m+1} - \lambda_j)(\lambda_j - \lambda_i) A_{ij} \tag{3.22}$$

vanishes by antisymmetry.

4. The case of a Schrödinger like operator

Let $H = A + V$ be an operator defined on $\mathcal{D} \subset \mathcal{H}$, where A and V are self-adjoint operators, $A = -\sum_{k=1}^{N} T_k^2$, and the T_k's are skew-symmetric with domains $T_k(\mathcal{D})$ satisfying $T_k(\mathcal{D}) \subset \mathcal{D}(T_k)$ and $\mathcal{D}(V) \supset \mathcal{D}(T_k) \supset \mathcal{D}(A) \equiv \mathcal{D}$. The operator H is modeled after the Schrödinger operator. We assume that the spectrum of H is discrete consisting of eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots$, and we let $\{u_i\}_{i=1}^{\infty}$ be a complete orthonormal basis of eigenvectors corresponding to $\{\lambda_i\}_{i=1}^{\infty}$. We take a family of symmetric operators $\{B_k : H(\mathcal{D}) \to \mathcal{H}\}_{k=1}^{N}$ that leave $\mathcal{D}$ invariant, such that $[T_\ell, B_k]u_i = \delta_{\ell k} u_i$. The quantities ρ_i and Λ_i are given by

$$\rho_i = \sum_{k=1}^{N} \big\langle [H, B_k]u_i, B_k u_i \big\rangle \quad \text{and} \quad \Lambda_i = \sum_{k=1}^{N} \big\| [H, B_k]u_i \big\|^2.$$

In obvious notation we write $\rho_i = \rho_i^A + \rho_i^V$, corresponding to the decomposition $H = A + V$. We have the following generalization of Corollary 2.4.

Theorem 4.1. *Suppose* $[V, B_k] = 0$ *for* $1 \leq k \leq N$. *Then* $\rho_i = N$, $\Lambda_i = 4\big(\lambda_i - \langle V u_i, u_i \rangle\big)$, *and*

$$\lambda_{m+1} - \lambda_m \leq \frac{4}{Nm} \sum_{i=1}^{m} \big(\lambda_i - \langle V u_i, u_i \rangle\big), \tag{4.1}$$

$$\frac{Nm}{4} \leq \sum_{i=1}^{m} \frac{\lambda_i - \langle V u_i, u_i \rangle}{\lambda_{m+1} - \lambda_i}, \tag{4.2}$$

$$\sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^2 \leq \frac{4}{N} \sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)\big(\lambda_i - \langle V u_i, u_i \rangle\big). \tag{4.3}$$

Proof. For $[V, B_k] = 0$, $\rho_i^V = 0$. Therefore $\rho_i = \rho_i^A = N$, by Corollary 2.4. For an eigenvector u, $[H, B_k]u = [A, B_k]u = -2T_k u$, by Lemma 2.3. Hence,

$$\big\| [H, B_k]u_i \big\|^2 = 4 \| T_k u_i \|^2$$

and

$$\begin{aligned} \Lambda_i &= \sum_{k=1}^{N} \big\| [H, B_k]u_i \big\|^2 = 4 \sum_{k=1}^{N} \| T_k u_i \|^2 \\ &= 4 \left\langle \sum_{k=1}^{N} -T_k^2 \ u_i, u_i \right\rangle = 4 \langle A u_i, u_i \rangle \\ &= 4 \langle (H - V) u_i, u_i \rangle = 4 \Big(\lambda_i - \langle V u_i, u_i \rangle \Big). \end{aligned} \tag{4.4}$$

The result then follows from previous considerations. $\square$

Remark. Suppose $V \geq M > 0$. Then the theorem reduces to

$$\lambda_{m+1} - \lambda_m \leq \frac{4}{Nm} \sum_{i=1}^{m} \big(\lambda_i - M\big), \tag{4.5}$$

$$\frac{Nm}{4} \leq \sum_{i=1}^{m} \frac{\lambda_i - M}{\lambda_{m+1} - \lambda_i}, \tag{4.6}$$

$$\sum_{i=1}^{m} (\lambda_{m+1} - \lambda_i)^2 \leq \frac{4}{N} \sum_{i=1}^{m} \big(\lambda_{m+1} - \lambda_i\big)\big(\lambda_i - M\big). \tag{4.7}$$

Therefore, the classical PPW, HP, and H.C. Yang inequalities (without M) hold with *strict* inequalities. This fact (and more) was noted by Ashbaugh and Benguria [**7**] for the Hile–Protter type inequality and was first observed in the case of the Payne–Pólya–Weinberger inequality by Allegretto [**1**], Harrell [unpublished] (in [**16**] the PPW type inequality (4.1) is interpreted as a family of pointwise bounds for the potential V), and Singer, Wong, Yau, and Yau [**33**]. Hook [**23**] gives the HP type results applying to various second-order elliptic operators. The Yang type bound (4.7) is stronger than that proved in [**20**] though we essentially use the same assumptions (see Theorem 1 and Theorem 5 therein).

5. Comparing the bounds

In this section, we are interested in comparing the bounds on λ_{m+1} that arise from each of the three classical inequalities of Payne–Pólya–Weinberger, Hile–Protter, and H. C. Yang, including some simpler variants of these bounds. In his paper, H. C. Yang [**35**] stated that the inequality he obtained implied that of an "averaged" version of PPW that in turn implied that of Hile–Protter (which is known to imply the classical PPW inequality). However, while the first point is clear, he did not give specifics for the remaining points, so we find it instructive to do so here. We will in fact provide a proof valid for the general setting of this work, with applicability to the Laplacian and various other operators. Similar methods were used in [**10**] to treat the Harrell–Stubbe type inequalities given as (1.5) and (1.6). H.C. Yang's inequality will be proved to be the strongest of the existing classical universal inequalities.

Throughout this section, we assume that the operators A and B_k, for $1 \leq k \leq N$, satisfy the conditions (2.16) and (2.17), namely

$$\gamma \leq [B_k, [A, B_k]] \quad \text{and} \quad -\sum_{k=1}^{N} [B_k, A]^2 \leq \beta A$$

for some $\beta, \gamma > 0$. For simplicity we set

$$\bar{\lambda} = \frac{\sum_{i=1}^m \lambda_i}{m}.$$

We let

$$\sigma_{\mathrm{PPW}} = \lambda_m + \frac{2\beta}{mN\gamma} \sum_{i=1}^m \lambda_i = \lambda_m + \frac{2\beta}{N\gamma} \bar{\lambda}. \tag{5.1}$$

We define

$$g_m(\sigma) = \sum_{i=1}^m \frac{\lambda_i}{\sigma - \lambda_i}, \tag{5.2}$$

and, for a fixed index ℓ, $1 \le \ell \le m$,

$$\widetilde{g}_{m\ell}(\sigma) = \sum_{i=1}^{\ell} \frac{\lambda_i}{\sigma - \lambda_\ell} + \sum_{i=\ell+1}^{m} \frac{\lambda_i}{\sigma - \lambda_m}. \tag{5.3}$$

We define σ_{HP} to be the unique solution of the equation

$$g_m(\sigma) = \frac{mN\gamma}{2\beta} \tag{5.4}$$

and $\widetilde{\sigma}_{\mathrm{HP},\ell}$ the unique solution of

$$\widetilde{g}_{m\ell}(\sigma) = \frac{mN\gamma}{2\beta}, \tag{5.5}$$

both on (λ_m, ∞).

The uniqueness of σ_{HP} and $\widetilde{\sigma}_{\mathrm{HP},\ell}$ follows from the monotonicity of $g_m(\sigma)$ and $\widetilde{g}_{m\ell}(\sigma)$, respectively, both of which decrease from ∞ to zero as σ varies on (λ_m, ∞). Since $g_m(\sigma_{\mathrm{HP}}) = \widetilde{g}_{m\ell}(\widetilde{\sigma}_{\mathrm{HP},\ell})$ and $g_m(\sigma) \le \widetilde{g}_{m\ell}(\sigma)$ for all $\sigma > \lambda_m$, we obtain $\widetilde{g}_{m\ell}(\widetilde{\sigma}_{\mathrm{HP},\ell}) \le \widetilde{g}_{m\ell}(\sigma_{\mathrm{HP}})$ from which we have $\sigma_{\mathrm{HP}} \le \widetilde{\sigma}_{\mathrm{HP},\ell}$ (since $\widetilde{g}_{m\ell}$ is decreasing). Hence $\widetilde{\sigma}_{\mathrm{HP},\ell}$ is an upper estimate for σ_{HP}. On the other hand, since, by Theorem 2.5, $g_m(\sigma_{\mathrm{HP}}) \le g_m(\lambda_{m+1})$, we have $\lambda_{m+1} \le \sigma_{\mathrm{HP}}$. In fact $\widetilde{\sigma}_{\mathrm{HP},\ell}$ is given explicitly by

$$\begin{aligned} \widetilde{\sigma}_{\mathrm{HP},\ell} = {} & \frac{\lambda_m + \lambda_\ell}{2} + \frac{\beta}{N\gamma}\bar{\lambda} \\ & + \left(\left(\frac{\lambda_m - \lambda_\ell}{2} + \frac{\beta}{N\gamma}\bar{\lambda} \right)^2 - \frac{2\beta}{mN\gamma} (\lambda_m - \lambda_\ell) \sum_{i=1}^{\ell} \lambda_i \right)^{1/2}. \end{aligned} \tag{5.6}$$

Obviously, the case $\ell = m$ gives us the bound $\widetilde{\sigma}_{\mathrm{HP},m} = \sigma_{\mathrm{PPW}}$, which is thus weaker than σ_{HP}.

Next we set

$$(5.7) \qquad h_m(\sigma) = \sum_{i=1}^m (\sigma - \lambda_i)^2 - \frac{2\beta}{N\gamma} \sum_{i=1}^m (\sigma - \lambda_i)\lambda_i$$
$$= m\sigma^2 - 2\Big(1 + \frac{\beta}{N\gamma}\Big)\Big(\sum_{i=1}^m \lambda_i\Big)\sigma + \Big(1 + \frac{2\beta}{N\gamma}\Big)\sum_{i=1}^m \lambda_i^2,$$

and let $\sigma_Y^{\pm}$ denote the two roots of the quadratic equation $h_m(\sigma) = 0$ (that $h_m(\sigma)$ has two real roots follows from the fact that $h_m(\lambda_{m+1}) \le 0$ and

$$\lim_{\sigma\to\infty} h_m(\sigma) = \infty$$

(see Prop. 6, p. 1802 of [**20**] for an alternative proof of this fact)). We have

$$(5.8) \quad \sigma_Y^{\pm} = \Big(1 + \frac{\beta}{N\gamma}\Big)\bar{\lambda} \pm \left(\Big(\Big(1 + \frac{\beta}{N\gamma}\Big)\bar{\lambda}\Big)^2 - \Big(1 + \frac{2\beta}{N\gamma}\Big)\frac{1}{m}\sum_{i=1}^m \lambda_i^2\right)^{1/2},$$

which can be rewritten in the form

$$(5.9) \quad \sigma_Y^{\pm} = \Big(1 + \frac{\beta}{N\gamma}\Big)\bar{\lambda} \pm \left(\Big(\frac{\beta}{N\gamma}\bar{\lambda}\Big)^2 - \Big(1 + \frac{2\beta}{N\gamma}\Big)\frac{1}{m}\sum_{i=1}^m (\lambda_i - \bar{\lambda})^2\right)^{1/2}.$$

Clearly,

$$\sigma_Y^+ \le \widetilde{\sigma}_Y,$$

where

$$(5.10) \qquad \widetilde{\sigma}_Y \equiv \Big(1 + \frac{2\beta}{N\gamma}\Big)\bar{\lambda}.$$

Since $h_m(\lambda_{m+1}) \le 0$, we conclude that $\sigma_Y^- \le \lambda_{m+1} \le \sigma_Y^+$. Also, since

$$h_m(\lambda_m) = h_{m-1}(\lambda_m) \le 0,$$

we have $\sigma_Y^- \le \lambda_m$ (the quadratic $h_m(\sigma)$ is negative between its roots $\sigma_Y^{\pm}$). The σ_Y^+ bound on λ_{m+1} gives

$$(5.11) \qquad \lambda_{m+1} \le \widetilde{\sigma}_Y.$$

Inequality (5.11) can be written in the form

$$(5.12) \qquad \frac{N\gamma}{2\beta} \le \frac{\bar{\lambda}}{\lambda_{m+1} - \bar{\lambda}}.$$

The function $f(x) = \dfrac{x}{\lambda_{m+1} - x}$ is convex on $(0, \lambda_{m+1})$ since

$$f''(x) = 2\lambda_{m+1}(\lambda_{m+1} - x)^{-3} > 0 \quad \text{for} \quad x < \lambda_{m+1}.$$

Hence,

$$f(\bar{\lambda}) = f\Big(\frac{1}{m}\sum_{i=1}^m \lambda_i\Big) \le \frac{1}{m}\sum_{i=1}^m f(\lambda_i)$$

and thus (from (5.12))

$$\frac{mN\gamma}{2\beta} \le \sum_{i=1}^{m} \frac{\lambda_i}{\lambda_{m+1} - \lambda_i}, \tag{5.13}$$

from which we conclude that Yang's weaker bound implies the statement of the HP bound. An elementary proof of the fact that $\widetilde{\sigma}_Y \le \sigma_{HP}$ will be given in the proof of Theorem 5.2 below.

Remark. These calculations show, as in the case of the Dirichlet Laplacian [**35**], that the generalized H.C. Yang inequality implies that of Hile–Protter, which in turn implies that of Payne–Pólya–Weinberger.

We now prove a lemma which we will need for the next theorem.

Lemma 5.1 (Reverse Chebyshev Inequality)**.** *Suppose* $\{a_i\}_{i=1}^m$ *and* $\{b_i\}_{i=1}^m$ *are two real sequences with* $\{a_i\}$ *increasing and* $\{b_i\}$ *decreasing. Then the following inequality holds:*

$$\sum_{i=1}^{m} a_i \sum_{i=1}^{m} b_i \ge m \sum_{i=1}^{m} a_i b_i. \tag{5.14}$$

Proof. We use the simplified notation

$$\sum a = \sum_{i=1}^{m} a_i, \ \sum b = \sum_{i=1}^{m} b_i, \ \sum ab = \sum_{i=1}^{m} a_i b_i.$$

Starting with the fact that $\big(a_i - a_j\big)\big(b_i - b_j\big) \le 0$ for $i, j = 1, \ldots, m$, we sum over both indices to arrive at

$$\begin{aligned} 0 &\ge \sum_i \sum_j \big(a_i - a_j\big)\big(b_i - b_j\big) \\ &= \sum_i \sum_j \big(a_i b_i - a_i b_j - a_j b_i + a_j b_j\big) \\ &= m \sum_i a_i b_i - \sum_i a_i \sum_j b_j - \sum_j a_j \sum_i b_i + m \sum_j a_j b_j \\ &= 2\left(m \sum ab - \sum a \ \sum b\right), \end{aligned} \tag{5.15}$$

from which the result is immediate. ☐

Remark. A weighted version of this inequality can also be proved. See, for example, p. 43 of [**15**].

Theorem 5.2.

$$\sigma_Y^- \le \lambda_m \le \lambda_{m+1} \le \sigma_Y^+ \le \widetilde{\sigma}_Y \le \sigma_{\mathrm{HP}} \le \widetilde{\sigma}_{\mathrm{HP},\ell} \le \sigma_{\mathrm{PPW}}. \tag{5.16}$$

Proof. That $\sigma_Y^- \le \lambda_m \le \lambda_{m+1} \le \sigma_Y^+$ and $\sigma_{\mathrm{HP}} \le \widetilde{\sigma}_{\mathrm{HP},\ell}$ have already been proved in the previous discussion. That $\sigma_Y^+ \le \widetilde{\sigma}_Y$ follows from the definitions of σ_Y^+ and $\widetilde{\sigma}_Y$ (Equations (5.9) and (5.10)). That $\widetilde{\sigma}_{\mathrm{HP},\ell} \le \sigma_{\mathrm{PPW}}$ is an immediate consequence of (5.6). To complete the chain of inequalities we just need to prove that $\widetilde{\sigma}_Y \le \sigma_{\mathrm{HP}}$. This is equivalent to showing that

$$g_m(\widetilde{\sigma}_Y) \ge g_m(\sigma_{\mathrm{HP}})$$

(since g_m is decreasing). From (5.10), we have

$$\frac{N\gamma}{2\beta} = \frac{\bar{\lambda}}{\widetilde{\sigma}_Y - \bar{\lambda}}. \tag{5.17}$$

Since

$$g_m(\sigma_{\mathrm{HP}}) = \frac{mN\gamma}{2\beta},$$

we simply need to prove that

$$\sum_{i=1}^m \frac{\lambda_i}{\widetilde{\sigma}_Y - \lambda_i} \ge \frac{mN\gamma}{2\beta}. \tag{5.18}$$

Using (5.17) and incorporating the definition of $\bar{\lambda}$, this is equivalent to showing that

$$\sum_{i=1}^m \frac{\lambda_i}{\widetilde{\sigma}_Y - \lambda_i} \ge \frac{m\sum_{i=1}^m \lambda_i}{\sum_{i=1}^m (\widetilde{\sigma}_Y - \lambda_i)}, \tag{5.19}$$

or

$$\sum_{i=1}^m \frac{\lambda_i}{\widetilde{\sigma}_Y - \lambda_i} \sum_{i=1}^m (\widetilde{\sigma}_Y - \lambda_i) \ge m \sum_{i=1}^m \lambda_i. \tag{5.20}$$

The sequence $a_i = \dfrac{\lambda_i}{\widetilde{\sigma}_Y - \lambda_i}$ is increasing, while $b_i = \widetilde{\sigma}_Y - \lambda_i$ is decreasing. The result of the theorem then follows by applying Lemma 5.1.

Corollary 5.3.

$$\sigma_Y^+ \le \sigma_{\mathrm{HP}}.$$

The result is an obvious consequence of Theorem 5.2. We argue below that this statement can be directly proven without recourse to the intermediate inequalities $\sigma_Y^+ \le \widetilde{\sigma}_Y$ and $\widetilde{\sigma}_Y \le \sigma_{\mathrm{HP}}$. Since $g_m(\sigma)$ is decreasing, the statement would follow from the inequality $g_m(\sigma_Y^+) \ge g_m(\sigma_{\mathrm{HP}})$. Since $g_m(\sigma_{\mathrm{HP}}) = mN\gamma/(2\beta)$, the statement is equivalent to

$$g_m(\sigma_Y^+) \ge \frac{mN\gamma}{2\beta}. \tag{5.21}$$

But $h(\sigma_Y^+) = 0$, i.e.,

$$\sum_{i=1}^{m} (\sigma_Y^+ - \lambda_i)^2 = \frac{2\beta}{N\gamma} \sum_{i=1}^{m} (\sigma_Y^+ - \lambda_i)\ \lambda_i. \tag{5.22}$$

Hence, eliminating $2\beta/(N\gamma)$, (5.21) is equivalent to

$$\sum_{i=1}^{m} \frac{\lambda_i}{\sigma_Y^+ - \lambda_i} \geq \frac{m \sum_{i=1}^{m} (\sigma_Y^+ - \lambda_i)\ \lambda_i}{\sum_{i=1}^{m} (\sigma_Y^+ - \lambda_i)^2}, \tag{5.23}$$

or

$$\sum_{i=1}^{m} \frac{\lambda_i}{\sigma_Y^+ - \lambda_i} \ \sum_{i=1}^{m} (\sigma_Y^+ - \lambda_i)^2 \geq m \sum_{i=1}^{m} (\sigma_Y^+ - \lambda_i)\ \lambda_i. \tag{5.24}$$

The sequence $a_i = \lambda_i/(\sigma_Y^+ - \lambda_i)$ is increasing, while $b_i = (\sigma_Y^+ - \lambda_i)^2$ is decreasing and hence statement (5.24) is true by Lemma 5.1, and the alternative proof of the corollary is complete. □

References

[1] W. Allegretto, *Lower bounds on the number of points in the lower spectrum of elliptic operators*, Canad. J. Math., **31** (1979), 419–426, MR 0528821 (81a:35078), Zbl 0368.35049.

[2] M.S. Ashbaugh, *Isoperimetric and universal inequalities for eigenvalues*, in 'Spectral Theory and Geometry' (Edinburgh, 1998, E. B. Davies and Yu. Safarov, eds.), London Math. Soc. Lecture Note Series, **273**, Cambridge University Press, Cambridge, 1999, 95–139, MR 1736867 (2001a:35131), Zbl 0937.35114.

[3] M.S. Ashbaugh, *The universal eigenvalue bounds of Payne–Pólya–Weinberger, Hile–Protter, and H.C. Yang*, Proc. Indian Acad. Sci. (Math. Sci.), **112** (2002), 3–30, MR 1894540 (2004c:35302).

[4] M.S. Ashbaugh and R.D. Benguria, *Proof of the Payne–Pólya–Weinberger conjecture*, Bull. Amer. Math. Soc., **25** (1991), 19–29, MR 1085824 (91m:35173), Zbl 0736.35075.

[5] M.S. Ashbaugh and R.D. Benguria, *A sharp bound for the ratio of the first two eigenvalues of Dirichlet Laplacians and extensions*, Ann. of Math., **135** (1992), 601–628, MR 1166646 (93d:35105), Zbl 0757.35052.

[6] M.S. Ashbaugh and R.D. Benguria, *More bounds on eigenvalue ratios for Dirichlet Laplacians in n dimensions*, SIAM J. Math. Anal., **24** (1993), 1622–1651, MR 1241161 (94i:35139), Zbl 0809.35067.

[7] M.S. Ashbaugh and R.D. Benguria, *Isoperimetric inequalities for eigenvalue ratios*, in 'Partial Differential Equations of Elliptic Type' (Cortona, 1992, A. Alvino, E. Fabes, and G. Talenti, eds.), Symposia Mathematica, **35**, Cambridge University Press, Cambridge, 1994, 1–36, MR 1297771 (95h:35158), Zbl 0814.35081.

[8] M.S. Ashbaugh and R.D. Benguria, *Bounds for ratios of the first, second, and third membrane eigenvalues*, in 'Nonlinear Problems in Applied Mathematics: In Honor of Ivar Stakgold on his Seventieth Birthday' (T.S. Angell, L. Pamela Cook, R.E. Kleinman, and W.E. Olmstead, eds.), Society for Industrial and Applied Mathematics, Philadelphia, 1996, 30–42.

[9] M.S. Ashbaugh and L. Hermi, *On extending the inequalities of Payne, Pólya, and Weinberger using spherical harmonics*, accepted.

[10] M.S. Ashbaugh and L. Hermi, *On Harrell–Stubbe type inequalities for the discrete spectrum of a self-adjoint operator*, submitted.

[11] M.S. Ashbaugh and L. Hermi, *On Yang type bounds for eigenvalues with applications to physical and geometric problems*, preprint.

[12] M.S. Ashbaugh and L. Hermi, *Universal inequalities for higher-order elliptic operators*, preprint.

[13] C. Bandle, *Isoperimetric inequality for some eigenvalues of an inhomogeneous, free membrane*, SIAM J. Appl. Math., **22** (1972), 142–147, MR 0313648 (47 #2202), Zbl 0237.35069.

[14] S.-Y. Cheng, *Eigenfunctions and eigenvalues of Laplacian*, in 'Differential Geometry' (Stanford Univ., Stanford, Calif., 1973, S. S. Chern and R. Osserman, eds.), Proc. Symp. Pure Math., **27**, Part 2, American Mathematical Society, Providence, Rhode Island, 1975, 185–193, MR 0378003 (51 #14172), Zbl 0308.35076.

[15] G.H. Hardy, J.E. Littlewood, and G. Pólya, *Inequalities* (second edition), Cambridge University Press, Cambridge, 1952, MR 0944909 (89d:26016), Zbl 0634.26008.

[16] E.M. Harrell II, *General bounds for the eigenvalues of Schrödinger operators*, in 'Maximum Principles and Eigenvalue Problems in Partial Differential Equations' (Knoxville, TN, 1987, P. W. Schaefer, ed.), Pitman Res. Notes Math. Ser., **175**, Longman Scientific and Technical, Harlow, Essex, United Kingdom, 1988, 146–166; see, in particular, the Appendix, 161–163, MR 0963465 (90d:35220), Zbl 0678.35075.

[17] E.M. Harrell II, *Some geometric bounds on eigenvalue gaps*, Comm. Partial Differential Equations, **18** (1993), 179–198, MR 1211730 (94c:35135), Zbl 0810.35067.

[18] E.M. Harrell II and P.L. Michel, *Commutator bounds for eigenvalues, with applications to spectral geometry*, Comm. Partial Differential Equations, **19** (1994), 2037–2055, MR 1301181 (95i:58182), Zbl 0815.35078; Erratum: ibid., **20** (1995), 1453, MR 1335758 (95i:58182), Zbl 0828.35091.

[19] E.M. Harrell II and P.L. Michel, *Commutator bounds for eigenvalues of some differential operators*, in 'Evolution Equations' (Baton Rouge, LA, 1992, G. Ferreyra, G. R. Goldstein, and F. Neubrander, eds.), Lecture Notes in Pure and Appl. Math. Series, **168**, Marcel Dekker, New York, 1995, 235–244, MR 1300432 (95j:47028), Zbl 0816.35094.

[20] E.M. Harrell II and J. Stubbe, *On trace identities and universal eigenvalue estimates for some partial differential operators*, Trans. Amer. Math. Soc., **349** (1997), 1797–1809, MR 1401772 (97i:35129), Zbl 0887.35111.

[21] G.N. Hile and M.H. Protter, *Inequalities for eigenvalues of the Laplacian*, Indiana Univ. Math. J., **29** (1980), 523–538, MR 0578204 (82c:35052), Zbl 0454.35064.

[22] G.N. Hile and R.Z. Yeh, *Inequalities for eigenvalues of the biharmonic operator*, Pacific J. Math., **112** (1984), 115–133, MR 0739143 (85k:35170), Zbl 0541.35059.

[23] S.M. Hook, *Domain-independent upper bounds for eigenvalues of elliptic operators*, Trans. Amer. Math. Soc., **318** (1990), 615–642, MR 0994167 (90h:35075), Zbl 0727.35096.

[24] J.M. Lee, *The gaps in the spectrum of the Laplace–Beltrami operator*, Houston J. Math., **17** (1991), 1–24, MR 1107183 (92f:58186), Zbl 0736.58041.

[25] P.-F. Leung, *On the consecutive eigenvalues of the Laplacian of a compact minimal submanifold in a sphere*, J. Austral. Math. Soc. Ser. A, **50** (1991), 409–416, MR 1096895 (92d:58212), Zbl 0728.53035.

[26] P. Li, *Eigenvalue estimates on homogeneous manifolds*, Comment. Math. Helv., **55** (1980), 347–363, MR 0593051 (81k:58067), Zbl 0451.53036.

[27] P.L. Michel, *Eigenvalue Gaps for Self-Adjoint Operators*, Ph.D. Thesis, Georgia Institute of Technology, 1994.

[28] L.E. Payne, G. Pólya, and H.F. Weinberger, *Sur le quotient de deux fréquences propres consécutives*, Comptes Rendus Acad. Sci. Paris, **241** (1955), 917–919 (reprinted as pp. 410–412 of [**30**] with comments by J. Hersch on p. 518), MR 0073046 (17,372d), Zbl 0065.08801.

[29] L.E. Payne, G. Pólya, and H.F. Weinberger, *On the ratio of consecutive eigenvalues*, J. of Math. and Phys., **35** (1956), 289–298 (reprinted as pp. 420–429 of [**30**] with comments by J. Hersch on p. 521), MR 0084696 (18,905c), Zbl 0073.08203.

[30] G. Pólya, *Collected Papers,* III: *Analysis* (J. Hersch, G.-C. Rota, M.C. Reynolds and R.M. Shortt, eds.), Mathematicians of Our Time, **21**, MIT Press, Cambridge, Massachusetts, 1984, MR 0758989 (85m:01108a), Zbl 0561.01029.

[31] M.H. Protter, *Universal inequalities for eigenvalues*, in 'Maximum Principles and Eigenvalue Problems in Partial Differential Equations' (Knoxville, TN, 1987, P. W. Schaefer, ed.), Pitman Research Notes in Mathematics Series, **175**, Longman Group UK Limited, 1988, 111–120, MR 0963462 (89k:35167), Zbl 0663.35052.

[32] M.H. Protter, *Upper bounds for eigenvalues of elliptic operators*, in 'Partial Differential Equations and Applications' (P. Marcellini, G. Talenti, and E. Vesentini, eds.), Lecture Notes in Pure and Appl. Math., **177**, Marcel Dekker, New York, 1996, 271–277, MR 1371598, Zbl 0845.35075.

[33] I.M. Singer, B. Wong, S.-T. Yau, and S.S.T. Yau, *An estimate of the gap of the first two eigenvalues in the Schrödinger operator*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), **12** (1985), 319–333, MR 0829055 (87j:35280), Zbl 0603.35070.

[34] C.J. Thompson, *On the ratio of consecutive eigenvalues in N-dimensions*, Stud. Appl. Math., **48** (1969), 281–283, MR 0257592 (41 #2242), Zbl 0183.11005.

[35] H.C. Yang, *Estimates of the difference between consecutive eigenvalues*, preprint, 1995 (revision of International Centre for Theoretical Physics preprint IC/91/60, Trieste, Italy, April 1991).

[36] P.C. Yang and S.-T. Yau, *Eigenvalues of the Laplacian of compact Riemann surfaces and minimal submanifolds*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), **7** (1980), 55–63, MR 0577325 (81m:58084), Zbl 0446.58017.

Received April 20, 2002. The first author was partially supported by National Science Foundation (USA) grant DMS–9870156.

Department of Mathematics
University of Missouri
Columbia MO 65211
E-mail address: mark@math.missouri.edu

Department of Mathematics
University of Arizona
617 Santa Rita
Tucson AZ 85721
E-mail address: hermi@math.arizona.edu