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We construct a Jacobian of dimension three whose theta divisor contains an elliptic curve. We work over an algebraically closed field of characteristic zero.

Let E be an elliptic curve and F a principally polarized abelian variety of dimension 3. Let \mathcal{L} and \mathcal{N} be their principal polarizations.

Lemma 1. *There exist E' and F' such that we have isogenies $\psi_E : E' \rightarrow E$ and $\psi_F : F' \rightarrow F$ of degree two.*

Proof. Let $F' = (F^\vee/\mathbf{Z}/2\mathbf{Z})^\vee$ and the same with E . □

Let $\mathcal{L}' = \psi_E^* \mathcal{L}$ and $\mathcal{N}' = \psi_F^* \mathcal{N}$. Then $\varphi_{\mathcal{L}'} : E' \rightarrow (E')^\vee$ and $\varphi_{\mathcal{N}'} : F' \rightarrow (F')^\vee$ have degree four. Let $H_{\mathcal{L}'}$ and $H_{\mathcal{N}'}$ be their kernels. Then we have theta groups

$$1 \rightarrow \mathbf{G}_m \rightarrow G_{\mathcal{L}'} \rightarrow H_{\mathcal{L}'} \rightarrow 0$$

and

$$1 \rightarrow \mathbf{G}_m \rightarrow G_{\mathcal{N}'} \rightarrow H_{\mathcal{N}'} \rightarrow 0.$$

By Mumford theory we have two torsion elements $\alpha_{\mathcal{L}'}$ and $\beta_{\mathcal{L}'}$ of $G_{\mathcal{L}'}$ such that $\alpha_{\mathcal{L}'} \cdot \beta_{\mathcal{L}'} = (-1) \beta_{\mathcal{L}'} \cdot \alpha_{\mathcal{L}'}$ and the same with \mathcal{N} . Here the images of $\alpha_{\mathcal{L}'}$ and $\beta_{\mathcal{L}'}$ generate $H_{\mathcal{L}'}$. Consider $\mathcal{M} = \pi_{E'}^* \mathcal{L}' \otimes \pi_{F'}^* \mathcal{N}'$. Then $H_{\mathcal{M}} = H_{\mathcal{L}'} \times H_{\mathcal{N}'}$.

Lemma 2. *We have an inclusion $K = (\mathbf{Z}/2\mathbf{Z})^2 \subset G_{\mathcal{M}}$.*

Proof. $\alpha_{\mathcal{L}'} \otimes \alpha_{\mathcal{N}'}$ and $\beta_{\mathcal{L}'} \otimes \beta_{\mathcal{N}'}$ generate the group. □

Let $X = E' \otimes F' / \text{Im } K$ and let R be the quotient of \mathcal{M} by $(\mathbf{Z}/2\mathbf{Z})^2$. Then R gives a principal polarization on X . Let γ be a nonzero section of X . Let θ be the zeroes of γ .

Lemma 3. *θ contains some translate of $\text{Im } E'$.*

Proof. γ corresponds to a section of \mathcal{M} that is invariant under K . Let τ and μ be nonzero sections of \mathcal{L}' and \mathcal{N}' invariant under $\alpha_{\mathcal{L}'}$ and $\alpha_{\mathcal{N}'}$. Let $\tau' = \beta_{\mathcal{L}'}(\tau)$ and $\mu' = \beta_{\mathcal{N}'}(\mu)$. Then τ' and μ' are anti-invariant under $\alpha_{\mathcal{L}'}$ and $\alpha_{\mathcal{N}'}$. Consider the section $\eta = \tau \otimes \mu + \tau' \otimes \mu' \neq 0$ of \mathcal{M} . Then η is invariant under $(\mathbf{Z}/2\mathbf{Z})^2$. Then the inverse image of θ is the zeroes $\eta \supset E \times (\mu = \mu' = 0)$, where the second set is nonempty as \mathcal{N}^2 is ample. □

Assume that F contains no elliptic curve.

Lemma 4. $(\mu = \mu' = 0)$ is a finite set.

Proof. Let D be the largest divisor in the intersection. Then D is invariant under the group P generated by the image of $\alpha_{\mathcal{N}}$ and $\beta_{\mathcal{N}}$. Then D comes from an effective divisor D' on F'/P where $\#P = 4$. So $\frac{(D^2)}{4} = 4\frac{(D')^2}{2}$ and $\frac{(D^2)}{2} \leq \frac{(\mu=0)^2}{2} = 2$. So by [1], D' comes from a divisor on a quotient of F'/D which is a point. So D' is empty. \square

Lemma 5. (X, θ) is a Jacobian.

Proof. We need to see that θ is irreducible. If θ is reducible, we have $X = E \oplus R$ by [1], where θ is the sum of divisors depending on the factors. Thus $\theta \supset E \times x$ for a curve x . But this contradicts Lemma 4. \square

References

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