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ALEX KUMJIAN

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In memory of Gert K. Pedersen.

Let A be a separable unital C^* -algebra. Let $\pi : A \rightarrow \mathcal{L}(\mathfrak{H})$ be a faithful representation of A on a separable Hilbert space \mathfrak{H} such that $\pi(A) \cap \mathcal{K}(\mathfrak{H}) = \{0\}$. We show that \mathcal{O}_E , the Cuntz–Pimsner algebra associated to the Hilbert A -bimodule $E = \mathfrak{H} \otimes_{\mathbb{C}} A$, is simple and purely infinite. If A is nuclear and belongs to the bootstrap class to which the UCT applies, the same applies to \mathcal{O}_E . Hence by the Kirchberg–Phillips Theorem the isomorphism class of \mathcal{O}_E only depends on the K -theory of A and the class of the unit.

In his seminal paper [Pm], Pimsner constructed a C^* -algebra \mathcal{O}_E from a Hilbert bimodule over a C^* -algebra A as a quotient of a concrete C^* -algebra \mathcal{T}_E , an analogue of the Toeplitz algebra, acting on the Fock space associated to E . There has recently been much interest in these Cuntz–Pimsner algebras (or Cuntz–Krieger–Pimsner algebras), which generalize both crossed products by \mathbb{Z} and Cuntz–Krieger algebras, as well as the associated Toeplitz algebras. The structure of these C^* -algebras is not yet fully understood, though considerable progress has been made. For example, Pimsner found a six-term exact sequence for the K -theory of \mathcal{O}_E that generalizes the Pimsner–Voiculescu exact sequence (see [Pm, Theorem 4.8]); conditions for simplicity were found in [Sc2, MS, KPW1, DPZ] and for pure infiniteness in [Z].

The purpose of the present note is to analyze the structure of Cuntz–Pimsner algebras associated to a certain class of Hilbert bimodules. Let A be a separable unital C^* -algebra and let $\pi : A \rightarrow \mathcal{L}(\mathfrak{H})$ be a faithful nondegenerate representation of A on a separable Hilbert space \mathfrak{H} such that $\pi(A) \cap \mathcal{K}(\mathfrak{H}) = \{0\}$. Then $E = \mathfrak{H} \otimes_{\mathbb{C}} A$ is a Hilbert bimodule over A in a natural way. We show that \mathcal{O}_E is separable, simple and purely infinite. If A is nuclear and in the bootstrap class, then the same holds for \mathcal{O}_E and thus by the Kirchberg–Phillips theorem the isomorphism class of \mathcal{O}_E is completely determined by the K -theory of A together with the class of the unit (since \mathcal{O}_E is KK -equivalent to A).

Many examples of Cuntz–Pimsner algebras found in the literature arise from Hilbert bimodules that are finitely generated and projective; in such

cases the left action must consist entirely of compact operators. Our examples do not fall in this class; in fact, the left action has trivial intersection with the compacts. And this has some interesting consequences: $\mathcal{O}_E \cong \mathcal{T}_E$ (see [Pm, Corollary 3.14]) and the natural embedding $A \hookrightarrow \mathcal{O}_E$ induces a KK -equivalence (see [Pm, Corollary 4.5]).

In §1 we review some basic facts concerning the construction of \mathcal{T}_E as operators on the Fock space of E and the gauge action $\lambda : \mathbb{T} \rightarrow \text{Aut}(\mathcal{T}_E)$. We assume that the left action of A does not meet the compacts $\mathcal{K}(E)$ and identify \mathcal{O}_E with \mathcal{T}_E . The fixed point algebra \mathcal{F}_E , the analogue of the AF-core of a Cuntz–Krieger algebra, contains a canonical descending sequence of essential ideals indexed by \mathbb{N} with trivial intersection. The crossed product $\mathcal{O}_E \rtimes_{\lambda} \mathbb{T}$ has a similar collection of essential ideals indexed by \mathbb{Z} on which the dual group of automorphisms acts in a natural way. By Takesaki–Takai duality,

$$\mathcal{O}_E \otimes \mathcal{K}(L^2(\mathbb{T})) \cong (\mathcal{O}_E \rtimes_{\lambda} \mathbb{T}) \rtimes_{\widehat{\lambda}} \mathbb{Z};$$

hence, much of the structure of \mathcal{O}_E is revealed through an analysis of the double crossed product.

In §2 we show that if E is the Hilbert bimodule over A associated to a representation as described above, then for every nonzero positive element $d \in \mathcal{O}_E$ there is a $z \in \mathcal{O}_E$ such that $z^*dz = 1$; it follows that \mathcal{O}_E is simple and purely infinite (see Theorem 2.8). The proof of this proceeds through a sequence of lemmas and is patterned on the proof of [Rø, Theorem 2.1], which is in turn based on a key lemma of Kishimoto (see [Ks, Lemma 3.2]). Our argument uses the version of this lemma found in [OP3, Lemma 7.1] and this requires that we show that the Connes spectrum of the dual action is full (this is also an ingredient in the proof of simplicity found in [DPZ]). We invoke a version of a key lemma of Rørdam for crossed products by \mathbb{Z} that arise from automorphisms with full Connes spectrum. The fact that \mathcal{O}_E embeds equivariantly into $(\mathcal{O}_E \rtimes_{\lambda} \mathbb{T}) \rtimes_{\widehat{\lambda}} \mathbb{Z}$ allows us to apply this lemma to \mathcal{O}_E .

In §3 we use the Kirchberg–Phillips theorem to collect some consequences of this theorem as indicated above and discuss certain connections with reduced (amalgamated) free products.

We fix some notation and terminology. Given a C^* -algebra B we let \widehat{B} denote its spectrum, that is, the collection of irreducible representations modulo unitary equivalence endowed with the Jacobson topology (see [Pd, §4.1]). If I is an ideal in a C^* -algebra B , every irreducible representation of I extends uniquely to an irreducible representation of B . This allows one to identify \widehat{I} with an open subset of \widehat{B} , the complement of which consists of the classes of irreducible representations that vanish on I . Given a $*$ -automorphism β of a C^* -algebra B , let $\Gamma(\beta)$ denote the Connes spectrum

of β (see [O, Co] or [Pd, §8.8]); recall that

$$\Gamma(\beta) = \bigcap_H \text{Sp}(\beta|_H)$$

where the intersection is taken over all nonzero β -invariant hereditary subalgebras H . A C^* -algebra is said to be purely infinite if every nonzero hereditary subalgebra contains an infinite projection.

1. Preliminaries

We review some basic facts concerning Cuntz–Pimsner algebras; we shall be mainly interested in those that arise from bimodules for which the left action has trivial intersection with the compacts (see Remark 1.3). Let A be a C^* -algebra.

Definition 1.1 (see [L, pp. 2–4], [Ka, pp. 134, 135] and [Ri1, Def. 2.1]). Let E be a right A -module. Then E is said to be a (right) pre-Hilbert A -module if it is equipped with an A -valued inner product $\langle \cdot, \cdot \rangle_A$ satisfying the following conditions for all $\xi, \eta, \zeta \in E$, $s, t \in \mathbb{C}$, and $a \in A$:

- (i) $\langle \xi, s\eta + t\zeta \rangle_A = s\langle \xi, \eta \rangle_A + t\langle \xi, \zeta \rangle_A$.
- (ii) $\langle \xi, \eta a \rangle_A = \langle \xi, \eta \rangle_A a$.
- (iii) $\langle \eta, \xi \rangle_A = \langle \xi, \eta \rangle_A^*$.
- (iv) $\langle \xi, \xi \rangle_A \geq 0$ and $\langle \xi, \xi \rangle_A = 0$ only if $\xi = 0$.

E is said to be a (right) Hilbert A -module if it is complete in the norm $\|\xi\| = \|\langle \xi, \xi \rangle_A\|^{1/2}$.

A Hilbert A -module E is said to be *full* if the span of the values of the inner product is dense. The collection of bounded adjointable operators on E , $\mathcal{L}(E)$, is a C^* -algebra. The closure of the span of operators of the form $\theta_{\xi, \eta}$ for $\xi, \eta \in E$ (where $\theta_{\xi, \eta}(\zeta) = \xi \langle \eta, \zeta \rangle_A$ for $\zeta \in E$) forms an essential ideal in $\mathcal{L}(E)$, denoted by $\mathcal{K}(E)$. A Hilbert space is a Hilbert module over \mathbb{C} .

Definition 1.2. Let E be a Hilbert A -module and let $\varphi : A \rightarrow \mathcal{L}(E)$ be an injective $*$ -homomorphism. The pair (E, φ) is said to be a Hilbert bimodule over A (or a Hilbert A -bimodule).

Pimsner defines the Cuntz–Pimsner algebra \mathcal{O}_E as a quotient of the analogue of the Toeplitz algebra, \mathcal{T}_E , generated by creation operators on the Fock space of E (see [Pm]). The injectivity of φ is not really necessary (see [Pm, Remark 1.2(1)]). We will henceforth assume that E is full (see [Pm, Remark 1.2(3)]).

The Fock space of E is the Hilbert A -module

$$\mathcal{E}_+ = \bigoplus_{n=0}^{\infty} E^{\otimes n}$$

where $E^{\otimes 0} = A$, $E^{\otimes 1} = E$ and for $n > 1$, $E^{\otimes n}$ is the n -fold tensor product:

$$E^{\otimes n} = E \otimes_A \cdots \otimes_A E.$$

The tensor product used here is called the inner tensor product by Lance (see [L, p. 41], but note Lance uses different notation; see also Theorem 5.9 of [Ri1]). Observe that \mathcal{E}_+ is also a Hilbert A -bimodule with left action defined by $\varphi_+(a)b = ab$ for $a, b \in A = E^{\otimes 0}$ and

$$\varphi_+(a)(\xi_1 \otimes \cdots \otimes \xi_n) = \varphi(a) \xi_1 \otimes \cdots \otimes \xi_n$$

for $a \in A$ and $\xi_1 \otimes \cdots \otimes \xi_n \in E^{\otimes n}$.

Then $\mathcal{T}_E \subset \mathcal{L}(\mathcal{E}_+)$ is the C^* -algebra generated by the creation operators T_ξ for $\xi \in E$ where $T_\xi(a) = \xi a$ and

$$T_\xi(\xi_1 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n.$$

Note that $T_\xi^* T_\eta = \varphi_+(\langle \xi, \eta \rangle_A)$ for $\xi, \eta \in E$. Since E is full, $\varphi_+(A) \subset \mathcal{T}_E$; let $\iota : A \hookrightarrow \mathcal{T}_E$ denote the embedding. One may also define T_ξ for $\xi \in E^{\otimes n}$ in an analogous manner and we have $T_\xi^* T_\eta = \iota(\langle \xi, \eta \rangle_A)$ for $\xi, \eta \in E^{\otimes n}$.

There is an embedding $\iota_n : \mathcal{K}(E^{\otimes n}) \hookrightarrow \mathcal{T}_E$ (identify $\mathcal{K}(E^{\otimes 0})$ with A), given for $n > 0$ by $\iota_n(\theta_{\xi, \eta}) = T_\xi T_\eta^*$ for $\xi, \eta \in E^{\otimes n}$. Such operators preserve the grading of \mathcal{E}_+ and there is an embedding $\mathcal{K}(E^{\otimes n}) \hookrightarrow \mathcal{L}(E^{\otimes m})$ for $m \geq n$. Let C_n denote the C^* -subalgebra of \mathcal{T}_E generated by operators of the form $T_\xi T_\eta^*$ for $\xi, \eta \in E^{\otimes k}$ with $k \leq n$ (by convention $C_0 = \iota(A)$). Then the C_n form an ascending family of C^* -subalgebras.

Remark 1.3. With notation as above the natural map $C_n \rightarrow \mathcal{L}(E^{\otimes m})$ is an embedding for $m \geq n$. Suppose $\varphi(A) \cap \mathcal{K}(E) = \{0\}$; then by [Pm, Corollary 3.14] $\mathcal{T}_E \cong \mathcal{O}_E$ and the inclusion $A \hookrightarrow \mathcal{O}_E$ induces a KK -equivalence (see [Pm, Corollary 4.5]). Under the isomorphism of \mathcal{T}_E with \mathcal{O}_E , $\bigcup_n C_n$ is mapped to \mathcal{F}_E , the analog of the AF core of a Cuntz–Krieger algebra.

For the remainder of this section we shall assume that $\varphi(A) \cap \mathcal{K}(E) = \{0\}$ and identify \mathcal{T}_E with \mathcal{O}_E .

Proposition 1.4. *For each $n \in \mathbb{N}$ the C^* -subalgebra J_n generated by the $\iota_k(\mathcal{K}(E^{\otimes k}))$ for $k \geq n$ is an essential ideal in \mathcal{F}_E . We obtain a descending sequence of ideals*

$$J_0 \supset J_1 \supset J_2 \supset \cdots$$

with $J_0 = \mathcal{F}_E$ and $\bigcap_n J_n = \{0\}$. Furthermore, $J_n/J_{n+1} \cong \mathcal{K}(E^{\otimes n})$ (thus J_n/J_{n+1} is strongly Morita equivalent to A) and the restriction of the quotient map yields an isomorphism $C_n \cong \mathcal{F}_E/J_{n+1}$.

Proof. Given $n \in \mathbb{N}$ it is clear that J_n is an ideal (see [Pm, Definition 2.1]). To see that J_n is essential it suffices to show that for every m and nonzero element $c \in C_m$ there is an element $d \in \mathcal{K}(E^{\otimes k})$ for some $k \geq n$ such that $\iota_k(d) \neq 0$. Let k be an integer with $k \geq \max(m, n)$; since the map from

C_m to $\mathcal{L}(E^{\otimes k})$ is an embedding for $k \geq m$, $c\xi \neq 0$ for some $\xi \in E^{\otimes k}$. Then $cT_\xi T_\xi^* \neq 0$ and we take $d = \theta_{\xi, \xi}$.

The J_n form a descending sequence of ideals by construction. Since $\varphi(A)$ and $\mathcal{K}(E)$ have trivial intersection and $\mathcal{K}(E) \hookrightarrow \mathcal{L}(E^{\otimes k})$ is nondegenerate for $k \geq 1$, the image of A in $\mathcal{L}(E^{\otimes k})$ has trivial intersection with $\mathcal{K}(E^{\otimes k})$ for $k \geq 1$; it follows that

$$\iota_m(\mathcal{K}(E^{\otimes m})) \cap \iota_n(\mathcal{K}(E^{\otimes n})) = \{0\}$$

and, hence, $C_m \cap J_n = \{0\}$ for $m < n$. Thus, $\bigcap_n J_n = \{0\}$, for \mathcal{F}_E is the inductive limit of the C_m . Further, for each n we have

$$J_n = \iota_n(\mathcal{K}(E^{\otimes n})) + J_{n+1} \quad \text{and} \quad \iota_n(\mathcal{K}(E^{\otimes n})) \cap J_{n+1} = \{0\};$$

it follows that $J_n/J_{n+1} \cong \mathcal{K}(E^{\otimes n})$. Finally, since

$$\mathcal{F}_E = C_n + J_{n+1} \quad \text{and} \quad C_n \cap J_{n+1} = \{0\},$$

we have $C_n \cong \mathcal{F}_E/J_{n+1}$. □

There is a strongly continuous action

$$\lambda : \mathbb{T} \rightarrow \text{Aut}(\mathcal{O}_E)$$

such that $\lambda_t(T_\xi) = tT_\xi$. The fixed point algebra under this action is \mathcal{F}_E and we have a faithful conditional expectation $P_E : \mathcal{O}_E \rightarrow \mathcal{F}_E$ given by

$$P_E(x) = \int_{\mathbb{T}} \lambda_t(x) dt.$$

Consider the spectral subspaces of \mathcal{O}_E under this action: for $n \in \mathbb{Z}$

$$(\mathcal{O}_E)_n = \{x \in \mathcal{O}_E : \lambda_t(x) = t^n x \text{ for all } t \in \mathbb{T}\}.$$

Remark 1.5. Note that $(\mathcal{O}_E)_n$ is the closure of the span of elements of the form $T_\xi T_\eta^*$, where $\xi \in E^{\otimes k}$ and $\eta \in E^{\otimes l}$ with $n = k - l$. For $n \geq 0$ and $x \in (\mathcal{O}_E)_n$ we have $x^*x \in \mathcal{F}_E$ and $xx^* \in J_n$. We may regard $(\mathcal{O}_E)_n$ as a J_n - \mathcal{F}_E -equivalence bimodule (or J_n - \mathcal{F}_E -imprimitivity bimodule; see [Ri1, Def. 6.10]). Hence, J_n is strongly Morita equivalent to \mathcal{F}_E for each $n \geq 0$ (see [Ri2, Def. 1.1], [L, p. 74]). If we regard $(\mathcal{O}_E)_1$ as a Hilbert \mathcal{F}_E -bimodule, we have

$$E \otimes_A \mathcal{F}_E \cong (\mathcal{O}_E)_1,$$

where the isomorphism is implemented by the map $\xi \otimes a \mapsto T_\xi a$ (the Hilbert \mathcal{F}_E -module $E \otimes_A \mathcal{F}_E$ is denoted E_∞ in [Pm, §2]). The crossed product $\mathcal{O}_E \rtimes_\lambda \mathbb{T}$ may be identified with the closure of the subalgebra of $\mathcal{O}_E \otimes \mathcal{K}(\ell^2(\mathbb{Z}))$ consisting of finite sums of the form

$$\sum x_{ij} \otimes e_{ij},$$

where e_{ij} are the standard rank-one partial isometries in $\mathcal{K}(\ell^2(\mathbb{Z}))$ and $x_{ij} \in (\mathcal{O}_E)_{j-i}$.

Let $\widehat{\lambda} : \mathbb{Z} \rightarrow \text{Aut}(\mathcal{O}_E \rtimes_{\lambda} \mathbb{T})$ denote the dual automorphism group.

Proposition 1.6. *There is an embedding $\epsilon : \mathcal{F}_E \hookrightarrow \mathcal{O}_E \rtimes_{\lambda} \mathbb{T}$ onto a corner and a collection of essential ideals $\{I_n\}_{n \in \mathbb{Z}}$ in $\mathcal{O}_E \rtimes_{\lambda} \mathbb{T}$ satisfying the following conditions:*

- (i) *For all $n \in \mathbb{Z}$, \mathcal{F}_E is strongly Morita equivalent to I_n and A is strongly Morita equivalent to I_n/I_{n+1} .*
- (ii) *For all $n \geq 0$, $\epsilon(J_n) = \epsilon(1)I_n\epsilon(1)$.*
- (iii) *$I_n \subset I_m$ if $m \leq n$.*
- (iv) *$\bigcap_n I_n = \{0\}$.*
- (v) *$\overline{\bigcup_n I_n} = \mathcal{O}_E \rtimes_{\lambda} \mathbb{T}$.*
- (vi) *$\widehat{\lambda}_k(I_n) = I_{n+k}$.*

Proof. We use the identification, given in Remark 1.5, between $\mathcal{O}_E \rtimes_{\lambda} \mathbb{T}$ with a C^* -subalgebra of $\mathcal{O}_E \otimes \mathcal{K}(\ell^2(\mathbb{Z}))$. For each n let I_n be the ideal generated by $p_n = 1 \otimes e_{nn}$. Since $\mathcal{F}_E = (\mathcal{O}_E)_0$, it follows that \mathcal{F}_E is isomorphic to the corner determined by p_n and thus is strongly Morita equivalent to I_n . The desired embedding $\epsilon : \mathcal{F}_E \hookrightarrow \mathcal{O}_E \rtimes_{\lambda} \mathbb{T}$ is given by $\epsilon(a) = a \otimes e_{00}$.

Given an element of the form $a_{mn} = x_{mn} \otimes e_{mn}$ in $\mathcal{O}_E \rtimes_{\lambda} \mathbb{T}$ with $m \leq n$, we have

$$a_{mn}^* a_{mn} = x_{mn}^* x_{mn} \otimes e_{nn} \quad \text{and} \quad a_{mn} a_{mn}^* = x_{mn} x_{mn}^* \otimes e_{mm},$$

with $x_{mn} x_{mn}^* \in J_{n-m}$; since p_n may be expressed as a finite sum of elements of the form $a_{mn}^* a_{mn}$, it follows that $I_n \subset I_m$ and that

$$(*) \quad p_m I_n p_m = J_{n-m} \otimes e_{mm}.$$

Moreover, I_n is essential in I_m , since J_{n-m} is an essential ideal in \mathcal{F}_E (by Proposition 1.4). Since $q_n = \sum_{i=-n}^n p_i \in I_n$ and $\{q_n\}_n$ forms an approximate identity, we have $\overline{\bigcup_n I_n} = \mathcal{O}_E \rtimes_{\lambda} \mathbb{T}$. Thus I_n is an essential ideal in $\mathcal{O}_E \rtimes_{\lambda} \mathbb{T}$ for all $n \in \mathbb{Z}$. Assertion (ii) follows immediately from (*). Assertion (vi) follows from the fact that $\widehat{\lambda}_k(p_n) = 1 \otimes p_{n+k}$. The remaining assertions follow from Proposition 1.4. □

2. \mathcal{O}_E is simple and purely infinite

Let A be a separable unital C^* -algebra and let $\pi : A \rightarrow \mathcal{L}(\mathfrak{H})$ be a faithful nondegenerate representation of A on a separable nontrivial Hilbert space \mathfrak{H} ; since π is nondegenerate we have $\pi(1) = 1$.

Proposition 2.1. *With A and $\pi : A \rightarrow \mathcal{L}(\mathfrak{H})$ as above,*

$$E = \mathfrak{H} \otimes_{\mathbb{C}} A$$

is a full Hilbert bimodule over A under the operations

$$\langle \xi \otimes a, \eta \otimes b \rangle_A = \langle \xi, \eta \rangle a^* b, \quad \varphi(a)(\xi \otimes b) = \pi(a)\xi \otimes b$$

for all $\xi, \eta \in \mathfrak{H}$ and $a, b \in A$. Moreover, if $\pi(A) \cap \mathcal{K}(\mathfrak{H}) = \{0\}$, then $\varphi(A) \cap \mathcal{K}(E) = \{0\}$ and $\mathcal{O}_E \cong \mathcal{T}_E$.

Proof. $E = \mathfrak{H} \otimes_{\mathbb{C}} A$ is the tensor product of the Hilbert A - \mathbb{C} -bimodule \mathfrak{H} and the Hilbert \mathbb{C} - A -bimodule A as defined by Rieffel in [Ri1, Theorem 5.9] (see also [L, p. 41]). The natural map from $\mathcal{L}(\mathfrak{H})$ to $\mathcal{L}(E) = \mathcal{L}(\mathfrak{H} \otimes_{\mathbb{C}} A)$ induces an embedding $\mathcal{L}(\mathfrak{H})/\mathcal{K}(\mathfrak{H}) \hookrightarrow \mathcal{L}(E)/\mathcal{K}(E)$ (since $\mathcal{K}(\mathfrak{H})$ is mapped into $\mathcal{K}(E)$ and the Calkin algebra $\mathcal{L}(\mathfrak{H})/\mathcal{K}(\mathfrak{H})$ is simple). Hence, if $\pi(A) \cap \mathcal{K}(\mathfrak{H}) = \{0\}$, then $\varphi(A) \cap \mathcal{K}(E) = \{0\}$. The last assertion, $\mathcal{O}_E \cong \mathcal{T}_E$, follows by [Pm, Corollary 3.14]. \square

Henceforth, we assume that $\pi(A) \cap \mathcal{K}(\mathfrak{H}) = \{0\}$ and identify \mathcal{O}_E with \mathcal{T}_E . The aim of this section is to show that \mathcal{O}_E is simple and purely infinite. Simplicity may be proven directly by invoking [Sc2, Theorem 3.9]: if A is unital and E is full, then \mathcal{O}_E is simple if and only if E is minimal and nonperiodic. Lemma 2.3 would then be a consequence of [OP1, Theorem 6.5]. We follow a more indirect route patterned on the proof of [Rø, Theorem 2.1]; this will also show that \mathcal{O}_E is purely infinite.

Remark 2.2. With $E = \mathfrak{H} \otimes_{\mathbb{C}} A$ as above, we have $E^{\otimes n} \cong \mathfrak{H}^{\otimes n} \otimes_{\mathbb{C}} A$ via the map

$$(\xi_1 \otimes a_1) \otimes (\xi_2 \otimes a_2) \otimes \cdots \otimes (\xi_n \otimes a_n) \mapsto (\xi_1 \otimes \pi(a_1)\xi_2 \otimes \cdots \otimes \pi(a_{n-1})\xi_n) \otimes a_n.$$

If $\sigma : A \rightarrow \mathcal{L}(\mathfrak{K})$ is a nondegenerate representation of A on a Hilbert space \mathfrak{K} , then

$$E \otimes_A \mathfrak{K} \cong \mathfrak{H} \otimes_{\mathbb{C}} A \otimes_A \mathfrak{K} \cong \mathfrak{H} \otimes_{\mathbb{C}} \mathfrak{K}$$

and, hence,

$$E^{\otimes n} \otimes_A \mathfrak{K} \cong E^{\otimes n-1} \otimes_A E \otimes_A \mathfrak{K} \cong E^{\otimes n-1} \otimes_A \mathfrak{H} \otimes_{\mathbb{C}} \mathfrak{K}.$$

Recall that the action of \mathcal{F}_E on Fock space preserves the natural grading. Let $\tilde{\sigma}_n$ denote the representation of \mathcal{F}_E on $E^{\otimes n} \otimes_A \mathfrak{K}$ given by left action on $E^{\otimes n}$. Then the restriction of $\tilde{\sigma}_n$ to C_{n-1} is faithful: indeed, this follows from the facts that the natural map

$$\mathcal{L}(E^{\otimes n-1}) \rightarrow \mathcal{L}(E^{\otimes n-1} \otimes_A \mathfrak{H} \otimes_{\mathbb{C}} \mathfrak{K}) \cong \mathcal{L}(E^{\otimes n} \otimes_A \mathfrak{K})$$

is an embedding (since π is faithful) and that $\tilde{\sigma}_n|_{\mathcal{K}(E^{\otimes n-1})}$ factors through $\mathcal{L}(E^{\otimes n-1})$. Note that $\tilde{\sigma}_n$ is equivalent to the representation of \mathcal{F}_E obtained from σ as follows: use the strong Morita equivalence between A and J_n/J_{n+1} to obtain a representation of J_n/J_{n+1} and extend this to a representation of \mathcal{F}_E . Since the restriction of $\tilde{\sigma}_n$ to C_{n-1} is faithful, $\ker \tilde{\sigma}_n \subset J_n$ (see Proposition 1.4). It follows that the closure of a point in $\widehat{J_n} - \widehat{J_{n+1}}$ contains the complement of $\widehat{J_n}$. A similar assertion holds for $\mathcal{O}_E \rtimes_{\lambda} \mathbb{T}$: for any $n \in \mathbb{Z}$ the closure of a point in $\widehat{I_n} - \widehat{I_{n+1}}$ contains the complement of $\widehat{I_n}$.

Lemma 2.3. *With A and E as above, $\Gamma(\widehat{\lambda}_1) = \mathbb{T}$, where $\widehat{\lambda}$ is the dual action of \mathbb{Z} on $\mathcal{O}_E \rtimes_{\lambda} \mathbb{T}$.*

Proof. By [OP2, Theorem 4.6] it suffices to find a dense invariant subset of $(\mathcal{O}_E \rtimes_{\lambda} \mathbb{T})^{\widehat{}}$ on which $\widehat{\lambda}_1^*$ acts freely. That is, we must find an irreducible representation σ of $\mathcal{O}_E \rtimes_{\lambda} \mathbb{T}$ such that $\{[\sigma \circ \widehat{\lambda}_n] : n \in \mathbb{Z}\}$, the orbit of the unitary equivalence class of σ under $\widehat{\lambda}^*$, is dense in $(\mathcal{O}_E \rtimes_{\lambda} \mathbb{T})^{\widehat{}}$ and satisfies $[\sigma \circ \widehat{\lambda}_m] \neq [\sigma \circ \widehat{\lambda}_n]$ if $m \neq n$. Let σ_0 be an irreducible representation of A and use the strong Morita equivalence between A and I_0/I_1 to obtain an irreducible representation σ' of I_0/I_1 . Then σ , the extension of σ' to $\mathcal{O}_E \rtimes_{\lambda} \mathbb{T}$, is also irreducible. The classes $[\sigma \circ \widehat{\lambda}_n]$ are distinct, for if $m < n$, $\sigma \circ \widehat{\lambda}_m$ vanishes on I_n . Moreover, for each $n \in \mathbb{Z}$ the closure of $[\sigma \circ \widehat{\lambda}_n]$ in $(\mathcal{O}_E \rtimes_{\lambda} \mathbb{T})^{\widehat{}}$ includes the classes of all irreducible representations that vanish on I_n (since $[\sigma \circ \widehat{\lambda}_n] \in \widehat{I_n - I_{n+1}}$; see Remark 2.2). Hence, $\{[\sigma \circ \widehat{\lambda}_n] : n \in \mathbb{Z}\}$ is dense in $(\mathcal{O}_E \rtimes_{\lambda} \mathbb{T})^{\widehat{}}$. \square

Using Takesaki–Takai duality we show below that a C^* -algebra D equipped with an action α of \mathbb{T} may be embedded equivariantly as a corner in $(D \rtimes_{\alpha} \mathbb{T}) \rtimes_{\widehat{\alpha}} \mathbb{Z}$. This fact is related to Rosenberg’s observation that the fixed point algebra under a compact group action embeds as a corner in the crossed product (see [Ro]).

Proposition 2.4. *Given a unital C^* -algebra D and a strongly continuous action $\alpha : \mathbb{T} \rightarrow \text{Aut}(D)$, there is an isomorphism ψ of D onto a full corner of $(D \rtimes_{\alpha} \mathbb{T}) \rtimes_{\widehat{\alpha}} \mathbb{Z}$ which is equivariant in the sense that $\widehat{\alpha}_t \circ \psi = \psi \circ \alpha_t$ for all $t \in \mathbb{T}$. Moreover, $\psi(1) \in D \rtimes_{\alpha} \mathbb{T}$.*

Proof. By Takesaki–Takai duality [Pd, 7.9.3] there is an isomorphism

$$\gamma : D \otimes \mathcal{K}(L^2(\mathbb{T})) \cong (D \rtimes_{\alpha} \mathbb{T}) \rtimes_{\widehat{\alpha}} \mathbb{Z},$$

which is equivariant with respect to $\alpha \otimes \text{Ad } \rho$ and $\widehat{\alpha}$ (where ρ is the right regular representation of \mathbb{T} on $L^2(\mathbb{T})$). The desired embedding is obtained by finding an $\text{Ad } \rho$ invariant minimal projection p in $\mathcal{K}(L^2(\mathbb{T}))$ (cf. [Ro]): set $\psi(d) = \gamma(d \otimes p)$ for $d \in D$. Since ψ is equivariant, $\psi(1)$ is in the fixed point algebra of $\widehat{\alpha}$; hence, $\psi(1) \in D \rtimes_{\alpha} \mathbb{T}$. \square

The following lemma is adapted from [Rø, Lemma 2.4]; the proof follows Rørdam’s but we substitute [OP3, Lemma 7.1] for [Ks, Lemma 3.2].

Lemma 2.5. *Let B be a C^* -algebra, let β be an automorphism of B such that $\Gamma(\beta) = \mathbb{T}$, and let P denote the canonical conditional expectation from $B \rtimes_{\beta} \mathbb{Z}$ to B . For every positive element $y \in B \rtimes_{\beta} \mathbb{Z}$ and $\varepsilon > 0$ there are positive elements $x, b \in B$ such that*

$$\|b\| > \|P(y)\| - \varepsilon, \quad \|x\| \leq 1 \quad \text{and} \quad \|xyx - b\| < \varepsilon.$$

If y is in the corner determined by a projection $p \in B$, then x, b may also be chosen to be in the corner.

Proof. As in the proof of [RØ, Lemma 2.4] we may assume (by perturbing y if necessary) that y is of the form

$$y = y_{-n}u^{-n} + \dots + y_{-1}u^{-1} + y_0 + y_1u + \dots + y_nu^n$$

for some n , where $y_j \in B$ and u is the canonical unitary in $B \rtimes_{\beta} \mathbb{Z}$ implementing the automorphism β ; note that $y_0 = P(y)$ is positive.

By [OP3, Theorem 10.4] β^k is properly outer for all $k \neq 0$. Hence, by [OP3, Lemma 7.1] there is a positive element x with $\|x\| = 1$ such that

$$\|xy_0x\| > \|y_0\| - \varepsilon \quad \text{and} \quad \|xy_ku^kx\| = \|xy_k\beta^k(x)\| < \varepsilon/2n \quad \text{for } 0 < |k| \leq n.$$

Set $b = xy_0x$; then a straightforward calculation yields $\|xyx - b\| < \varepsilon$. We now verify the last assertion. Suppose that y is in the corner determined by a projection $p \in B$; we may again assume that y is of the above form. Since P is a conditional expectation onto B , $y_0 = P(y)$ is also in the corner determined by p . In the proof of [OP3, Lemma 7.1] the positive element x is constructed in the hereditary subalgebra determined by y_0 ; hence we may assume that x and therefore also $b = xy_0x$ lies in the same corner. \square

Recall that C_n is the C^* -subalgebra of \mathcal{F}_E generated by operators of the form $T_{\xi}T_{\eta}^*$ for $\xi, \eta \in E^{\otimes k}$ with $k \leq n$ and that they form an ascending family of C^* -subalgebras with dense union. The subspace $E^{\otimes n}$ is left invariant by C_n and there is an embedding $C_n \hookrightarrow \mathcal{L}(E^{\otimes n})$.

Lemma 2.6. *Given a positive element $c \in C_n$ and $\varepsilon > 0$, there is $\xi \in E^{\otimes n}$ with $\|\xi\| = 1$ such that $T_{\xi}^*cT_{\xi} \in C_0$ and $\|T_{\xi}^*cT_{\xi}\| > \|c\| - \varepsilon$.*

Proof. The first assertion follows from a straightforward calculation: given $c \in C_n$ and $\xi \in E^{\otimes n}$, we have $c\xi \in E^{\otimes n}$ and

$$T_{\xi}^*cT_{\xi} = T_{\xi}^*T_{c\xi} = \iota(\langle \xi, c\xi \rangle_A) \in C_0.$$

The second assertion follows from the embedding $C_n \hookrightarrow \mathcal{L}(E^{\otimes n})$ and the fact that

$$\|d\| = \sup \{ \|\langle \xi, d\xi \rangle_A\| : \xi \in E^{\otimes n}, \|\xi\| = 1 \}$$

for $d \in \mathcal{L}(E^{\otimes n})$ positive. \square

Lemma 2.7. *Given a positive element $a \in A$ and $\varepsilon > 0$ with $\|a\| > \varepsilon$, there is $\eta \in E$ with $\|\eta\| \leq (\|a\| - \varepsilon)^{-1/2}$ such that $T_{\eta}^*\iota(a)T_{\eta} = 1$.*

Proof. Let f be a continuous nonzero real-valued function supported on the interval $[\|a\| - \varepsilon, \|a\|]$ and choose a vector $\zeta \in \pi(f(a))\mathfrak{H}$ such that $\langle \zeta, \pi(a)\zeta \rangle = 1$; we have

$$(\|a\| - \varepsilon)\|\zeta\|^2 \leq \|\langle \zeta, \pi(a)\zeta \rangle\| = 1.$$

Then $\eta = \zeta \otimes 1 \in E$ satisfies the desired conditions. \square

It will now follow that \mathcal{O}_E is simple and purely infinite (compare the proof of [Rø, Theorem 2.1]).

Theorem 2.8. *For every nonzero positive element $d \in \mathcal{O}_E$ there is a $z \in \mathcal{O}_E$ such that $z^*dz = 1$. Hence, \mathcal{O}_E is simple and purely infinite.*

Proof. Let $d \in \mathcal{O}_E$ be a nonzero positive element and choose ε such that $0 < \varepsilon < \frac{1}{4}\|P(d)\|$. By Proposition 2.4 there is a \mathbb{T} -equivariant isomorphism ψ from \mathcal{O}_E onto a corner of $(\mathcal{O}_E \rtimes_\lambda \mathbb{T}) \rtimes_{\widehat{\lambda}} \mathbb{Z}$ determined by a projection $p \in \mathcal{O}_E \rtimes_\lambda \mathbb{T}$. We now apply Lemma 2.5 to the element $y = \psi(d)$ and the automorphism $\beta = \widehat{\lambda}_1$ (note that $\Gamma(\widehat{\lambda}_1) = \mathbb{T}$ by Lemma 2.3). We identify \mathcal{O}_E with the corner determined by p ; under this identification \mathcal{F}_E is identified with $p(\mathcal{O}_E \rtimes_\lambda \mathbb{T})p$. There are then positive elements $x, b \in \mathcal{F}_E$ such that

$$\|b\| > \|P(d)\| - \varepsilon, \quad \|x\| \leq 1 \quad \text{and} \quad \|xdx - b\| < \varepsilon.$$

Since $\bigcup_n C_n$ is dense in \mathcal{F}_E we may assume that $b \in C_n$ for some n . Hence, by Lemma 2.6 there is $\xi \in E^{\otimes n}$ with $\|\xi\| = 1$ such that

$$T_\xi^*bT_\xi \in C_0 \quad \text{and} \quad \|T_\xi^*bT_\xi\| > \|b\| - \varepsilon.$$

Let a denote the unique element of A such that $\iota(a) = T_\xi^*bT_\xi$; then $\|a\| > \|P(d)\| - 2\varepsilon$ and

$$\|T_\xi^*xdxT_\xi - \iota(a)\| = \|T_\xi^*(xdx - b)T_\xi\| < \varepsilon.$$

By Lemma 2.7 there is $\eta \in E$ such that $T_\eta^*\iota(a)T_\eta = 1$ and

$$\|\eta\| \leq (\|a\| - \varepsilon)^{-1/2} < (\|P(d)\| - 3\varepsilon)^{-1/2} < \varepsilon^{-1/2}.$$

It follows that

$$\begin{aligned} \|T_\eta^*T_\xi^*xdxT_\xi T_\eta - 1\| &= \|T_\eta^*(T_\xi^*xdxT_\xi - \iota(a))T_\eta\| \\ &\leq \|T_\xi^*xdxT_\xi - \iota(a)\|(\varepsilon^{-1/2})^2 < 1. \end{aligned}$$

Therefore, $c = T_\eta^*T_\xi^*xdxT_\xi T_\eta$ is an invertible positive element and we take $z = xT_\xi T_\eta c^{-1/2}$. □

3. Applications and concluding remarks

We collect some applications of the theorem above and consider certain connections with the theory of reduced (amalgamated) free product C^* -algebras. First we consider criteria under which the Kirchberg–Phillips Theorem applies (see [Kr, Theorem C], [Ph, Corollary 4.2.2]).

Theorem 3.1. *Let A be a separable nuclear unital C^* -algebra belonging to the bootstrap class to which the UCT applies (see [RS]); let $\pi : A \rightarrow \mathcal{L}(\mathfrak{H})$ be a faithful nondegenerate representation of A on a nontrivial separable Hilbert space \mathfrak{H} such that $\pi(A) \cap \mathcal{K}(\mathfrak{H}) = \{0\}$ and let E denote the Hilbert A -bimodule $\mathfrak{H} \otimes_{\mathbb{C}} A$. Then \mathcal{O}_E is a unital Kirchberg algebra (simple, purely*

infinite, separable and nuclear) belonging to the bootstrap class. Hence, the Kirchberg–Phillips Theorem applies and the isomorphism class of \mathcal{O}_E only depends on $(K_*(A), [1_A])$ and not on the choice of representation π .

Proof. By Theorem 2.8, \mathcal{O}_E is simple and purely infinite. If A is nuclear, the argument given in the proof of [DS, Theorem 2.1] shows that \mathcal{O}_E must also be nuclear (alternatively, the nuclearity of \mathcal{O}_E follows from the structural results discussed in §1). Hence, \mathcal{O}_E is a unital Kirchberg algebra. Recall that the inclusion $A \hookrightarrow \mathcal{O}_E$ defines a KK -equivalence (see [Pm, Corollary 4.5]) that induces a unit-preserving isomorphism $K_*(A) \cong K_*(\mathcal{O}_E)$. Hence, if A is in the bootstrap class, so is \mathcal{O}_E . Therefore, the Kirchberg–Phillips Theorem applies and the isomorphism class of \mathcal{O}_E only depends on $(K_*(A), [1_A])$. \square

Let X be a second countable compact Hausdorff space, let μ be a nonatomic Borel measure with full support and let

$$\pi : C(X) \rightarrow \mathcal{L}(L^2(X, \mu))$$

be the representation given by multiplication of functions. Then π is faithful and

$$\pi(C(X)) \cap \mathcal{K}(L^2(X, \mu)) = \{0\}.$$

Hence, we may apply Theorem 3.1 with $A = C(X)$ and $\mathfrak{H} = L^2(X, \mu)$.

Corollary 3.2. *Let X and μ be as above. Then*

$$E = L^2(X, \mu) \otimes_{\mathbb{C}} C(X)$$

is a Hilbert bimodule over $C(X)$ and \mathcal{O}_E is a unital Kirchberg algebra. The embedding $C(X) \hookrightarrow \mathcal{O}_E$ induces a (unit preserving) KK -equivalence. Hence, the isomorphism class of \mathcal{O}_E only depends on $(K_(C(X)), [1_{C(X)}])$ (and not on μ); moreover, if X is contractible, then $\mathcal{O}_E \cong \mathcal{O}_\infty$.*

The next proposition is Theorem 5.6 of [L] (see also [Ka, Theorem 3]); Lance calls this the Kasparov–Stinespring–Gelfand–Naimark–Segal construction.

Proposition 3.3. *Let B and C be C^* -algebras, let F be a Hilbert C -module and let $f : B \rightarrow \mathcal{L}(F)$ be a completely positive map. Then there is a Hilbert C -module E_f , a $*$ -homomorphism $\varphi_f : B \rightarrow \mathcal{L}(E_f)$ and an element $v_f \in \mathcal{L}(F, E_f)$ such that $f(b) = v_f^* \varphi_f(b) v_f$ and $\varphi_f(B) v_f F$ is dense in E_f .*

I am grateful to D. Shlyakhtenko for the following observation. Let \mathcal{T} be the “usual” Toeplitz algebra (\mathcal{T}_E , where E is the 1-dimensional Hilbert bimodule over \mathbb{C}) and let g denote the vacuum state on \mathcal{T} .

Proposition 3.4. *Let A be a separable unital C^* -algebra and let $\pi : A \rightarrow \mathcal{L}(\mathfrak{H})$ be a faithful representation of A on a separable Hilbert space \mathfrak{H} such that π has a cyclic vector $\xi \in \mathfrak{H}$. Let f denote the vector state $\langle \xi, \pi(\cdot) \xi \rangle$ and let \tilde{f} denote the corresponding completely positive map from A to $\mathcal{L}(A)$*

(given by $\tilde{f}(a) = f(a)1$). Then $E = E_{\tilde{f}} \cong \mathfrak{H} \otimes A$ and \mathcal{T}_E may be realized as a reduced free product (see [A, V]):

$$(\mathcal{T}_E, h) \cong (A, f) * (\mathcal{T}, g) \quad \text{for some state } h \text{ on } \mathcal{T}_E.$$

Proof. This follows from [Sh, Theorem 2.3, Corollary 2.5]. \square

As a result of this observation, at least part of Corollary 3.2 follows from the existing literature on reduced free products. The simplicity follows from a theorem of Dykema [Dy, Theorem 2]. Criteria for when reduced free products are purely infinite have been found by Choda, Dykema and Rørdam in a series of papers [DR1, DR2, DC]; but none seem to apply generally to the case considered in the corollary.

A theorem of Speicher (see [Sp]) on reduced amalgamated free products (see [V, §5]) and Toeplitz algebras associated to Hilbert bimodules yields a curious stability property of the algebras we have been considering. The following is the version given in [BDS, Theorem 2.4].

Proposition 3.5. *Suppose that E_1 and E_2 are full Hilbert bimodules over the C^* -algebra A . Then*

$$\mathcal{T}_{E_1 \oplus E_2} = \mathcal{T}_{E_1} *_A \mathcal{T}_{E_2}.$$

Corollary 3.6. *Let A be a separable nuclear unital C^* -algebra belonging to the bootstrap class to which the UCT applies (see [RS]) and let $\pi : A \rightarrow \mathcal{L}(\mathfrak{H})$ be a faithful representation of A on a separable Hilbert space \mathfrak{H} such that $\pi(A) \cap \mathcal{K}(\mathfrak{H}) = \{0\}$. Let E be the Hilbert bimodule $\mathfrak{H} \otimes_{\mathbb{C}} A$. Then*

$$\mathcal{O}_E \cong \mathcal{O}_E *_A \mathcal{O}_E.$$

Proof. Observe that $E \oplus E = (\mathfrak{H} \oplus \mathfrak{H}) \otimes_{\mathbb{C}} A$. Since $\pi \oplus \pi : A \rightarrow \mathcal{L}(\mathfrak{H} \oplus \mathfrak{H})$ is a faithful representation and $(\pi \oplus \pi)(A) \cap \mathcal{K}(\mathfrak{H} \oplus \mathfrak{H}) = \{0\}$, the result follows from Theorem 3.1 and the above proposition. \square

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