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In memory of Gert K. Pedersen.

Let A be a separable unital C*-algebra. Let $\pi : A \to \mathcal{L}(\mathfrak{H})$ be a faithful representation of A on a separable Hilbert space $\mathfrak{H}$ such that $\pi(A) \cap \mathcal{K}(\mathfrak{H}) = \{0\}$. We show that $\mathcal{O}_E$, the Cuntz–Pimsner algebra associated to the Hilbert A-bimodule $E = \mathfrak{H} \otimes_{\mathbb{C}} A$, is simple and purely infinite. If A is nuclear and belongs to the bootstrap class to which the UCT applies, the same applies to $\mathcal{O}_E$. Hence by the Kirchberg–Phillips Theorem the isomorphism class of $\mathcal{O}_E$ only depends on the K-theory of A and the class of the unit.

In his seminal paper [**Pm**], Pimsner constructed a C*-algebra $\mathcal{O}_E$ from a Hilbert bimodule over a C*-algebra A as a quotient of a concrete C*-algebra $\mathcal{T}_E$, an analogue of the Toeplitz algebra, acting on the Fock space associated to E. There has recently been much interest in these Cuntz–Pimsner algebras (or Cuntz–Krieger–Pimsner algebras), which generalize both crossed products by $\mathbb{Z}$ and Cuntz–Krieger algebras, as well as the associated Toeplitz algebras. The structure of these C*-algebras is not yet fully understood, though considerable progress has been made. For example, Pimsner found a six-term exact sequence for the K-theory of $\mathcal{O}_E$ that generalizes the Pimsner–Voiculescu exact sequence (see [**Pm**, Theorem 4.8]); conditions for simplicity were found in [**Sc2**, **MS**, **KPW1**, **DPZ**] and for pure infiniteness in [**Z**].

The purpose of the present note is to analyze the structure of Cuntz–Pimsner algebras associated to a certain class of Hilbert bimodules. Let A be a separable unital C*-algebra and let $\pi : A \to \mathcal{L}(\mathfrak{H})$ be a faithful nondegenerate representation of A on a separable Hilbert space $\mathfrak{H}$ such that $\pi(A) \cap \mathcal{K}(\mathfrak{H}) = \{0\}$. Then $E = \mathfrak{H} \otimes_{\mathbb{C}} A$ is a Hilbert bimodule over A in a natural way. We show that $\mathcal{O}_E$ is separable, simple and purely infinite. If A is nuclear and in the bootstrap class, then the same holds for $\mathcal{O}_E$ and thus by the Kirchberg–Phillips theorem the isomorphism class of $\mathcal{O}_E$ is completely determined by the K-theory of A together with the class of the unit (since $\mathcal{O}_E$ is KK-equivalent to A).

Many examples of Cuntz–Pimsner algebras found in the literature arise from Hilbert bimodules that are finitely generated and projective; in such cases the left action must consist entirely of compact operators. Our examples do not fall in this class; in fact, the left action has trivial intersection with the compacts. And this has some interesting consequences: $\mathcal{O}_E \cong \mathcal{T}_E$ (see [**Pm**, Corollary 3.14]) and the natural embedding $A \hookrightarrow \mathcal{O}_E$ induces a KK-equivalence (see [**Pm**, Corollary 4.5]).

In §1 we review some basic facts concerning the construction of $\mathcal{T}_E$ as operators on the Fock space of E and the gauge action $\lambda : \mathbb{T} \to \mathrm{Aut}\,(\mathcal{T}_E)$. We assume that the left action of A does not meet the compacts $\mathcal{K}(E)$ and identify $\mathcal{O}_E$ with $\mathcal{T}_E$. The fixed point algebra $\mathcal{F}_E$, the analogue of the AF-core of a Cuntz–Krieger algebra, contains a canonical descending sequence of essential ideals indexed by $\mathbb{N}$ with trivial intersection. The crossed product $\mathcal{O}_E \rtimes_\lambda \mathbb{T}$ has a similar collection of essential ideals indexed by $\mathbb{Z}$ on which the dual group of automorphisms acts in a natural way. By Takesaki–Takai duality,

$$\mathcal{O}_E \otimes \mathcal{K}(L^2(\mathbb{T})) \cong (\mathcal{O}_E \rtimes_\lambda \mathbb{T}) \rtimes_{\widehat{\lambda}} \mathbb{Z}\,;$$

hence, much of the structure of $\mathcal{O}_E$ is revealed through an analysis of the double crossed product.

In §2 we show that if E is the Hilbert bimodule over A associated to a representation as described above, then for every nonzero positive element $d \in \mathcal{O}_E$ there is a $z \in \mathcal{O}_E$ such that $z^*dz = 1$; it follows that $\mathcal{O}_E$ is simple and purely infinite (see Theorem 2.8). The proof of this proceeds through a sequence of lemmas and is patterned on the proof of [**Rø**, Theorem 2.1], which is in turn based on a key lemma of Kishimoto (see [**Ks**, Lemma 3.2]). Our argument uses the version of this lemma found in [**OP3**, Lemma 7.1] and this requires that we show that the Connes spectrum of the dual action is full (this is also an ingredient in the proof of simplicity found in [**DPZ**]). We invoke a version of a key lemma of Rørdam for crossed products by $\mathbb{Z}$ that arise from automorphisms with full Connes spectrum. The fact that $\mathcal{O}_E$ embeds equivariantly into $(\mathcal{O}_E \rtimes_\lambda \mathbb{T}) \rtimes_{\widehat{\lambda}} \mathbb{Z}$ allows us to apply this lemma to $\mathcal{O}_E$.

In §3 we use the Kirchberg–Phillips theorem to collect some consequences of this theorem as indicated above and discuss certain connections with reduced (amalgamated) free products.

We fix some notation and terminology. Given a C*-algebra B we let $\widehat{B}$ denote its spectrum, that is, the collection of irreducible representations modulo unitary equivalence endowed with the Jacobson topology (see [**Pd**, §4.1]). If I is an ideal in a C*-algebra B, every irreducible representation of I extends uniquely to an irreducible representation of B. This allows one to identify $\widehat{I}$ with an open subset of $\widehat{B}$, the complement of which consists of the classes of irreducible representations that vanish on I. Given a *-automorphism β of a C*-algebra B, let $\Gamma(\beta)$ denote the Connes spectrum of β (see [**O**, **Co**] or [**Pd**, §8.8]); recall that

$$\Gamma(\beta) = \bigcap_H \mathrm{Sp}\,(\beta|_H)$$

where the intersection is taken over all nonzero β-invariant hereditary subalgebras H. A C*-algebra is said to be purely infinite if every nonzero hereditary subalgebra contains an infinite projection.

1. Preliminaries

We review some basic facts concerning Cuntz–Pimsner algebras; we shall be mainly interested in those that arise from bimodules for which the left action has trivial intersection with the compacts (see Remark 1.3). Let A be a C*-algebra.

Definition 1.1 (see [**L**, pp. 2–4], [**Ka**, pp. 134, 135] and [**Ri1**, Def. 2.1])**.** Let E be a right A-module. Then E is said to be a (right) pre-Hilbert A-module if it is equipped with an A-valued inner product $\langle\,\cdot\,,\cdot\,\rangle_A$ satisfying the following conditions for all $\xi,\eta,\zeta \in E$, $s,t \in \mathbb{C}$, and $a \in A$:

(i) $\langle \xi, s\eta + t\zeta\rangle_A = s\langle\xi,\eta\rangle_A + t\langle\xi,\zeta\rangle_A$.
(ii) $\langle\xi,\eta a\rangle_A = \langle\xi,\eta\rangle_A a$.
(iii) $\langle\eta,\xi\rangle_A = \langle\xi,\eta\rangle_A^*$.
(iv) $\langle\xi,\xi\rangle_A \geq 0$ and $\langle\xi,\xi\rangle_A = 0$ only if $\xi = 0$.

E is said to be a (right) Hilbert A-module if it is complete in the norm $\|\xi\| = \|\langle\xi,\xi\rangle_A\|^{1/2}$.

A Hilbert A-module E is said to be *full* if the span of the values of the inner product is dense. The collection of bounded adjointable operators on E, $\mathcal{L}(E)$, is a C*-algebra. The closure of the span of operators of the form $\theta_{\xi,\eta}$ for $\xi,\eta \in E$ (where $\theta_{\xi,\eta}(\zeta) = \xi\langle\eta,\zeta\rangle_A$ for $\zeta \in E$) forms an essential ideal in $\mathcal{L}(E)$, denoted by $\mathcal{K}(E)$. A Hilbert space is a Hilbert module over $\mathbb{C}$.

Definition 1.2. Let E be a Hilbert A-module and let $\varphi : A \to \mathcal{L}(E)$ be an injective $*$-homomorphism. The pair (E,φ) is said to be a Hilbert bimodule over A (or a Hilbert A-bimodule).

Pimsner defines the Cuntz–Pimsner algebra $\mathcal{O}_E$ as a quotient of the analogue of the Toeplitz algebra, $\mathcal{T}_E$, generated by creation operators on the Fock space of E (see [**Pm**]). The injectivity of φ is not really necessary (see [**Pm**, Remark 1.2(1)]). We will henceforth assume that E is full (see [**Pm**, Remark 1.2(3)]).

The Fock space of E is the Hilbert A-module

$$\mathcal{E}_+ = \bigoplus_{n=0}^{\infty} E^{\otimes n}$$

where $E^{\otimes 0} = A$, $E^{\otimes 1} = E$ and for $n > 1$, $E^{\otimes n}$ is the n-fold tensor product:

$$E^{\otimes n} = E \otimes_A \cdots \otimes_A E.$$

The tensor product used here is called the inner tensor product by Lance (see [**L**, p. 41], but note Lance uses different notation; see also Theorem 5.9 of [**Ri1**]). Observe that $\mathcal{E}_+$ is also a Hilbert A-bimodule with left action defined by $\varphi_+(a)b = ab$ for $a, b \in A = E^{\otimes 0}$ and

$$\varphi_+(a)(\xi_1 \otimes \cdots \otimes \xi_n) = \varphi(a)\, \xi_1 \otimes \cdots \otimes \xi_n$$

for $a \in A$ and $\xi_1 \otimes \cdots \otimes \xi_n \in E^{\otimes n}$.

Then $\mathcal{T}_E \subset \mathcal{L}(\mathcal{E}_+)$ is the C*-algebra generated by the creation operators T_ξ for $\xi \in E$ where $T_\xi(a) = \xi a$ and

$$T_\xi(\xi_1 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n.$$

Note that ${T_\xi}^* T_\eta = \varphi_+(\langle \xi, \eta \rangle_A)$ for $\xi, \eta \in E$. Since E is full, $\varphi_+(A) \subset \mathcal{T}_E$; let $\iota : A \hookrightarrow \mathcal{T}_E$ denote the embedding. One may also define T_ξ for $\xi \in E^{\otimes n}$ in an analogous manner and we have ${T_\xi}^* T_\eta = \iota(\langle \xi, \eta \rangle_A)$ for $\xi, \eta \in E^{\otimes n}$.

There is an embedding $\iota_n : \mathcal{K}(E^{\otimes n}) \hookrightarrow \mathcal{T}_E$ (identify $\mathcal{K}(E^{\otimes 0})$ with A), given for $n > 0$ by $\iota_n(\theta_{\xi,\eta}) = T_\xi {T_\eta}^*$ for $\xi, \eta \in E^{\otimes n}$. Such operators preserve the grading of $\mathcal{E}_+$ and there is an embedding $\mathcal{K}(E^{\otimes n}) \hookrightarrow \mathcal{L}(E^{\otimes m})$ for $m \geq n$. Let C_n denote the C*-subalgebra of $\mathcal{T}_E$ generated by operators of the form $T_\xi {T_\eta}^*$ for $\xi, \eta \in E^{\otimes k}$ with $k \leq n$ (by convention $C_0 = \iota(A)$). Then the C_n form an ascending family of C*-subalgebras.

Remark 1.3. With notation as above the natural map $C_n \to \mathcal{L}(E^{\otimes m})$ is an embedding for $m \geq n$. Suppose $\varphi(A) \cap \mathcal{K}(E) = \{0\}$; then by [**Pm**, Corollary 3.14] $\mathcal{T}_E \cong \mathcal{O}_E$ and the inclusion $A \hookrightarrow \mathcal{O}_E$ induces a KK-equivalence (see [**Pm**, Corollary 4.5]). Under the isomorphism of $\mathcal{T}_E$ with $\mathcal{O}_E$, $\overline{\bigcup_n C_n}$ is mapped to $\mathcal{F}_E$, the analog of the AF core of a Cuntz–Krieger algebra.

For the remainder of this section we shall assume that $\varphi(A) \cap \mathcal{K}(E) = \{0\}$ and identify $\mathcal{T}_E$ with $\mathcal{O}_E$.

Proposition 1.4. *For each $n \in \mathbb{N}$ the C*-subalgebra J_n generated by the $\iota_k(\mathcal{K}(E^{\otimes k}))$ for $k \geq n$ is an essential ideal in $\mathcal{F}_E$. We obtain a descending sequence of ideals*

$$J_0 \supset J_1 \supset J_2 \supset \cdots$$

with $J_0 = \mathcal{F}_E$ and $\bigcap_n J_n = \{0\}$. Furthermore, $J_n/J_{n+1} \cong \mathcal{K}(E^{\otimes n})$ (thus J_n/J_{n+1} is strongly Morita equivalent to A) and the restriction of the quotient map yields an isomorphism $C_n \cong \mathcal{F}_E/J_{n+1}$.

Proof. Given $n \in \mathbb{N}$ it is clear that J_n is an ideal (see [**Pm**, Definition 2.1]). To see that J_n is essential it suffices to show that for every m and nonzero element $c \in C_m$ there is an element $d \in \mathcal{K}(E^{\otimes k})$ for some $k \geq n$ such that $c\iota_k(d) \neq 0$. Let k be an integer with $k \geq \max(m, n)$; since the map from

C_m to $\mathcal{L}(E^{\otimes k})$ is an embedding for $k \geq m$, $c\xi \neq 0$ for some $\xi \in E^{\otimes k}$. Then $cT_\xi {T_\xi}^* \neq 0$ and we take $d = \theta_{\xi,\xi}$.

The J_n form a descending sequence of ideals by construction. Since $\varphi(A)$ and $\mathcal{K}(E)$ have trivial intersection and $\mathcal{K}(E) \hookrightarrow \mathcal{L}(E^{\otimes k})$ is nondegenerate for $k \geq 1$, the image of A in $\mathcal{L}(E^{\otimes k})$ has trivial intersection with $\mathcal{K}(E^{\otimes k})$ for $k \geq 1$; it follows that

$$\iota_m(\mathcal{K}(E^{\otimes m})) \cap \iota_n(\mathcal{K}(E^{\otimes n})) = \{0\}$$

and, hence, $C_m \cap J_n = \{0\}$ for $m < n$. Thus, $\bigcap_n J_n = \{0\}$, for $\mathcal{F}_E$ is the inductive limit of the C_m. Further, for each n we have

$$J_n = \iota_n(\mathcal{K}(E^{\otimes n})) + J_{n+1} \quad \text{and} \quad \iota_n(\mathcal{K}(E^{\otimes n})) \cap J_{n+1} = \{0\};$$

it follows that $J_n/J_{n+1} \cong \mathcal{K}(E^{\otimes n})$. Finally, since

$$\mathcal{F}_E = C_n + J_{n+1} \quad \text{and} \quad C_n \cap J_{n+1} = \{0\},$$

we have $C_n \cong \mathcal{F}_E/J_{n+1}$. □

There is a strongly continuous action

$$\lambda : \mathbb{T} \to \text{Aut}\,(\mathcal{O}_E)$$

such that $\lambda_t(T_\xi) = tT_\xi$. The fixed point algebra under this action is $\mathcal{F}_E$ and we have a faithful conditional expectation $P_E : \mathcal{O}_E \to \mathcal{F}_E$ given by

$$P_E(x) = \int_{\mathbb{T}} \lambda_t(x)\, dt.$$

Consider the spectral subspaces of $\mathcal{O}_E$ under this action: for $n \in \mathbb{Z}$

$$(\mathcal{O}_E)_n = \{x \in \mathcal{O}_E : \lambda_t(x) = t^n x \text{ for all } t \in \mathbb{T}\}.$$

Remark 1.5. Note that $(\mathcal{O}_E)_n$ is the closure of the span of elements of the form $T_\xi {T_\eta}^*$, where $\xi \in E^{\otimes k}$ and $\eta \in E^{\otimes l}$ with $n = k - l$. For $n \geq 0$ and $x \in (\mathcal{O}_E)_n$ we have $x^*x \in \mathcal{F}_E$ and $xx^* \in J_n$. We may regard $(\mathcal{O}_E)_n$ as a J_n-$\mathcal{F}_E$-equivalence bimodule (or J_n-$\mathcal{F}_E$-imprimitivity bimodule; see [**Ri1**, Def. 6.10]). Hence, J_n is strongly Morita equivalent to $\mathcal{F}_E$ for each $n \geq 0$ (see [**Ri2**, Def. 1.1], [**L**, p. 74]). If we regard $(\mathcal{O}_E)_1$ as a Hilbert $\mathcal{F}_E$-bimodule, we have

$$E \otimes_A \mathcal{F}_E \cong (\mathcal{O}_E)_1,$$

where the isomorphism is implemented by the map $\xi \otimes a \mapsto T_\xi a$ (the Hilbert $\mathcal{F}_E$-module $E \otimes_A \mathcal{F}_E$ is denoted E_∞ in [**Pm**, §2]). The crossed product $\mathcal{O}_E \rtimes_\lambda \mathbb{T}$ may be identified with the closure of the subalgebra of $\mathcal{O}_E \otimes \mathcal{K}(\ell^2(\mathbb{Z}))$ consisting of finite sums of the form

$$\sum x_{ij} \otimes e_{ij},$$

where e_{ij} are the standard rank-one partial isometries in $\mathcal{K}(\ell^2(\mathbb{Z}))$ and $x_{ij} \in (\mathcal{O}_E)_{j-i}$.

Let $\widehat{\lambda} : \mathbb{Z} \to \mathrm{Aut}\,(\mathcal{O}_E \rtimes_\lambda \mathbb{T})$ denote the dual automorphism group.

Proposition 1.6. *There is an embedding $\epsilon : \mathcal{F}_E \hookrightarrow \mathcal{O}_E \rtimes_\lambda \mathbb{T}$ onto a corner and a collection of essential ideals $\{I_n\}_{n\in\mathbb{Z}}$ in $\mathcal{O}_E \rtimes_\lambda \mathbb{T}$ satisfying the following conditions:*

(i) *For all $n \in \mathbb{Z}$, $\mathcal{F}_E$ is strongly Morita equivalent to I_n and A is strongly Morita equivalent to I_n/I_{n+1}.*
(ii) *For all $n \geq 0$, $\epsilon(J_n) = \epsilon(1)I_n\epsilon(1)$.*
(iii) *$I_n \subset I_m$ if $m \leq n$.*
(iv) *$\bigcap_n I_n = \{0\}$.*
(v) *$\overline{\bigcup_n I_n} = \mathcal{O}_E \rtimes_\lambda \mathbb{T}$.*
(vi) *$\widehat{\lambda}_k(I_n) = I_{n+k}$.*

Proof. We use the identification, given in Remark 1.5, between $\mathcal{O}_E \rtimes_\lambda \mathbb{T}$ with a C*-subalgebra of $\mathcal{O}_E \otimes \mathcal{K}(\ell^2(\mathbb{Z}))$. For each n let I_n be the ideal generated by $p_n = 1 \otimes e_{nn}$. Since $\mathcal{F}_E = (\mathcal{O}_E)_0$, it follows that $\mathcal{F}_E$ is isomorphic to the corner determined by p_n and thus is strongly Morita equivalent to I_n. The desired embedding $\epsilon : \mathcal{F}_E \hookrightarrow \mathcal{O}_E \rtimes_\lambda \mathbb{T}$ is given by $\epsilon(a) = a \otimes e_{00}$.

Given an element of the form $a_{mn} = x_{mn} \otimes e_{mn}$ in $\mathcal{O}_E \rtimes_\lambda \mathbb{T}$ with $m \leq n$, we have

$$a_{mn}{}^* a_{mn} = x_{mn}{}^* x_{mn} \otimes e_{nn} \quad \text{and} \quad a_{mn} a_{mn}{}^* = x_{mn} x_{mn}{}^* \otimes e_{mm},$$

with $x_{mn}x_{mn}{}^* \in J_{n-m}$; since p_n may be expressed as a finite sum of elements of the form $a_{mn}{}^* a_{mn}$, it follows that $I_n \subset I_m$ and that

$$(*) \qquad p_m I_n p_m = J_{n-m} \otimes e_{mm}.$$

Moreover, I_n is essential in I_m, since J_{n-m} is an essential ideal in $\mathcal{F}_E$ (by Proposition 1.4). Since $q_n = \sum_{i=-n}^{n} p_i \in I_n$ and $\{q_n\}_n$ forms an approximate identity, we have $\overline{\bigcup_n I_n} = \mathcal{O}_E \rtimes_\lambda \mathbb{T}$. Thus I_n is an essential ideal in $\mathcal{O}_E \rtimes_\lambda \mathbb{T}$ for all $n \in \mathbb{Z}$. Assertion (ii) follows immediately from ($*$). Assertion (vi) follows from the fact that $\widehat{\lambda}_k(p_n) = 1 \otimes p_{n+k}$. The remaining assertions follow from Proposition 1.4. □

2. $\mathcal{O}_E$ is simple and purely infinite

Let A be a separable unital C*-algebra and let $\pi : A \to \mathcal{L}(\mathfrak{H})$ be a faithful nondegenerate representation of A on a separable nontrivial Hilbert space $\mathfrak{H}$; since π is nondegenerate we have $\pi(1) = 1$.

Proposition 2.1. *With A and $\pi : A \to \mathcal{L}(\mathfrak{H})$ as above,*

$$E = \mathfrak{H} \otimes_{\mathbb{C}} A$$

is a full Hilbert bimodule over A under the operations

$$\langle \xi \otimes a, \eta \otimes b\rangle_A = \langle \xi, \eta\rangle a^* b, \qquad \varphi(a)(\xi \otimes b) = \pi(a)\xi \otimes b$$

for all $\xi, \eta \in \mathfrak{H}$ *and* $a, b \in A$. *Moreover, if* $\pi(A) \cap \mathcal{K}(\mathfrak{H}) = \{0\}$, *then* $\varphi(A) \cap \mathcal{K}(E) = \{0\}$ *and* $\mathcal{O}_E \cong \mathcal{T}_E$.

Proof. $E = \mathfrak{H} \otimes_{\mathbb{C}} A$ is the tensor product of the Hilbert A-$\mathbb{C}$-bimodule $\mathfrak{H}$ and the Hilbert $\mathbb{C}$-A-bimodule A as defined by Rieffel in [**Ri1**, Theorem 5.9] (see also [**L**, p. 41]). The natural map from $\mathcal{L}(\mathfrak{H})$ to $\mathcal{L}(E) = \mathcal{L}(\mathfrak{H} \otimes_{\mathbb{C}} A)$ induces an embedding $\mathcal{L}(\mathfrak{H})/\mathcal{K}(\mathfrak{H}) \hookrightarrow \mathcal{L}(E)/\mathcal{K}(E)$ (since $\mathcal{K}(\mathfrak{H})$ is mapped into $\mathcal{K}(E)$ and the Calkin algebra $\mathcal{L}(\mathfrak{H})/\mathcal{K}(\mathfrak{H})$ is simple). Hence, if $\pi(A) \cap \mathcal{K}(\mathfrak{H}) = \{0\}$, then $\varphi(A) \cap \mathcal{K}(E) = \{0\}$. The last assertion, $\mathcal{O}_E \cong \mathcal{T}_E$, follows by [**Pm**, Corollary 3.14]. □

Henceforth, we assume that $\pi(A) \cap \mathcal{K}(\mathfrak{H}) = \{0\}$ and identify $\mathcal{O}_E$ with $\mathcal{T}_E$. The aim of this section is to show that $\mathcal{O}_E$ is simple and purely infinite. Simplicity may be proven directly by invoking [**Sc2**, Theorem 3.9]: if A is unital and E is full, then $\mathcal{O}_E$ is simple if and only if E is minimal and nonperiodic. Lemma 2.3 would then be a consequence of [**OP1**, Theorem 6.5]. We follow a more indirect route patterned on the proof of [**Rø**, Theorem 2.1]; this will also show that $\mathcal{O}_E$ is purely infinite.

Remark 2.2. With $E = \mathfrak{H} \otimes_{\mathbb{C}} A$ as above, we have $E^{\otimes n} \cong \mathfrak{H}^{\otimes n} \otimes_{\mathbb{C}} A$ via the map

$$(\xi_1 \otimes a_1) \otimes (\xi_2 \otimes a_2) \otimes \cdots \otimes (\xi_n \otimes a_n) \mapsto (\xi_1 \otimes \pi(a_1)\xi_2 \otimes \cdots \otimes \pi(a_{n-1})\xi_n) \otimes a_n.$$

If $\sigma : A \to \mathcal{L}(\mathfrak{K})$ is a nondegenerate representation of A on a Hilbert space $\mathfrak{K}$, then

$$E \otimes_A \mathfrak{K} \cong \mathfrak{H} \otimes_{\mathbb{C}} A \otimes_A \mathfrak{K} \cong \mathfrak{H} \otimes_{\mathbb{C}} \mathfrak{K}$$

and, hence,

$$E^{\otimes n} \otimes_A \mathfrak{K} \cong E^{\otimes n-1} \otimes_A E \otimes_A \mathfrak{K} \cong E^{\otimes n-1} \otimes_A \mathfrak{H} \otimes_{\mathbb{C}} \mathfrak{K}.$$

Recall that the action of $\mathcal{F}_E$ on Fock space preserves the natural grading. Let $\widetilde{\sigma}_n$ denote the representation of $\mathcal{F}_E$ on $E^{\otimes n} \otimes_A \mathfrak{K}$ given by left action on $E^{\otimes n}$. Then the restriction of $\widetilde{\sigma}_n$ to C_{n-1} is faithful: indeed, this follows from the facts that the natural map

$$\mathcal{L}(E^{\otimes n-1}) \to \mathcal{L}(E^{\otimes n-1} \otimes_A \mathfrak{H} \otimes \mathfrak{K}) \cong \mathcal{L}(E^{\otimes n} \otimes_A \mathfrak{K})$$

is an embedding (since π is faithful) and that $\widetilde{\sigma}_n|_{\mathcal{K}(E^{\otimes n-1})}$ factors through $\mathcal{L}(E^{\otimes n-1})$. Note that $\widetilde{\sigma}_n$ is equivalent to the representation of $\mathcal{F}_E$ obtained from σ as follows: use the strong Morita equivalence between A and J_n/J_{n+1} to obtain a representation of J_n/J_{n+1} and extend this to a representation of $\mathcal{F}_E$. Since the restriction of $\widetilde{\sigma}_n$ to C_{n-1} is faithful, $\ker \widetilde{\sigma}_n \subset J_n$ (see Proposition 1.4). It follows that the closure of a point in $\widehat{J_n} - \widehat{J_{n+1}}$ contains the complement of $\widehat{J_n}$. A similar assertion holds for $\mathcal{O}_E \rtimes_\lambda \mathbb{T}$: for any $n \in \mathbb{Z}$ the closure of a point in $\widehat{I_n} - \widehat{I_{n+1}}$ contains the complement of $\widehat{I_n}$.

Lemma 2.3. *With A and E as above, $\Gamma(\widehat{\lambda}_1) = \mathbb{T}$, where $\widehat{\lambda}$ is the dual action of $\mathbb{Z}$ on $\mathcal{O}_E \rtimes_\lambda \mathbb{T}$.*

Proof. By [**OP2**, Theorem 4.6] it suffices to find a dense invariant subset of $(\mathcal{O}_E \rtimes_\lambda \mathbb{T})\widehat{\ }$ on which $\widehat{\lambda}_1^*$ acts freely. That is, we must find an irreducible representation σ of $\mathcal{O}_E \rtimes_\lambda \mathbb{T}$ such that $\{[\sigma \circ \widehat{\lambda}_n] : n \in \mathbb{Z}\}$, the orbit of the unitary equivalence class of σ under $\widehat{\lambda}^*$, is dense in $(\mathcal{O}_E \rtimes_\lambda \mathbb{T})\widehat{\ }$ and satisfies $[\sigma \circ \widehat{\lambda}_m] \neq [\sigma \circ \widehat{\lambda}_n]$ if $m \neq n$. Let σ_0 be an irreducible representation of A and use the strong Morita equivalence between A and I_0/I_1 to obtain an irreducible representation σ' of I_0/I_1. Then σ, the extension of σ' to $\mathcal{O}_E \rtimes_\lambda \mathbb{T}$, is also irreducible. The classes $[\sigma \circ \widehat{\lambda}_n]$ are distinct, for if $m < n$, $\sigma \circ \widehat{\lambda}_m$ vanishes on I_n. Moreover, for each $n \in \mathbb{Z}$ the closure of $[\sigma \circ \widehat{\lambda}_n]$ in $(\mathcal{O}_E \rtimes_\lambda \mathbb{T})\widehat{\ }$ includes the classes of all irreducible representations that vanish on I_n (since $[\sigma \circ \widehat{\lambda}_n] \in \widehat{I_n} - \widehat{I_{n+1}}$; see Remark 2.2). Hence, $\{[\sigma \circ \widehat{\lambda}_n] : n \in \mathbb{Z}\}$ is dense in $(\mathcal{O}_E \rtimes_\lambda \mathbb{T})\widehat{\ }$. □

Using Takesaki–Takai duality we show below that a C*-algebra D equipped with an action α of $\mathbb{T}$ may be embedded equivariantly as a corner in $(D \rtimes_\alpha \mathbb{T}) \rtimes_{\widehat{\alpha}} \mathbb{Z}$. This fact is related to Rosenberg's observation that the fixed point algebra under a compact group action embeds as a corner in the crossed product (see [**Ro**]).

Proposition 2.4. *Given a unital C*-algebra D and a strongly continuous action $\alpha : \mathbb{T} \to \mathrm{Aut}\,(D)$, there is an isomorphism ψ of D onto a full corner of $(D \rtimes_\alpha \mathbb{T}) \rtimes_{\widehat{\alpha}} \mathbb{Z}$ which is equivariant in the sense that $\widehat{\widehat{\alpha}}_t \circ \psi = \psi \circ \alpha_t$ for all $t \in \mathbb{T}$. Moreover, $\psi(1) \in D \rtimes_\alpha \mathbb{T}$.*

Proof. By Takesaki–Takai duality [**Pd**, 7.9.3] there is an isomorphism

$$\gamma : D \otimes \mathcal{K}(L^2(\mathbb{T})) \cong (D \rtimes_\alpha \mathbb{T}) \rtimes_{\widehat{\alpha}} \mathbb{Z},$$

which is equivariant with respect to $\alpha \otimes \mathrm{Ad}\,\rho$ and $\widehat{\widehat{\alpha}}$ (where ρ is the right regular representation of $\mathbb{T}$ on $L^2(\mathbb{T})$). The desired embedding is obtained by finding an $\mathrm{Ad}\,\rho$ invariant minimal projection p in $\mathcal{K}(L^2(\mathbb{T}))$ (cf. [**Ro**]): set $\psi(d) = \gamma(d \otimes p)$ for $d \in D$. Since ψ is equivariant, $\psi(1)$ is in the fixed point algebra of $\widehat{\widehat{\alpha}}$; hence, $\psi(1) \in D \rtimes_\alpha \mathbb{T}$. □

The following lemma is adapted from [**Rø**, Lemma 2.4]; the proof follows Rørdam's but we substitute [**OP3**, Lemma 7.1] for [**Ks**, Lemma 3.2].

Lemma 2.5. *Let B be a C*-algebra, let β be an automorphism of B such that $\Gamma(\beta) = \mathbb{T}$, and let P denote the canonical conditional expectation from $B \rtimes_\beta \mathbb{Z}$ to B. For every positive element $y \in B \rtimes_\beta \mathbb{Z}$ and $\varepsilon > 0$ there are positive elements $x, b \in B$ such that*

$$\|b\| > \|P(y)\| - \varepsilon, \quad \|x\| \leq 1 \quad \textit{and} \quad \|xyx - b\| < \varepsilon.$$

If y is in the corner determined by a projection $p \in B$, then x, b may also be chosen to be in the corner.

Proof. As in the proof of [**Rø**, Lemma 2.4] we may assume (by perturbing y if necessary) that y is of the form

$$y = y_{-n}u^{-n} + \cdots + y_{-1}u^{-1} + y_0 + y_1u + \cdots + y_nu^n$$

for some n, where $y_j \in B$ and u is the canonical unitary in $B \rtimes_\beta \mathbb{Z}$ implementing the automorphism β; note that $y_0 = P(y)$ is positive.

By [**OP3**, Theorem 10.4] β^k is properly outer for all $k \neq 0$. Hence, by [**OP3**, Lemma 7.1] there is a positive element x with $\|x\| = 1$ such that

$$\|xy_0x\| > \|y_0\| - \varepsilon \quad \text{and} \quad \|xy_ku^kx\| = \|xy_k\beta^k(x)\| < \varepsilon/2n \text{ for } 0 < |k| \leq n.$$

Set $b = xy_0x$; then a straightforward calculation yields $\|xyx - b\| < \varepsilon$. We now verify the last assertion. Suppose that y is in the corner determined by a projection $p \in B$; we may again assume that y is of the above form. Since P is a conditional expectation onto B, $y_0 = P(y)$ is also in the corner determined by p. In the proof of [**OP3**, Lemma 7.1] the positive element x is constructed in the hereditary subalgebra determined by y_0; hence we may assume that x and therefore also $b = xy_0x$ lies in the same corner. □

Recall that C_n is the C*-subalgebra of $\mathcal{F}_E$ generated by operators of the form $T_\xi T_\eta{}^*$ for $\xi, \eta \in E^{\otimes k}$ with $k \leq n$ and that they form an ascending family of C*-subalgebras with dense union. The subspace $E^{\otimes n}$ is left invariant by C_n and there is an embedding $C_n \hookrightarrow \mathcal{L}(E^{\otimes n})$.

Lemma 2.6. *Given a positive element $c \in C_n$ and $\varepsilon > 0$, there is $\xi \in E^{\otimes n}$ with $\|\xi\| = 1$ such that $T_\xi{}^* c T_\xi \in C_0$ and $\|T_\xi{}^* c T_\xi\| > \|c\| - \varepsilon$.*

Proof. The first assertion follows from a straightforward calculation: given $c \in C_n$ and $\xi \in E^{\otimes n}$, we have $c\xi \in E^{\otimes n}$ and

$$T_\xi{}^* c T_\xi = T_\xi{}^* T_{c\xi} = \iota(\langle \xi, c\xi \rangle_A) \in C_0.$$

The second assertion follows from the embedding $C_n \hookrightarrow \mathcal{L}(E^{\otimes n})$ and the fact that

$$\|d\| = \sup \left\{ \|\langle \xi, d\xi \rangle_A\| : \xi \in E^{\otimes n}, \|\xi\| = 1 \right\}$$

for $d \in \mathcal{L}(E^{\otimes n})$ positive. □

Lemma 2.7. *Given a positive element $a \in A$ and $\varepsilon > 0$ with $\|a\| > \varepsilon$, there is $\eta \in E$ with $\|\eta\| \leq (\|a\| - \varepsilon)^{-1/2}$ such that $T_\eta{}^* \iota(a) T_\eta = 1$.*

Proof. Let f be a continuous nonzero real-valued function supported on the interval $\left[\|a\| - \varepsilon,\, \|a\|\right]$ and choose a vector $\zeta \in \pi(f(a))\mathfrak{H}$ such that $\langle \zeta, \pi(a)\zeta \rangle = 1$; we have

$$(\|a\| - \varepsilon)\|\zeta\|^2 \leq \left\| \langle \zeta, \pi(a)\zeta \rangle \right\| = 1.$$

Then $\eta = \zeta \otimes 1 \in E$ satisfies the desired conditions. □

It will now follow that $\mathcal{O}_E$ is simple and purely infinite (compare the proof of [**Rø**, Theorem 2.1]).

Theorem 2.8. *For every nonzero positive element $d \in \mathcal{O}_E$ there is a $z \in \mathcal{O}_E$ such that $z^*dz = 1$. Hence, $\mathcal{O}_E$ is simple and purely infinite.*

Proof. Let $d \in \mathcal{O}_E$ be a nonzero positive element and choose ε such that $0 < \varepsilon < \frac{1}{4}\|P(d)\|$. By Proposition 2.4 there is a $\mathbb{T}$-equivariant isomorphism ψ from $\mathcal{O}_E$ onto a corner of $(\mathcal{O}_E \rtimes_\lambda \mathbb{T}) \rtimes_{\widehat{\lambda}} \mathbb{Z}$ determined by a projection $p \in \mathcal{O}_E \rtimes_\lambda \mathbb{T}$. We now apply Lemma 2.5 to the element $y = \psi(d)$ and the automorphism $\beta = \widehat{\lambda}_1$ (note that $\Gamma(\widehat{\lambda}_1) = \mathbb{T}$ by Lemma 2.3). We identify $\mathcal{O}_E$ with the corner determined by p; under this identification $\mathcal{F}_E$ is identified with $p(\mathcal{O}_E \rtimes_\lambda \mathbb{T})p$. There are then positive elements $x, b \in \mathcal{F}_E$ such that

$$\|b\| > \|P(d)\| - \varepsilon, \quad \|x\| \le 1 \quad \text{and} \quad \|xdx - b\| < \varepsilon.$$

Since $\bigcup_n C_n$ is dense in $\mathcal{F}_E$ we may assume that $b \in C_n$ for some n. Hence, by Lemma 2.6 there is $\xi \in E^{\otimes n}$ with $\|\xi\| = 1$ such that

$$T_\xi{}^* b T_\xi \in C_0 \quad \text{and} \quad \|T_\xi{}^* b T_\xi\| > \|b\| - \varepsilon.$$

Let a denote the unique element of A such that $\iota(a) = T_\xi{}^* b T_\xi$; then $\|a\| > \|P(d)\| - 2\varepsilon$ and

$$\|T_\xi{}^* xdx T_\xi - \iota(a)\| = \|T_\xi{}^*(xdx - b)T_\xi\| < \varepsilon.$$

By Lemma 2.7 there is $\eta \in E$ such that $T_\eta{}^* \iota(a) T_\eta = 1$ and

$$\|\eta\| \le (\|a\| - \varepsilon)^{-1/2} < (\|P(d)\| - 3\varepsilon)^{-1/2} < \varepsilon^{-1/2}.$$

It follows that

$$\begin{aligned} \|T_\eta{}^* T_\xi{}^* xdx T_\xi T_\eta - 1\| &= \|T_\eta{}^*(T_\xi{}^* xdx T_\xi - \iota(a))T_\eta\| \\ &\le \|T_\xi{}^* xdx T_\xi - \iota(a)\| (\varepsilon^{-1/2})^2 < 1. \end{aligned}$$

Therefore, $c = T_\eta{}^* T_\xi{}^* xdx T_\xi T_\eta$ is an invertible positive element and we take $z = xT_\xi T_\eta c^{-1/2}$. □

3. Applications and concluding remarks

We collect some applications of the theorem above and consider certain connections with the theory of reduced (amalgamated) free product C*-algebras. First we consider criteria under which the Kirchberg–Phillips Theorem applies (see [**Kr**, Theorem C], [**Ph**, Corollary 4.2.2]).

Theorem 3.1. *Let A be a separable nuclear unital* C*-*algebra belonging to the bootstrap class to which the* UCT *applies (see* [**RS**]*); let $\pi : A \to \mathcal{L}(\mathfrak{H})$ be a faithful nondegenerate representation of A on a nontrivial separable Hilbert space $\mathfrak{H}$ such that $\pi(A) \cap \mathcal{K}(\mathfrak{H}) = \{0\}$ and let E denote the Hilbert A-bimodule $\mathfrak{H} \otimes_{\mathbb{C}} A$. Then $\mathcal{O}_E$ is a unital Kirchberg algebra (simple, purely* *infinite, separable and nuclear) belonging to the bootstrap class. Hence, the Kirchberg–Phillips Theorem applies and the isomorphism class of $\mathcal{O}_E$ only depends on $(K_*(A), [1_A])$ and not on the choice of representation π.*

Proof. By Theorem 2.8, $\mathcal{O}_E$ is simple and purely infinite. If A is nuclear, the argument given in the proof of [**DS**, Theorem 2.1] shows that $\mathcal{O}_E$ must also be nuclear (alternatively, the nuclearity of $\mathcal{O}_E$ follows from the structural results discussed in §1). Hence, $\mathcal{O}_E$ is a unital Kirchberg algebra. Recall that the inclusion $A \hookrightarrow \mathcal{O}_E$ defines a KK-equivalence (see [**Pm**, Corollary 4.5]) that induces a unit-preserving isomorphism $K_*(A) \cong K_*(\mathcal{O}_E)$. Hence, if A is in the bootstrap class, so is $\mathcal{O}_E$. Therefore, the Kirchberg–Phillips Theorem applies and the isomorphism class of $\mathcal{O}_E$ only depends on $(K_*(A), [1_A])$. □

Let X be a second countable compact Hausdorff space, let μ be a nonatomic Borel measure with full support and let

$$\pi : C(X) \to \mathcal{L}(L^2(X, \mu))$$

be the representation given by multiplication of functions. Then π is faithful and

$$\pi(C(X)) \cap \mathcal{K}(L^2(X, \mu)) = \{0\}.$$

Hence, we may apply Theorem 3.1 with $A = C(X)$ and $\mathfrak{H} = L^2(X, \mu)$.

Corollary 3.2. *Let X and μ be as above. Then*

$$E = L^2(X, \mu) \otimes_{\mathbb{C}} C(X)$$

is a Hilbert bimodule over $C(X)$ and $\mathcal{O}_E$ is a unital Kirchberg algebra. The embedding $C(X) \hookrightarrow \mathcal{O}_E$ induces a (unit preserving) KK-equivalence. Hence, the isomorphism class of $\mathcal{O}_E$ only depends on $(K_*(C(X)), [1_{C(X)}])$ (and not on μ); moreover, if X is contractible, then $\mathcal{O}_E \cong \mathcal{O}_\infty$.

The next proposition is Theorem 5.6 of [**L**] (see also [**Ka**, Theorem 3]); Lance calls this the Kasparov–Stinespring–Gelfand–Naimark–Segal construction.

Proposition 3.3. *Let B and C be* C*-*algebras, let F be a Hilbert C-module and let $f : B \to \mathcal{L}(F)$ be a completely positive map. Then there is a Hilbert C-module E_f, a $*$-homomorphism $\varphi_f : B \to \mathcal{L}(E_f)$ and an element $v_f \in \mathcal{L}(F, E_f)$ such that $f(b) = {v_f}^* \varphi_f(b) v_f$ and $\varphi_f(B) v_f F$ is dense in E_f.*

I am grateful to D. Shlyakhtenko for the following observation. Let $\mathcal{T}$ be the "usual" Toeplitz algebra ($\mathcal{T}_E$, where E is the 1-dimensional Hilbert bimodule over $\mathbb{C}$) and let g denote the vacuum state on $\mathcal{T}$.

Proposition 3.4. *Let A be a separable unital* C*-*algebra and let $\pi : A \to \mathcal{L}(\mathfrak{H})$ be a faithful representation of A on a separable Hilbert space $\mathfrak{H}$ such that π has a cyclic vector $\xi \in \mathfrak{H}$. Let f denote the vector state $\langle \xi, \pi(\,\cdot\,)\xi \rangle$ and let $\widetilde{f}$ denote the corresponding completely positive map from A to $\mathcal{L}(A)$*

(given by $\widetilde{f}(a) = f(a)1$*). Then* $E = E_{\widetilde{f}} \cong \mathfrak{H} \otimes A$ *and* $\mathcal{T}_E$ *may be realized as a reduced free product* (*see* [**A**, **V**]):

$$(\mathcal{T}_E, h) \cong (A, f) * (\mathcal{T}, g) \quad \textit{for some state } h \textit{ on } \mathcal{T}_E.$$

Proof. This follows from [**Sh**, Theorem 2.3, Corollary 2.5]. □

As a result of this observation, at least part of Corollary 3.2 follows from the existing literature on reduced free products. The simplicity follows from a theorem of Dykema [**Dy**, Theorem 2]. Criteria for when reduced free products are purely infinite have been found by Choda, Dykema and Rørdam in a series of papers [**DR1**, **DR2**, **DC**]; but none seem to apply generally to the case considered in the corollary.

A theorem of Speicher (see [**Sp**]) on reduced amalgamated free products (see [**V**, §5]) and Toeplitz algebras associated to Hilbert bimodules yields a curious stability property of the algebras we have been considering. The following is the version given in [**BDS**, Theorem 2.4].

Proposition 3.5. *Suppose that* E_1 *and* E_2 *are full Hilbert bimodules over the* C*-*algebra* A*. Then*

$$\mathcal{T}_{E_1 \oplus E_2} = \mathcal{T}_{E_1} *_A \mathcal{T}_{E_2}.$$

Corollary 3.6. *Let* A *be a separable nuclear unital* C*-*algebra belonging to the bootstrap class to which the* UCT *applies* (*see* [**RS**]) *and let* $\pi : A \to \mathcal{L}(\mathfrak{H})$ *be a faithful representation of* A *on a separable Hilbert space* $\mathfrak{H}$ *such that* $\pi(A) \cap \mathcal{K}(\mathfrak{H}) = \{0\}$*. Let* E *be the Hilbert bimodule* $\mathfrak{H} \otimes_{\mathbb{C}} A$*. Then*

$$\mathcal{O}_E \cong \mathcal{O}_E *_A \mathcal{O}_E.$$

Proof. Observe that $E \oplus E = (\mathfrak{H} \oplus \mathfrak{H}) \otimes_{\mathbb{C}} A$. Since $\pi \oplus \pi : A \to \mathcal{L}(\mathfrak{H} \oplus \mathfrak{H})$ is a faithful representation and $(\pi \oplus \pi)(A) \cap \mathcal{K}(\mathfrak{H} \oplus \mathfrak{H}) = \{0\}$, the result follows from Theorem 3.1 and the above proposition. □

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References

[A] D. Avitzour, *Free products of* C*-*algebras*, Trans. Amer. Math. Soc., **271** (1982), 423–435, MR 0654842 (83h:46070), Zbl 0513.46037.

[BDS] N. Brown, K. Dykema and D. Shlyakhtenko, *Topological entropy of free product automorphisms*, Acta Math., **189** (2002), 1–35, MR 1946916 (2003i:46073).

[Co] A. Connes, *Une classification des facteurs de type* III, Ann. Sci. École Norm. Sup., **6** (1973), 133–252, MR 0341115 (49 #5865), Zbl 0274.46050.

[Cu] J. Cuntz, *Simple* C*-*algebras generated by isometries*, Comm. Math. Phys., **57** (1977), 173–185, MR 0467330 (57 #7189), Zbl 0399.46045.

[CK] J. Cuntz and W. Krieger, *A class of* C*-*algebras and topological Markov chains*, Invent. Math., **56** (1980), 251–268, MR 0561974 (82f:46073a), Zbl 0434.46045.

[De] V. Deaconu, *Generalized solenoids and* C*-*algebras*, Pacific J. Math., **190** (1999), 247–260, MR 1722892 (2000j:46103).

[DPZ] Doplicher, C. Pinzari and R. Zuccante, *The* C*-*algebra of a Hilbert bimodule*, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8), **1** (1998), 263–281, MR 1638139 (99i:46040), Zbl 0916.46053.

[Dy] K. Dykema, *Simplicity and the stable rank of some free product* C*-*algebras*, Trans. Amer. Math. Soc., **351** (1999), 1–40, MR 1473439 (99g:46078), Zbl 0908.46037.

[DC] K. Dykema and M. Choda, *Purely infinite, simple* C*-*algebras arising from free product constructions*, III, Proc. Amer. Math. Soc., **128** (2000), 3269–3273, MR 1707512 (2001b:46095), Zbl 0977.46037.

[DR1] K. Dykema and M. Rørdam, *Purely infinite, simple* C*-*algebras arising from free product constructions*, Canad. J. Math., **50** (1998), 323–341, MR 1618318 (99d:46074), Zbl 0919.46040.

[DR2] K. Dykema and M. Rørdam, *Purely infinite, simple* C*-*algebras arising from free product constructions*, II, Math. Scand., **90**(**1**) (2002), 73–86, MR 1887095 (2003d:46074), Zbl 1030.46073.

[DS] K. Dykema and D. Shlyakhtenko, *Exactness of Cuntz–Pimsner* C*-*algebras*, Proc. Edinburgh Math. Soc. (2), **44**(**2**) (2001), 425–444, MR 1880402 (2003a:46084), Zbl 0985.46034.

[E] G.A. Elliott, *Some simple* C*-*algebras constructed as crossed products with discrete outer automorphism groups*, Publ. Res. Inst. Math. Sci., **16** (1980), 299–311, MR 0574038 (81j:46102), Zbl 0438.46044.

[FR] N.J. Fowler and I. Raeburn, *The Toeplitz algebra of a Hilbert bimodule*, Indiana Univ. Math. J., **48** (1999), 155–181, MR 1722197 (2001b:46093), Zbl 0938.47052.

[FRM] N.J. Fowler, I. Raeburn and P.S. Muhly, *Representations of Cuntz–Pimsner algebras*, Indiana Univ. Math. J., **52** (2003), 569–605, MR 1986889.

[KPW1] T. Kajiwara, C. Pinzari and Y. Watatani, *Ideal structure and simplicity of the* C*-*algebras generated by Hilbert bimodules*, J. Funct. Anal., **159** (1998), 295–322, MR 1658088 (2000a:46094), Zbl 0942.46035.

[KPW2] T. Kajiwara, C. Pinzari and Y. Watatani, *Hilbert* C*-*bimodules and countably generated Cuntz–Krieger algebras*, J. Operator Theory, **45** (2001), 3–18, MR 1823059 (2002b:46095), Zbl 1005.46028.

[Ka] G.G. Kasparov, *Hilbert* C*-*modules: theorems of Stinespring and Voiculescu*, J. Operator Theory, **4** (1980), 133–150, MR 0587371 (82b:46074), Zbl 0456.46059.

[Kr] E. Kirchberg, *The classification of purely infinite* C*-*algebras using Kasparov's theory*, Fields Institute Communications, to appear.

[Ks] A. Kishimoto, *Outer automorphisms and reduced crossed products of simple* C*-*algebras*, Comm. Math. Phys., **81** (1981), 429–435, MR 0634163 (83c:46061), Zbl 0467.46050.

[L] E.C. Lance, *Hilbert* C*-*modules. A toolkit for operator algebraists*, London Math. Soc. Lecture Note Series, **210**, Cambridge University Press, Cambridge, 1995, MR 1325694 (96k:46100), Zbl 0822.46080.

[MS] P.S. Muhly and B. Solel, *On the simplicity of some Cuntz–Pimsner algebras*, Math. Scand., **83** (1998), 53–73, MR 1662076 (99m:46140), Zbl 0940.46034.

[O] D. Olesen, *Inner *-automorphisms of simple* C**-algebras*, Comm. Math. Phys., **44** (1975), 175–190, MR 0388113 (52 #8950), Zbl 0323.46061.

[OP1] D. Olesen and G.K. Pedersen, *Applications of the Connes spectrum to* C**-dynamical systems*, J. Funct. Anal., **30** (1978), 179–197, MR 0515224 (81i:46076a), Zbl 0391.46051.

[OP2] D. Olesen and G.K. Pedersen, *Applications of the Connes spectrum to* C**-dynamical systems*, II, J. Funct. Anal., **36** (1980), 18–32, MR 0568973 (81i:46076b), Zbl 0422.46054.

[OP3] D. Olesen and G.K. Pedersen, *Applications of the Connes spectrum to* C**-dynamical systems*, III, J. Funct. Anal., **45** (1982), 357–390, MR 0650187 (83i:46080), Zbl 0511.46064.

[Pa] W.L. Paschke, *The crossed product of a* C**-algebra by an endomorphism*, Proc. Amer. Math. Soc., **80** (1980), 113–118, MR 0574518 (81m:46081), Zbl 0435.46045.

[Pd] G.K. Pedersen, C**-algebras and their automorphism groups*, London Mathematical Society Monographs, **14**, Academic Press, London, 1979, MR 0548006 (81e:46037), Zbl 0416.46043.

[Ph] N.C. Phillips, *A classification theorem for nuclear purely infinite simple* C**-algebras*, Doc. Math., **5** (2000), 49–114, MR 1745197 (2001d:46086b), Zbl 0943.46037.

[Pm] M. Pimsner, *A class of* C**-algebras generalizing both Cuntz–Krieger algebras and crossed products by* $\mathbb{Z}$, in 'Free probability theory' (Waterloo, ON, 1995), 189–212, Fields Institute Communications, **12**, Amer. Math. Soc., Providence, 1997, MR 1426840 (97k:46069), Zbl 0871.46028.

[PV] M. Pimsner and D. Voiculescu, *Exact sequences for K-groups and Ext-groups of certain cross-product* C**-algebras*, J. Operator Theory, **4** (1980), 93–118, MR 0587369 (82c:46074), Zbl 0474.46059.

[Pn] C. Pinzari, *The ideal structure of Cuntz–Krieger–Pimsner algebras and Cuntz–Krieger algebras over infinite matrices*, in 'Operator Algebras and Quantum Field Theory' (Rome, 1996), 136–150, International Press, 1997, MR 1491114 (98m:46074), Zbl 0912.46071.

[Ri1] M.A. Rieffel, *Induced representations of* C**-algebras*, Advances in Math., **13** (1974), 176-257, MR 0353003 (50 #5489), Zbl 0284.46040.

[Ri2] M.A. Rieffel, *Strong Morita equivalence of certain transformation group* C**-algebras*, Math. Ann., **222** (1976), 7–22, MR 0419677 (54 #7695), Zbl 0328.22013.

[Rø] M. Rørdam, *Classification of certain infinite simple* C**-algebras*, III, in 'Operator algebras and their applications' (Waterloo, ON, 1994/1995), 257–283, Fields Institute Communications, **13**, Amer. Math. Soc., Providence, 1997, MR 1424966 (98j:46059), Zbl 0894.46045.

[Ro] J. Rosenberg, *Appendix to: 'Crossed products of* UHF *algebras by product type actions'*, by O. Bratteli, Duke Math. J., **46** (1979), 25–26, MR 0523599 (82a:46064), Zbl 0395.46049.

[RS] J. Rosenberg and C. Schochet, *The Künneth theorem and the universal coefficient theorem for Kasparov's generalized K-functor*, Duke Math. J., **55** (1987), 431–474, MR 0894590 (88i:46091), Zbl 0644.46051.

[Sc1] J. Schweizer, *Crossed products by* C^**-correspondences and Cuntz–Pimsner algebras*, in 'C*-algebras' (Münster, 1999), 203–226, Springer-Verlag, Berlin, 2000, MR 1798598 (2002f:46133), Zbl 0995.46045.

[Sc2] J. Schweizer, *Dilations of* C^**-correspondences and the simplicity of Cuntz–Pimsner algebras*, J. Funct. Anal., **180** (2001), 404–425, MR 1814994 (2002f:46113), Zbl 0985.46033.

[Sh] D. Shlyakhtenko, *Some applications of freeness with amalgamation*, J. Reine Angew. Math., **500** (1998), 191–212, MR 1637501 (99j:46079), Zbl 0926.46046.

[Sp] R. Speicher, *Combinatorial theory of the free product with amalgamation and operator-valued free probability theory*, Mem. Amer. Math. Soc., **132**(**627**), 1998, MR 1407898 (98i:46071), Zbl 0935.46056.

[T] H. Takai, *On a duality for crossed products of* C^**-algebras*, J. Funct. Anal., **19** (1975), 25–39, MR 0365160 (51 #1413), Zbl 0295.46088.

[V] D. Voiculescu, *Symmetries of some reduced free product* C^**-algebras*, in 'Operator algebras and their connections with topology and ergodic theory' (Buşteni, 1983), 556–588, Lecture Notes in Math., **1132**, Springer, Berlin, 1985, MR 0799593 (87d:46075), Zbl 0618.46048.

[Z] J. Zacharias, *Quasi-free automorphisms of Cuntz–Krieger–Pimsner algebras*, in 'C*-algebras' (Münster 1999), 262–272, Springer-Verlag, Berlin, 2000, MR 1798602 (2002a:46079), Zbl 0976.46046.

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