Pacific Journal of Mathematics

OPERATORS AND DIVERGENT SERIES

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Volume 217 No. 2 December 2004

PACIFIC JOURNAL OF MATHEMATICS
Vol. 217, No. 2, 2004

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We give a natural extension of the classical definition of Césaro convergence of a divergent sequence/function. This involves understanding the spectrum of eigenvalues and eigenvectors of a certain Césaro operator on a suitable space of functions or sequences. The essential idea is applicable in identical fashion to other summation methods such as Borel's. As an example we show how to obtain the analytic continuation of the Riemann zeta function $\zeta(z)$ for $\operatorname{Re} z \leq 1$ directly from generalised Césaro summation of its divergent defining series. We discuss a variety of analytic and symmetry properties of these generalised methods and some possible further applications.

1. Introduction

Methods for assigning generalised limits (sums) to divergent sequences (series) have been studied for centuries. A large variety of definitions exist, due to Césaro, Borel and others, each applicable to a different class of sequences/series.

In this paper we describe an approach which permits generalisation of many of these definitions, expanding the class of divergent sequences/series to which generalised limits/sums can be attached by each method. In describing this approach we will principally consider just one method, that of Césaro. In the final two sections, however, we will describe how this approach naturally generalises to other methods such as Borel's.

In more detail, in §2 we give our generalisation of the definition of Césaro convergence, both in a discrete (i.e., sequences and series) and continuous (i.e., functions and what we call pictures) setting. The key is to recast the existing definition in terms of a Césaro operator and use its spectrum of eigenvalues and eigenvectors. We state an alternative practical formulation of this definition and investigate its precise relation to our original version. We also discuss a notion of Césaro asymptotics and certain functorial properties which we will require.

In §3 we turn to the example of analytically continuing the Riemann zeta function $\zeta(z) \equiv \sum_{n=1}^{\infty} n^{-z}$ outside its half-plane of convergence $\operatorname{Re} z > 1$. We show how, using our continuous generalised Césaro scheme, we obtain this analytic continuation directly by analysing the divergent defining series for $\operatorname{Re} z \le 1$. Moreover we show how to obtain the location and residue of the simple pole of ζ at $z = 1$ in this framework.

In §4 we reconsider this example in the context of the discrete Césaro scheme where certain anomalous errors arise in trying to perform the analogous analytic continuation of ζ. We show how these relate to the singularity arguments of §3 and how to adapt those arguments to rectify the errors.

In §5 we discuss basic analyticity properties of our Césaro schemes. The main result clarifies why the extensions of ζ obtained in §3 and §4 must both *a priori* be the unique analytic continuation.

In §6 we then discuss, at varying levels of rigor, some possible implications of our Césaro analysis for Dirichlet series more generally. These include observations regarding scaling, dilation and translation invariances of our schemes, possible criteria for detecting poles and zeros of ordinary Dirichlet series, and example applications to other Dirichlet series arising from analysis of self-adjoint elliptic operators on manifolds.

In §7 we turn back to describing how our approach to extending Césaro's definition of generalised convergence can be applied more broadly to a whole class of definitions. We illustrate by introducing a new notion of Borel summation and comparing it with the existing definition.

Finally in §8 we outline briefly some possible extensions of this work in a variety of directions. We discuss higher-dimensional schemes for series or functions of several variables, the concept of "ratio eigenfunctions", schemes associated to arbitrary measures, and some speculative relations with recent dynamical-systems treatments of the zeta function.

2. Generalised Césaro convergence

2.1. Existing definitions. Let $\mathcal{S}$ be the space of all sequences $a = \{a_n\}_{n=1}^{\infty}$. The usual definition of the Césaro[1] limit of a can be phrased as follows: let $P_D : \mathcal{S} \to \mathcal{S}$ be the linear (discrete) Césaro operator given by $P_D[a]_n \equiv \frac{1}{n}\sum_{j=1}^{n} a_j$. We define a to have Césaro limit L, and write $C_D \lim_{n\to\infty} a_n = L$, if for some positive integer r the sequence $P_D^r[a]$ converges classically to L. Since P_D is a *regular* operator (meaning that if a is classically convergent then so is $P_D[a]$ with the same limit) Césaro convergence is a well-defined generalisation of the notion of classical convergence.

The discrete Césaro sum of a divergent series is then the Césaro limit of its sequence of partial sums: thus, for example, $\sum_{n=1}^{\infty}(-1)^{n-1}$ has discrete Césaro sum $\frac{1}{2}$ while $\sum_{n=1}^{\infty}(-1)^{n-1}n$ has discrete Césaro sum $\frac{1}{4}$.

We can also define corresponding notions of continuous Césaro convergence and Césaro integration of functions. Precisely, let $\mathcal{F}$ be the space of

[1] Technically this is Hölder's formulation ([**2**], §5) but the origin of the idea is Césaro's.

complex-valued functions on $[0, \infty)$ given by

$$\mathcal{F} = \left\{ f : \int_0^x |f(t)(\ln(t))^m| dt < \infty \text{ for all } x \geq 0 \text{ and for all } m \in \mathbb{Z}_{\geq 0} \right\}.$$

Then we define the (continuous) Césaro operator $P : \mathcal{F} \to \mathcal{F}$ by $P[f](x) \equiv \frac{1}{x}\int_0^x f(t)dt$ and say that f has Césaro limit L, written $\mathrm{Clim}_{x\to\infty} f(x) = L$, if for some positive integer r, $P^r[f]$ converges classically to L. The function space, $\mathcal{F}$, is defined to ensure that P sends $\mathcal{F}$ back into itself and hence that P^r is well-defined. This can be verified by integration by parts and an elementary estimate. Once again P is a regular operator and can be used to define values for certain divergent improper integrals, $\int_0^\infty f(t)dt$, by application to the associated partial-integral function $F(x) \equiv \int_0^x f(t)dt$.

P can in fact also be used as an alternative to P_D in analysing series $\sum_{n=1}^\infty a_n$, by considering not the associated sequence of partial sums $\{s_k\}_{k=1}^\infty$ but instead the partial sum function $s(x) \equiv \sum_{n\leq x} a_n$ and its continuous Césaro limit. This corresponds geometrically to viewing the terms in the series as being added in at the integer points along the positive real axis.

Although these definitions of discrete and continuous Césaro convergence enlarge the class of series/integrals to which we can attach values to include, for instance, the alternating series $\sum_{n=1}^\infty (-1)^{n-1}$ and $\sum_{n=1}^\infty (-1)^{n-1}n$ above, it is readily checked that they do not allow evaluation of nonalternating series like $\sum_{n=1}^\infty 1$ and $\sum_{n=1}^\infty n$, which arise as the formal defining series for $\zeta(0)$ and $\zeta(-1)$. We thus turn now to extending these definitions in order to handle these examples and obtain the correct values of $\zeta(0) = \frac{-1}{2}$ and $\zeta(-1) = \frac{-1}{12}$ as their generalised Césaro sums.

2.2. Generalised definitions. Consider first the definition of continuous Césaro convergence. Its key feature was the regularity of the operators P and hence P^r. In operator terms, however, the restriction to regular operators which are pure powers of P is clearly unnecessary. In particular it is natural to consider arbitrary regular polynomials in P, $q(P)$. Any such $q(P)$ is immediately well-defined as an operator (unlike a power series or more general function of P), and the condition of regularity is clearly equivalent to simply requiring $q(1) = 1$. We thus generalise the definition of continuous Césaro convergence as follows:

Definition 1. We say that $f \in \mathcal{F}$ has generalised Césaro limit L if there exists $q(P)$ a regular polynomial in P ($q(1) = 1$) such that $q(P)[f](x) \to L$ classically as $x \to \infty$. We continue to write $\mathrm{Clim}_{x\to\infty} f(x) = L$ in this case.

Note that L is uniquely determined in this definition: if $q_1(P)[f](x) \to L_1$ and $q_2(P)[f](x) \to L_2$, then by the regularity of each $q_i(P)$ we see that $L_2 = \lim_{x\to\infty} q_1(P)q_2(P)[f](x) = \lim_{x\to\infty} q_2(P)q_1(P)[f](x) = L_1$.

The generalisation of the definition of discrete Césaro convergence follows identical lines. For the remainder of this section and §3, however, we now restrict attention solely to the setting of functions and our continuous Césaro definitions. We shall refer simply to the Césaro operator (meaning P not P_D) and Césaro convergence and summation (meaning their continuous versions).

2.3. Interpretation of definition. To identify what the generalisation in Definition 1 achieves, note that P is a linear operator and consider its spectrum of eigenvalues and eigenfunctions. If $f \in \mathcal{F}$ is an eigenfunction of P with eigenvalue $\lambda \in \mathbb{C}$ then by definition $(P-\lambda)[f] \equiv 0$. Although $(P-\lambda)$ is not a regular operator, for $\lambda \neq 1$ the constant multiple $\frac{1}{1-\lambda}(P-\lambda)$ is. Taking $q(P)$ as $\frac{1}{1-\lambda}(P-\lambda)$ in our definition, we thus obtain the following:

Lemma 1. *If $f \in \mathcal{F}$ is any eigenfunction of P with eigenvalue $\lambda \neq 1$ then we have* $\operatorname{Clim}_{x\to\infty} f(x) = 0$.

Note that the exclusion of the case $\lambda = 1$ is to be expected; constant functions, which are eigenfunctions of P with eigenvalue 1, should have their limits preserved by regular polynomials $q(P)$ instead of having generalised Césaro limit 0.

Lemma 1 does, however, extend to *generalised* eigenfunctions with eigenvalue $\lambda \neq 1$, that is functions $f \in \mathcal{F}$ such that $(P-\lambda)^n[f] \equiv 0$ for some $n \in \mathbb{Z}_{>1}$. In this case, taking $q(P) = (\frac{1}{1-\lambda})^n (P-\lambda)^n$ we obtain likewise:

Lemma 2. *If $f \in \mathcal{F}$ is any generalised eigenfunction of P with eigenvalue $\lambda \neq 1$ then* $\operatorname{Clim}_{x\to\infty} f(x) = 0$.

Linear combinations of eigenfunctions and generalised eigenfunctions of P with eigenvalues all not equal to 1 must also have generalised Césaro limit 0. This follows immediately from the following easy observation:

Lemma 3. *If* $\operatorname{Clim}_{x\to\infty} f_1(x) = L_1$ *and* $\operatorname{Clim}_{x\to\infty} f_2(x) = L_2$ *and $c \in \mathbb{C}$ then* $\operatorname{Clim}_{x\to\infty} cf_i(x) = cL_i$ *for each $i = 1, 2$ and* $\operatorname{Clim}_{x\to\infty}(f_1+f_2)(x) = L_1 + L_2$.

Proof. By definition there exist regular polynomials $q_1(P)$ and $q_2(P)$ such that $q_i(P)[f_i](x) \to L_i$ classically for each $i = 1, 2$. The first result follows trivially by linearity of the $q_i(P)$. The second follows on using the regular polynomial $q(P) = q_1(P)q_2(P)$, since, by commuting the $q_i(P)$ as required, we have

$$q(P)[f_1+f_2](x) = q_2(P)q_1(P)[f_1](x) + q_1(P)q_2(P)[f_2](x) \to L_1 + L_2. \qquad \square$$

In light of this last lemma we have now proved at least the following proposition as a consequence of our generalised definition of Césaro convergence.

Lemma 4. *Suppose $f \in \mathcal{F}$ can be written as $f(x) = \sum_{j=1}^{n} c_j f_j(x) + R(x)$ where each $c_j \in \mathbb{C}$, each f_j is an eigenfunction or generalised eigenfunction of P with eigenvalue $\lambda_j \neq 1$, and $R(x)$ is a remainder function satisfying $P^r[R](x) \to L$ classically as $x \to \infty$ for some nonnegative integer r. Then* $\operatorname{Clim}_{x\to\infty} f(x) = L$.

Two questions immediately arise. First, whether the converse of Lemma 4 also holds, thus giving a characterisation of generalised Césaro convergence, or whether Definition 1 is strictly stronger. This is the question of whether the new class of functions to which we can now assign generalised Césaro limit 0 consists precisely just of the eigenfunctions and generalised eigenfunctions of P with eigenvalue $\lambda \neq 1$, or whether it also includes other types of functions. The second is the more immediate question of identifying explicitly the eigenfunctions and generalised eigenfunctions of P. The following lemma answers this question first.

Lemma 5.

(i) *The functions x^ρ, $\rho \in \mathbb{C}$, $\operatorname{Re}\rho > -1$ are all eigenfunctions of P in $\mathcal{F}$ with eigenvalue $\frac{1}{\rho+1}$. Each spans a one-dimensional eigenspace of P.*

(ii) *For each eigenvalue $\frac{1}{\rho+1}$ the corresponding generalised eigenfunctions of P are then the functions $x^\rho(\ln(x))^m$, $m = 1, 2, 3, \ldots$.*

Proof. (i) It is trivial that $P[\widetilde{x}^\rho](x) \equiv \frac{1}{\rho+1}x^\rho$ for any $\operatorname{Re}\rho > -1$ (where here and throughout we adopt a convention of attaching tildes to dummy variables used in defining functions). Now consider the eigenvalue equation $P[f] = \frac{1}{\rho+1}f$. This means that $\int_0^x f(t)dt \equiv \frac{1}{\rho+1}xf(x)$ as functions on $(0, \infty)$, and differentiating with respect to x then implies $x\frac{df}{dx} = \rho f(x)$. Since this is a homogeneous first-order linear ODE its solution space must be one-dimensional as claimed.

(ii) It is easily verified by an induction argument based on repeated integration by parts that each $x^\rho(\ln(x))^m$ satisfies $(P - \frac{1}{\rho+1})^{m+1}[\widetilde{x}^\rho(\ln(\widetilde{x}))^m] \equiv 0$. That the generalised eigenspace of solutions of $(P - \frac{1}{\rho+1})^{m+1}[f] \equiv 0$ in $\mathcal{F}$ is precisely of dimension $m + 1$ (hence spanned by the functions $x^\rho, x^\rho \ln(x), \ldots, x^\rho(\ln(x))^m$) is then established inductively along the lines of the argument in (i), by translating to an equivalent first-order linear ODE. □

Lemmas 1, 2 and 5, together with the observation that for $\operatorname{Re}\rho \leq -1$ the functions $x^\rho(\ln(x))^m$ already converge classically to 0, establish that for any $\rho \neq 0$ and any nonnegative integer m, $\operatorname{Clim}_{x\to\infty} x^\rho(\ln(x))^m = 0$. Lemma 4 thus translates into the following more explicit proposition:

Lemma 6. *Suppose $f \in \mathcal{F}$ can be written as $f(x) = \sum_{j=1}^n c_j x^{\rho_j}(\ln(x))^{m_j} + R(x)$ for some collection of constants $c_j \in \mathbb{C}$, $\rho_j \in \mathbb{C} \setminus \{0\}$ and $m_j \in \mathbb{Z}_{\geq 0}$, and some remainder function $R(x)$ satisfying $P^r[R](x) \to L$ classically as $x \to \infty$ for some nonnegative integer r. Then $\operatorname{Clim}_{x\to\infty} f(x) = L$.*

Note. The relationship here between the collection of constants ρ_j, m_j appearing in the expansion of f and the simplest polynomial $q(P)$ satisfying $q(P)[f](x) \to L$ is as follows: take the list $\rho_1, \ldots \rho_n$. For each distinct value,

ρ, in this list consider those ρ_j with $\rho_j = \rho$ and let m be the largest of the corresponding values of m_j. Then include in the construction of $q(P)$ a regular factor of the form $(\frac{\rho+1}{\rho})^{m+1}(P - \frac{1}{\rho+1})^{m+1}$. The product of these regular factors over all distinct ρ-values, together with a final factor of P^r, gives a polynomial $q(P)$ with the required property.

Definition 1 thus extends the existing definition of Césaro convergence by allowing us to assign generalised limits not just to functions which become classically convergent upon repeated application of P, but also to ones which have additional power and power-log divergences. For example, we can now correctly evaluate the previously intractable formal defining series for $\zeta(0)$ and $\zeta(-1)$ mentioned in §2.1. Let $x = k + \alpha$ with $k = \lfloor x \rfloor$ and $\alpha \in [0, 1)$. Then:

(i) For $\sum_{n=1}^{\infty} 1$ the partial sum function is $s(x) = k = x - \alpha$. Since the saw-tooth function $R(x) = \alpha$ clearly satisfies $P[R](x) \to \frac{1}{2}$ as $x \to \infty$ it follows immediately from Lemma 6 that $\mathrm{Clim}_{x\to\infty} s(x) = -\frac{1}{2}$, i.e., $\sum_{n=1}^{\infty} 1 = -\frac{1}{2}$ in a generalised Césaro sense, as desired.

(ii) For $\sum_{n=1}^{\infty} n$ we have

$$s(k+\alpha) = \frac{1}{2}(k^2+k) = \frac{1}{2}(k+\alpha)^2 + \left(\frac{1}{2} - \alpha\right)k - \frac{1}{2}\alpha^2 = \frac{1}{2}x^2 + R(x)$$

where $R(k+\alpha) = (\frac{1}{2} - \alpha)k - \frac{1}{2}\alpha^2$. Now

$$\begin{aligned} P[R](k+\alpha) &= \frac{1}{k+\alpha}\Bigg(\sum_{j=0}^{k-1}\left(\frac{1}{2} - \int_0^1 \beta\, d\beta\right)j - \left(\frac{1}{2}\int_0^1 \beta^2\, d\beta\right)k \\ &\qquad + \left(\int_0^\alpha \frac{1}{2} - \beta\, d\beta\right)k - \frac{1}{2}\int_0^\alpha \beta^2\, d\beta\Bigg) \\ &= \left(\frac{-1}{6} + \frac{\alpha}{2} - \frac{\alpha^2}{2}\right) + O\left(\frac{1}{k}\right) \end{aligned}$$

and so as, $k \to \infty$, $P^2[R](k+\alpha) \to -\frac{1}{6} + \frac{1}{4} - \frac{1}{6} = -\frac{1}{12}$. In Lemma 6 we thus obtain that $\sum_{n=1}^{\infty} n = -\frac{1}{12}$ in a generalised Césaro sense, again as desired.

Returning to the first of our earlier two questions now, it turns out that Lemma 6 is in fact slightly weaker than Definition 1. This is due to the existence, for certain eigenvalues, of nontrivial asymptotic eigenfunctions in addition to the exact eigenfunctions calculated in Lemma 5. Here we are using the following:

Definition 2. A function $f \in \mathcal{F}$ is an asymptotic eigenfunction of P with eigenvalue λ if $(P - \lambda)[f](x) = o(1)$.

A nontrivial asymptotic eigenfunction is one that does not merely differ from an exact eigenfunction by a $o(1)$-function. Since we clearly still have $\mathrm{Clim}_{x\to\infty} f(x) = 0$ for any asymptotic eigenfunction with eigenvalue $\lambda \neq 1$, nontrivial such functions will be ones which can be assigned generalised Césaro limits under Definition 1, but which are not simply of the form described in Lemma 6. The following Tauberian-type lemma and corollary prove (constructively) the existence of such functions, and clarify more precisely the relationship between Lemma 6 and Definition 1.

Lemma 7. *Let $S_{\frac{1}{2},\frac{1}{2}}$ be the circle in $\mathbb{C}$ with centre $\frac{1}{2}$ and radius $\frac{1}{2}$. Note that $S_{\frac{1}{2},\frac{1}{2}} \setminus \{0\}$ is the image of the imaginary axis under the mapping $\rho \mapsto \frac{1}{\rho+1}$.*

(i) *Suppose that $\mathrm{Re}\,\rho > -1$, $\mathrm{Re}\,\rho \neq 0$, and that $f \in \mathcal{F}$ satisfies $\left(\frac{\rho+1}{\rho}\right)\left(P - \frac{1}{\rho+1}\right)[f](x) \to 0$ classically as $x \to \infty$. Then $f(x) = Cx^\rho + o(1)$ for some constant C. The converse of Lemma 6 thus holds at least for $q(P)$ of degree 1 with root not lying on $S_{\frac{1}{2},\frac{1}{2}} \setminus \{0\}$.*

(ii) *However, for any $\mathrm{Re}\,\rho = 0$, $\rho \neq 0$, there exist functions $f \in \mathcal{F}$ such that $\left(\frac{\rho+1}{\rho}\right)\left(P - \frac{1}{\rho+1}\right)[f](x) \to 0$ classically as $x \to \infty$, but f is not of the form $f(x) = Cx^\rho + o(1)$. Thus the converse of Lemma 6 fails when $q(P)$ has a root lying on $S_{\frac{1}{2},\frac{1}{2}} \setminus \{0, 1\}$.*

Proof. (i) The proof is principally due to Andrew Stone. The given eigenvalue equation states that $\frac{1}{x}\int_0^x f(t)dt - \frac{1}{\rho+1} f(x) = r(x)$ where $r(x) = o(1)$. Writing $F(x) \equiv \int_0^x f(t)dt$ this becomes the asymptotic differential equation $\frac{1}{x}F(x) - \frac{1}{\rho+1}F'(x) = r(x)$, which can be rewritten as $\frac{d}{dx}(x^{-(\rho+1)}F(x)) = -(\rho+1)x^{-(\rho+1)}r(x)$. Integrating implies $x^{-(\rho+1)}F(x) = F(1) - (\rho+1)\phi(x)$ where $\phi(x) \equiv \int_1^x t^{-(\rho+1)}r(t)dt$. Consider two cases separately.

Case (a): $\mathrm{Re}\,\rho > 0$. In this case $\lim_{x\to\infty}\phi(x)$ exists. Denoting it by $\phi_\infty(\rho)$ we obtain

$$F(x) = C_\rho x^{\rho+1} + (\rho+1)x^{\rho+1}\int_x^\infty t^{-(\rho+1)}\, r(t)\, dt$$

where $C_\rho = F(1) - (\rho+1)\phi_\infty(\rho)$ is a constant, and differentiating then yields

$$f(x) = C_\rho(\rho+1)x^\rho + (\rho+1)^2 x^\rho \int_x^\infty t^{-(\rho+1)}\, r(t)\, dt - (\rho+1)r(x).$$

Since $r(x) = o(1)$ the result thus follows immediately if we can show also that $x^\rho \int_x^\infty t^{-(\rho+1)} r(t)dt = o(1)$. But to see this let X_ϵ, for any $\epsilon > 0$, be such that $|r(x)| \leq \epsilon$ whenever $x > X_\epsilon$. Then, for any $x > X_\epsilon$ we have

$$\left| x^\rho \int_x^\infty t^{-(\rho+1)}\, r(t)\, dt \right| \leq x^{\mathrm{Re}\,\rho}\int_x^\infty t^{-(\mathrm{Re}\,\rho+1)}\,\epsilon\, dt \leq \frac{\epsilon}{\mathrm{Re}\,\rho}$$

and the result follows.

Case (b): $-1 < \operatorname{Re}\rho < 0$. In this case we need to show that $f(x)$ itself is $o(1)$ so we directly consider

$$f(x) = F'(x) = (\rho+1)F(1)x^\rho - (\rho+1)^2 x^\rho \phi(x) - (\rho+1)r(x).$$

But here the first and third terms are immediately $o(1)$ and the term $x^\rho\phi(x) = x^\rho \int_1^x t^{-(\rho+1)} r(t)dt$ is also seen to be $o(1)$ by a similar argument to the one just given, on writing the integral over $[1, x]$ as a sum of integrals over $[1, X_\epsilon]$ and $[X_\epsilon, x]$. Hence the result follows in this case also.

(ii) For any $\rho = i\beta$ with $\beta \neq 0$ real, it is easy to check that, for example,

$$f(x) \equiv \begin{cases} 0, & 0 \le x \le e \\ x^{i\beta}\ln(\ln x), & x > e \end{cases}$$

is a function with the required properties and thus a nontrivial asymptotic eigenfunction of P with eigenvalue $\frac{1}{1+i\beta} \in S_{\frac{1}{2},\frac{1}{2}} \setminus \{0, 1\}$. □

Note. In the cases of eigenvalue 0 or 1 ($\rho = \infty$ or 0) excluded in (ii), P certainly does still have nontrivial asymptotic eigenfunctions. For eigenvalue 0 this is of course the reason the original definition of Césaro convergence is stronger than simply classical convergence. In the case of eigenvalue 1 the example in the proof still yields a suitable eigenfunction, on taking $\beta = 0$. We omitted this case only because factors of $(P - 1)$ cannot arise in a regular polynomial $q(P)$, so that it need not be considered initially in analysing Definition 1.

The following corollary, which follows easily from part (i) of Lemma 7, then extends the discussion there to polynomials of arbitrary degree.

Corollary 1. *Suppose $q(P)$ is any regular polynomial in P none of whose roots lie on $S_{\frac{1}{2},\frac{1}{2}} \setminus \{0\}$, and that $f \in \mathcal{F}$ satisfies $q(P)[f](x) \to L$ classically as $x \to \infty$. Then $f(x)$ must be of the form $f(x) = \sum_{j=1}^n c_j x^{\rho_j}(\ln(x))^{m_j} + R(x)$ where the relationship between the exponents ρ_j, m_j and the roots of $q(P)$ is precisely as outlined in the note subsequent to Lemma 6, and where $P^r[R](x) \to L$ with r being the multiplicity of 0 as a root of $q(P)$.*

Thus the converse of Lemma 6 holds subject to these root conditions on $q(P)$.

We now consider finally in §2.3 the important case of eigenvalue $\lambda = 1$ ($\rho = 0$) omitted throughout Lemmas 1–4 and Lemma 6. By Lemma 5, the exact eigenfunctions with eigenvalue 1 are the constant functions and the $\lambda = 1$ generalised eigenspace is spanned by the functions $(\ln x)^m$, $m = 1, 2, \ldots$. As constant functions, the eigenfunctions with eigenvalue 1 of course have classical limits, not just generalised Césaro ones. This is not so, however, for the generalised eigenfunctions with eigenvalue 1. For example, since $P[\ln](x) = \ln x - 1$ and P is regular it follows that no generalised Césaro limit, L, can be assigned to $\ln x$, since L would have to satisfy $L = L - 1$.

Working inductively, the same conclusion holds for all $(\ln x)^m$, $m \in \mathbb{Z}_{>0}$. We thus arrive at the following observation which we shall use frequently.

Lemma 8. *For any integer $m \geq 1$ the generalised eigenfunction, $(\ln x)^m$, of P with eigenvalue 1 cannot be assigned a generalised Césaro limit.*

2.4. Césaro asymptotics. We conclude §2 by defining a notion which will prove useful in §3 and elsewhere.

Definition 3. We say that two functions f and g in $\mathcal{F}$ are Césaro asymptotic, and write $f \overset{C}{\sim} g$, if $\operatorname{Clim}_{x\to\infty}(f-g)(x) = 0$.

This definition satisfies the following basic functorial properties:

Lemma 9. *For any functions in $\mathcal{F}$ we have:*

(i) *If $f \overset{C}{\sim} g$ and $\operatorname{Clim}_{x\to\infty} f(x) = L$ then $\operatorname{Clim}_{x\to\infty} g(x) = L$.*
(ii) *$\overset{C}{\sim}$ is an equivalence relation.*
(iii) *If $f_1 \overset{C}{\sim} f_2$ and $g_1 \overset{C}{\sim} g_2$ then $f_1 + g_1 \overset{C}{\sim} f_2 + g_2$.*

The proofs of these properties are all elementary, involving commutation arguments as in the proof of Lemma 3, the regularity and linearity of P, and the closure of the space of regular polynomials in P under multiplication.

3. The Riemann zeta function

In this section we illustrate the scope of our generalised definition of Césaro convergence by sketching how it yields the analytic continuation of the Riemann zeta function, $\zeta(z)$, directly from analysis of its divergent defining series.

Let ζ^{ext} be the function on $\mathbb{C}$ whose value at any z is the generalised Césaro sum of the series $\sum_{n=1}^{\infty} n^{-z}$. Clearly $\zeta^{\text{ext}}(z) = \zeta(z)$ for $\operatorname{Re} z > 1$. To show that $\zeta^{\text{ext}}(z) = \zeta(z)$ for all $\operatorname{Re} z \leq 1$ also (verified explicitly for $z = 0, -1$ in §2.3) we will need to interpret both ζ and ζ^{ext} in terms of the Euler–MacLaurin sum formula.

As a preliminary, however, note that ζ^{ext} does have a singularity at $z = 1$ as it should. At $z = 1$ the defining series for ζ^{ext} is $\sum_{n=1}^{\infty} \frac{1}{n}$ with partial sum function $s(x) = \ln x + \gamma + o(1)$ where γ is Euler's constant. But by §2.3, Lemma 8, no generalised Césaro limit can be assigned to the function $\ln x$ and so $\zeta^{\text{ext}}(1)$ is undefined. We will show later that $z = 1$ is the only singularity of ζ^{ext} and is in fact a simple pole with residue 1 as required for agreement with ζ.

We now turn, however, to considering the general case. Throughout the next sections let $z \neq 1$ be a fixed complex number with $\operatorname{Re} z \leq 1$.

3.1. The Euler–MacLaurin sum formula. The version we use here is essentially the formulation in [**2**], §13.

Theorem 1 (Euler–MacLaurin Sum Formula). *Suppose that* $f \in C^\infty(0,\infty) \cap L^1_{\mathrm{loc}}[0,\infty)$ *and that* f *and its successive derivatives form an asymptotic scale. Then we have*

$$\sum_{n=1}^{k} f(n) \sim \int_0^k f(x)\,dx + C_f + \frac{1}{2}f(k) + \sum_{r=1}^{\infty}(-1)^{r-1}\frac{B_r}{(2r)!}f^{(2r-1)}(k). \tag{1}$$

Here C_f *is a constant, the* B_r *are the Bernoulli numbers* $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, $B_3 = \frac{1}{42}, \ldots$ *and the expansion is asymptotic in the usual sense that truncating the infinite sum at any point yields a remainder which can be estimated in little-o terms by the last term retained.*

Applying this to the case of $f(x) = x^{-z}$ we find that the partial sum function for the divergent series defining $\zeta(z)$, $\sum_{n=1}^{\infty} n^{-z}$, is given by

$$\begin{aligned} s_{\zeta,z}(k+\alpha) = {} & \frac{k^{-z+1}}{1-z} + C_{\zeta,z} + \frac{1}{2}k^{-z} \\ & + \sum_{r=1}^{\infty}\frac{(-1)^r B_r}{(2r)!} z(z+1)\cdots(z+2r-2)k^{-z-2r+1}. \end{aligned} \tag{2}$$

This expression truncates after some finite number of terms (depending on z), with a remainder which is $o(1)$ as $k \to \infty$ and can therefore be neglected in evaluating $\mathrm{Clim}_{k\to\infty} s_{\zeta,z}(k+\alpha)$. This observation, strengthened by noting the local uniformity of the $o(1)$-estimate in z-neighbourhoods in $\mathbb{C}$, allows one to deduce easily (see e.g., [**2**], §13.10) the following simple formula for $\zeta(z)$ in terms of expansion (2):

Theorem 2. *For any* $z \neq 1$ *the value of* $\zeta(z)$ *is given by*

$$\zeta(z) = C_{\zeta,z} \tag{3}$$

It remains to prove that the same formula holds for $\zeta^{\mathrm{ext}}(z)$. We will do this by re-expressing Equation (2) in simpler form in terms of Césaro asymptotics:

Lemma 10. *For any* $\mathrm{Re}\, z \leq 1$, $z \neq 1$,

$$s_{\zeta,z}(k+\alpha) \overset{C}{\sim} \frac{(k+\alpha)^{-z+1}}{1-z} + C_{\zeta,z}. \tag{4}$$

The desired formula for $\zeta^{\mathrm{ext}}(z)$ will follow at once from this together with Lemmas 6 and 9, completing the proof that $\zeta^{\mathrm{ext}} = \zeta$.

3.2. The proof of Lemma 10. Proving Lemma 10 involves obtaining a general Césaro asymptotic expression for a term of the form $(k+\alpha)^\gamma$, $\mathrm{Re}\,\gamma \geq 0$. Indeed taking $\gamma \equiv 1 - z$ in Equation (2), Lemma 10 is precisely equivalent to verifying the following such expression:

Lemma 11. *For any* $\operatorname{Re}\gamma \geq 0$ *we have*

$$(5)\quad (k+\alpha)^\gamma \overset{C}{\sim} k^\gamma + \frac{1}{2}\gamma k^{\gamma-1} + \sum_{r=1}^{\infty}(-1)^{r-1}\frac{B_r}{(2r)!}\gamma(\gamma-1)\cdots(\gamma-2r+1)k^{\gamma-2r}.$$

When $0 \leq \operatorname{Re}\gamma < 1$ this says simply that $(k+\alpha)^\gamma \overset{C}{\sim} k^\gamma$, while for $1 \leq \operatorname{Re}\gamma < 2$ it states that $(k+\alpha)^\gamma \overset{C}{\sim} k^\gamma + \frac{1}{2}\gamma k^{\gamma-1}$. Both cases are of course easily verified, the first holding classically, the second because $P[(\widetilde{k}+\widetilde{\alpha})^\gamma - \widetilde{k}^\gamma - \frac{1}{2}\gamma\widetilde{k}^{\gamma-1}](k+\alpha) = o(1)$. For $\operatorname{Re}\gamma$ arbitrarily large, however, we need to work indirectly, starting with the Taylor series expansion

$$(6)\qquad (k+\alpha)^\gamma = k^\gamma + \sum_{l=1}^{\lfloor \operatorname{Re}\gamma \rfloor} \frac{\gamma(\gamma-1)\cdots(\gamma-l+1)}{l!}\, k^{\gamma-l}\alpha^l + o(1).$$

Our strategy involves first obtaining a Césaro asymptotic formula for expressions of the form $k^\delta\alpha^r$, for any $\operatorname{Re}\delta \geq 0$ and $r \in \mathbb{Z}_{>0}$. We shall then apply this to each term in Equation (6) to obtain Lemma 11.

Lemma 12. *For any* $\operatorname{Re}\delta \geq 0$ *and any nonnegative integer* r *we have*

$$(7)\qquad k^\delta\alpha^r \overset{C}{\sim} \frac{1}{r+1}k^\delta + \sum_{j=1}^{\lfloor \operatorname{Re}\delta \rfloor} c_j(\delta,r)k^{\delta-j}$$

where

$$(8)\qquad c_j(\delta,r) = \delta(\delta-1)\cdots(\delta-j+1)d_j(r)$$

and

$$(9)\quad d_j(r) = \frac{(-1)^{\lfloor j/2\rfloor-1}}{(r+1)\cdots(r+j+1)}\left(\sum_{l=0}^{\lfloor j/2\rfloor-1}(-1)^l\binom{r+j+1}{2(\lfloor j/2\rfloor-l)}B_{\lfloor j/2\rfloor-l} + \frac{(-1)^{\lfloor j/2\rfloor}}{2}(r+j-1)\right).$$

Proof. We argue by induction on $\operatorname{Re}\delta$, verifying (7) for $0 \leq \operatorname{Re}\delta < 1$ (and arbitrary r) and then proceeding to $1 \leq \operatorname{Re}\delta < 2, \ldots$.

In the base case $0 \leq \operatorname{Re}\delta < 1$, (7) reduces to the formula

$$(10)\qquad k^\delta\alpha^r \overset{C}{\sim} \frac{1}{r+1}k^\delta$$

which is immediate, as for any r, $P\big[(\widetilde{\alpha}^r - \frac{1}{r+1})\widetilde{k}^\delta\big](k+\alpha) = O(k^{\delta-1}) = o(1)$.

For the inductive step, suppose Equation (7) holds for all $\operatorname{Re}\delta < l$ (and arbitrary r) for some positive integer l. To show that it continues to hold for $l \leq \operatorname{Re}\delta < l+1$ (and arbitrary r again), we work in a sequence of steps.

Step (i). First we observe that

$$\begin{aligned} P\left[\left(\widetilde{\alpha}^r - \frac{1}{r+1}\right)\widetilde{k}^\delta\right](k+\alpha) &= \frac{1}{k+\alpha}\left\{k^\delta\left(\frac{\alpha^{r+1}}{r+1} - \frac{\alpha}{r+1}\right)\right\} \\ &= k^{\delta-1}\left(\frac{\alpha^{r+1}}{r+1} - \frac{\alpha}{r+1}\right)\left(1 - \frac{\alpha}{k} + \frac{\alpha^2}{k^2} - \cdots\right) \\ &= k^{\delta-1}\left(\frac{\alpha^{r+1}}{r+1} - \frac{\alpha}{r+1}\right) - k^{\delta-2}\left(\frac{\alpha^{r+2}}{r+1} - \frac{\alpha^2}{r+1}\right) \\ &\qquad + \quad k^{\delta-3}\left(\frac{\alpha^{r+3}}{r+1} - \frac{\alpha^3}{r+1}\right) + \cdots . \end{aligned}$$

Step (ii). Next we apply the inductive hypothesis to each term on the right-hand side in this expression. Using Lemma 9 we thus rewrite it as a Césaro asymptotic equation involving a linear combination of terms $k^{\delta-j}$ with certain real constant coefficients,

$$P\left[\left(\widetilde{\alpha}^r - \frac{1}{r+1}\right)\widetilde{k}^\delta\right](k+\alpha) \overset{C}{\sim} a_1^{(0)}k^{\delta-1} + a_2^{(0)}k^{\delta-2} + a_3^{(0)}k^{\delta-3} + \cdots .$$

We now need to write the entire expression on the right here as the image of some other expression under P, at least Césaro asymptotically. We do this by iteratively "inverting P at top order".

Step (iii). At top order we have

$$a_1^{(0)}k^{\delta-1} = P\big[a_1^{(0)}\delta\widetilde{k}^{\delta-1}\big](k+\alpha) + \pi_2^{(0)}(\alpha)k^{\delta-2} + \pi_3^{(0)}(\alpha)k^{\delta-3} + \cdots + o(1)$$

where each $\pi_j^{(0)}(\alpha)$ is some constant-coefficient polynomial in α arising from the sub-leading terms in $P\big[a_1^{(0)}\delta\widetilde{k}^{\delta-1}\big](k+\alpha)$ (whose evaluation entails using the Euler–MacLaurin sum formula).

Step (iv). We thus obtain

$$\begin{aligned} P\left[\left(\widetilde{\alpha}^r - \frac{1}{r+1}\right)\widetilde{k}^\delta\right](k+\alpha) &\overset{C}{\sim} P\big[a_1^{(0)}\delta\widetilde{k}^{\delta-1}\big](k+\alpha) + \big(a_2^{(0)} + \pi_2^{(0)}(\alpha)\big)k^{\delta-2} \\ &\qquad + \big(a_3^{(0)} + \pi_3^{(0)}(\alpha)\big)k^{\delta-3} + \cdots \end{aligned}$$

and invoking the inductive hypothesis again, this can in turn be rewritten as

$$\begin{aligned} P\left[\left(\widetilde{\alpha}^r - \frac{1}{r+1}\right)\widetilde{k}^\delta\right](k+\alpha) &\overset{C}{\sim} P\big[a_1^{(0)}\delta\widetilde{k}^{\delta-1}\big](k+\alpha) \\ &\qquad + a_2^{(1)}k^{\delta-2} + a_3^{(1)}k^{\delta-3} + \cdots \end{aligned}$$

for some new collection of constant coefficients $a_j^{(1)}, j = 2, 3, \ldots$.

Step (v). Iterating Steps (iii) and (iv), starting next at order $k^{\delta-2}$ and dropping successively by one order in k at each iteration, we ultimately obtain a complete Césaro asymptotic expression for the right-hand side as the image of some expression under P:

$$\begin{aligned} P\left[\left(\widetilde{\alpha}^r - \frac{1}{r+1}\right)\widetilde{k}^\delta\right](k+\alpha) \overset{C}{\sim} P\big[a_1^{(0)}\delta\widetilde{k}^{\delta-1}\big](k+\alpha) \\ + P\big[a_2^{(1)}(\delta-1)\widetilde{k}^{\delta-2}\big](k+\alpha) \\ + P\big[a_3^{(2)}(\delta-2)\widetilde{k}^{\delta-3}\big](k+\alpha) + \cdots. \end{aligned}$$

Step (vi) This then finally yields the desired Césaro asymptotic expression for $k^\delta\alpha^r$,

$$k^\delta\alpha^r \overset{C}{\sim} \frac{1}{r+1}k^\delta + a_1^{(0)}\delta k^{\delta-1} + a_2^{(1)}(\delta-1)k^{\delta-2} + \cdots$$

and it remains only to verify that the coefficients in this expansion continue to be given by formulae (7)–(9) to complete the inductive step and hence the proof of Lemma 12.

Unfortunately detailed verification of this appears to be combinatorially messy, owing to the need to iteratively invoke the inductive hypothesis and keep track of the lower-order correction terms in each top-order inversion of P. We will only show how the strategy proceeds in the two simplest situations, going from the base case to the case $1 \le \operatorname{Re}\delta < 2$, and then from this to the case $2 \le \operatorname{Re}\delta < 3$. We note, however, that such computations as far as $\operatorname{Re}\delta < 7$ have been performed and were essential in guessing the correct form of Lemma 12 in the first place.

For $1 \le \operatorname{Re}\delta < 2$, in Step (i) we have simply

$$P\left[\left(\widetilde{\alpha}^r - \frac{1}{r+1}\right)\widetilde{k}^\delta\right](k+\alpha) = k^{\delta-1}\left(\frac{\alpha^{r+1}}{r+1} - \frac{\alpha}{r+1}\right) + o(1)$$

and, by the base case Equation (10), this yields in Step (ii) that

$$P\left[\left(\widetilde{\alpha}^r - \frac{1}{r+1}\right)\widetilde{k}^\delta\right](k+\alpha) \overset{C}{\sim} \frac{-r}{2(r+1)(r+2)}\, k^{\delta-1}.$$

Inverting at top order, and noting all sub-leading terms are $o(1)$ in this case, we thus obtain in Step (iii) that

$$P\left[\left(\widetilde{\alpha}^r - \frac{1}{r+1}\right)\widetilde{k}^\delta\right](k+\alpha) \overset{C}{\sim} P\left[\frac{-r\delta}{2(r+1)(r+2)}\widetilde{k}^{\delta-1}\right](k+\alpha)$$

and this means immediately in Step (vi) that

$$k^\delta\alpha^r \overset{C}{\sim} \frac{1}{r+1}\, k^\delta - \frac{r\delta}{2(r+1)(r+2)}\, k^{\delta-1}.$$

This verifies Lemma 12 for this case.

Now suppose $2 \le \operatorname{Re}\delta < 3$. Then in Step (i) we have that

$$\begin{aligned} &P\left[\left(\widetilde{\alpha}^r - \frac{1}{r+1}\right)\widetilde{k}^\delta\right](k+\alpha) \\ &= k^{\delta-1}\left(\frac{\alpha^{r+1}}{r+1} - \frac{\alpha}{r+1}\right) - k^{\delta-2}\left(\frac{\alpha^{r+2}}{r+1} - \frac{\alpha^2}{r+1}\right) + o(1) \end{aligned}$$

and by Equations (7)–(9) this becomes, in Step (ii), the equation

$$\begin{aligned} P\left[\left(\widetilde{\alpha}^r - \frac{1}{r+1}\right)\widetilde{k}^\delta\right](k+\alpha) \overset{C}{\sim} &\frac{-r}{2(r+1)(r+2)}\, k^{\delta-1} \\ &+ \left\{\frac{(r^2-r)\delta + (3r^2+9r)}{12(r+1)(r+2)(r+3)}\right\} k^{\delta-2}. \end{aligned}$$

Inverting at top order in Step (iii), note that

$$\begin{aligned} &P\left[\frac{r\delta}{2(r+1)(r+2)}\widetilde{k}^{\delta-1}\right](k+\alpha) \\ &= \left(\frac{r\delta}{2(r+1)(r+2)}\right)\frac{1}{k+\alpha}\left(\sum_{j=0}^{k-1} j^{\delta-1} + k^{\delta-1}\alpha\right) \\ &= \left(\frac{r\delta}{2(r+1)(r+2)}\right)\left(\frac{(k-1)^\delta}{\delta} + \frac{1}{2}(k-1)^{\delta-1}\right. \\ &\qquad\qquad \left. + k^{\delta-1}\alpha + O(k^{\delta-2})\right)\frac{1}{k}\left(1 - \frac{\alpha}{k} + \dots\right) \\ &= \left(\frac{r}{2(r+1)(r+2)}\right)k^{\delta-1} + \left(\frac{-r\delta}{4(r+1)(r+2)} + \frac{r\delta\alpha}{2(r+1)(r+2)}\right. \\ &\qquad\qquad \left. - \frac{r\alpha}{2(r+1)(r+2)}\right)k^{\delta-2} + o(1). \end{aligned}$$

We thus obtain initially in Step (iv) that

$$\begin{aligned} &P\left[\left(\widetilde{\alpha}^r - \frac{1}{r+1}\right)\widetilde{k}^\delta\right](k+\alpha) \\ &\quad\overset{C}{\sim} P\left[\frac{-r\delta}{2(r+1)(r+2)}\widetilde{k}^{\delta-1}\right](k+\alpha) + \left(\frac{(r^2-r)\delta + (3r^2+9r)}{12(r+1)(r+2)(r+3)}\right. \\ &\qquad\qquad \left. - \frac{r\delta}{4(r+1)(r+2)} + \frac{r\delta\alpha}{2(r+1)(r+2)} - \frac{r\alpha}{2(r+1)(r+2)}\right)k^{\delta-2} \end{aligned}$$

and, on invoking Equations (7)–(9) again, this reduces to simply

$$P\left[\left(\widetilde{\alpha}^r - \frac{1}{r+1}\right)\widetilde{k}^\delta\right](k+\alpha) \overset{C}{\sim} P\left[\frac{-r\delta}{2(r+1)(r+2)}\widetilde{k}^{\delta-1}\right](k+\alpha) + \left(\frac{(r^2-r)\delta}{12(r+1)(r+2)(r+3)}\right)k^{\delta-2}.$$

But then iterating Step (iii) by inverting now at order $k^{\delta-2}$ we deduce that

$$\left(\frac{(r^2-r)\delta}{12(r+1)(r+2)(r+3)}\right)k^{\delta-2} = P\left[\frac{r(r-1)\delta(\delta-1)}{12(r+1)(r+2)(r+3)}\widetilde{k}^{\delta-2}\right](k+\alpha) + o(1)$$

and hence overall (Step (v)) that

$$P\left[\left(\widetilde{\alpha}^r - \frac{1}{r+1}\right)\widetilde{k}^\delta\right](k+\alpha) \overset{C}{\sim} P\left[\frac{-r\delta}{2(r+1)(r+2)}\widetilde{k}^{\delta-1}\right](k+\alpha) + P\left[\frac{r(r-1)\delta(\delta-1)}{12(r+1)(r+2)(r+3)}\widetilde{k}^{\delta-2}\right](k+\alpha).$$

It follows immediately in Step (vi) that

$$k^\delta\alpha^r \overset{C}{\sim} \frac{1}{r+1}\,k^\delta - \frac{r\delta}{2(r+1)(r+2)}\,k^{\delta-1} + \frac{r(r-1)\delta(\delta-1)}{12(r+1)(r+2)(r+3)}\,k^{\delta-2}$$

and this again verifies Lemma 12, for the case $2 \le \operatorname{Re}\delta < 3$. □

Having sketched a proof of Lemma 12 it now remains to verify that it does in turn yield Lemma 11. We turn to this now.

Invoking Lemma 12 term by term in the Taylor series expansion (6), we obtain the Césaro asymptotic equation

$$(k+\alpha)^\gamma \overset{C}{\sim} k^\gamma + \sum_{l=1}^{\lfloor\gamma\rfloor}\sum_{j=0}^{\lfloor\gamma\rfloor-l} \frac{\gamma(\gamma-1)\cdots(\gamma-l+1)}{l!}\,c_j(\gamma-l,l)k^{\gamma-l-j}$$

where here we have extended the definition of $c_j(\delta,r)$ in Equations (8) and (9) by setting $c_0(\delta,r) \equiv \frac{1}{r+1}$ so Equation (7) becomes simply

$$k^\delta\alpha^r \overset{C}{\sim} \sum_{j=0}^{\lfloor\delta\rfloor} c_j(\delta,r)k^{\delta-j}.$$

Letting $p = j+l$ and swapping the order of summation this becomes

$$(k+\alpha)^\gamma \overset{C}{\sim} k^\gamma + \sum_{p=1}^{\lfloor\gamma\rfloor}\beta_p(\gamma)k^{\gamma-p} \tag{11}$$

where

$$\beta_p(\gamma) = \sum_{l=1}^{p} \frac{\gamma(\gamma-1)\cdots(\gamma-l+1)}{l!}\, c_{p-l}(\gamma-l,l). \tag{12}$$

Comparing Equations (11) and (5) we see that proving Lemma 11 reduces to showing on the one hand that

$$\beta_1(\gamma) = \frac{1}{2}\gamma \quad \text{and} \quad \beta_p(\gamma) = 0 \;\text{ for all } p \text{ odd}, \; p \geq 3 \tag{13}$$

and on the other that

$$\beta_p(\gamma) = \frac{(-1)^{\frac{p}{2}-1}}{p!}\, B_{\frac{p}{2}}\, \gamma(\gamma-1)\cdots(\gamma-p+1) \;\text{ for } p \text{ even}, \; p \geq 2. \tag{14}$$

Consider Equation (13) first. It is easy to see that $\beta_1(\gamma) = \frac{1}{2}\gamma$ so it remains to show that $\beta_{2m+1}(\gamma) = 0$ for any positive integer m.

Now in Equation (12) for $\beta_{2m+1}(\gamma)$ consider first just the $l = 1$ term. This is simply $\gamma\, c_{2m}(\gamma-1,1)$ and so is a multiple of $d_{2m}(1)$ where, by Equation (9),

$$\begin{aligned} d_{2m}(1) &= \frac{(-1)^{m-1}}{(2m+2)!}\left(\sum_{i=0}^{m-1}(-1)^i \binom{2m+2}{2(m-i)} B_{m-i} + (-1)^m m\right) \\ &= \frac{-1}{(2m+2)!}\left(\sum_{q=1}^{m}(-1)^q \binom{2m+2}{2q} B_q + m\right). \end{aligned}$$

But this expression is in fact identically zero for any $m \geq 1$. To see this we use the Bernoulli polynomials which, for even index, are given by

$$B_{2n}(x) \equiv x^{2n} - nx^{2n-1} + \sum_{q=1}^{n}(-1)^{q-1}\binom{2n}{2q} B_q\, x^{2n-2q} \tag{15}$$

([4], §9.6, adjusting for a different convention regarding the indexing of the Bernoulli numbers). Letting $n = m+1$, splitting off the $q = m+1$ term from the sum and rearranging, this becomes the equation

$$\begin{aligned} \sum_{q=1}^{m}(-1)^{q-1}\binom{2m+2}{2q} B_q\, x^{2m+2-2q} = {} & B_{2m+2}(x) - x^{2m+2} \\ & + (m+1)x^{2m+1} + (-1)^{m-1}B_{m+1}. \end{aligned}$$

But now recall ([4], §9.6]) that $B_{2m+2}(1) = (-1)^m B_{m+1}$. Substituting $x = 1$ into our expression it follows that

$$\sum_{q=1}^{m}(-1)^{q-1}\binom{2m+2}{2q} B_q = m \tag{16}$$

and hence we obtain at once that $d_{2m}(1) = 0$ for all $m \geq 1$ as claimed.

It follows that for any $m \geq 1$ Equation (12) for $\beta_{2m+1}(\gamma)$ becomes simply

$$\beta_{2m+1}(\gamma) = \sum_{l=2}^{2m+1} \frac{\gamma(\gamma-1)\cdots(\gamma-l+1)}{l!}\, c_{2m+1-l}(\gamma-l,l).$$

Splitting the terms in this sum into pairs , we can next rewrite this as

$$\beta_{2m+1}(\gamma) = \sum_{q=1}^{m} A_q(\gamma)$$

where

$$A_q(\gamma) = \frac{\gamma(\gamma-1)\cdots(\gamma-2q+1)}{(2q)!}\, c_{2m-2q+1}(\gamma-2q,2q) + \frac{\gamma(\gamma-1)\cdots(\gamma-2q)}{(2q+1)!}\, c_{2m-2q}(\gamma-2q-1,2q+1).$$

But using Equations (8) and (9) we find that, for any $1 \leq q \leq m-1$, both terms in this expression for $A_q(\gamma)$ in fact collapse to the same quantity, giving

$$A_q(\gamma) = 2\left(\frac{\gamma(\gamma-1)\cdots(\gamma-2m)}{(2m+2)!}\right)\left(\sum_{s=1}^{m-q}(-1)^{s-1}\binom{2m+2}{2s} B_s - m\right)$$

while for the case $q=m$, recalling that $c_0(\delta,r) \equiv \frac{1}{r+1}$, we obtain easily that

$$A_m(\gamma) = \frac{\gamma(\gamma-1)\cdots(\gamma-2m)}{(2m+2)!}\,(1-m).$$

Substituting these expressions for $A_q(\gamma)$ we thus obtain overall that

$$\begin{aligned}\beta_{2m+1}(\gamma) &= 2\left(\frac{\gamma(\gamma-1)\cdots(\gamma-2m)}{(2m+2)!}\right)\\ &\quad\cdot\left(\left(\frac{1-m}{2}\right) + \sum_{q=1}^{m-1}\left(\sum_{s=1}^{m-q}(-1)^{s-1}\binom{2m+2}{2s} B_s - m\right)\right)\\ &= 2\left(\frac{\gamma(\gamma-1)\cdots(\gamma-2m)}{(2m+2)!}\right)\\ &\quad\cdot\left(-\frac{(2m+1)(m-1)}{2} + \sum_{s=1}^{m-1}(-1)^{s-1}(m-s)\binom{2m+2}{2s} B_s\right)\end{aligned}$$

where, in the last step, we have combined all terms not involving Bernoulli numbers and reduced the double sum by reversing the order of summation and noting that the summand is independent of q.

But now consider again the Bernoulli polynomials defined by Equation (15). Differentiating with respect to x and setting $x = 1$ yields that

$$2\sum_{q=1}^{n}(-1)^{q-1}(n-q)\left(\begin{array}{c}2n\\2q\end{array}\right)B_q = B_{2n}'(1) + 2n^2 - 3n$$

and since $B_{2n}'(1) = 0$ for any n ([**4**], §9.6 again) this in turn reduces to the equation

$$\sum_{q=1}^{n-1}(-1)^{q-1}(n-q)\left(\begin{array}{c}2n\\2q\end{array}\right)B_q = \frac{2n^2-3n}{2}. \tag{17}$$

But setting $n = m+1$ it then follows directly from our earlier calculation (16) that

$$\sum_{q=1}^{m}(-1)^{q-1}(m-q)\left(\begin{array}{c}2m+2\\2q\end{array}\right)B_q = \frac{(2m+1)(m-1)}{2}. \tag{18}$$

Substituting into our last formula for $\beta_{2m+1}(\gamma)$ this immediately implies $\beta_{2m+1}(\gamma) = 0$ for any $m \geq 1$, as claimed, and this completes the proof of identity (13).

The proof of the second identity (14) follows in similar fashion.

But this then completes our sketch of the proof of Lemma 11 using Lemma 12; hence of Lemma 10; and hence ultimately, as noted, of our central result in §3, namely that $\zeta^{ext} = \zeta$ on all of $\mathbb{C} \setminus \{1\}$.

3.3. The singularity. We conclude §3 by showing how we can also determine the nature of the singularity of $\zeta^{\text{ext}} = \zeta$ at $z = 1$ within our generalised Césaro framework.

The key is to observe that in the proof of Lemma 11, the Césaro asymptotic formula (5) is in fact obtained by applying a pure power of P, namely $P^{\lfloor \mathrm{Re}\,\gamma \rfloor}$. Recalling our identification $\gamma \equiv 1 - z$ it thus follows from Lemmas 10 and 6 that, for any $z \neq 1$, the regular polynomial $q(z, P)$ needed to evaluate $\zeta^{\text{ext}}(z)$ as the generalised Césaro limit of $s_{\zeta,z}(k+\alpha)$ (i.e., to obtain $q(z, P)[s_{\zeta,z}](k+\alpha) \to C_{\zeta,z}$ as $k \to \infty$) is given explicitly by

$$q(z,P) = \left(\frac{2-z}{1-z}\right)\left(P - \frac{1}{2-z}\right)P^{\lfloor -\mathrm{Re}\,z \rfloor + 1}. \tag{19}$$

This observation immediately leads to a deeper explanation for the presence of the singularity of ζ^{ext} at $z = 1$: it occurs because of the breakdown of regularity and analyticity of $q(z, P)$ at $z = 1$, arising from the presence of the factor $(\frac{2-z}{1-z})$.

To see the precise form of this singularity, consider $\lim_{z\to 1}(z-1)\zeta^{\text{ext}}(z)$ within our Césaro framework. We have

$$\begin{aligned}\lim_{z\to 1}(z-1)\zeta^{\text{ext}}(z) &= \lim_{z\to 1}(z-1)\lim_{k\to\infty} q(z,P)[s_{\zeta,z}](k+\alpha)\\ &= -\lim_{z\to 1}\lim_{k\to\infty}((2-z)P-1)P^{\lfloor -\operatorname{Re} z\rfloor+1}[s_{\zeta,z}](k+\alpha)\\ &= -\lim_{k\to\infty}(P-1)[s_{\zeta,1}](k+\alpha)\\ &= -\lim_{k\to\infty}(P-1)[\ln\widetilde{x}+\gamma+o(1)](k+\alpha)=1\end{aligned}$$

on swapping limits in the third line and recalling that $P[\ln](x)=\ln x-1$. It follows that, for z near 1, $\zeta^{\text{ext}}(z)=\frac{1}{z-1}+\textit{analytic}$. Thus $z=1$ is a simple pole of $\zeta^{\text{ext}}=\zeta$ with residue 1.

4. Discrete Césaro summation and ζ

It is interesting to briefly reconsider the example of ζ in the context of our original *discrete* Césaro scheme for sequences/series.

4.1. Basic spectrum of P_D. We first need to mimic our analysis of P and identify the spectrum of eigenvalues and eigensequences/generalised eigensequences of P_D.

Clearly the unique eigensequences of P_D with eigenvalue $\lambda=1$ are the constant eigensequences, and by analogy with §2 it seems natural, for $\lambda\neq 1$, to look at sequences of the form $\{j^\rho\}_{j=1}^\infty$ for arbitrary $\operatorname{Re}\rho\geq 0$, $\rho\neq 0$ (we again ignore $\operatorname{Re}\rho<0$ since then $\{j^\rho\}_{j=1}^\infty$ is already classically convergent to 0). Here, however, we have to split into two cases:

Case (i). For $\rho\in\mathbb{Z}_{>0}$ a simple induction argument shows that the exact eigensequence, $\{a_j\}_{j=1}^\infty$, of P_D with eigenvalue $\lambda=\frac{1}{\rho+1}$ is given not simply by $a_j=j^\rho$ but rather by

$$a_j=\prod_{i=1}^{\rho}(j-i)=(j-1)(j-2)\cdots(j-\rho). \tag{20}$$

Case (ii). For $\rho\notin\mathbb{Z}_{>0}$ we need to work in steps. Consider first sequences $\{j^\rho\}_{j=1}^\infty$ with $0\leq\operatorname{Re}\rho<1$, $\rho\neq 0$. By the Euler–MacLaurin sum formula

$$P_D[\{j^\rho\}]_k=\frac{1}{k}\left(\frac{k^{\rho+1}}{\rho+1}+O(k^\rho)\right)=\frac{k^\rho}{\rho+1}+o(1)$$

and it follows that in this case $\{j^\rho\}_{j=1}^\infty$ is an asymptotic eigensequence of P_D (in the obvious sense analogous to Definition 2) with eigenvalue $\frac{1}{\rho+1}$. This means at once (cf. §2, Lemma 1) that

$$C_D\lim_{k\to\infty}\{k^\rho\}=0\quad\text{for any}\quad 0\leq\operatorname{Re}\rho<1,\ \rho\neq 0. \tag{21}$$

Next consider the case $1 \le \operatorname{Re}\rho < 2$, $\rho \ne 1$. Here we obtain

$$P_D[\{j^\rho\}]_k = \frac{1}{\rho+1}\, k^\rho + \frac{1}{2}\, k^{\rho-1} + o(1)$$

so $\{j^\rho\}_{j=1}^\infty$ is no longer an asymptotic eigensequence of P_D. It is easy to turn it into one, however, simply by adding a suitable lower-order correction term. A short computation (similar to our top-order inversions of P in §3) yields that in fact in this case the desired asymptotic eigensequence with eigenvalue $\frac{1}{\rho+1}$ is $\{j^\rho - \frac{\rho(\rho+1)}{2} j^{\rho-1}\}_{j=1}^\infty$ and we deduce that

$$C_D \lim_{k\to\infty} \left\{ k^\rho - \frac{\rho(\rho+1)}{2} k^{\rho-1} \right\} = 0 \quad \text{for any} \quad 1 \le \operatorname{Re}\rho < 2,\ \rho \ne 1.$$

In light of Equation (21) (and the discrete analogue of §2, Lemma 9), however, this still implies simply

$$C_D \lim_{k\to\infty} \{k^\rho\} = 0 \quad \text{for any} \quad 1 \le \operatorname{Re}\rho < 2,\ \rho \ne 1. \tag{22}$$

In the same way, for $2 \le \operatorname{Re}\rho < 3$, $\rho \ne 2$ we find that the asymptotic eigensequence of P_D is now $\{j^\rho - \frac{\rho(\rho+1)}{2} j^{\rho-1} + \frac{\rho(\rho+1)(\rho-1)(3\rho+2)}{24} j^{\rho-2}\}_{j=1}^\infty$, but, in light of Equations (21) and (22), this clearly still implies

$$C_D \lim_{k\to\infty} \{k^\rho\} = 0 \quad \text{for any} \quad 2 \le \operatorname{Re}\rho < 3,\ \rho \ne 2 \tag{23}$$

and continuing in this fashion we see that in general

$$C_D \lim_{k\to\infty} \{k^\rho\} = 0 \quad \text{for any} \quad \operatorname{Re}\rho \ge 0,\ \rho \notin \mathbb{Z}. \tag{24}$$

We shall limit our analysis of Case (ii) to this observation. Results (20) and (24) can then be combined, after an elementary computation, into a single lemma summarising our cursory spectral analysis of P_D so far:

Lemma 13. *For any* $\operatorname{Re}\rho \ge 0$ *we have*

$$C_D \lim_{k\to\infty} \{k^\rho\} = \begin{cases} 1 & \text{if } \rho \in \mathbb{Z}_{\ge 0} \\ 0 & \text{otherwise.} \end{cases} \tag{25}$$

(Cf. the continuous Césaro scheme for which $\operatorname{Clim}_{k\to\infty}\{k^\rho\} = 0$ for $\rho \notin \mathbb{Z}_{\ge 0}$ but $\operatorname{Clim}_{k\to\infty}\{k^\rho\} = (-1)^\rho \frac{1}{\rho+1}$ for $\rho \in \mathbb{Z}_{\ge 0}$.)

4.2. $\zeta(z)$ and the discrete Césaro scheme. Defining $\zeta^{\text{ext},D}(z)$ as the generalised discrete Césaro sum of $\sum_{n=1}^\infty n^{-z}$, we now investigate, as in §3, whether $\zeta^{\text{ext},D} = \zeta$ for all $\operatorname{Re} z \le 1$.

By working almost identical to that in §3, we may first easily verify that $z = 1$ is a singularity of $\zeta^{\text{ext},D}$ also, as we want. Turning to $z \ne 1$, the results of Lemma 13 suggest that we should divide our analysis into two cases:

Case (i) $[z \notin \mathbb{Z}_{\leq 0}]$. In this case the partial sum sequence for $\sum_{n=1}^{\infty} n^{-z}$, which is given (cf. Equation (2)) by

$$(s_{\zeta,z})_k = \frac{k^{-z+1}}{1-z} + C_{\zeta,z} + \frac{1}{2}k^{-z} - \frac{1}{12}zk^{-z-1} + \cdots + o(1)$$

has only noninteger powers of k in the terms other than $C_{\zeta,z}$. It follows at once from Lemma 13 that $C_D \lim_{k\to\infty}\{(s_{\zeta,z})_k\} = C_{\zeta,z}$ and, in light of Theorem 2, this means immediately that we do indeed have

$$\zeta^{\text{ext},D}(z) = \zeta(z) \quad \text{for all} \quad \operatorname{Re} z \leq 1,\ z \notin \mathbb{Z}_{\leq 0}. \tag{26}$$

Case (ii) $[z \in \mathbb{Z}_{\leq 0}]$. In this case, however, we obtain the following result:

Lemma 14. *For any* $z \in \mathbb{Z}_{\leq 0}$

$$\zeta^{\text{ext},D}(z) = 1. \tag{27}$$

Proof. Writing $z = -r$ we need to show that $C_D \lim_{k\to\infty}\{s_k^{(r)}\} = 1$ for all nonnegative integers r, where $s_k^{(r)} \equiv \sum_{j=1}^{k} j^r$. But $\sum_{j=1}^{k} j^r$ can be expanded as a linear combination of nonnegative integer powers of k, whose generalised discrete Césaro limits are all 1 by Lemma 13. It follows that $C_D \lim_{k\to\infty}\{s_k^{(r)}\}$ can be computed simply by setting $k = 1$ in the expression $\sum_{j=1}^{k} j^r$ for $s_k^{(r)}$. This trivially implies Equation (27) since $\sum_{j=1}^{1} j^r = 1$. □

Lemma 14 means of course that

$$\zeta^{\text{ext},D}(z) \neq \zeta(z) \quad \text{for all} \quad z \in \mathbb{Z}_{\leq 0}. \tag{28}$$

Thus in the discrete setting $\zeta^{\text{ext},D}$ yields a countable collection of anomalous evaluations of the zeta function at $z = 0, -1, \ldots$. Closer analysis of our working, however, explains this problem.

4.3. Correcting the anomalies. We would like a diagnostic way of identifying in advance that the values of $\zeta^{\text{ext},D}(z)$ are potentially anomalous at $z = 0, -1, \ldots$ and, in turn, a way of computing the true values of ζ at these points.

The key, as in §3.3, is to consider explicitly the polynomials $q(z, P_D)$ used to evaluate $\zeta^{\text{ext},D}(z)$ as $\lim_{k\to\infty}(q(z, P_D)[\{(s_{\zeta,z})_{\widetilde{k}}\}])_k$. We examine these in successive vertical strips moving leftwards in the complex plane.

Consider first the strip $-1 < \operatorname{Re} z < 1$ containing 0. Here $(s_{\zeta,z})_k = \frac{k^{-z+1}}{1-z} + C_{\zeta,z} + \frac{1}{2}\, k^{-z} + o(1)$, and writing this in terms of asymptotic eigensequences of P_D we have

$$(s_{\zeta,z})_k = \frac{1}{1-z}\left(k^{1-z} - \frac{(1-z)(2-z)}{2}\, k^{-z}\right) + \frac{3-z}{2}\, k^{-z} + C_{\zeta,z} + o(1). \tag{29}$$

It follows that in this strip the regular polynomial we need to use is

$$q(z, P_D) = -\left(\frac{2-z}{z}\right)\left(P_D - \frac{1}{2-z}\right)\left(P_D - \frac{1}{1-z}\right), \tag{30}$$

the first factor here annihilating the eigensequence $\{k^{1-z} - \frac{(1-z)(2-z)}{2} k^{-z}\}_{k=1}^{\infty}$ and the second the other eigensequence $\{k^{-z}\}_{k=1}^{\infty}$ in the expression (29).

But it is now clear why the calculation of $\zeta^{\mathrm{ext},D}(0)$ was anomalous: the analyticity of $q(z, P_D)$ breaks down at $z = 0$ due to the regularising factor $(\frac{2-z}{z})$. As in §3.3, this signals the presence of a singularity in $\zeta^{\mathrm{ext},D}$ at $z = 0$. In this case, however, $z = 0$ is not a pole but a removable singularity and we can obtain the correct value of $\zeta(0)$ simply by applying L'Hôpital's law within our discrete Césaro scheme:

$$\begin{aligned}
&\zeta(0)\\
&= \lim_{z\to 0} \zeta^{\mathrm{ext},D}(z) = \lim_{z\to 0}\lim_{k\to\infty} q(z, P_D)[\{(s_{\zeta,z})_{\widetilde{k}}\}]_k\\
&= -\lim_{k\to\infty}\lim_{z\to 0} \frac{((2-z)P_D - 1)(P_D - \frac{1}{1-z})[\{(s_{\zeta,z})_{\widetilde{k}}\}]_k}{z}\\
&= -\lim_{k\to\infty}\lim_{z\to 0} \Bigg\{((2-z)P_D - 1)\left(P_D - \frac{1}{1-z}\right)\left[\left\{\frac{d}{dz}(s_{\zeta,z})_{\widetilde{k}}\right\}\right]_k\\
&\qquad + \left(\frac{-1}{(1-z)^2}\right)((2-z)P_D - 1)[\{(s_{\zeta,z})_{\widetilde{k}}\}]_k\\
&\qquad - P_D\left(P_D - \frac{1}{1-z}\right)[\{(s_{\zeta,z})_{\widetilde{k}}\}]_k\Bigg\}\\
&= -\lim_{k\to\infty}\Bigg\{(2P_D - 1)(P_D - 1)\left[\left\{-\widetilde{k}\ln\widetilde{k} + \widetilde{k} + \left(\frac{d}{dz}C_{\zeta,z}\right)_{z=0} - \frac{1}{2}\ln\widetilde{k}\right\}\right]_k\\
&\qquad - (2P_D - 1)[\{\widetilde{k}\}]_k - P_D(P_D - 1)[\{\widetilde{k}\}]_k\Bigg\}\\
&= -\lim_{k\to\infty}\left\{\left(\frac{5}{4} - \frac{1}{4}k + \frac{1}{2}\right) - 1 + \frac{1}{4}(k-1)\right\} = -\frac{1}{2}.
\end{aligned}$$

In the final steps here we have used that $(2P_D - 1)[\{\widetilde{k}\ln\widetilde{k}\}]_k = \ln k - \frac{1}{2}\, k + o(1)$ and $(P_D - 1)[\{\ln\widetilde{k}\}]_k = -1 + o(1)$.

A more careful analysis within our discrete Césaro scheme thus explains the anomalous discrepancy between $\zeta^{\mathrm{ext},D}$ and ζ at $z = 0$ and how to correct it. The same explanatory framework is easily seen to apply also at $z = -1, -2, \ldots$.

Indeed, if we work in an open neighbourhood of any $z_0 \in \mathbb{Z}_{<0}$ and express $(s_{\zeta,z})_k$ as a linear combination of asymptotic eigensequences of P_D as in Equation (29), we find that the analyticity of the regular polynomial $q(z, P_D)$ we need to use in this neighbourhood breaks down as $z \to z_0$. This occurs because the lowest-order eigensequence in this expansion becomes a constant sequence at $z = z_0$. Since constant sequences are eigensequences of P_D with eigenvalue 1, one of the factors in $q(z, P_D)$ thus degenerates into $(P - 1)$ as $z \to z_0$, and since $q(z, P_D)$ must be regular, this leads immediately to a breakdown of analyticity in z. However since, at $z = z_0$, $\{(s_{\zeta,z})_k\}_{k=1}^{\infty}$ is acquiring just an eigensequence of P_D with eigenvalue 1 rather than a generalised eigensequence of the form $\{(\ln k)^m\}_{k=1}^{\infty}$ (as occurs at $z = 1$), the singularity at z_0 is a removable singularity rather than a pole. A L'Hôpital's calculation can thus again be used to correct the anomaly in $\zeta^{\mathrm{ext},D}(z_0)$ and yield the correct value of $\zeta(z_0)$.

In this fashion then it is possible to obtain the correct analytic continuation of the zeta function to all of $\mathbb{C} \setminus \{1\}$ within our discrete Césaro scheme also.

4.4. Final remarks. Our definitions of generalised Césaro convergence give new notions of convergence for sequences and functions and hence, in some sense, new topologies on $\mathbb{C}$. However, as the presence of the anomalies in our calculations in §4 shows, when applied to a sequence/family of functions of a complex variable z, evaluation of generalised limits pointwise does not guarantee analyticity of the limit function. As with Weierstrass' classical theorem, we will see in the next section that to guarantee analyticity of generalised limits one needs to work in open neighbourhoods, using families of polynomials $q(z, P_D)$ or $q(z, P)$ which are analytic in z and regular throughout these entire neighbourhoods, and obtaining convergence of the transformed sequences of functions which is appropriately uniform. We begin concretely by reconsidering the cases of ζ^{ext} and $\zeta^{\mathrm{ext},D}$.

5. Analyticity and generalised convergence

In §3 and §4 we deduced the analyticity of ζ^{ext} on $\mathbb{C} \setminus \{1\}$ and of $\zeta^{\mathrm{ext},D}$ on $\mathbb{C} \setminus \mathbb{Z}_{\leq 1}$ only "after the fact" by deriving formulas for them in agreement with the known formula (3) for the analytic continuation of ζ. If we could prove these extensions were *a priori* analytic, however, their coincidence with ζ for $\mathrm{Re}\, z > 1$ would imply that they are in fact both the unique analytic continuation of ζ and formula (3) would follow as a corollary. We consider $\zeta^{\mathrm{ext},D}$ first.

5.1. A priori analyticity of $\boldsymbol{\zeta^{\mathrm{ext},D}}$. We shall need a pair of lemmas, the first of which is as follows:

Lemma 15. *Suppose $\{r_k(z)\}_{k=1}^{\infty}$ is a sequence of analytic functions on $U \subseteq \mathbb{C}$, converging uniformly to zero on compact subsets of U, and suppose $q(z, P_D)$ is a family of polynomials in P_D analytic in z and regular throughout U. Then the transformed sequence $\{q(z, P_D)[\{r_{\tilde{k}}(z)\}]_k\}_{k=1}^{\infty}$ of analytic functions also converges uniformly to zero on compact subsets of U.*

This follows by reducing to the case $q(z, P_D) = P_D$ and elementary estimates.

The second lemma we need concerns local uniformity of convergence in our eigenvalue equations for P_D. For any $\mathrm{Re}\,\rho \geq 0$, let $\{a_k(\rho)\}_{k=1}^{\infty}$ be the eigensequence (asymptotic for $\rho \notin \mathbb{Z}_{\geq 0}$) of P_D with eigenvalue $\frac{1}{\rho+1}$ described in §4.1; i.e., $a_k(\rho) = k^{\rho}$ for $0 \leq \mathrm{Re}\,\rho < 1$, $a_k(\rho) = k^{\rho} - \frac{\rho(\rho+1)}{2}k^{\rho-1}$ for $1 \leq \mathrm{Re}\,\rho < 2$, and so on.

Now consider first any ρ_0 in the interior of one of these strips $l \leq \mathrm{Re}\,\rho < l+1$. If we take $\delta_0 = \min\{\frac{\mathrm{Re}\,\rho_0 - l}{2}, \frac{l+1-\mathrm{Re}\,(\rho_0)}{2}\}$, the open ball $B(\rho_0, \delta_0)$ lies within this strip and at strictly positive distance at least $\frac{l+1-\mathrm{Re}\,\rho_0}{2}$ from its right-hand side. By standard Euler–MacLaurin remainder analysis like that in [**2**], §13.10, it is then easy to see that the remainder sequences $\{r_k(\rho)\}_{k=1}^{\infty}$ in the eigenvalue equation

$$\left(P_D - \frac{1}{\rho+1}\right)[\{a_{\widetilde{k}}(\rho)\}]_k \equiv r_k(\rho) = o(1) \tag{31}$$

in fact converge uniformly to zero for all $\rho \in B(\rho_0, \delta_0)$.

The same is not true, however, for ρ_0 on the edge of a strip, say $\mathrm{Re}\,\rho_0 = l$, $l \in \mathbb{Z}_{\geq 0}$. For such ρ_0 it is readily verified that there is no local neighbourhood on which the remainders $\{r_k(\rho)\}_{k=1}^{\infty}$ in Equation (31) all converge uniformly to zero.

To rectify this we change to an alternative family of asymptotic eigensequences. Specifically, for any $l \in \mathbb{Z}_{\geq 0}$ let $\{a_k^{(l)}(\rho)\}_{k=1}^{\infty}$ be the family of sequences defined, for $-1 < \mathrm{Re}\,\rho < l+1$, by simply applying the formula for $\{a_k(\rho)\}_{k=1}^{\infty}$ in the strip $l \leq \mathrm{Re}\,\rho < l+1$ throughout the region $-1 < \mathrm{Re}\,\rho < l$ also. This yields an equally valid family of eigensequences for $-1 < \mathrm{Re}\,\rho < l+1$ since the leading terms in the formula for $\{a_k(\rho)\}_{k=1}^{\infty}$ in any strip always coincide with the formulae for the $\{a_k(\rho)\}_{k=1}^{\infty}$ in strips further left, so that for $-1 < \mathrm{Re}\,(\rho) < l$ each $\{a_k^{(l)}(\rho)\}_{k=1}^{\infty}$ differs from the known asymptotic eigensequence $\{a_k(\rho)\}_{k=1}^{\infty}$ only by a sequence which is classically $o(1)$.

Using these adapted families of eigensequences instead in Equation (31) the problems above with obtaining locally uniform convergence of remainders at integer values of $\mathrm{Re}\,\rho_0$ disappear and we obtain the following simple uniformity result:

Lemma 16. *For any* $\mathrm{Re}\,\rho_0 > -1$*, set* $\delta_0 = \min\{\frac{\mathrm{Re}\,\rho_0+1}{2}, \frac{l+1-\mathrm{Re}\,\rho_0}{2}\}$ *where* $\lfloor \mathrm{Re}\,\rho_0 \rfloor = l$*. Then the remainder sequences* $\{r_k^{(l)}(\rho)\}_{k=1}^{\infty}$ *in the alternative eigenvalue equation*

$$\left(P_D - \frac{1}{\rho+1}\right)\left[\left\{a_{\widetilde{k}}^{(l)}(\rho)\right\}\right]_k \equiv r_k^{(l)}(\rho) = o(1) \tag{32}$$

converge uniformly to zero for all $\rho \in B(\rho_0, \delta_0)$*.*

With Lemmas 15 and 16 it finally becomes easy to show the desired *a priori* analyticity of $\zeta^{\text{ext},D}$ on $\mathbb{C}\setminus\mathbb{Z}_{\leq 1}$. Let z_0 be any point in $\mathbb{C}$ with $\operatorname{Re} z_0 \leq 1$, $z_0 \notin \mathbb{Z}_{\leq 1}$. Let $l = \lfloor 1 - \operatorname{Re} z_0 \rfloor$ and take $\delta_0 = \min\{\frac{\operatorname{Re} z_0 + l}{2}, \frac{|1-z_0-l|}{2}\}$. Here the first possibility for δ_0 arises from setting $\rho_0 = 1 - z_0$ in the formula for δ_0 in Lemma 16, while the second guarantees that $B(z_0, \delta_0)$ stays a strictly positive distance from the set $\mathbb{Z}_{\leq 1}$ where anomalies/regularity breakdowns occur.

Using expression (2) for $(s_{\zeta,z})_k$, it is clear that, for $z \in B(z_0, \delta_0)$, we can write $(s_{\zeta,z})_k$ as a linear combination of adapted asymptotic eigensequences of P_D (cf. §4.2 Equation (29))

$$(s_{\zeta,z})_k = \sum_{i=0}^{l} \lambda_i(z) a_k^{(l-i)}(1 - z - i) + C_{\zeta,z} + (R_{\zeta,z})_k \tag{33}$$

where $\lambda_0(z) = \frac{1}{1-z}$, the other coefficients $\lambda_i(z)$ are all simply polynomials in z, and the remainders $(R_{\zeta,z})_k$ are uniformly $o(1)$ on $B(z_0, \delta_0)$.

But now consider the transformed family of sequences

$$\{q(z, P_D)[\{(s_{\zeta,z})_{\tilde{k}}\}]_k\}_{k=1}^{\infty}$$

on $B(z_0, \delta_0)$ where, in light of expression (33) (cf. §4.3), we take

$$q(z, P_D) = \prod_{j=0}^{l} \left(\frac{2 - z - j}{1 - z - j}\right) \left(P_D - \frac{1}{2 - z - j}\right). \tag{34}$$

By our choice of δ_0, $q(z, P_D)$ is clearly both analytic and regular throughout $B(z_0, \delta_0)$, so the transformed remainder sequences

$$\{q(z, P_D)[\{(R_{\zeta,z})_{\tilde{k}}\}]_k\}_{k=1}^{\infty}$$

remain uniformly $o(1)$ on compact subsets of $B(z_0, \delta_0)$ by Lemma 15.

Take next the terms $\lambda_i(z) a_k^{(l-i)}(1 - z - i)$, $i = 0, \ldots, l$ in expression (33). For $i = 0$ it follows directly from the way we chose δ_0 that Lemma 16 applies, and we obtain that $(\frac{2-z}{1-z})(P_D - \frac{1}{2-z})[\{\lambda_0(z) a_{\tilde{k}}^{(l)}(1 - z)\}]_k$ is uniformly $o(1)$ on $B(z_0, \delta_0)$. By Lemma 15 this uniformity is then preserved on compact subsets of $B(z_0, \delta_0)$ by the other factors in the product for $q(z, P_D)$, and so $q(z, P_D)[\{\lambda_0(z) a_{\tilde{k}}^{(l)}(1 - z)\}]_k$ is uniformly $o(1)$ on compact subsets of $B(z_0, \delta_0)$. As for $i > 0$, in this case the variables $1 - z - i$ in the eigensequences $a_k^{(l-i)}(1 - z - i)$ lie in integer translates of our ball, but since we are using the $(l - i)$-family of eigensequences we can still immediately invoke Lemma 16. It follows that $(\frac{2-z-i}{1-z-i})(P_D - \frac{1}{2-z-i})[\{\lambda_i(z) a_{\tilde{k}}^{(l-i)}(1 - z - i)\}]_k$ is likewise uniformly $o(1)$ on $B(z_0, \delta_0)$ for all $i = 1, \ldots, l$, and this is again preserved on compact subsets of $B(z_0, \delta_0)$ by the other factors in $q(z, P_D)$.

The transformed sequence of analytic functions $\{(q(z,P_D)[\{(s_{\zeta,z})_{\tilde{k}}\}])_k\}_{k=1}^{\infty}$ thus in fact converges uniformly to $C_{\zeta,z} = \zeta^{\text{ext},D}(z)$ on compact subsets of

$B(z_0, \delta_0)$ and, by the classical Weierstrass theorem, this implies the analyticity of $\zeta^{\mathrm{ext},D}$ on $B(z_0, \delta_0)$. The desired *a priori* analyticity of $\zeta^{\mathrm{ext},D}$ on all of $\mathbb{C} \setminus \mathbb{Z}_{\leq 1}$ follows at once.

5.2. A priori analyticity of $\zeta^{\mathbf{ext}}$. For ζ^{ext} the approach to proving *a priori* analyticity is broadly similar, but with certain nontrivial variations. We will provide only a schematic overview.

The first step remains obtaining a continuous analogue of Lemma 15 and is straightforward. The only variant needed is an extra condition on the 1-parameter family of functions $r(x, z)$, that for any compact $K \subseteq U$ and fixed $x \geq 0$ there exists $M > 0$ with $\int_0^x \mid r(\widetilde{x}, z) \mid d\widetilde{x} \leq M$ for all $z \in K$. If the $r(x, z)$ are remainders in partial sum functions, as in §3, this holds trivially.

The other ingredient now, however, is not an analogue of Lemma 16 (our eigenfunctions of P are already exact rather than asymptotic). Instead observe that, while $q(z, P_D)$ never involved pure powers of P_D but acquired progressively more nontrivial factors $(P_D - \lambda)$ leading to anomalies and necessitating the analysis of Lemma 16, in the continuous case $q(z, P)$ only ever has one nontrivial factor $(\frac{2-z}{1-z})(P - \frac{1}{2-z})$ but requires successively higher pure powers of P to obtain the necessary Césaro asymptotic behaviour in Lemma 10. The second result we need is thus a proof that the pointwise Césaro asymptotic relationship in Lemma 10 can in fact be obtained locally uniformly.

It turns out this can be achieved simply by including one extra factor of P beyond what was used in the pointwise arguments of §3. More precisely, for any $z_0 \in \mathbb{C} \setminus \{1\}$ with, say, $\mathrm{Re}\, z_0 < \frac{5}{3}$, there exists an open neighbourhood of z_0 such that

$$P^{\lfloor -\mathrm{Re}\,(z_0)+2 \rfloor} \left[s_{\zeta,z}(\widetilde{k} + \widetilde{\alpha}) - \frac{(\widetilde{k} + \widetilde{\alpha})^{-z+1}}{1-z} - C_{\zeta,z} \right] (k + \alpha) = o(1)$$

uniformly on compact subsets of this neighbourhood. We call this Lemma $10'$.

To prove this we need first to prove a corresponding locally uniform version, Lemma $12'$, of Lemma 12 by including an extra power of P, then work backwards via Lemma 11. We shall only sketch the argument.

Consider first the case of $k^\delta \alpha^r$ for $-\frac{2}{3} < \mathrm{Re}\,\delta < \frac{2}{3}$. We have

$$P \left[\widetilde{k}^\delta \left(\widetilde{\alpha}^r - \frac{1}{r+1} \right) \right] (k + \alpha) = k^{\delta - 1} \left(\frac{\alpha^{r+1}}{r+1} - \frac{\alpha}{r+1} \right) + \ldots$$

and the desired locally uniform version Lemma $12'$ follows immediately in this strip by Euler–MacLaurin remainder analysis.

Next suppose $\frac{1}{3} < \mathrm{Re}\,\delta < \frac{5}{3}$. Our working from the case of $1 \leq \mathrm{Re}\,\delta < 2$ in the proof of Lemma 12 implies, on keeping careful track of powers of P

in Steps (i) and (ii), that we likewise have

$$P^2\left[\widetilde{k}^\delta\left(\widetilde{\alpha}^r-\frac{1}{r+1}\right)\right](k+\alpha)=P\left[\frac{-r}{2(r+1)(r+2)}\,\widetilde{k}^{\delta-1}\right](k+\alpha)+o(1)$$

uniformly throughout this strip. Moreover, the top-order inversion (Step (iii)) giving

$$P\left[\frac{-r\delta}{2(r+1)(r+2)}\,\widetilde{k}^{\delta-1}\right](k+\alpha)=\frac{-r}{2(r+1)(r+2)}\,k^{\delta-1}+o(1)$$

also holds uniformly for $\frac{1}{3}<\operatorname{Re}\delta<\frac{5}{3}$. It follows at once that we have

$$P^2\left[\widetilde{k}^\delta\widetilde{\alpha}^r-\frac{1}{r+1}\widetilde{k}^\delta+\frac{r\delta}{2(r+1)(r+2)}\,\widetilde{k}^{\delta-1}\right](k+\alpha)=o(1)$$

uniformly for $\frac{1}{3}<\operatorname{Re}\delta<\frac{5}{3}$ and this is the desired locally uniform version Lemma 12′ for this strip as well.

Continuing in this way, working in successive overlapping open strips, we obtain the requisite locally uniform version Lemma 12′ for any δ with $\operatorname{Re}\delta>\frac{-2}{3}$.

The transition back via Lemma 11 to Lemma 10′ is then elementary except for one observation. When we apply Lemma 12′ term by term in the Taylor series expansion (6) to obtain a uniform version of Lemma 11, the different terms come with different powers of P associated with them from Lemma 12′. We need, however, to apply the single power, $P^{\lfloor\operatorname{Re}\gamma+2\rfloor}$, to all terms in expression (5). We thus need to apply extra powers of P to each of the terms in expression (6). That uniformity is preserved in doing this is due to our continuous analogue of Lemma 15 and this explains the restriction to compact subsets of our local neighbourhoods in Lemma 10′.

With Lemma 10′ it is finally trivial, on invoking the analogue of Lemma 15 once more in relation to the factor $(\frac{2-z}{1-z})(P-\frac{1}{2-z})$, to deduce the desired local uniform convergence of

$$\left(\frac{2-z}{1-z}\right)\left(P-\frac{1}{2-z}\right)P^{\lfloor-\operatorname{Re}(z_0)+2\rfloor}[s_{\zeta,z}(\widetilde{k}+\widetilde{\alpha})](k+\alpha)$$

to $C_{\zeta,z}=\zeta^{\text{ext}}(z)$ for all z in some sufficiently small neighbourhood of any $z_0\neq 1$. This yields at once the claimed *a priori* analyticity of ζ^{ext} throughout $\mathbb{C}\setminus\{1\}$.

5.3. Commuting differentiation and generalised convergence. In §5.1 and §5.2 we proved the *a priori* analyticity of the generalised limit, f, of a sequence of analytic functions $\{f_k(z)\}_{k=1}^\infty$ (or family $f(x,z)$) by finding local regular analytic families of polynomials $q(z,P_D)$ (or $q(z,P)$) for which the transformed sequence/family of functions converges to f locally uniformly, and then applying the classical Weierstrass theorem. The following result shows that under such circumstances the derivative f' is then also the generalised limit of the sequence $\{f_k'(z)\}_{k=1}^\infty$ (or family $\frac{\partial}{\partial z}f(x,z)$) and hence that, in general, differentiation with respect to z commutes with taking generalised limits. We only state the result in the discrete Césaro setting, but translation to the continuous setting is trivial, modulo minor details regarding differentiation under integrals.

Lemma 17. *Suppose $U \subseteq \mathbb{C}$ is open, $\{f_k(z)\}_{k=1}^\infty$ is a sequence of analytic functions on U, and $q(z,P_D)$ is a family of polynomials in P_D, analytic in z and regular throughout U, such that $q(z,P_D)[\{f_{\widetilde{k}}(z)\}]_k$ converges to $f(z)$ uniformly on compact subsets of U. Then f is analytic in U and $(q(z,P_D))^2[\{f_{\widetilde{k}}'(z)\}]_k$ converges to $f'(z)$ uniformly on compact subsets of U.*

Proof. Fix $K \subseteq U$ compact. We have that $q(z,P_D)[\{f_{\widetilde{k}}(z)\}]_k = f(z)+R_k(z)$ with $R_k(z)$ uniformly convergent to zero on K. Since all terms in this equation are analytic in U we may differentiate throughout with respect to z. We obtain

$$q(z,P_D)[\{f_{\widetilde{k}}'(z)\}]_k + q'(z,P_D)[\{f_{\widetilde{k}}(z)\}]_k = f'(z) + R_k'(z)$$

and applying $q(z,P_D)$ again yields that

$$(q(z,P_D))^2[\{f_{\widetilde{k}}'(z)\}]_k + q'(z,P_D)q(z,P_D)[\{f_{\widetilde{k}}(z)\}]_k \\ = f'(z) + q(z,P_D)[\{R_{\widetilde{k}}'(z)\}]_k$$

on noting trivially that $q(z,P_D)$ is regular and $f'(z)$ is independent of k. Now since $\{R_k(z)\}_{k=1}^\infty$ is uniformly convergent to zero on compact subsets of U, the classical Weierstrass theorem implies that the same is true for $\{R_k'(z)\}_{k=1}^\infty$. By Lemma 15 this remains true for $\{q(z,P_D)[\{R_k'(z)\}]_k\}_{k=1}^\infty$ and it remains only to prove that

$$q'(z,P_D)q(z,P_D)[\{f_{\widetilde{k}}(z)\}]_k = q'(z,P_D)[\{f(z)+R_{\widetilde{k}}(z)\}]_k$$

also converges uniformly to zero on K. But note that since $q(z,1)=1$ for all $z \in U$, by regularity, we have $q'(z,1)=0$ for all $z \in U$. Thus $q'(z,P_D)$ is a polynomial in P_D whose coefficients are analytic functions of z with sum zero throughout U. It follows that $q'(z,P_D)[\{f(z)\}]_k$ is actually identically zero on U, while the uniform convergence of $q'(z,P_D)[\{R_{\widetilde{k}}(z)\}]_k$ to zero on K follows by invoking Lemma 15 repeatedly for each term P^i in $q'(z,P_D)$ in turn, and noting the boundedness of its analytic coefficient on the compact set K. □

Applied to a sequence of partial sums, this lemma of course represents an analogue, for generalised convergence, of Weierstrass' theorem on term-by-term differentiation of power series inside their circles of convergence.

Note that if $q(z,P_D)$ produces generalised convergence of $\{f_k(z)\}_{k=1}^\infty$ to f, then it is $(q(z,P_D))^2$ that yields generalised convergence of $\{f_k'(z)\}_{k=1}^\infty$ to f' and, iterating the arguments in the proof, $(q(z,P_D))^{n+1}$ that gives generalised convergence of $\{f_k^{(n)}(z)\}_{k=1}^\infty$ to $f^{(n)}$. To understand more concretely why this is so, consider again $\zeta^{\text{ext},D}(z)$ as the generalised limit of $\{(s_{\zeta,z})_k\}_{k=1}^\infty$ for $z \notin \mathbb{Z}_{\le 1}$.

By Equation (2), the sequence $\{(s_{\zeta,z})'_k\}_{k=1}^\infty$ (which is the partial sum sequence for the derivative defining series $-\sum_{n=1}^\infty n^{-z}\ln n$) is given by

(35)
$$(s_{\zeta,z})'_k = \left(-\frac{1}{1-z}k^{1-z}\ln k + \frac{1}{(1-z)^2}k^{1-z}\right) + C'_{\zeta,z} - \frac{1}{2}k^{-z}\ln k + \dots$$

where the terms omitted are all products of analytic functions of z with terms of the form either $k^{-z-2r+1}$ or $k^{-z-2r+1}\ln k$. The terms involving only powers of k can be grouped and rewritten as a linear combination in the fashion of Equation (33), and hence can be uniformly locally asymptotically annihilated by $q(z, P_D)$ alone from Equation (34).

To handle the terms of the form $k^{-z-2r+1}\ln k$, however, note that these arise not in eigensequences of P_D but in generalised eigensequences of P_D. We omitted discussion of this in §4.1 but it is readily verified that multiplying the formulae from §4.1 for our eigensequences $\{a_k(\rho)\}_{k=1}^\infty$ by factors $(\ln k)^m$, $m \in \mathbb{Z}_{\ge 1}$, yields generalised asymptotic eigensequences of P_D with the same eigenvalue, at least after including further lower-order correction terms (e.g., for $-1 < \operatorname{Re}\rho < 1$, $\{k^\rho \ln k\}_{k=1}^\infty$ is at once a generalised eigensequence of P_D, while for $1 \le \operatorname{Re}\rho < 2$ we need to take $\left\{\left(k^\rho - \frac{\rho(\rho+1)}{2}k^{\rho-1}\right)\ln k - \frac{(2\rho+1)}{2}k^{\rho-1}\right\}_{k=1}^\infty$, and so on).

It follows that when we group these terms in Equation (35) and write them as a corresponding linear combination of generalised asymptotic eigensequences, their uniform local asymptotic annihilation requires using $(q(z, P_D))^2$ rather than just $q(z, P_D)$, so that each factor $\left(\frac{2-z-j}{1-z-j}\right)\left(P_D - \frac{1}{2-z-j}\right)$ occurs with exponent one higher.

Each further differentiation of Equation (2) in turn leads to terms with one more factor of $\ln k$, whose annihilation requires one higher power on each factor of $q(z, P_D)$, and this explains, at least for this example, the need to take one higher power each time of the polynomial $q(z, P_D)$.

In fact such behaviour occurs much more generally. In §7 we shall define arbitrary (non-Césaro) convergence schemes and consider examples using them, but in these cases too we shall find that where eigenvectors of the operator in the scheme must be annihilated in order to extend a series outside its domain of classical convergence, it is generalised eigenvectors that need to be annihilated in order to treat the derivative series and ensure that the resulting extension is analytic. An analogue of Lemma 17 will thus hold for convergence schemes in general.

We conclude §5 now, however, by returning briefly to the example of ζ and showing how the ideas of this subsection lead to a new explanation of why anomalies/removable singularities arise at the nonpositive integer points for $\zeta^{\text{ext},D}$ but not for ζ^{ext}.

Consider the partial sum sequence/function for the derivative defining series $-\sum_{n=1}^{\infty} n^{-z}\ln n$ at, for example, $z=0$:

$$(s_{\zeta,z})'_k|_{z=0} = (s_{\zeta,z})'(k+\alpha)|_{z=0} = -\left(k+\frac{1}{2}\right)\ln k + k + C'_{\zeta,z}|_{z=0}. \tag{36}$$

Within the continuous scheme it is readily verified that we have

$$(s_{\zeta,z})'(k+\alpha)|_{z=0} \overset{C}{\sim} -(k+\alpha)\ln(k+\alpha) + (k+\alpha) + C'_{\zeta,z}|_{z=0}$$

and hence that $(s_{\zeta,z})'(k+\alpha)|_{z=0}$ converges in a generalised continuous Césaro sense (via the polynomial $(2P-1)^2$ as per Lemma 17) to the correct value $C'_{\zeta,z}|_{z=0}$ (which may be computed explicitly as $-\frac{1}{2}\ln(2\pi)$ using Sterling's theorem).

Within the discrete scheme, however, no generalised Césaro limit can be attached to $\{(s_{\zeta,z})'_k|_{z=0}\}_{k=1}^{\infty}$, because any attempt to write expression (36) in terms of eigensequences and generalised eigensequences of P_D leaves pure factors of $\ln k$ left over. This gives a new indication that our original pointwise evaluation of $\zeta^{\text{ext},D}(0)$ must represent an anomaly/removable singularity, since it shows that although $\zeta^{\text{ext},D}$ can be successfully evaluated at $z=0$ as a pointwise generalised limit of the defining series for ζ, the derivative series cannot be handled there by the same pointwise approach.

Of course in §4.3 we saw how to identify the location of such anomalies, and correct them, by considering explicitly the polynomial we use, $q(z,P_D)$. It is noteworthy, however, that they can be detected even within a naive pointwise approach and without ever needing to identify $q(z,P_D)$, simply by considering not just the original defining series, but also its derivative series whose behaviour tells us about the analyticity of our extension.

6. Conjectures, results for Dirichlet series

We now consider new directions that can be explored using our Césaro schemes. We begin with some notions still related to ζ and its number-theoretic role.

6.1. Pictures, dilation and scaling. An additional basic question arises in using the continuous Césaro scheme. In §2.1 we supplied a necessary procedure for going from series to partial sum functions of the continuous variable x by adding in the terms at the integer points along the positive axis. This choice, however, was arbitrary; we could have chosen to add in the terms at points λ_n for any monotonically increasing, unbounded sequence $\{\lambda_n\}_{n=1}^{\infty}$.

It follows that our analysis of ζ using P in §3 really involved analysing not just the defining series $\sum_{n=1}^{\infty} n^{-z}$, but rather a defining "picture" for

ζ, namely this series together with a geometric prescription of the points $\lambda_n = n$ at which the terms of the series were added in.

This raises an obvious question: had we chosen a different defining picture for ζ (i.e., different $\{\lambda_n\}_{n=1}^{\infty}$), would we still have obtained the correct analytic continuation of ζ from the corresponding $\widetilde{\zeta}^{\text{ext}}$?

The initial answer is no. For example if we take $\lambda_n = e^{n-1}$ or $\lambda_n = \ln n$ then we are unable even to evaluate $\widetilde{\zeta}^{\text{ext}}(z)$ within our continuous Césaro scheme for any $\operatorname{Re} z \leq 1$, and so obtain no extension at all.

However, for choices of λ_n given just by linear combinations of powers of n, i.e., $\lambda_n = \sum_{i=1}^{r} c_i n^{\rho_i}$ with each $\rho_i \in \mathbb{R}_{\geq 0}$ and $\max\{\rho_i\} > 0$, the answer turns out to be effectively yes, reflecting the fact that the eigenfunctions of P are themselves powers of x. We defer a proof of this claim to another paper ([7]) but we make here two remarks.

First, for such pictures the extension $\widetilde{\zeta}^{\text{ext}}$ generically involves a countable family of anomalies/removable singularities like those which arose in our discrete analysis in §4, and these must of course be corrected for in order to obtain the full analytic continuation of ζ. Thus, although we have a broad class of pictures for which we can still correctly perform continuous Césaro extension of ζ, the standard picture $\lambda_n = n$ in §3 remains special in having no anomalies/removable singularities and thus requiring only pointwise analysis. Within the continuous Césaro scheme this picture is thus, in some sense, especially well adapted to analysing ζ. We conjecture that this is related to the basic role of the integers in the definition of ζ itself, and with this in mind, it would seem interesting to see to what degree this conjecture, and indeed our whole Césaro approach, can be carried over to the setting of more general algebraic zeta functions.

Our second remark then concerns the following corollary question: within this class of pictures with $\lambda_n = \sum_{i=1}^{r} c_i n^{\rho_i}$, is the case $\lambda_n = n$ the only one where no anomalies arise, or is there some subclass for which this is true?

In fact there is such a subclass, arising from consideration of the one-parameter dilation and scaling groups, $\{D_r\}_{r>0}$ and $\{S_r\}_{r>0}$, given by

$$D_r[f](x) \equiv f(rx) \quad \text{and} \quad S_r[f](x) \equiv f(x^r)$$

respectively. Again we defer proof until [7], but we will show there that the generalised continuous Césaro limit of any function in $\mathcal{F}$ is preserved under the actions of D_r and S_r for any $r > 0$. It follows that for any picture related to the standard one by either a simple dilation ($\lambda_n = rn$) or scaling ($\lambda_n = n^r$) we also obtain the analytic continuation of ζ directly from pointwise calculations without anomalies.

It is interesting to consider whether such invariances could possibly be exploited to understand ζ (and its zeros) better, by viewing it as a $\widetilde{\zeta}^{\text{ext}}$ with the associated picture dilated or rescaled in a way depending on z.

6.2. Detecting poles and zeros. The simple pole of ζ at $z = 1$ coincides with the presence of a pure log divergence in the partial sum function $s_{\zeta,1}(x) = \ln x + \gamma + o(1)$. Similarly the m^{th} derivative of ζ has its pole of order $m+1$ at $z = 1$ signaled by a pure log divergence $(\ln x)^{m+1}$ in $s_{\zeta^{(m)},1}(x)$. As generalised eigenfunctions of P with eigenvalue 1, the functions $(\ln x)^{m+1}$ arise only at z-values where the regularity/analyticity of $q(z, P)$ breaks down and, moreover, cannot be ascribed generalised limits within our continuous Césaro scheme.

Based on these examples we might conjecture that for any function, $f(z)$, obtained by using the continuous (or discrete) Césaro scheme to analytically continue a defining picture outside its domain of classical convergence, its poles should coincide with z-values where the associated partial sum function $s_{f,z}(x)$ contains a pure logarithmic divergence, the order of the pole coinciding with the power of the log divergence.

This, however, is false. For example the series $\sum_{n=1}^{\infty} zn^{-1-z^2}$, with the standard picture $\lambda_n = n$, gives rise to the function $z\zeta(1 + z^2)$ which has a simple pole at $z = 0$ even though the associated partial sum function there is identically zero.

To avoid this difficulty we restrict our focus now exclusively to Dirichlet series of the form $\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$, considering initially the case where the coefficients a_n and the exponents λ_n satisfy the following conditions: the λ_n are logarithms of real linear combinations of real nonnegative powers of n, and the a_n are real linear combinations of products of nonnegative integer powers of such logarithmic expressions with real nonnegative powers of n. This class of Dirichlet series is obviously closed under differentiation with respect to z. Moreover, on using Taylor expansions to rewrite the series in any local z-neighbourhood in terms of pure powers of n, we see that any such series can be extended successfully by either the discrete or continuous (in, say, the standard picture) Césaro schemes.

We might hope that our conjecture is valid at least within this restricted setting. Unfortunately this is still false. Consider, for example, the series $\sum_{n=1}^{\infty}(n+1)^2 \ln n(n^2+2n)^{-z}$, classically convergent for $\operatorname{Re} z > \frac{3}{2}$. In a neighbourhood of $z = 1$ this has expansion

$$\sum_{n=1}^{\infty}\{n^{2-2z}\ln n + 2(1-z)n^{1-2z}\ln n + (2z^2-2z+1)n^{-2z}\ln n + \dots\}.$$

Near $z = 1$ it thus represents $-2(1-z)\zeta'(2z-1) + f(z)$, f analytic, and our extension should have a simple pole with residue $\frac{-1}{2}$ at $z = 1$. But the partial sum function at $z = 1$ again has no pure log divergence at all, owing to the factor of $(1-z)$ in front of the $n^{1-2z}\ln n$ term (the $(k+\frac{1}{2})\ln k$ arising from the leading order term is not a pure log divergence — as in §5.3, it comes from a generalised eigenfunction of P with eigenvalue $\frac{1}{2}$ rather than 1 in the continuous Césaro scheme).

In this case, however, note that if we take the derivative series, its partial sum function at $z = 1$ does have a pure $(\ln x)^2$ divergence. This suggests amending our conjecture for this class of Dirichlet series/pictures to the following claim: poles occur precisely at those z-values where pure log divergences arise in the partial sum function for either the original series or its derivative series, with the order of the pole at such z given by the highest power of pure log-divergence appearing minus the order of the derivative with respect to z taken.

This form of the conjecture is finally true and is actually relatively easy to prove by combining the sorts of Taylor expansion arguments used above with our basic results from the case of ζ. In fact the proof shows that the result may be extended slightly to include both removable singularities as poles of order 0 (e.g., the case of $\zeta^{\mathrm{ext},D}$ at $z = 0$ in §5.3), and zeros as poles of negative order (e.g., $\sum_{n=1}^{\infty}(z-1)^2 n^{-z}$ yields an $(\ln x)^1$ divergence at $z = 1$ after taking two derivatives, reflecting a "pole of order -1" or order 1 zero of $(z-1)^2\zeta(z)$ at $z = 1$). In addition, the proof also yields a simple relationship between the coefficient of the highest pure log divergence in the partial sum function at any given z-value and the residue of the associated pole there.

Rather than pursue these issues here, however, we instead turn briefly now to consider a different class of number-theoretic Dirichlet series which arise in the study of the Riemann zeta function. These are not generally of the type we have just considered. For instance, the well-known Dirichlet series for $-\frac{\zeta'}{\zeta}$ is given by $\sum_{n=1}^{\infty}\Lambda(n)n^{-z}$ where $\Lambda(n)$ is the von Mangoldt function

$$\Lambda(n) = \begin{cases} \ln p, & n = p^r \text{ and } p \text{ prime}, \\ 0, & \text{otherwise} \end{cases}$$

and clearly this series does not lie within the class just discussed.

For series like this the irregular behaviour of the coefficients makes it unclear whether they even can be extended outside their domains of classical convergence by Césaro methods. Let us assume for a moment, however, that they can, and moreover that, as in the result we have just discussed, poles of their analytic continuations coincide with z-values where the associated partial sum functions for either the original series or their derivatives contain pure log divergences.

Then since the nontrivial zeros of ζ correspond precisely to simple poles of $\frac{\zeta'}{\zeta}$ in the critical strip $0 < \mathrm{Re}\,(z) < 1$, it would follow that these zeros should occur at z-values where the particular partial sum functions

$$s_{\left(\frac{\zeta'}{\zeta}\right)^{(m)},z}(k+\alpha) = (-1)^m \sum_{n=1}^{k} \Lambda(n)(\ln n)^m n^{-z}$$

have a pure log divergence for some $m \in \mathbb{Z}_{\geq 0}$. We would thus obtain an explicit pointwise criterion for the locations of the nontrivial zeros of ζ, purely in terms of the asymptotic behaviour of the partial sum functions for the single Dirichlet series $\sum_{n=1}^{\infty} \Lambda(n)n^{-z}$ and its derivatives.

Now of course this is all highly speculative. Nevertheless it is interesting to ask to what degree Césaro methods can be applied to such number-theoretic Dirichlet series and conjectures regarding their poles confirmed, falsified or refined. The central difficulty is in obtaining expressions for the associated partial sum functions in the first place, but at an anecdotal level we may at least note a few intriguing "experimental" observations.

First, for example, $-\frac{\zeta'}{\zeta}$ certainly has a simple pole at $z = 1$, consistent with the fact that $\sum_{n=1}^{k} \Lambda(n)n^{-1} \sim \ln k$ ([**5**], pp12). Next, consider the so-called "explicit formula" for $-\frac{\zeta'(0)}{\zeta(0)}$ ([**5**], §3.9):

$$\sum_{n \leq X} \epsilon_X(n)\Lambda(n) = X - \sum_{\rho \in Z} \frac{X^\rho}{\rho} - \ln 2\pi - \frac{1}{2}\ln(1 - X^{-2})$$

where $\epsilon_X(n)$ is 1 if $n < X$, $\frac{1}{2}$ if $n = X$, and 0 if $n > X$, and where Z is the set of critical zeros of ζ. At a qualitative level this too seems to fit our Césaro framework, at least loosely, in that if we assign the generalised Césaro limit 0 to all the powers of X (and ignore for the moment that there are infinitely many of them) we obtain the correct generalised limit of $-\ln 2\pi$ for the partial sum function.

Observations like this (see also the discussion in §8.1 of product series and multi-dimensional schemes) suggest that Césaro methods may in fact be applicable to the analysis of number-theoretic Dirichlet series.

6.3. Zeta functions for elliptic operators. We conclude §6 by considering zeta functions of positive self-adjoint elliptic differential operators on compact manifolds. The zeta function of such an operator, A, of order m on a compact manifold of dimension n, is defined by $\zeta_A(z) = \sum_{j=1}^{\infty} M_j {\lambda_j}^{-z}$ where λ_j is the j^{th} distinct eigenvalue of A and M_j its multiplicity, the series being classically convergent for $\operatorname{Re} z > \frac{n}{m}$ (see e.g., [**6**], §13).

Such zeta functions can often, like the Riemann zeta function, be analytically continued by our Césaro schemes. For example, for the Laplacian $\triangle$ on any sphere S^n, both M_j and λ_j are real polynomials, of degrees $n-1$ and 2 respectively, and so $\zeta_{\triangle_{S^n}}$ lies within the Césaro-amenable class discussed in §6.2. Indeed in this case these facts alone are sufficient to deduce from the Césaro approach certain universal qualitative features ([**6**], §13) of such zeta functions.

To begin with, it follows that $\zeta_{\triangle_{S^n}}$ extends to a meromorphic function on $\mathbb{C}$ whose only poles lie among the points $z_i = \frac{n-i}{2}$, $i \in \mathbb{Z}_{\geq 0}$ and certainly not at the points $0, -1, -2, \ldots$. This follows from the result of §6.2 since, on performing Taylor expansion in descending powers of j, such z_i are the only points where terms of the form $\frac{a}{j}$ can arise in the summand $M_j{\lambda_j}^{-z}$, and hence the only points where the partial sum sequence/function can have a pure log divergence. This log-divergence test can, moreover, be applied explicitly in any given case to decide whether a prospective point z_i is, in fact, a pole. For example, for $\triangle$ on S^3 we have $\lambda_j = (j^2 + 2j)$ and $M_j = (j+1)^2$. Thus at $z = 1$ we have $(s_{\zeta_{\triangle_{S^3}},1})_k = k + \frac{3}{4} + o(1)$ and there is actually no pole at $z = 1$.

Further, the fact that the M_j have no $\ln j$ factors (which we have actually just used implicitly) implies immediately that any poles in $\zeta_{\triangle_{S^n}}$ must be simple. For any such simple pole, finally, the result outlined in §6.2 implies simply that its residue is half the coefficient of the associated log-divergence, and this yields the explicit formula for the residue of the simple pole at $z_0 = \frac{n}{2}$ given in [**6**], §13.

Unfortunately, for self-adjoint elliptic differential operators A more generally, things are not as straightforward. For example, even for the Laplacian on the 2-torus, T, its eigenvalues $\{p^2 + q^2\}$ form a 2-parameter family which is hard to order into a 1-parameter sequence $\{\lambda_j\}$, making Césaro analysis problematic (in this case it may turn out to be better to use a two-dimensional Césaro scheme of the type we will discuss briefly in §8). We nonetheless remain hopeful that Césaro methods should still be applicable to such zeta functions. In conjecturing this we conclude simply by noting that, at least for any such continuous Césaro analysis, our computations suggest one should use a picture adapted to A in the sense that the j^{th} term $M_j{\lambda_j}^{-z}$ should be added in not at the point $x = j$ but rather at $x = \lambda_j$. This appears to prevent the occurrence of anomalous removable singularities in the continuous Césaro extensions ζ_A^{ext}.

7. General schemes, the Borel scheme

So far we have considered only generalised notions of convergence based on the continuous or discrete Césaro operator. Clearly, however, we may define similar generalised convergence schemes using any regular, linear operator, A, defined either on the space, $\mathcal{S}$, of arbitrary sequences or some suitable function space, $\mathcal{F}_A$, analogous to $\mathcal{F}$. Given any sequence $s \in \mathcal{S}$ (resp. $f \in \mathcal{F}_A$) define s (resp. f) to have generalised A-limit L, and write $A\lim_{k\to\infty} s_k = L$ (resp. $A\lim_{x\to\infty} f(x) = L$), if there exists a regular polynomial, $q(A)$, such that $\lim_{k\to\infty}(q(A)[\{s\}])_k = L$ (resp. $\lim_{x\to\infty} q(A)[f](x) = L$). Regularity of q is again equivalent to the condition $q(1) = 1$.

Such alternative schemes can be used to analytically continue series not amenable to Césaro methods, such as ones diverging more rapidly than the ordinary (or nearly ordinary) Dirichlet series considered so far.

For example, consider the simple geometric series $g(z) \equiv \sum_{n=1}^{\infty} z^{n-1}$, classically convergent for $|z| < 1$. Its partial sum sequence is given, for $z \neq 1$, by

$$(s_{g,z})_k = \left(\frac{z}{z-1}\right) z^{k-1} + \frac{1}{1-z} \tag{37}$$

and for any fixed $|z| > 1$ this diverges exponentially in k. It thus cannot be handled by our Césaro schemes, whose eigensequences/eigenfunctions involve only power divergences.

Traditionally (e.g., [**2**], §4.12) this series has been analysed instead using the classical Borel operator, B_{cl}, from sequences, $\{a_n\}_{n=0}^{\infty}$, to functions, $B_{\mathrm{cl}}[\{a\}](x)$ given by

$$B_{\mathrm{cl}}[\{a\}](x) = \frac{1}{e^x} \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}.$$

Applied to $\{(s_{g,z})_n\}$ (trivially reindexed to start at $n = 0$) this yields a function which converges to $\frac{1}{1-z}$ for all $\operatorname{Re} z < 1$. Classical Borel summation thus yields the correct analytic continuation of g to the half-plane $\operatorname{Re} z < 1$, but does not succeed in extending to $\operatorname{Re} z \geq 1$.

By instead using a suitably adapted generalised convergence scheme of the type just described, however, we may obtain at once the correct analytic continuation $g(z) = \frac{1}{1-z}$ throughout the whole complex plane.

Define a new Borel operator $B : \mathcal{S} \to \mathcal{S}$ by

$$B[\{a\}]_k \equiv \frac{1}{e^{k-1}} \left\{ a_1 + \sum_{j=2}^{k} (e^{j-1} - e^{j-2}) a_j \right\}. \tag{38}$$

Clearly B is both linear and regular, so we may consider the (discrete) "Borel scheme" with this as its fundamental operator. For this, we have the following lemma, which may be verified directly by elementary computations.

Lemma 18. *The asymptotic eigensequences of B are the sequences $\{z^{n-1}\}_{n=1}^{\infty}$, $z \in \mathbb{C} \setminus \{\frac{1}{e}\}$, each with eigenvalue $\frac{(e-1)z}{ez-1}$ (for $z = \frac{1}{e}$, the sequence $\{e^{-n}\}_{n=1}^{\infty}$ is already classically convergent to zero). The sequences $\{n^{\rho}\}_{n=1}^{\infty}$ are asymptotic generalised eigensequences of B with eigenvalue 1 for any $\rho \in \mathbb{C}$ and, for any $z \in \mathbb{C} \setminus \{\frac{1}{e}\}$, the asymptotic generalised eigensequences of B with eigenvalue $\frac{(e-1)z}{ez-1}$ are the sequences $\{z^{n-1} n^m\}_{n=1}^{\infty}$, $m \in \mathbb{Z}_{\geq 1}$.*

Since eigensequences and generalised eigensequences of B with eigenvalue $\lambda \neq 1$ all have generalised B-limit 0 in the usual fashion, it follows immediately from this lemma and Equation (37) that, in the example of the geometric series, we have $g^{\mathrm{ext},B}(z) = \frac{1}{1-z}$ for any $z \neq 1$. We thus do obtain the correct analytic continuation of g to the whole complex plane via this new Borel scheme as promised. That $z = 1$ is *a priori* a simple pole with residue -1 follows, moreover, in similar fashion to our earlier singularity computations in §3.3 — the polynomial we have used is $q(z, B) = \frac{ez-1}{z-1}(B - \frac{(e-1)z}{ez-1})$, and its analyticity breaks down at $z = 1$ with $\lim_{z\to 1} \lim_{k\to\infty}(z-1)q(z, B)[\{s_{g,z}\}]_k = (e-1) \lim_{k\to\infty}(B-1)[\{\widetilde{k}\}]_k = -1$.

Although the analytic continuation in this example is, of course, trivial by other means, it nonetheless shows how an effective generalised convergence scheme may be constructed in general using an operator whose choice is adapted to the type of series requiring extension. An infinite variety of such schemes exists (see e.g., §8.3) and they may easily be adapted to other natural settings such as integrals (with complex parameter) having divergences at finite points rather than just as $x \to \infty$. We conclude §7, however, with two further brief remarks regarding the example of the geometric series and Borel scheme.

First, if we write $z = e^{-w}$ then $\sum_{n=1}^{\infty} z^{n-1}$ becomes a Dirichlet series in the variable w, $\sum_{n=1}^{\infty} a_n e^{-\lambda_n w}$, with $a_n = 1$ and $\lambda_n = (n-1)$. Comparing with Equation (38) we conjecture that a Dirichlet series $\sum_{n=1}^{\infty} a_n e^{-\lambda_n w}$ (with all $\lambda_n \geq 0$) should in general be analysed by a discrete generalised convergence scheme with fundamental operator $A : \mathcal{S} \to \mathcal{S}$ given by

$$A[\{a\}]_k \equiv \frac{1}{e^{\lambda_k}} \left\{ e^{\lambda_1} a_1 + \sum_{j=2}^{k} (e^{\lambda_j} - e^{\lambda_{j-1}}) a_j \right\}. \tag{39}$$

The earlier case of P_D for the Dirichlet series $\sum_{n=1}^{\infty} n^{-z}$ in §4 fits within this framework.

Secondly, this new Borel scheme may be useful more broadly in defining generalised Fourier series to handle periodic functions with nonintegrable singularities and convergence off the real line. We investigate this more fully in a separate paper, in preparation ([8]).

8. Final observations

We conclude by schematically discussing some further topics of potential interest related to the notions of generalised convergence introduced.

8.1. Higher-dimensional schemes. One is often interested (e.g., $\zeta_{\triangle_T}$ in §6.3) in sums of arrays of numbers indexed by two or more parameters. They arise for example when taking a product of two series, $\sum_i a_i$ and $\sum_j b_j$, producing a double series $\sum_{i,j} a_i b_j$. It is thus natural to seek to construct higher-dimensional convergence schemes for defining generalised limits of functions/sequences of several variables. We shall work only in the context of the continuous Césaro scheme and two dimensions but the treatment of other schemes and higher dimensions is obviously analogous.

The initial difficulties are to decide what it should mean even at the classical level to say that a function, $f(x, y)$, is convergent to limit L, and then what the function space and operators should be for a 2-d Césaro scheme. Two requirements are natural — first, if $f_1(x)$ converges classically to L_1 and $f_2(y)$ converges classically to L_2 then $f_1(x)f_2(y)$ should converge classically to L_1L_2, and secondly, this should continue to hold for generalised Césaro limits.

Based on the first of these conditions we define $f(x, y)$ to converge classically to limit L if and only if for all $\epsilon > 0$ there exist $M, N > 0$ such that $|f(x, y) - L| < \epsilon$ whenever both $x > M$ and $y > N$.

Turning to the second, consider operators P_1 and P_2 defined by

$$P_1[f](x, y) \equiv \frac{1}{x}\int_0^x f(t, y)\, dt \quad \text{and} \quad P_2[f](x, y) \equiv \frac{1}{y}\int_0^y f(x, t)\, dt.$$

If P_1 and P_2 were regular operators permitted within our 2-d Césaro scheme we would obtain at once the desired result on generalised limits of products.

Now in order for each P_i even to map back into the same function space, $\mathcal{F}^{(2)}$, for our 2-d scheme we need functions f in this space to satisfy the condition in our definition of the 1-d Césaro space, $\mathcal{F}$, on each slice parallel to the x or y axes. That is, for any fixed y we need that $\int_0^x |f(t, y)(\ln t)^m| dt < \infty$ for all $x \geq 0$ and all $m \in \mathbb{Z}_{\geq 0}$, and similarly for any fixed x.

These conditions alone, however, are insufficient to guarantee that each P_i is regular. For example, if we take

$$f(x, y) = \begin{cases} 1, & x \leq 1,\ y \leq 1 \\ y, & x \leq 1,\ y > 1 \\ x, & x > 1,\ y \leq 1 \\ x^{2-y}y^{2-x}, & x > 1,\ y > 1 \end{cases}$$

then f is continuous, satisfies the above slice conditions, and has classical limit 0 under our definition, but neither $P_1[f]$ nor $P_2[f]$ is classically convergent under this definition.

To overcome this we restrict $\mathcal{F}^{(2)}$ further by imposing uniformity requirements in our slice conditions. Our final definition is that f lies in $\mathcal{F}^{(2)}$ if:

(a) For all y we have $\int_0^x |f(t, y)(\ln t)^m| dt < \infty$ for all $x \geq 0$ and $m \in \mathbb{Z}_{\geq 0}$, and for any fixed $X > 0$ and $m \in \mathbb{Z}_{\geq 0}$ there exist $Y, C > 0$ such that $\int_0^X |f(t, y)(\ln t)^m| dt < C$ for all $y > Y$.

(b) The same holds with the roles of x and y reversed.

With this definition it is readily verified that each P_i still maps $\mathcal{F}^{(2)}$ back into itself but is now also a regular operator as wanted. We obtain our desired 2-d Césaro scheme finally by defining $f \in \mathcal{F}^{(2)}$ to have generalised $C^{(2)}$-limit L if there exists a regular polynomial $q(P_1, P_2)$ (i.e., $q(1, 1) = 1$) such that $q(P_1, P_2)[f](x, y)$ converges classically to L. Since P_1 commutes with P_2, the same argument as in the 1-d case shows that this formulation is well-defined, and, as remarked earlier, the 2-d generalised Césaro limit of a product of functions is now clearly equal to the product of their generalised 1-d Césaro limits.

With this in mind, consider briefly again our discussion from §6.2 about the amenability of number-theoretic Dirichlet series to Césaro analysis. Take for example the series $\sum_{n=1}^{\infty} d_2(n)n^{-z}$, absolutely convergent for $\operatorname{Re} z > 1$, where $d_2(n)$ is the cardinality of the set of ordered pairs (a, b) such that $ab = n$. This is precisely of the kind discussed in §6.2 where the irregularity of the coefficients makes it unclear whether 1-d Césaro analysis is possible. For $\operatorname{Re} z > 1$, however, this series is simply the product $\zeta(z)^2 = (\sum_{n=1}^{\infty} n^{-z})^2$ ([**5**], §1). It is therefore certainly amenable to successful 2-d Césaro extension by treating the double-series $\sum_{i,j=1}^{\infty} i^{-z}j^{-z}$.

The fact that Césaro methods thus can be employed to analytically continue a Dirichlet series like this, albeit by rewriting it as a product and using a 2-d scheme, suggests perhaps that Césaro analysis may indeed be applicable to number-theoretic Dirichlet series more generally.

8.2. Ratio eigenfunctions. In §5.3 we saw, in the context of the discrete Césaro scheme, that where generalised convergence of $\{s_n(z)\}_{n=1}^{\infty}$ is obtained by annihilating eigensequences of P_D, the generalised convergence of $\{\frac{d}{dz}s_n(z)\}_{n=1}^{\infty}$ requires annihilation of generalised eigensequences of P_D, and similarly for further derivatives. The same pattern holds for arbitrary convergence schemes of the type defined in §7 such as the Borel scheme.

Taking antiderivatives is also interesting. Consider, for example, the family of functions $s(x, z) = x^z$. As $x \to \infty$, $s(x, z)$ converges in a generalised continuous Césaro sense to the zero function on $\mathbb{C} \setminus \{0\}$ with a removable singular value 1 at $z = 0$. This occurs via polynomials $q(z, P)$ whose coefficients are uniformly bounded on any set a strictly positive distance from the origin.

Taking such a set U of the form $\mathbb{C} \setminus D_r$, where D_r is a disk of radius r, consider now the antiderivative family $\widetilde{s}(x, z) = \frac{x^z}{\ln x}$ on U (of course for any z the function $\frac{x^z}{\ln x}$ is not strictly in $\mathcal{F}$, but $\frac{x^z-1}{\ln x}$ is, and since $\frac{1}{\ln x}$ converges classically to 0 we may ignore this technicality — alternatively we could simply redefine both families s and $\widetilde{s}$ as zero for $x < 2$). We would hope that on U antidifferentiation should commute with generalised convergence just as differentiation did. Since the antiderivative of the zero function is a constant function and $\frac{x^z}{\ln x}$ already converges classically to zero for all z in U with $\operatorname{Re} z < 0$, this means we should have

Conjecture 1. $\operatorname{Clim}_{x\to\infty} \frac{x^z}{\ln x} = 0 \quad$ for all $z \in U$.

Unfortunately it is easy to see that, for any $\operatorname{Re} z > 0$ in U, no polynomial $q(z, P)$ exists which annihilates $\frac{x^z}{\ln x}$ even asymptotically. For if we consider

$P[\frac{\widetilde{x}^z}{\ln \widetilde{x}}](x)$ we obtain an infinite asymptotic expansion

$$P\left[\frac{\widetilde{x}^z}{\ln \widetilde{x}}\right](x) \sim \frac{x^z}{(z+1)\ln x} \cdot \left\{1 + \frac{1!}{(z+1)\ln x} + \frac{2!}{(z+1)^2(\ln x)^2} + \frac{3!}{(z+1)^3(\ln x)^3} + \dots\right\}$$

and the presence of classically divergent terms $\frac{x^z}{(\ln x)^k}$ for k arbitrarily large makes it impossible to construct any annihilating polynomial of finite degree.

It follows that our current definition of generalised Césaro convergence is inadequate to validate Conjecture 1, and the same is true for the families $\frac{x^z}{(\ln x)^m}$, $m \in \mathbb{Z}_{>1}$, which arise from further antidifferentiation on U. An identical situation arises for convergence schemes in general when we consider antiderivatives leading to ratios of arbitrary eigenfunctions/sequences over powers of generalised eigenfunctions/sequences with eigenvalue 1 (or indeed products of arbitrary eigenfunctions/sequences with ρ^{th} powers of generalised eigenfunctions/sequences with eigenvalue 1 for $\rho \notin \mathbb{Z}_{\geq 0}$).

The functions $\widetilde{s}(x,z) = \frac{x^z}{\ln x}$ do, however, behave in a similar way to eigenfunctions of P with eigenvalue $\frac{1}{z+1}$ in one sense. Although they don't satisfy either the exact eigenfunction equation $(P - \frac{1}{z+1})[\widetilde{s}(\cdot, z)] = 0$ or its asymptotic counterpart, they do satisfy an infinite descending chain of asymptotic relations involving the operator $(P - \frac{1}{z+1})$. Writing $\widetilde{s}(x,z) = \widetilde{s}_0(x,z)$ we have

$$\begin{array}{rclcrcl} (P - \frac{1}{z+1})[\widetilde{s}_0(\widetilde{x},z)](x) & = & \widetilde{s}_1(x,z) & \text{with} & \widetilde{s}_1(x,z) & = & o(\widetilde{s}_0(x,z)), \\ (P - \frac{1}{z+1})[\widetilde{s}_1(\widetilde{x},z)](x) & = & \widetilde{s}_2(x,z) & \text{with} & \widetilde{s}_2(x,z) & = & o(\widetilde{s}_1(x,z)), \\ & & \vdots & & & & \vdots \end{array}$$

Since each relation in this chain can be rewritten (at least loosely) as the ratio

$$\frac{(P - \frac{1}{z+1})[\widetilde{s}_i(\widetilde{x},z)](x)}{\widetilde{s}_i(x,z)} = o(1),$$

we shall call the function $\widetilde{s}_0(x,z) = \frac{x^z}{\ln x}$ a "ratio eigenfunction" of P with eigenvalue $\frac{1}{z+1}$. The functions $\frac{x^z}{(\ln x)^2}$, $\frac{x^z}{(\ln x)^3}$, ..., are all also ratio eigenfunctions of P with eigenvalue $\frac{1}{z+1}$.

It remains, however, to determine how to extend our definition of generalised Césaro convergence to obtain the desired generalised limit 0 for such ratio eigenfunctions.

At present we do not have a fully satisfactory answer, but one possibility, again developed jointly with Andrew Stone, consists of extending to permit polynomials constructed not just from the basic operator P, but also from conjugates of the form $A_m = M_{(\ln x)^{-m}} \circ P \circ M_{(\ln x)^m}$, where $M_{f(x)}$ is the operator of multiplication by $f(x)$ on $\mathcal{F}$. Since the functions $(\ln x)^m$ are generalised eigenfunctions of P with eigenvalue 1, these conjugate operators are all regular, and each function $\frac{x^z}{(\ln x)^r}$ is now an exact eigenfunction of A_r and generalised eigenfunction of A_m, $m > r$. It is possible to frame an extended definition of generalised Césaro convergence, using such A_m, which handles linear combinations of such ratio eigenfunctions and satisfies uniqueness of generalised limits, although this now requires some care due to the noncommutativity of A_m and A_n for $m \neq n$. We shall not go into details here, however, and instead close with one remark.

It is that more exotic ratio eigenfunctions than just $\frac{x^z}{(\ln x)^m}$ also arise naturally in many applications and may necessitate extending further still, to permit conjugation by multiplication operators $M_{f(x)}$ where f is an arbitrary asymptotic generalised eigenfunction of P with eigenvalue 1. Such a further generalisation has yet to be fully worked out, but we note that an ability to handle ratio eigenfunctions (for arbitrary schemes) is essential for the application to generalised notions of Fourier theory which we mentioned at the end of §7 and shall discuss in [**8**].

8.3. Schemes and measures. The continuous Césaro convergence scheme is only one example of a class of schemes associated to measures on $[0, \infty)$. If we take a measure $\mu(t)dt$ where μ is any positive locally integrable function with $\lim_{x\to\infty} \int_0^x \mu(t)dt = \infty$, we can define an associated convergence scheme with fundamental regular operator P_μ given by

$$P_\mu[f](x) = \frac{1}{\int_0^x \mu(t)dt} \int_0^x f(t)\mu(t)dt. \tag{40}$$

Continuous Césaro corresponds to ordinary Lebesgue measure with $\mu(t) \equiv 1$.

Letting $\nu(x) \equiv \int_0^x \mu(t)dt$ it is trivial to verify that, for any such scheme, the functions $(\nu(x))^z$ are the eigenfunctions of P_μ with eigenvalue $\frac{1}{1+z}$, with the generalised eigenfunctions being simply $(\nu(x))^z(\ln(\nu(x)))^m$, $m \in \mathbb{Z}_{>0}$.

If we have some family of divergent series or integrals to which we want to ascribe generalised limits, in order say to perform an explicit analytic continuation, we now see that one approach is to choose a $\mu(t)$ with the eigenfunctions/generalised eigenfunctions of the resulting scheme involving divergences of the same type as the ones we have to deal with. This is essentially what we did in using the continuous Césaro scheme (where $\nu(x) = x$) to handle the power divergences in the ordinary Dirichlet series for ζ. In [**8**] we shall likewise use a scheme with $\mu(t) = e^t$ adapted to handling exponential divergences which arise there in treating Fourier transforms.

For the remainder of this section, however, we now briefly consider just one family of schemes arising from taking μ as $\mu_r(t) = t^r$, $r > -1$. The schemes in this class are all closely related to the single continuous Césaro case $r = 0$; for example, they all have the same eigenfunctions and generalised eigenfunctions, albeit with varying eigenvalues. Indeed the operators

$P_r \equiv P_{t^r}$ are all conjugates of P by rescaling operators of the kind described in §6.1:

$$P_r = S^{-1}_{\frac{1}{r+1}} \circ P \circ S_{\frac{1}{r+1}} \tag{41}$$

(if we view all functions as partial sum functions arising from some underlying picture on $[0, \infty)$, this relationship just says that we may equivalently think either of the picture as fixed and the measure varying, or alternatively of the measure as fixed but the underlying picture being suitably rescaled).

Now since, as remarked in §6.1, generalised continuous Césaro limits are invariant under rescalings, Equation (41) means that the generalised limits of functions are the same for all the schemes in our family. Let us consider, however, the limiting cases $r = -1$ and $r = \infty$.

For $r = -1$ a technicality arises since the function $\frac{1}{t}$ is not integrable at zero, but if instead we take $\mu_{-1}(t) = 0$ for $0 \le t < 1$ and $\mu_{-1}(t) = \frac{1}{t}$ for $t \ge 1$ this problem disappears and we can still consider μ_{-1} as a limiting case of a slightly amended family in which we similarly truncate the functions $\mu_r(t)$, $r > -1$. The case $r = -1$ then yields the operator P_{-1} given by

$$P_{-1}[f](x) = \frac{1}{\ln x} \int_1^x f(t)\frac{1}{t}dt. \tag{42}$$

This is the continuous analogue of Riesz' logarithmic mean, as discussed in [**3**].[2] In terms of rescalings we have $P_{-1} = S_{\ln} \circ P \circ S_{\exp}$ where $S_{\exp}[f](x) \equiv f(e^x)$ and $S_{\ln}[f](x) \equiv f(\ln x)$.

An interesting phenomenon occurs with the eigenfunctions and generalised eigenfunctions of our schemes in this limit. Under P_r for any $r > -1$ the eigenfunction of P with eigenvalue λ (namely $x^{\frac{1}{\lambda}-1}$) remains an eigenfunction but with eigenvalue $\frac{(r+1)\lambda}{r\lambda+1}$. As $r \to -1$ these eigenvalues all flow towards zero, except for $\lambda = 1$ which stays fixed. For $\lambda = 1$, however, there is still variation, with the associated generalised eigenvalue equation becoming r-dependent:

$$(P_r - 1)[\ln \widetilde{x}](x) = \frac{-1}{r+1}.$$

What happens in the actual limiting case $r = -1$? To begin with, the functions $(\ln x)^z$, which were formerly all generalised eigenfunctions with eigenvalue 1, now become a full array of eigenfunctions of P_{-1} with eigenvalues $\frac{1}{1+z}$. Some nontrivial asymptotic eigenfunctions (not generalised eigenfunctions) with eigenvalue 1, such as $\ln(\ln x)$, become generalised eigenfunctions of P_{-1} with eigenvalue 1. Others like $\sin(\sqrt{\ln x})$ now fall directly in the asymptotic kernel of P_{-1}. Passing to the limiting case $r = -1$ thus separates the asymptotic eigenfunctions and generalised eigenfunctions of

[2] P_{-1} is also useful in giving an alternative way of obtaining the gamma function, $\Gamma(z)$, starting from the divergent product $\prod_{n=1}^{\infty}(1 + \frac{z}{n})$.

P with eigenvalue 1 into different hierarchies according to their behaviour under P_{-1}.

As for the other eigenfunctions and generalised eigenfunctions of P (and hence of P_r for any $r > -1$), these no longer generally remain eigenfunctions/generalised eigenfunctions of P_{-1}. But interestingly the action of P_{-1} intertwines them with the ratio eigenfunctions of P discussed in the previous subsection. For example we have

$$P_{-1}[\widetilde{x}^z](x) = \frac{1}{z}\frac{x^z - 1}{\ln x} = \frac{1}{z}\frac{x^z}{\ln x} + o(1)$$

for any $z \in \mathbb{C} \setminus \{0\}$. Only for $\operatorname{Re} z = 0$ does this still represent a nontrivial asymptotic eigenvalue equation, with eigenvalue 0 for all such z in line with our earlier eigenvalue flow remarks. Note that this intertwining means that, rather than conjugating P by multiplication operators $M_{(\ln x)^m}$ to handle ratio eigenfunctions in §8.2, we could alternatively conjugate by powers of the operator P_{-1}.

As for the case $r \to \infty$, it is not immediately obvious how to interpret this limit in terms of a limiting measure. In this case, however, the eigenvalues $\lambda \neq 1$ all flow towards 1. This suggests that any limiting operator, P_∞, should have the functions x^z as eigenfunctions or generalised eigenfunctions with eigenvalue 1 for all $z \in \mathbb{C}$. Such an operator is easily manufactured by taking $P_\infty \equiv S_{\exp} \circ P \circ S_{\ln}$. More explicitly, P_∞ is then just the operator

$$P_\infty[f](x) = \frac{1}{e^x} \int_{-\infty}^{x} f(t) e^t dt \tag{43}$$

which is very close to being the operator P_μ with $\mu(t) = e^t$ mentioned earlier in this subsection. Its eigenfunctions are the exponentials e^{zx}, $z \in \mathbb{C}$, with generalised eigenfunctions $e^{zx}x^m$, $m \in \mathbb{Z}_{>0}$, and they are intertwined with the associated ratio eigenfunctions of P_∞ by the action of P.

We see that we can construct schemes using P_{-1} and P_∞, corresponding to logarithmic and exponential underlying rescalings of our original continuous Césaro scheme, which are, in some sense, limits of our invariant family $\{P_r\}_{r>-1}$. Passage to these limiting cases involves interesting behaviour in the eigenvalues and eigenfunctions/generalised eigenfunctions of the associated operators.

8.4. Dynamical systems, quantisation and symmetries for $\boldsymbol{\zeta}$. The inverse of the Césaro operator, P, used in §3 to analyse ζ, is the differential operator $P^{-1} = x\frac{d}{dx} + 1$. There has been tremendous recent interest (e.g., [**1**] and many others) in trying to understand the zeros of ζ in terms of the spectrum of the quantised Hamiltonian operator of some chaotic dynamical system. In this context it is interesting that P^{-1} is in fact the canonical quantisation, modulo ordering choices and a factor of $-i$, of precisely the classical Hamiltonian $H_{\text{cl}} = XP$ suggested for this role in [1][3] . In light of the possible role also of the dilation group D_r, discussed in §6.1, in this dynamical systems approach (e.g., [1], §6), it seems interesting to ask whether this relationship between P^{-1} and the putative H_{cl} could be of significance. In particular, it is interesting to speculate whether the scaling group, S_r, which has played a role in this paper, could also be of value in the dynamical systems approach.

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Received December 22, 1999 and revised October 6, 2000.

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[3]In [1] it is actually the symmetrised expression $\frac{1}{2}(XP + PX)$ which is quantised to guarantee hermiticity, but this just corresponds, up to a factor of $-i/2$, to considering the inverse of the related rescaled operator $P_{-\frac{1}{2}}$, defined in §8.3, rather than P