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Using asymptotic morphisms between graded C^*-algebras, we construct for every open m-dimensional spin manifold M a fundamental class in the m-th analytic K-homology group of M. This class is associated to the not necessarily essentially self-adjoint Dirac operator on M. A careful treatment is given of the main properties of these fundamental K-homology classes.

Introduction

Atiyah [1970] suggested a way of constructing K-homology — the dual theory to topological K-theory — using functional analysis and the properties of elliptic pseudo-differential operators. He defined the cycles in the K-homology group $K_0(X)$ of a compact polyhedron X, but left open the problem of finding the appropriate equivalence relation that will turn these cycles into an abelian group. The problem was solved by Kasparov [1976]. Later Kasparov [1981] constructed a *bivariant* theory, which for any separable C^*-algebras A and B associates abelian groups $KK_n(A, B)$, $n \in \mathbb{Z}$. If $A = C(X)$ and $B = \mathbb{C}$, then $KK_n(C(X), \mathbb{C}) = K_n(X)$, $n = 0, 1, 2, \ldots$, are the K-homology groups of X. Connes and Higson [1989; 1990] concretely realized the *universal* bivariant theory — which they called E-theory — in terms of asymptotic morphisms between C^*-algebras. One recovers again K-homology through the groups $E^{-n}(C(X), \mathbb{C}) \simeq KK_n(C(X), \mathbb{C})$.

In our paper we use the E-theoretical description of K-homology to give full details of the construction of the K-homology cycles corresponding to Dirac operators on open manifolds. The idea is that some complicated technical details in Kasparov's approach [Higson and Roe 2000, Chapters 9–11] become easier when using asymptotic morphisms instead of Kasparov cycles. The other novelty of this article is the consistent use of graded objects, which simplifies the presentation. Ideas contained in this paper first appeared in [Higson 1988; 1991; 1993].

In Section 1 we present three equivalent ways of defining asymptotic morphisms, one being new (Definition 1.11); we discuss some constructions involving graded C^*-algebras that are used in the context of E-theory; and we introduce the analytic

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K-homology groups using the language of E-theory. In Section 2 we briefly cover the needed notions from spin geometry. The main construction of the paper is contained in Section 3: to an m-dimensional Spin-manifold M we associate an asymptotic morphism $\mathcal{S}\widehat{\otimes}C_0(M) \dashrightarrow \mathcal{K}\widehat{\otimes}\mathcal{C}_{-m}$, where $\mathcal{C}_{-m}$ is the Clifford algebra of $\mathbb{R}^m$. This asymptotic morphism is based on the not necessarily essentially self-adjoint Dirac operator on M. The technical tool in its definition is the family $\{D_t\}_t$, and we show that its class does not depend on the various choices that enter into the construction. It is the K-homology class of this asymptotic morphism what we call *the fundamental K-homology class of M*. In Section 4 we present the main properties of the fundamental K-homology classes: behavior under external product, homotopy invariance, and invariance under the boundary map. To illustrate the usefulness of these properties, we end the article by presenting in Section 5 a short proof of the cobordism invariance of the index for compact manifolds.

1. Review of E-theory

Asymptotic morphisms. The entire presentation of the paper is in the category of graded C^*-algebras: all the C^*-algebras will be $\mathbb{Z}/2$-graded and separable; all $*$-homomorphisms and asymptotic morphisms will map homogeneous elements to homogeneous elements, the grading degree being preserved. We shall use ∂x to denote the degree of the element x belonging to the graded object (C^*-algebra, Hilbert module, bundle, bundle of C^*-algebras) under consideration. All the commutators that appear are graded ones: $[a, b] = ab - (-1)^{\partial a\, \partial b} ba$. See [Blackadar 1998, Section 14] for more details about graded C^*-algebras. The C^*-algebra $C_0(X)$, of complex valued vanishing at infinity functions on a locally compact space X, will always be trivially graded, for any X.

Definition 1.1. A graded complex C^*-algebra $A_{\mathbb{C}}$ is a *Real C^*-algebra* if there is a grading-preserving map $\overline{} : A_{\mathbb{C}} \to A_{\mathbb{C}}$ satisfying $\overline{a+\alpha a'} = \bar{a} + \bar{\alpha}\overline{a'}$, $\overline{aa'} = \bar{a}\overline{a'}$, $\overline{a^*} = (\bar{a})^*$, and $\bar{\bar{a}} = a$, for all $a, a' \in A$ and $\alpha \in \mathbb{C}$. Any $*$-homomorphism $\varphi : A \to B$ between Real C^*-algebras must satisfy the additional relation $\varphi(\bar{a}) = \overline{\varphi(a)}$, for all $a \in A$.

Remarks 1.2. Given a Real C^*-algebra $A_{\mathbb{C}}$, the fixed-point algebra of $\overline{}$, namely $A_{\mathbb{R}} = \{a \in A_{\mathbb{C}} \mid a = \bar{a}\}$, is what is known as a *real C^*-algebra*. A real C^*-algebra is a real Banach algebra with involution that is $*$-isometrically isomorphic to a norm-closed subalgebra of $\mathcal{B}(H_{\mathbb{R}})$, where $H_{\mathbb{R}}$ is a graded real Hilbert space. From a real C^*-algebra $A_{\mathbb{R}}$ we can form its complexification $A_{\mathbb{C}} = A_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$, which is a Real C^*-algebra with the involution $\overline{}$ given by complex conjugation in the second variable. (Here $\mathbb{C}$ is trivially graded.) Realification and complexification induce an equivalence between the category of real C^*-algebras and the category of Real C^*-algebras. Spectral theory and functional calculus for elements in a real C^*-algebra are by definition the corresponding ones in its complexification. Our

focus is real C^*-algebras, but the facts above show that a careful treatment of Real C^*-algebras will suffice. From now on the term "C^*-algebra" will refer to either complex or Real graded C^*-algebras.

We will present three equivalent definitions for asymptotic morphisms between two C^*-algebras. The first one requires some preliminaries:

Definition 1.3. Let B be a Real C^*-algebra. The *asymptotic algebra* $\mathfrak{A}B$ of B is the quotient Real C^*-algebra

$$\mathfrak{A}B = \frac{C_b([1,\infty), B)}{C_0([1,\infty), B)},$$

where $C_b([1,\infty), B)$ is the space of bounded continuous functions from $[1,\infty)$ into B, and $C_0([1,\infty), B)$ is the closed ideal of functions vanishing at infinity. (See [Connes and Higson 1989, Section 1, Def. 2], or [Dădărlat 1994, following Lemma 2].)

Definition 1.4. Let A and B be Real C^*-algebras. An *asymptotic morphism φ from A to B* is a grading-preserving $*$-homomorphism $\varphi : A \to \mathfrak{A}B$. We write $\varphi : A \dashrightarrow B$, with a broken arrow. Let $\mathrm{Asym}_1(A, B)$ be the set of asymptotic morphisms from A to B.

Example 1.5. Any grading-preserving $*$-homomorphism $\varphi : A \to B$ determines an asymptotic morphism assigning to $a \in A$ the class of the constant function $\varphi(a)$.

We turn to the second approach:

Definition 1.6. Let A and B be Real C^*-algebras. An *asymptotic family from A to B* [Connes and Higson 1990, Def. 1] is a family of grading-preserving functions $\{\varphi_t\}_{t\in[1,\infty)} : A \to B$ such that

$$t \mapsto \varphi_t(a) \text{ is bounded and norm-continuous for all } a \in A, \tag{1–1}$$

and

$$\lim_{t\to\infty} \left\{ \begin{array}{l} \varphi_t(a+\alpha a') - \varphi_t(a) - \alpha\varphi_t(a') \\ \varphi_t(aa') - \varphi_t(a)\varphi_t(a') \\ \varphi_t(a^*) - \varphi_t(a)^* \\ \varphi_t(\bar{a}) - \overline{\varphi_t(a)} \end{array} \right\} = 0, \quad \text{for all } a, a' \in A,\ \alpha \in \mathbb{C}. \tag{1–2}$$

We use the same broken arrow notation: $\{\varphi_t\} : A \dashrightarrow B$.

Remarks 1.7. (i) Despite the weak continuity conditions in this definition, (1–2) assures "automatic continuity" in the sense that $\limsup_t \|\varphi_t(a)\| \le \|a\|$, for all $a \in A$. See [Samuel 1997, Lemma 1.2] for a proof.

(ii) The composition of an asymptotic morphism or family with a $*$-morphism (to the left or to the right) is again an asymptotic morphism or family.

Definition 1.8. Two asymptotic families $\{\varphi_t\}_{t\in[1,\infty)}, \{\psi_t\}_{t\in[1,\infty)} : A \dashrightarrow B$ are said to be *asymptotically equivalent* if $\lim_{t\to\infty}\big(\varphi_t(a) - \psi_t(a)\big) = 0$, for all $a \in A$. Asymptotic equivalence is an equivalence relation on the set of asymptotic families, and we denote by $\mathrm{Asym}_2(A, B)$ the set of asymptotic equivalence classes of asymptotic families.

Lemma 1.9. $\mathrm{Asym}_2(A, B)$ *and* $\mathrm{Asym}_1(A, B)$ *are in bijective correspondence.*

Remark 1.10. Because of this lemma we shall use the term "asymptotic morphism" to refer also to an asymptotic family (or its asymptotic equivalence class).

Proof of Lemma 1.9. Let $\{\varphi_t\}_{t\in[1,\infty)}$ be an asymptotic family. Consider the map $\tilde{\varphi} : A \to C_b([1,\infty), B)$ defined by $(\tilde{\varphi}(a))(t) = \varphi_t(a)$, and let $\varphi = q \circ \tilde{\varphi}$ be the $*$-homomorphism given by the composition with the quotient map

$$q : C_b([1,\infty), B) \to \mathfrak{A}B.$$

The desired bijection is given by $\{\varphi_t\} \mapsto \varphi$. The inverse map is obtained by considering any set theoretic section $s : \mathfrak{A}B \to C_b([1,\infty), B)$. It sends a $*$-homomorphism $\varphi : A \to \mathfrak{A}B$ to the family $\{(s \circ \varphi(\cdot))(t)\}_{t\in[1,\infty)}$. □

One can relax things even more:

Definition 1.11. Let $\mathscr{A}$ be a dense $*$-subalgebra of A. A family of maps $\{\varphi_t\}_{t\in[1,\infty)} : \mathscr{A} \to B$ is called *asymptotically continuous* if (1–2) holds true for all $a, a' \in \mathscr{A}$, and (1–1) is replaced by the following condition: for every $a \in \mathscr{A}$ and for every $\varepsilon > 0$, there exists $T = T(\varepsilon, a)$ such that

$$t \geq T \implies \|\varphi_t(a)\| < \|a\| + \varepsilon \tag{1–3}$$

and

$$\forall t_0 \geq T \quad \exists \delta = \delta(\varepsilon, a, t_0) \quad \text{s.t. } |t - t_0| < \delta \implies \|\varphi_t(a) - \varphi_{t_0}(a)\| < \varepsilon. \tag{1–4}$$

Let $\mathrm{Asym}(\mathscr{A}, B)$ be the set of all asymptotic equivalence classes of such densely defined asymptotically continuous families. The next result shows that one can use these less stringent requirements when working with asymptotic morphisms. This characterization will be used later.

Lemma 1.12. *For every dense $*$-subalgebra $\mathscr{A}$ of A, the inclusion of* $\mathrm{Asym}_2(A, B)$ *into* $\mathrm{Asym}(\mathscr{A}, B)$ *is a bijection.*

Proof. That every nonzero element of $\mathrm{Asym}_2(A, B)$ restricts to a nonzero element of $\mathrm{Asym}(\mathscr{A}, B)$ is obvious. For surjectivity, let $\{\varphi_t\}_{t\in[1,\infty)}$ be as in Definition 1.11 and fix $a \in \mathscr{A}$. For every $n = 0, 1, 2, \ldots$, we can find $k(n)$ real numbers $T(2^{-n}, a) = t_{n,1} < t_{n,2} < \cdots < t_{n,k(n)} = T(2^{-(n+1)}, a)$ such that (1–4) is satisfied and the union of the open intervals $(t_{n,i} - \delta_i,\ t_{n,i} + \delta_i)$, for $\delta_i = \delta_i(2^{-n}, a, t_{n,i})$ and $i = 1, \ldots, k(n)$, covers $\big[T(2^{-n}, a), T(2^{-(n+1)}, a)\big]$. We ask also that $\lim_{n\to\infty} T(2^{-n}, a) = \infty$.

Construct a norm-continuous function $\widehat{\varphi}_t(a) : [1,\infty) \to B$ by joining linearly the consecutive values $\varphi_{t_{n,i}}(a)$, for all possible n and i. From these functions, form

an asymptotic morphism $\{\widehat{\varphi}_t\}_{t\in[1,\infty)} : A \to B$ as in Definition 1.6, by using property (1–3) to perform a construction of $\widehat{\varphi}_t(a)$ for $a \in A \setminus \mathscr{A}$. The restriction of $\{\widehat{\varphi}_t\}$ to $\mathscr{A}$ is asymptotically equivalent to the family $\{\varphi_t\}$ that we started with. □

Remark 1.13. The lemma shows that the choice of the dense $*$-subalgebra $\mathscr{A}$ of A in Definition 1.11 is not so important, and we shall use the notation $\mathrm{Asym}_3(A, B)$ to designate $\mathrm{Asym}(\mathscr{A}, B)$, for any "convenient" $\mathscr{A}$.

Graded C*-algebras and the structure of the graded category. An important and useful operation when working with C^*-algebras is the tensor product. For two graded C^*-algebras A_1 and A_2, we denote by $A_1 \widehat{\otimes} A_2$ their *maximal* graded tensor product. Because in this paper at least one of the C^*-algebras that appear in tensor products is nuclear, we do not have to worry about other possible completions. We recall the following rules that are to be satisfied in graded tensor products: $(a_1 \widehat{\otimes} a_2)(b_1 \widehat{\otimes} b_2) = (-1)^{\partial a_2\, \partial b_1} a_1 b_1 \widehat{\otimes} a_2 b_2$, $(a_1 \widehat{\otimes} a_2)^* = (-1)^{\partial a_1\, \partial a_2} a_1^* \widehat{\otimes} a_2^*$, and $\overline{a_1 \widehat{\otimes} a_2} = \overline{a_1} \widehat{\otimes} \overline{a_2}$, for all $a_1, b_1 \in A_1$, and $a_2, b_2 \in A_2$. The maximal graded tensor product is characterized by the following universal property:

Lemma 1.14 (Universal property of the maximal tensor product). *Let A_1, A_2, B be graded C^*-algebras, and let $\varphi : A_1 \to B$, $\psi : A_2 \to B$ be $*$-homomorphisms such that every element of $\varphi(A_1)$ gradedly commutes with every element of $\psi(A_2)$. Then there is a unique $*$-homomorphism $\varphi \widehat{\otimes} \psi : A_1 \widehat{\otimes} A_2 \to B$ such that*

$$\left(\varphi \widehat{\otimes} \psi\right)(a_1 \widehat{\otimes} a_2) = \varphi(a_1)\psi(a_2) = (-1)^{\partial a_1\, \partial a_2} \psi(a_2)\varphi(a_1) \ \textit{for all } a_i \in A_i,\ i = 1, 2.$$

We also mention that there is a transposition isomorphism $\tau : A_1 \widehat{\otimes} A_2 \to A_2 \widehat{\otimes} A_1$, and that the maximal graded tensor product is associative. External tensor products *of asymptotic morphisms* can be constructed in two situations:

Basic Lemma 1.15. (i) *Consider dense $*$-subalgebras $\mathscr{A}_1$ and $\mathscr{A}_2$ of the C^*-algebras A_1 and A_2. Let $\{\varphi_t\} : A_1 \dashrightarrow B$ and $\{\psi_t\} : A_2 \dashrightarrow B$ be asymptotic families such that $\lim_{t\to\infty}[\varphi_t(a_1), \psi_t(a_2)] = 0$ for all $a_1 \in \mathscr{A}_1$, $a_2 \in \mathscr{A}_2$, with $[\ ,\ ]$ denoting the commutator. Then there exists an asymptotic family $\{\varphi_t \widehat{\otimes} \psi_t\} : A_1 \widehat{\otimes} A_2 \dashrightarrow B$.*

(ii) *Let $\{\varphi_t\} : A_1 \dashrightarrow B_1$ and $\{\psi_t\} : A_2 \dashrightarrow B_2$ be asymptotic families. There exists an asymptotic family $\{\varphi_t \widehat{\otimes} \psi_t\} : A_1 \widehat{\otimes} A_2 \dashrightarrow B_1 \widehat{\otimes} B_2$.*

Proof. For (i), use Lemma 1.9 to construct the $*$-homomorphisms $\varphi : A_1 \to \mathfrak{A}B$ and $\psi : A_2 \to \mathfrak{A}B$ that correspond to $\{\varphi_t\}$ and $\{\psi_t\}$. The hypothesis implies that $[\varphi(a_1), \psi(a_2)] = 0$ for all $a_1 \in \mathscr{A}_1$, $a_2 \in \mathscr{A}_2$, and the universal property of the maximal tensor product ensures the existence of the tensor product $*$-homomorphism $\varphi \widehat{\otimes} \psi : A_1 \widehat{\otimes} A_2 \to \mathfrak{A}B$, satisfying $(\varphi \widehat{\otimes} \psi)(a_1 \widehat{\otimes} a_2) = \varphi(a_1)\psi(a_2)$. Use again Lemma 1.9 to construct now from $\varphi \widehat{\otimes} \psi$ an asymptotic family

$$\{\varphi_t \widehat{\otimes} \psi_t\} : A_1 \widehat{\otimes} A_2 \dashrightarrow B,$$

as required. Notice that $\lim_{t\to\infty}\big((\varphi_t \widehat{\otimes} \psi_t)(a_1 \widehat{\otimes} a_2) - \varphi_t(a_1)\psi_t(a_2)\big) = 0$ for all $a_1 \in A_1, a_2 \in A_2$.

To prove (ii), let $\{u_t\}_{t\in[1,\infty)}$ and $\{v_t\}_{t\in[1,\infty)}$ be approximate units for B_1 and B_2, respectively, consisting of even elements. The external tensor product from the statement is the external tensor product, as constructed in (i), of the asymptotic morphisms $\{\varphi'_t\} : A_1 \dashrightarrow B_1 \widehat{\otimes} B_2$ and $\{\psi'_t\} : A_2 \dashrightarrow B_1 \widehat{\otimes} B_2$, defined by $\varphi'_t(a_1) = \varphi_t(a_1) \otimes v_t$ and $\psi'_t(a_2) = u_t \otimes \psi_t(a_2)$. □

We show the usefulness of the Basic Lemma:

Example 1.16 (Basic example of asymptotic morphism). Consider the differentiation operator $D = -id/dx$ as an unbounded self-adjoint operator on $L^2(\mathbb{R})$. Let $M_g : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the bounded operator of pointwise multiplication by the function g. The family of functions

$$C_0(\mathbb{R}) \otimes C_0(\mathbb{R}) \longrightarrow \mathcal{K}(L^2(\mathbb{R})), \quad f \otimes g \mapsto f(t^{-1}D)M_g, \quad t \in [1, \infty)$$

(where $f(t^{-1}D)$ must be understood through the functional calculus) defines an asymptotic morphism from $C_0(\mathbb{R}) \otimes C_0(\mathbb{R})$ into the compact operators of $L^2(\mathbb{R})$. (The C^*-algebras $C_0(\mathbb{R})$ and $\mathcal{K}(L^2(\mathbb{R}))$ are trivially graded here, with $\overline{}$ given by complex conjugation.)

To see this we follow [Higson 1991]. Consider first $f(x) = (x \pm i)^{-1}$ and $g \in C_c^\infty(\mathbb{R})$. Then $f(t^{-1}D)M_g$ is compact, for every $t \in [1, \infty)$. Indeed,

$$\left\| -t^{-1} i \frac{d\xi}{dx} \pm i\xi \right\|_{L^2}^2 = t^{-2} \left\| \frac{d\xi}{dx} \right\|_{L^2}^2 + \|\xi\|_{L^2}^2 \geq t^{-2} \|\xi\|_{H^1_{\mathrm{loc}}}^2, \quad \text{for } \xi \in C_c^\infty(\mathbb{R}),$$

shows that $(t^{-1}D \pm i)$ is invertible, with $(t^{-1}D \pm i)^{-1} : L^2(\mathbb{R}) \to H^1_{\mathrm{loc}}(\mathbb{R})$. The compactness of $M_g(t^{-1}D \pm i)^{-1}$ now follows from Rellich's lemma, and the claimed result is obtained from this by taking adjoints. Next, $\lim_{t\to\infty}\big[(t^{-1}D \pm i)^{-1}, M_g\big] = 0$. Finally, notice that the C^*-subalgebras generated by $(x \pm i)^{-1}$ and $C_c^\infty(\mathbb{R})$ are dense in $C_0(\mathbb{R})$, hence we can apply Basic Lemma 1.15 to obtain the required result.

The advantage of using 1.6 as the definition for asymptotic morphisms should now be manifest: some computations are easier to perform when working with carefully chosen representatives of the equivalence classes. The advantage of using 1.11 as the definition for asymptotic morphisms will become clear in Section 3.

In this paper three graded Real C^*-algebras will play key roles: $\mathcal{K}$, the algebra of compact operators on a separable $\mathbb{Z}/2$-graded Hilbert space; $\mathcal{S}$, the algebra of complex-valued continuous functions on $\mathbb{R}$ that vanish at infinity; and $\mathcal{C}_{n,m}$, the Clifford algebra of $\mathbb{R}^{n+m}$. We present some of the properties of these C^*-algebras.

• $\mathcal{K} = \mathcal{K}(H)$ is the C^*-algebra of compact operators on a separable Real $\mathbb{Z}/2$-graded Hilbert space H, with the standard even grading [Blackadar 1998, 14.1.2], and with Real structure defined by:

$$\overline{T}(h) = \overline{T(\bar{h})} \quad \text{for } T \in \mathcal{K} \text{ and } h \in H.$$

Proposition 1.17. *There is a $*$-isomorphism $\mathcal{K}\widehat{\otimes}\mathcal{K} \xrightarrow{k} \mathcal{K}$, unique up to homotopy and unitary equivalence.*

• $\mathcal{S}$ denotes the C^*-algebra $C_0(\mathbb{R})$ of complex-valued continuous functions on $\mathbb{R}$ that vanish at infinity, with the grading given by even and odd functions; the map $\bar{\ }$ is complex conjugation. Two $*$-homomorphisms $\epsilon : \mathcal{S} \to \mathbb{C}$ and $\Delta : \mathcal{S} \to \mathcal{S}\widehat{\otimes}\mathcal{S}$ can be constructed in a natural way. For $f \in \mathcal{S}$, let $\epsilon(f) = f(0)$. To define Δ one needs to introduce the notion of unbounded multiplier. Whatever the definition, such a multiplier admits a functional calculus $*$-homomorphism; see [Higson et al. 1998, A.3]. Consequently, regarding $(x \otimes 1 + 1 \otimes x)$ as an essentially self-adjoint unbounded multiplier of $\mathcal{S}\widehat{\otimes}\mathcal{S}$, Δ is nothing but the functional calculus for it: $f \mapsto f(x \otimes 1 + 1 \otimes x)$. On the set of generators $\{e^{-x^2}, xe^{-x^2}\}$ of $\mathcal{S}$ we obtain $\Delta(e^{-x^2}) = e^{-x^2}\widehat{\otimes}e^{-x^2}$ and $\Delta(xe^{-x^2}) = xe^{-x^2}\widehat{\otimes}e^{-x^2} + e^{-x^2}\widehat{\otimes}xe^{-x^2}$.

Proposition 1.18. *The $*$-homomorphims $\Delta : \mathcal{S} \to \mathcal{S}\widehat{\otimes}\mathcal{S}$ and $\epsilon : \mathcal{S} \to \mathbb{C}$ form an associative, commutative comultiplication and a counit for $\mathcal{S}$, respectively.*

Remark 1.19. The philosophy behind this proposition is that $\mathcal{S}$ is an "organizer" for algebra, and should not be regarded merely as a space of continuous functions on $\mathbb{R}$.

• $\mathcal{C}_{m,0} = \mathcal{C}_{-m} = \mathrm{Cliff}(\mathbb{R}^m)$ is the Real Clifford algebra of $\mathbb{R}^m$, i.e., the universal Real algebra with odd generators $\{e_1, \dots, e_m\}$ satisfying $e_ie_j + e_je_i = -2\delta_{ij}$ for $1 \le i, j \le m$, plus $e_i{}^* = -e_i$, $\bar{e}_i = e_i$, and $\|e_i\| = 1$. (The grading is the standard one, and the notation agrees with [Kasparov 1976].) One can also define $\mathcal{C}_{m,n}$, the Real Clifford algebra of $\mathbb{R}^{m+n}$, with generators $\{e_1, \dots, e_m, \varepsilon_1, \dots, \varepsilon_n\}$ satisfying the additional properties $\varepsilon_i\varepsilon_j + \varepsilon_j\varepsilon_i = +2\delta_{ij}$, $\varepsilon_ie_j + e_j\varepsilon_i = 0$, $\varepsilon_i{}^* = \varepsilon_i$, $\bar{\varepsilon}_i = \varepsilon_i$, and $\|\varepsilon_i\| = 1$. We denote $\mathcal{C}_{0,n}$ by $\mathcal{C}_{+n}$. We shall need the following result [Lawson and Michelsohn 1989, I.1.5]:

Proposition 1.20. *Consider the inclusions*

$$\mathcal{C}_{-m} \to \mathcal{C}_{-(m+n)}, \quad e_i \mapsto e_i \quad \textit{and} \quad \mathcal{C}_{-n} \to \mathcal{C}_{-(m+n)}, \quad e_j \mapsto e_{m+j}.$$

The universal property of the maximal tensor product gives a $*$-isomorphism $\gamma : \mathcal{C}_{-m}\widehat{\otimes}\mathcal{C}_{-n} \to \mathcal{C}_{-(m+n)}$. Its inverse is given by $v \mapsto v_1\widehat{\otimes}1 + 1\widehat{\otimes}v_2$ for $v_1 \in \mathbb{R}^m$, $v_2 \in \mathbb{R}^n$, $v = v_1 + v_2 \in \mathbb{R}^{m+n} = \mathbb{R}^m \oplus \mathbb{R}^n$.

E-Theory. Given C^*-algebras A and B, a *homotopy* between two asymptotic morphisms from A to B is a $*$-homomorphism from A to $\mathfrak{A}\big(C([0,1], B)\big)$. We denote by $[\![A, B]\!]$ the set of homotopy classes of asymptotic morphisms from A to B, and by $[\![\varphi]\!]$ or $[\![\varphi_t]\!]$ the homotopy class of an asymptotic morphism. Homotopy is an essential equivalence relation for the definition of E-theory groups and for the construction of the composition of asymptotic morphisms. Asymptotically equivalent asymptotic morphisms are homotopic.

Definition 1.21. Let A and B be separable graded Real C^*-algebras. For any nonnegative integer n, the n-th *E-theory group of A and B*, denoted $E^{-n}(A, B)$, is the set of homotopy classes of asymptotic morphisms from $\mathcal{S}\widehat{\otimes} A$ to $\mathcal{K}\widehat{\otimes} B\widehat{\otimes}\mathcal{C}_{-n}$:

$$E^{-n}(A, B) = [\![\mathcal{S}\widehat{\otimes} A, \mathcal{K}\widehat{\otimes} B\widehat{\otimes}\mathcal{C}_{-n}]\!] \quad \text{for } n = 0, 1, 2, \dots.$$

This is the definition of E-theory groups introduced in [Higson and Kasparov 1997].

Definition 1.22. Let A be a separable graded Real C^*-algebra. The n-th *analytic K-homology group of A*, denoted by $E^{-n}(A)$, is

$$E^{-n}(A) = E^{-n}(A, \mathbb{C}) = [\![\mathcal{S}\widehat{\otimes} A, \mathcal{K}\widehat{\otimes}\mathcal{C}_{-n}]\!] \quad \text{for } n = 0, 1, 2, \dots.$$

For X a locally compact topological space, the n-th *analytic K-homology group of X* is

$$E_n(X) = E^{-n}(C_0(X)) = E^{-n}(C_0(X), \mathbb{C}) \quad \text{for } n = 0, 1, 2, \dots,$$

$C_0(X)$ being trivially graded.

In order to keep the paper self-contained, we shall not use the bivariant groups and the powerful composition product that relates them. We gave Definition 1.21 to make clear the context in which we define the K-homology groups. The main properties of these K-homology groups are summarized now:

Theorem 1.23. (1) *The $E^{-n}(A)$ are abelian groups for any separable graded Real C^*-algebra A and any $n = 0, 1, 2, \dots$. If A and B are Real C^*-algebras, any $*$-homomorphism $\alpha : A \to B$ induces group homomorphisms*

$$\alpha^* : E^{-n}(B) \to E^{-n}(A) \quad \textit{for } n = 0, 1, 2, \dots.$$

(2) (*Bott periodicity*) *$E^{-n}(A) \simeq E^{-n-b}(A)$ for all $n \in \mathbb{N}$, with $b = 2$ if A is a complex C^*-algebra and $b = 8$ if A is a Real C^*-algebra.*

(3) (*Half-exactness*) *For any short exact sequence of C^*-algebras*

$$0 \to J \to A \to A/J \to 0$$

there is an exact sequence

$$E^{-n}(J) \leftarrow E^{-n}(A) \leftarrow E^{-n}(A/J) \quad \textit{for } n = 0, 1, 2, \dots.$$

(4) (*Homotopy invariance*) *Let $\alpha_0, \alpha_1 : A \to B$ be homotopic $*$-homomorphisms. Then $\alpha_0^* = \alpha_1^* : E^{-n}(B) \to E^{-n}(A)$, for $n = 0, 1, 2, \dots$.*

(5) *The association $X \mapsto E_*(X)$ is a generalized homology theory in the category of locally compact metrizable spaces.*

(The last assertion implies among others the existence of a long exact sequence

$$\cdots \xrightarrow{\partial} E_n(Y) \to E_n(X) \to E_n(X, Y) \xrightarrow{\partial} E_{n-1}(Y) \to \cdots$$

for any compact pair (X, Y), and after *defining* $E_n(X, Y) = E_n(X \setminus Y)$.)

Example 1.24. Let $X = \bullet$ be a point. Then $E_0(\bullet) = E^0(\mathbb{C}) = [\![\mathcal{S}, \mathcal{K}]\!] = [\mathcal{S}, \mathcal{K}] = \mathbb{Z}$. The third equality is a generalization to the graded category of [Guentner et al. 2000, 2.19]. The result says that in each homotopy class of asymptotic morphisms from $\mathcal{S}$ to $\mathcal{K}$ there is a unique homomorphism up to homotopy. As explained in [Trout 2000], to any $*$-homomorphism $\varphi : \mathcal{S} \to \mathcal{K}(H)$ there corresponds a pair (H_0, D), where H_0 is a subspace of H and D is an operator on H_0 such that φ is equivalent to the functional calculus for D: $f \mapsto f(D)$. The isomorphism with $\mathbb{Z}$ is given by the Fredholm index of D. A generator can be described as the homotopy class of the $*$-homomorphism $f \mapsto f(0)e_{11}$, where e_{11} is a rank-one projection in $\mathcal{K}$, or as the pair with $H_0 = \mathbb{C}$ and $D = 0$.

We shall see in Section 4 that $H_n(\mathbb{R}^n) = \mathbb{Z}$.

2. Spinor bundles and Dirac operators

The spinor bundle of a Spin-manifold. Let M^m be an m-dimensional oriented Riemannian manifold, possibly noncompact, possibly with boundary. We adopt the following definition (see also [Higson and Roe 2000, 11.2], [Hitchin 1974, 4.2], or [Lawson and Michelsohn 1989, III.10]):

Definition 2.1. A *spinor bundle* $\mathbb{S} = \mathbb{S}_M$ *on* M^m is a Real vector bundle satisfying:

(i) $\mathbb{S}$ is $\mathbb{Z}/2$-graded.

(ii) $\mathbb{S}$ admits a left action of $\mathcal{C}(TM)$, the bundle with fibers

$$\mathcal{C}(TM)_x = \mathrm{Cliff}(TM_x);$$

we denote by $\cdot$ the left Clifford action.

(iii) Each fiber of $\mathbb{S}$ is a right Hilbert $\mathcal{C}_{-m}$-module. We denote by $\langle\, , \rangle_x$ the $\mathcal{C}_{-m}$-valued inner product on $\mathbb{S}_x$ and by $\centerdot$ the right action of elements of $\mathcal{C}_{-m}$. The following compatibility condition between the Clifford action and the Hilbert $\mathcal{C}_{-m}$-module structure is required: $\langle \xi \cdot s_1, s_2\rangle_x + \langle s_1, \xi \cdot s_2\rangle_x = 0$, for $\xi \in TM_x$, and $s_1, s_2 \in \mathbb{S}_x$.

(iv) For any $x \in M$ there exists an open set U_x containing x such that $\mathbb{S}|_{U_x}$ is isomorphic to $U_x \times \mathcal{C}_{-m}$, as bundles satisfying (i)–(iii) above, with the left module structure on $\mathbb{S}|_{U_x}$ determined by an oriented local orthonormal frame on TM.

The existence of a spinor bundle is an extra structure that the underlying manifold may or may not possess (see Definition 2.6). We give below three constructions associated to the existence of spinor bundles.

Construction 2.2 (Restriction to an open subset). Let U be an open subset of M and $\mathbb{S}$ a spinor bundle on M. Then $\mathbb{S}|_U$ satisfies properties (i)–(iv) of Definition 2.1 and is a spinor bundle on U.

Construction 2.3 (A spinor bundle on the product). Let $\mathbb{S}_M$ and $\mathbb{S}_N$ be spinor bundles on M and N, respectively. Then $T(M \times N) = TM \oplus TN$, and this gives the action of $\mathscr{C}(T(M \times N))$ on $\mathbb{S}_M \widehat{\otimes} \mathbb{S}_N$. Indeed, fiberwise this action is exactly the one given in the statement of Proposition 1.20. Consequently, $\mathbb{S}_M \widehat{\otimes} \mathbb{S}_N$ is a spinor bundle on $M \times N$.

Construction 2.4 (Restriction to the boundary). Let M^m be a manifold with boundary ∂M, and with collaring neighborhood of the boundary $\partial M^{m-1} \times [0, 1)$. Let $\nu \in TM|_{\partial M}$ be the outward-pointing normal vector field, and let $\mathbb{S}$ be a spinor bundle on M. Consider the following self-adjoint automorphism:

$$\sigma : \mathbb{S}\big|_{\partial M} \longrightarrow \mathbb{S}\big|_{\partial M}, \quad \sigma(s_x) = (-1)^{\partial s_x}\, \nu_x \cdot s_x \centerdot e_m,$$

for $x \in \partial M$, $s_x \in \mathbb{S}_x$ homogeneous, e_m the m-th generator of $\mathscr{C}_{-m}$, $\cdot$ denoting Clifford action, $\centerdot$ denoting the right $\mathscr{C}_{-m}$ action, and with the above formula extended by linearity to all the elements. The automorphism σ commutes with the Clifford action by vectors from $T(\partial M)$, with the right action of $\mathscr{C}_{-(m-1)}$, and obviously satisfies $\sigma^2 = \mathrm{id}$. Consequently, with the structure that it inherits from $\mathbb{S}$, the $+1$ eigenbundle of σ is a spinor bundle on ∂M, called the *the induced spinor bundle on the boundary*.

Definition 2.5. Let (M, g') and (M, g'') be two Riemannian manifolds with the same underlying smooth manifold M, and let $\mathbb{S}'$ and $\mathbb{S}''$ be spinor bundles on (M, g') and (M, g''), respectively. The triples $(M, g', \mathbb{S}')$ and $(M, g'', \mathbb{S}'')$ are *concordant* [Higson and Roe 2000, 11.2.6] if there is a pair consisting of a Riemannian metric g and a spinor bundle $\mathbb{S}$ on $\mathbb{R} \times M$ such that $\mathbb{S}\big|_{(a,b)\times M} = \mathbb{S}_{(a,b)} \widehat{\otimes} \mathbb{S}'$ and $\mathbb{S}\big|_{(c,d)\times M} = \mathbb{S}_{(c,d)} \widehat{\otimes} \mathbb{S}''$, where the intervals (a, b) and (c, d) of $\mathbb{R}$ are disjoint.

Definition 2.6. A *Spin-structure on M* is a concordance class of Riemannian metrics and spinor bundles on M. A *Spin-manifold* is a manifold with a given Spin-structure.

Finally, in order to describe the Dirac operator we introduce the following:

Definition 2.7. A *spinor connection* on a spinor bundle $\mathbb{S}$ is a bilinear map $\nabla : C^\infty(TM) \otimes_{\mathbb{C}} C^\infty(\mathbb{S}) \longrightarrow C^\infty(\mathbb{S}), \quad (X, s) \mapsto \nabla_X s$, compatible with the Real structure, with the right action, and with the Clifford action as follows: $\nabla_X(Y \cdot s) = (\nabla^{LC}_X Y) \cdot s + Y \cdot \nabla_X s$, where $X, Y \in C^\infty(TM)$ are vector fields on M, and ∇^{LC} is the Levi-Civita connection familiar from Riemannian geometry. Such a connection always exists and is unique for a given $\mathbb{S}$.

Definition 2.8. Let M be a Spin-manifold, with spinor bundle $\mathbb{S}$, and spinor connection ∇. The *Dirac operator on M* is the formally self-adjoint odd first order elliptic operator $D = D_M$ acting on sections of $\mathbb{S}$, which over a trivialization chart

U is given by:

$$D_M(s) = \sum_{i=1}^{m} V_i \cdot \nabla_{V_i} s,$$

for $s \in C^\infty(\mathbb{S}|_U)$, and $\{V_i\}_{i=1,m}$ an oriented local orthonormal frame for $TM|_U$. Taking into account the grading on $L^2(\mathbb{S})$ induced from the grading of $\mathbb{S}$, D_M has the following matrix form:

$$D_M = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}. \tag{2–1}$$

3. The K-homology class of an elliptic operator

Definition 3.1. Let M^m be a Spin-manifold without boundary, $\mathbb{S} = \mathbb{S}_M$ be a spinor bundle in the given concordance class, and $D = D_M$ be the corresponding Dirac operator, as explained in the previous section. A *localized family of Dirac-type operators on M* is a family $\{D_t\}_{t\in[1,\infty)}$ of essentially self-adjoint operators, acting on sections of $\mathbb{S}$, and satisfying the following condition: for every compact $K \subseteq M$, there is a real number $t(K)$ such that $D_t(s) = D(s)$, for every section s of $\mathbb{S}$ with support included in K, and for every $t > t(K)$.

Lemma 3.2. *Localized families $\{D_t\}_{t\in[1,\infty)}$ as in Definition 3.1 do exist.*

Proof. Indeed, start with a family $\{\chi_t\}_{t\in[1,\infty)}$ of cutoff functions on M, i.e., positive, smooth, compactly supported, with $\chi_{t_1} \le \chi_{t_2} \le 1$, for $t_1 \le t_2$, and such that for every $x \in M$ there exists $t = t(x)$ with $\chi_{t(x)}(x) = 1$. We construct a localized family of operators by defining $D_t = \chi_t\, D\, \chi_t$. □

Remark 3.3 (More on the existence of the family $\{D_t\}_{t\in[1,\infty)}$). The presence of cutoff functions guarantees that the operators $D_t = \chi_t\, D\, \chi_t$ are essentially self-adjoint. There are nevertheless situations when the self-adjointness can be obtained from a different source, and in such cases the construction given in the previous lemma can be simplified. For example, if M is a complete Spin-manifold then D is already essentially self-adjoint. One chooses $\{D_t\}_{t\in[1,\infty)}$ to be the constant family $\{D\}_{t\in[1,\infty)}$. The same choice of a constant family can obviously be made when M is compact, a compact manifold being complete. If M has boundary some boundary conditions must be taken into account, but we shall not enter into this here.

The main construction of the paper is the following: to a Spin-manifold without boundary M of dimension m, we associate an asymptotic morphism $\{\varphi_t^M\}_{t\in[1,\infty)} : \mathcal{S} \widehat{\otimes} C_0(M) \dashrightarrow \mathcal{K} \widehat{\otimes} \mathcal{C}_{-m}$ given by

$$\varphi_t^M(f \widehat{\otimes} g) = f\left(\tfrac{1}{t} D_t\right) \cdot g \quad \text{for } f \in \mathcal{S},\ g \in C_0(M). \tag{3–1}$$

Here $\{D_t\}_{t\in[1,\infty)}$ is a localized family as in Definition 3.1. The operator $f\left(\frac{1}{t} D_t\right)$ is then given by the functional calculus and is an element of $\mathcal{B}(L^2(\mathbb{S}))$, the space of bounded operators on the *Hilbert $\mathcal{C}_{-m}$-module* of square integrable sections of

$\mathbb{S}$. The operator denoted by g is the pointwise left multiplication operator with the function $(x \mapsto g(x)\,1)$, where 1 is the unit of $\mathscr{C}(TM)_x$, for $x \in M$; see Definition 2.1(ii). It is also an element of $\mathscr{B}(L^2(\mathbb{S}))$.

Theorem 3.4. *Let M^m be a Spin-manifold without boundary. The homotopy class of the asymptotic morphism $\{\varphi_t^M\}_{t\in[1,\infty)}$ of* (3–1) *gives an element*

$$[\![D_M]\!] \in E_m(M) = [\![\mathscr{S} \widehat{\otimes} C_0(M), \mathscr{K} \widehat{\otimes} \mathscr{C}_{-m}]\!]. \tag{3–2}$$

This element is called the fundamental K-homology class of M.

We collect the results needed to support the theorem above in the form of three lemmas.

Lemma 3.5. *For g compactly supported and t large,*

$$f\left(\tfrac{1}{t}\,D_t\right) \cdot g \in \mathscr{K}(L^2(\mathbb{S})) \simeq \mathscr{K} \widehat{\otimes} \mathscr{C}_{-m}.$$

Lemma 3.6. $\{\varphi_t^M\}_{t\in[1,\infty)}$ *is an asymptotic morphism of type* Asym_3.

Lemma 3.7. $[\![D_M]\!]$ *does not depend on the choice of the family* $\{D_t\}_{t\in[1,\infty)}$.

Proof of Lemma 3.5. Using $f(x) = (x \pm i)^{-1}$ and $g \in C_c^\infty(M)$, let $t > t(\operatorname{supp} g)$. Then $g \cdot (\frac{1}{t}D_t \pm i)^{-1} : L^2(\mathbb{S}) \to L^2(\mathbb{S})$, as a bounded operator between Hilbert $\mathscr{C}_{-m}$-modules, is defined by the composition

$$L^2(\mathbb{S}) \xrightarrow{\left(\frac{1}{t}D_t \pm i\right)^{-1}} H^1_{\mathrm{loc}}(\mathbb{S}) \xrightarrow{\;g\;} H^1_c(\mathbb{S}) \xhookrightarrow{\;I\;} L^2(\mathbb{S}).$$

The first map is functional calculus, but relies on the characterization of the domain of the Dirac operator. One can use the realization via cutoff functions of the localized family (Lemma 3.2) to show that $\operatorname{Dom}\left(\frac{1}{t}D_t \pm i\right)$ is contained in $H^1_{\mathrm{loc}}(\mathbb{S})$. (The Hilbert $\mathscr{C}_{-m}$-module structure permits the definition of the Sobolev spaces in a way similar to the classical case.) The proof of the classical theorem of Rellich also generalizes to this context, and it shows that I is a compact operator. Consequently $g \cdot \left(\frac{1}{t}D_t \pm i\right)^{-1} \in \mathscr{K}\left(L^2(\mathbb{S})\right)$. At this point one has to make an identification $\mathscr{K}\left(L^2(\mathbb{S})\right) \simeq \mathscr{K} \widehat{\otimes} \mathscr{C}_{-m}$. The desired result is obtained by taking adjoints. □

Proof of Lemma 3.6. The key fact is

$$\lim_{t\to\infty} \left[f\left(\tfrac{1}{t}D_t\right), g\right] = 0 \quad \text{for } f(x) = (x \pm i)^{-1},\ g \in C_c^\infty(M). \tag{3–3}$$

(Recall that $C_0(M)$ is trivially graded, so the commutator in (3–3) is ordinary.) Indeed,

$$\begin{aligned}
\left[\left(\tfrac{1}{t}D_t \pm i\right)^{-1}, g\right] &= \left(\tfrac{1}{t}D_t \pm i\right)^{-1} g - g\left(\tfrac{1}{t}D_t \pm i\right)^{-1} \\
&= \left(\tfrac{1}{t}D_t \pm i\right)^{-1} \left(g\left(\tfrac{1}{t}D_t \pm i\right) - \left(\tfrac{1}{t}D_t \pm i\right) g\right) \left(\tfrac{1}{t}D_t \pm i\right)^{-1} \\
&= \left(\tfrac{1}{t}D_t \pm i\right)^{-1} \tfrac{1}{t}\,[g, D] \left(\tfrac{1}{t}D_t \pm i\right)^{-1} \\
&= \left(\tfrac{1}{t}D_t \pm i\right)^{-1} \tfrac{1}{t}\,\nabla g \left(\tfrac{1}{t}D_t \pm i\right)^{-1} \xrightarrow{t\to\infty} 0.
\end{aligned}$$

This is a simple generalization of the proof from [Higson 1991] used for Example 1.16. (The explicit formula in local coordinates of the Dirac operator is also used to obtain the last equality above.) Part (i) of Basic Lemma 1.15 gives the extension to $\mathcal{S} \widehat{\otimes} C_0(M)$.

We use a similar computation to prove the asymptotic continuity of the map $t \mapsto \varphi_t(f \widehat{\otimes} g)$, for $f(x) = (x \pm i)^{-1}$, and $g \in C_c^\infty(M)$. Recall Definition 1.11. In our case $a = f \widehat{\otimes} g$ is the fixed element. Let $\varepsilon > 0$ be given. We have to show that there exists $T = T(\varepsilon)$ satisfying (1–3) and (1–4), which here means that $\| f(\frac{1}{t} D_t) g \| < \| f \| \| g \| + \varepsilon$ (this is obvious), and that for a given $t_0 \geq T$ there exists $\delta = \delta(\varepsilon, t_0)$ such that

$$\| f(\tfrac{1}{t} D_t) g - f(\tfrac{1}{t_0} D_{t_0}) g \|_{\mathcal{B}(L^2(\mathbb{S}))} < \varepsilon \quad \text{for } |t - t_0| < \delta.$$

Set $T_1 = (3/\varepsilon) \sup_{x \in M} \{ \| \nabla g(x) \|_{\mathscr{C}_{-m}} \}$, $T_2 = t(\operatorname{supp} g)$, and $T = 2 \max\{T_1, T_2\}$. For a given $t_0 > T$, put $\delta = \min\{T/2,\ \varepsilon T/(12 \|g\|)\}$. With these choices,

$$\begin{aligned}
&\| g \cdot f(\tfrac{1}{t} D_T) - g \cdot f(\tfrac{1}{t_0} D_T) \| \\
&\qquad = \| g(\tfrac{1}{t} D_T \pm i)^{-1} ((\tfrac{1}{t_0} - \tfrac{1}{t}) D_T) (\tfrac{1}{t_0} D_T \pm i)^{-1} \| \\
&\qquad \leq \| g(\tfrac{1}{t} D_T \pm i)^{-1} (1 - \tfrac{t_0}{t}) (\tfrac{1}{t_0} D_T \pm i) (\tfrac{1}{t_0} D_T \pm i)^{-1} \| \\
&\qquad\qquad + \| g(\tfrac{1}{t} D_T \pm i)^{-1} (1 - \tfrac{t_0}{t}) (\tfrac{1}{t_0} D_T \pm i)^{-1} \| \\
&\qquad \leq 2 \|g\| \left| 1 - \frac{t_0}{t} \right| < \frac{\varepsilon}{3}.
\end{aligned}$$

For the localized family $\{D_t\}_t$, as already noted when proving (3–3), we have $[g, D] = [g, D_t]$ for $t > t(\operatorname{supp} g)$. Putting things together, we get

$$\begin{aligned}
&\| f(\tfrac{1}{t} D_t) g - f(\tfrac{1}{t_0} D_{t_0}) g \| \\
&\qquad \leq \| f(\tfrac{1}{t} D_t) g - g f(\tfrac{1}{t} D_T) \| + \| g f(\tfrac{1}{t} D_T) - g f(\tfrac{1}{t_0} D_T) \| \\
&\qquad\qquad + \| g f(\tfrac{1}{t_0} D_T) - f(\tfrac{1}{t_0} D_{t_0}) g \| \\
&\qquad < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,
\end{aligned}$$

completing the proof. □

Proof of Lemma 3.7. Let $\{D_t\}_t$ and $\{D'_t\}_t$ be localized families as in Definition 3.1. For $f(x) = (x \pm i)^{-1}$, $g \in C_c^\infty(M)$, and $t > t(\operatorname{supp} g)$, we have

$$\begin{aligned}
f(\tfrac{1}{t} D_t) g - f(\tfrac{1}{t} D'_t) g &= f(\tfrac{1}{t} D_t) g - g f(\tfrac{1}{t} D'_t) - [(\tfrac{1}{t} D'_t \pm i)^{-1},\ g] \\
&= (\tfrac{1}{t} D_t \pm i)^{-1} \tfrac{1}{t} \nabla g (\tfrac{1}{t} D'_t \pm i)^{-1} - [(\tfrac{1}{t} D'_t \pm i)^{-1},\ g] \xrightarrow{t \to \infty} 0.
\end{aligned}$$

Thus the two asymptotic morphisms are asymptotically equivalent, showing that the class $[\![D_M]\!]$ does not depend on the choice of the family $\{D_t\}_{t \in [1, \infty)}$ □

Example 3.8. Let $M = \bullet$ be a point. Then $\mathbb{S}_\bullet = \mathscr{C}_0 = \mathbb{C}$, $D_\bullet = 0$, and $[\![D_\bullet]\!]$ is given by the class of the $*$–homomorphism $\mathcal{S} \to \mathcal{K}$, $f \mapsto f(0) e_{11}$, where e_{11} is

a rank one projection in $\mathcal{K}$. Recall from Example 1.24 that this is the generator of $E_0(\bullet)$.

Example 3.9. Let $M = \mathbb{R}$. Then $\mathbb{S}(\mathbb{R}) = \mathbb{R} \times \mathcal{C}_{-1} \simeq \mathbb{R} \times (\mathbb{C} \oplus e_1\mathbb{C})$, and the Dirac operator is

$$D_{\mathbb{R}} = e_1 \frac{d}{dx} = \begin{pmatrix} 0 & -d/dx \\ d/dx & 0 \end{pmatrix}.$$

We shall see in Corollary 4.5 that $E_1(\mathbb{R})$ is isomorphic to $\mathbb{Z}$, with generator $[\![D_{\mathbb{R}}]\!]$.

We end this section with a "contravariance" property satisfied by the K-homology classes that we have constructed:

Theorem 3.10. *Let M^m be an open Spin-manifold. For every open inclusion $r : U \hookrightarrow M$ we have a "wrong-way" group homomorphism $r^! : E_m(M) \to E_m(U)$, given by "extension by zero", such that $r^!([\![D_M]\!]) = [\![D_U]\!]$.*

Proof. The group homomorphism $r^!$ is induced by the 'extension by zero' $*$-homomorphism $r^* : C_0(U) \to C_0(M)$. Indeed, for $\varphi : \mathcal{S} \widehat{\otimes} C_0(M) \dashrightarrow \mathcal{K} \widehat{\otimes} \mathcal{C}_{-m}$, $r^!(\varphi)$ is given by the composition

$$\mathcal{S} \widehat{\otimes} C_0(U) \xrightarrow{\mathrm{id} \widehat{\otimes} r^*} \mathcal{S} \widehat{\otimes} C_0(M) \overset{\varphi}{\dashrightarrow} \mathcal{K} \widehat{\otimes} \mathcal{C}_{-m}.$$

See also [Guentner 1994, 6.2.13], or [Higson and Roe 2000, 11.1.7]. For the second part, let D_U be the Dirac operator corresponding to the restriction of the spinor bundle to U (see Construction 2.2), and consider $\{D_{U,t}\}_t$ to be a family used to define $[\![D_U]\!]$, and $\{D_{M,t}\}_t$ to be a family used to define $[\![D_M]\!]$. We show that the asymptotic morphisms $\left\{f\left(\frac{1}{t} D_{U,t}\right)g\right\}_{t\in[1,\infty)}$ and $\left\{f\left(\frac{1}{t} D_{M,t}\right)g\right\}_{t\in[1,\infty)}$, with $f \in C_0(\mathbb{R})$ and $g \in C_c^\infty(U)$, are asymptotically equivalent. Indeed, the same argument as in the proof of Lemma 3.7 applies for $\{D_t\}_t = \{D_{U,t}\}_t$ and $\{D'_t\}_t = \{D_{M,t}\}_t$. Consequently $[\![D_M]\!] \mapsto [\![D_U]\!]$ is well-defined and gives the desired homomorphism. □

4. Properties of the analytic K-homology groups

The external product. For any two C^*-algebras A and B and for any integers m and n, there is the external product map

$$E^{-m}(A) \otimes E^{-n}(B) \longrightarrow E^{-m-n}(A \widehat{\otimes} B), \quad \big([\![\varphi_t]\!], [\![\psi_t]\!]\big) \mapsto [\![\varphi_t]\!] \widehat{\boxtimes} [\![\psi_t]\!]. \tag{4–1}$$

$[\![\varphi_t]\!] \widehat{\boxtimes} [\![\psi_t]\!]$ is called the *external product* of the asymptotic morphisms

$$\{\varphi_t\} : \mathcal{S} \widehat{\otimes} A \dashrightarrow \mathcal{K} \widehat{\otimes} \mathcal{C}_{-m} \quad \text{and} \quad \{\psi_t\} : \mathcal{S} \widehat{\otimes} B \dashrightarrow \mathcal{K} \widehat{\otimes} \mathcal{C}_{-n},$$

and is the class of the asymptotic morphism obtained as the composition of asymptotic morphisms

$$\mathcal{S}\widehat{\otimes}A\widehat{\otimes}B \xrightarrow{\Delta} \mathcal{S}\widehat{\otimes}A\widehat{\otimes}\mathcal{S}\widehat{\otimes}B \overset{\varphi_t\widehat{\otimes}\psi_t}{\dashrightarrow} \mathcal{K}\widehat{\otimes}\mathcal{C}_{-m}\widehat{\otimes}\mathcal{K}\widehat{\otimes}\mathcal{C}_{-n} \xrightarrow{\simeq} \mathcal{K}\widehat{\otimes}\mathcal{C}_{-(m+n)}. \tag{4–2}$$

Here the first $*$-homomorphism incorporates the transposition isomorphism τ, and the last incorporates τ, k of Proposition 1.17, and γ of Proposition 1.20. (The map described in (4–1) and (4–2) is an example of product in E-theory .)

In the geometric context that interests us, given two Spin-manifolds without boundary M^m and N^n, by composing to the left with the isomorphism

$$\mathcal{S} \widehat{\otimes} C_0(M \times N) \to \mathcal{S} \widehat{\otimes} C_0(M) \widehat{\otimes} C_0(N)$$

in (4–2), the construction above makes possible an external multiplication of the fundamental K-homology classes of M and N:

$$E_m(M) \otimes E_n(N) \longrightarrow E_{m+n}(M \times N), \quad \big([\![D_M]\!], [\![D_N]\!]\big) \mapsto [\![D_M]\!] \widehat{\boxtimes} [\![D_N]\!]. \tag{4–3}$$

One of the important features of analytic K-homology is contained in the next result; see also [Higson and Roe 2000, 11.1.8].

Theorem 4.1. *In $E_{m+n}(M \times N)$, one has $[\![D_M]\!] \widehat{\boxtimes} [\![D_N]\!] = [\![D_{M\times N}]\!]$.*

Proof. Taking into account Construction 2.3, it is clear that $D_{M\times N} = D_M \widehat{\otimes} 1 + 1 \widehat{\otimes} D_N$. (We consider as the initial domain of $D_{M\times N}$ the algebraic tensor product of the domains of D_M and D_N.) There are two asymptotic morphisms: the "analytic" one, as described in (4–3) and (4–2),

$$\varphi_t^a(f \widehat{\otimes} h \widehat{\otimes} k) = \big([\![D_M]\!] \widehat{\boxtimes} [\![D_N]\!]\big)\,(f \widehat{\otimes} h \widehat{\otimes} k), \tag{4–4}$$

and the "geometric" one, as described in (3–1),

$$\varphi_t^g(f \widehat{\otimes} h \widehat{\otimes} k) = f\left(\tfrac{1}{t}\, D_{M\times N,t}\right) \cdot h \widehat{\otimes} k. \tag{4–5}$$

Here $f \in \mathcal{S}$, $h \in C_0(M)$, and $k \in C_0(N)$; moreover we identify $C_0(M \times N)$ with $C_0(M) \widehat{\otimes} C_0(N)$. Consequently we regard (4–4) and (4–5) as asymptotic morphisms $\mathcal{S}\widehat{\otimes} C_0(M\times N) \dashrightarrow \mathcal{K}(L^2(\mathbb{S}_{M\times N}))$, and the aim is to show that they are asymptotically equivalent, which will imply exactly the statement of the theorem. In other words, we want to show that

$$\lim_{t\mapsto\infty} \left\|\varphi_t^g(f \widehat{\otimes} h \widehat{\otimes} k) - \varphi_t^a(f \widehat{\otimes} h \widehat{\otimes} k)\right\| = 0 \tag{4–6}$$

for all $f \widehat{\otimes} h \widehat{\otimes} k \in \mathcal{S} \widehat{\otimes} C_0(M \times N)$, with $\|\ \|$ denoting the operator norm in $\mathcal{B}(L^2(\mathbb{S}_{M\times N}))$.

We have to be careful with the construction of the family $\{D_{M\times N,t}\}$. One way to do it is to consider cutoff functions $\{\chi_t^M\}_{t\in[1,\infty)}$ and $\{\chi_t^N\}_{t\in[1,\infty)}$ on M and N, and to define $D_{M\times N,t} = (\chi_t^M \widehat{\otimes} \chi_t^N)\, D_{M\times N}\, (\chi_t^M \widehat{\otimes} \chi_t^N)$. A simpler way is to use $D_{M\times N,t} = D_{M,t} \widehat{\otimes} 1 + 1 \widehat{\otimes} D_{N,t}$, over $C_c^\infty(M \times N)$. With this detail taken care of, we prove (4–6) by showing first that the two asymptotic morphisms are actually *equal* for $f = f(x) \in \{e^{-x^2}, xe^{-x^2}\}$ and h, k compactly supported. Everything is reduced to showing that

$$e^{-((1/t)(D_{M,t}\widehat{\otimes}1+1\widehat{\otimes}D_{N,t}))^2} = e^{-(1/t^2)D_{M,t}^2} \widehat{\otimes} e^{-(1/t^2)D_{N,t}^2},$$

and this is explained in [Higson et al. 1998, Appendix A]. A density argument finishes the proof. (Note that there is no hope to get equality as above for arbitrary f, h, and k.) $\square$

Corollary 4.2. $[\![D_{\mathbb{R}^k}]\!] \mathbin{\widehat{\boxtimes}} [\![D_{\mathbb{R}^l}]\!] = [\![D_{\mathbb{R}^{k+l}}]\!]$.

Having the external product at our disposal, we can now prove other properties of the K-homology groups. We start with Bott periodicity, already mentioned in Theorem 1.23(2). Its essence, in a form that will be used in the proof of Theorem 4.4, is given by the following lemma. (Compare also with [Blackadar 1998, 19.2].)

Lemma 4.3. *In E-theory, $\mathbb{C}$ and $C_0(\mathbb{R}) \mathbin{\widehat{\otimes}} \mathscr{C}_{+1}$ are equivalent as graded Real C^*-algebras.*

Proof. We shall construct two asymptotic morphisms, actually an asymptotic morphism and a $*$-homomorphism, such that their compositions are asymptotically equivalent to the identity. (These compositions are examples of products in E-theory.)

(i) The first asymptotic morphism, d, is the *Dirac element*. Let $D = D_{\mathbb{R}} = e_1 \cdot \dfrac{d}{dx}$ be the Dirac operator on $\mathbb{R}$. Then d is given by the composition

$$\begin{gathered}\mathscr{S} \mathbin{\widehat{\otimes}} C_0(\mathbb{R}) \mathbin{\widehat{\otimes}} \mathscr{C}_{+1} \overset{\varphi_t^{\mathbb{R}} \widehat{\otimes} \mathrm{id}}{\dashrightarrow} \mathscr{K} \mathbin{\widehat{\otimes}} \mathscr{C}_{-1} \mathbin{\widehat{\otimes}} \mathscr{C}_{+1} \xrightarrow{\simeq} \mathscr{K} \mathbin{\widehat{\otimes}} M_2(C) \xrightarrow{\simeq} \mathscr{K}, \\ f \mathbin{\widehat{\otimes}} g \mathbin{\widehat{\otimes}} v \mapsto f\big(\tfrac{1}{t} D\big) g \mathbin{\widehat{\otimes}} v.\end{gathered}$$

(ii) The second asymptotic morphism, β, is the *dual Dirac element*, and it is actually the $*$-homomorphism

$$\mathscr{S} \to C_0(\mathbb{R}) \mathbin{\widehat{\otimes}} \mathscr{C}_{+1}, \quad f \mapsto f(x \mathbin{\widehat{\otimes}} \varepsilon_1).$$

The composition $d \circ \beta$ is given by

$$\mathscr{S} \xrightarrow{\Delta} \mathscr{S} \mathbin{\widehat{\otimes}} \mathscr{S} \xrightarrow{\mathrm{id} \widehat{\otimes} \beta} \mathscr{S} \mathbin{\widehat{\otimes}} C_0(\mathbb{R}) \mathbin{\widehat{\otimes}} \mathscr{C}_{+1} \overset{d}{\dashrightarrow} \mathscr{K},$$

and it is asymptotically equivalent to the following family of $*$-homomorphisms; see [Higson et al. 1998, Appendix B]:

$$f \mapsto f\big(\tfrac{1}{t} B\big), \quad \text{where } B = D + (-1)^{\deg} x e_1. \tag{4–7}$$

Now $B^2 = -d^2/dx^2 + x^2 + (-1)^{1+\deg}$ has one-dimensional kernel, generated by $e^{-x^2/2}$. Denote by e_{11} the orthogonal projection onto the kernel of B^2. The family of $*$-homomorphisms (4–7) is asymptotically equivalent to the $*$-homomorphism $f \mapsto f(0) e_{11}$, which is exactly the identity.

The composition $\beta \circ d$ is given by

$$\mathscr{S} \mathbin{\widehat{\otimes}} C_0(\mathbb{R}) \mathbin{\widehat{\otimes}} \mathscr{C}_{+1} \xrightarrow{\Delta} \mathscr{S} \mathbin{\widehat{\otimes}} \mathscr{S} \mathbin{\widehat{\otimes}} C_0(\mathbb{R}) \mathbin{\widehat{\otimes}} \mathscr{C}_{+1} \overset{\beta \widehat{\otimes} d}{\dashrightarrow} C_0(\mathbb{R}) \mathbin{\widehat{\otimes}} \mathscr{C}_{+1} \mathbin{\widehat{\otimes}} \mathscr{K}.$$

To prove that it is asymptotically equivalent to the identity, one uses the rotation trick of Atiyah [Higson et al. 1998, Proof of Theorem 2.6 and Lemma 2.18] to reduce the computation to the one already performed for the composition $d \circ \beta$. $\square$

Theorem 4.4. *An isomorphism $E_m(M^m) \to E_{m+k}(\mathbb{R}^k \times M^m)$ is provided by the external tensor product with $[\![D_{\mathbb{R}^k}]\!]$.*

Proof. Induction reduces the problem to the case $k = 1$; denote this map by

$$b_M : E_m(M) \xrightarrow{[\![D_{\mathbb{R}}]\!]\widehat{\boxtimes}} E_{m+1}(\mathbb{R} \times M).$$

We shall attain our goal by constructing an inverse for b_M, the "suspension map" s_M. Let $[\![\psi_t]\!]$ be an element of $E_{m+1}(\mathbb{R} \times M)$, given by an asymptotic morphism $\{\psi_t\} : \mathcal{S} \widehat{\otimes} C_0(\mathbb{R} \times M) \dashrightarrow \mathcal{K} \widehat{\otimes} \mathcal{C}_{-(m+1)}$. The element $s_M([\![\psi_t]\!]) \in E_m(M)$ is the class of the asymptotic morphism obtained as result of the compositions

$$\mathcal{S}\widehat{\otimes}C_0(M) \xrightarrow{\beta\circ\Delta} \mathcal{S}\widehat{\otimes}C_0(\mathbb{R})\widehat{\otimes}C_0(M)\widehat{\otimes}\mathcal{C}_{+1} \overset{\psi_t\widehat{\otimes}\mathrm{id}}{\dashrightarrow} \mathcal{K}\widehat{\otimes}\mathcal{C}_{-(m+1)}\widehat{\otimes}\mathcal{C}_{+1} \xrightarrow{\simeq} \mathcal{K}\widehat{\otimes}\mathcal{C}_{-m}.$$

The remaining part now follows easily from Lemma 4.3. Indeed, given an element $[\![\phi_t]\!] \in E_m(M)$, $(s_M \circ b_M)([\![\phi_t]\!])$ is the class of the asymptotic morphism given by the composition

$$\begin{aligned} \mathcal{S} \widehat{\otimes} C_0(M) &\xrightarrow{\Delta^2\widehat{\otimes}\mathrm{id}} \mathcal{S} \widehat{\otimes} \mathcal{S} \widehat{\otimes} \mathcal{S} \widehat{\otimes} C_0(M) \\ &\xrightarrow{\mathrm{id}_{\mathcal{S}} \widehat{\otimes}\beta\widehat{\otimes}\mathrm{id}} \mathcal{S} \widehat{\otimes} C_0(\mathbb{R}) \widehat{\otimes} \mathcal{C}_{+1} \widehat{\otimes} \mathcal{S} \widehat{\otimes} C_0(M) \\ &\overset{[\![D_{\mathbb{R}}]\!]\widehat{\boxtimes}[\![\phi_t]\!]}{\dashrightarrow} \mathcal{K} \widehat{\otimes} \mathcal{C}_{-1} \widehat{\otimes} \mathcal{C}_{+1} \widehat{\otimes} \mathcal{K} \widehat{\otimes} \mathcal{C}_{-m} \simeq \mathcal{K} \widehat{\otimes} \mathcal{C}_{-m}. \end{aligned}$$

Consequently,

$$(s_M \circ b_M)\big([\![\phi_t]\!]\big) = \big[\!\big[(d \circ (\mathrm{id}_{\mathcal{S}} \widehat{\otimes}\beta) \circ \Delta) \widehat{\otimes} \{\phi_t\}\big]\!\big] = \big[\!\big[1 \widehat{\otimes} \{\phi_t\}\big]\!\big] = [\![\phi_t]\!],$$

the second equality following from Lemma 4.3.

Similarly, given $[\![\psi_t]\!] \in E_{m+1}(\mathbb{R} \times M)$, $(b_m \circ s_M)([\![\psi_t]\!])$ is the class of the asymptotic morphism given by the composition

$$\begin{aligned} \mathcal{S} \widehat{\otimes} C_0(\mathbb{R}) \widehat{\otimes} C_0(M) &\xrightarrow{\Delta} \mathcal{S} \widehat{\otimes} C_0(\mathbb{R}) \widehat{\otimes} \mathcal{S} \widehat{\otimes} C_0(M) \\ &\xrightarrow{(\mathrm{id}_{\mathcal{S}\widehat{\otimes}C_0(\mathbb{R})})\widehat{\otimes}(\beta\circ\Delta)} \mathcal{S} \widehat{\otimes} C_0(\mathbb{R}) \widehat{\otimes} \mathcal{S} \widehat{\otimes} C_0(\mathbb{R}) \widehat{\otimes} C_0(M) \widehat{\otimes} \mathcal{C}_{+1} \\ &\overset{[\![D_{\mathbb{R}}]\!]\widehat{\boxtimes}[\![\psi_t]\!]}{\dashrightarrow} \mathcal{K} \widehat{\otimes} \mathcal{C}_{-1} \widehat{\otimes} \mathcal{K} \widehat{\otimes} \mathcal{C}_{-(m+1)} \widehat{\otimes} \mathcal{C}_{+1} \simeq \mathcal{K} \widehat{\otimes} \mathcal{C}_{-(m+1)}. \end{aligned}$$

Again using Atiyah's rotation trick, this asymptotic morphism is asymptotically equivalent to the one with the two copies of $C_0(\mathbb{R})$ switched in the middle line of the composition above. In other words, its class is $\big[\!\big[(d \circ (\mathrm{id}_{\mathcal{S}} \widehat{\otimes}\beta) \circ \Delta) \widehat{\otimes} \{\psi_t\}\big]\!\big] = [\![\psi_t]\!]$. □

Corollary 4.5. *$[\![D_{\mathbb{R}}]\!]$ is nonzero and it generates $E_1(\mathbb{R})$.*

Homotopy invariance of the fundamental K-homology classes.

Theorem 4.6. *Let M be a Spin-manifold. Then $[\![D_M]\!]$ does not depend on the choice of the spinor bundle $\mathbb{S}_M$ in the given concordance class.*

Proof. The notation is that of Section 2, and the result follows from the commutative diagram

$$\begin{tikzcd}
 & E_{m+1}(\mathbb{R}\times M) \arrow[dl, "r_!"'] \arrow[dr, "r_!"] & \\
E_{m+1}\big((a,b)\times M\big) \arrow[d, equal, "\simeq"'] & & E_{m+1}\big((c,d)\times M\big) \arrow[d, equal, "\simeq"] \\
E_{m+1}(\mathbb{R}\times M) \arrow[dr, "s_M"'] & & E_{m+1}(\mathbb{R}\times M) \arrow[dl, "s_M"] \\
 & E_m(M) &
\end{tikzcd}$$

□

Remark 4.7. Consider a Spin-manifold M with metric g_0, and a spinor bundle $\mathbb{S}_0$ in the given concordance class. Let g_1 be another metric on M. The two metrics can be joined by a path of metrics $\{g_t\}_{t\in[0,1]}$ and there is a unique spinor bundle $\mathbb{S}_1$ over M with the metric g_1 and in the same concordance class with $\mathbb{S}_0$. Consequently, Theorem 4.6 applies and gives the invariance of $[\![D_M]\!]$ with respect to the metric on M.

The boundary map. In the remainder of this section we consider the case of manifolds with boundary. Let M^m be a Spin-manifold with boundary ∂M. We shall not enter into details about the relative E-theory groups; see [Guentner 1999] for a careful treatment of these matters. Instead, we adopt $E_*(M\setminus\partial M)$ as the definition for the relative groups $E_*(M,\partial M)$, and then Theorem 1.23(5) gives, in this particular case that interests us, the long exact sequence

$$\cdots\to E_p(M)\to E_p(M\setminus\partial M)\xrightarrow{\ \partial\ } E_{p-1}(\partial M)\to E_{p-1}(M)\to\cdots. \tag{4–8}$$

Our goal is Theorem 4.9, and we shall attain it by explicitly computing the boundary map ∂ in the sequence above. We start by abstractly defining ∂, following [Higson 1988].

(i) Given a sequence of half-exact contravariant functors F^{-n} with objects C^*-algebras, associated to a short exact sequence of C^*-algebras

$$0\to J\to A\xrightarrow{\ p\ } A/J\to 0,$$

there are maps $d: F^{-n}(J)\to F^{-n}(\Sigma(A/J))$, making the diagram

$$\begin{tikzcd}
F^{-n}(J) \arrow[rr, "d"] & & F^{-n}(\Sigma(A/J)) \\
 & F^{-n}(C_p) \arrow[ul, "h_1"] \arrow[ur, "\delta"'] &
\end{tikzcd}$$

commute, where for any C^*-algebra B we set $\Sigma B = C_0(\mathbb{R})\widehat{\otimes} B$ with $C_0(\mathbb{R})$ trivially graded, and where C_p is the mapping cone of the quotient $*$-homomorphism $p : A \to A/J$. The oblique maps come from the inclusions of J and $\Sigma(A/J)$ in C_p. The left oblique map h_1 is an isomorphism.

(ii) For any C^*-algebra B, there are concrete isomorphisms

$$F^{-n}(\Sigma B) \xrightarrow{s} F^{-n+1}(B).$$

The boundary map ∂ is by definition the composition of the two maps appearing in (i) and (ii):

$$\partial : F^{-n}(J) \xrightarrow{d} F^{-n}(\Sigma(A/J)) \xrightarrow{s} F^{-n+1}(A/J).$$

For the short exact sequence $0 \to C_0(M \setminus \partial M) \to C_0(M) \to C_0(\partial M) \to 0$, the map s of (ii) is the "suspension map" $s_{\partial M}$ introduced in the proof of Theorem 4.4. We need one more abstract fact. Let J be any C^*-algebra and let CJ denote the cone of J. For the short exact sequence $0 \to \Sigma J \to CJ \to J \to 0$, $d = \mathrm{id}$ in (i). As an immediate consequence we obtain the desired characterization of ∂:

Lemma 4.8. *If $M = [0, 1) \times \partial M$, then $\partial = s_{\partial M}$.*

We are now able to prove another property of the K-homology classes of Dirac operators:

Theorem 4.9. *Let M^m be a Spin-manifold with boundary ∂M, and let $\mathring{M} = M \setminus \partial M$. Then*

$$\partial([\![D_{\mathring{M}}]\!]) = [\![D_{\partial M}]\!].$$

Proof. Let U be a collaring neighborhood of ∂M, and assume that it is diffeomorphic to $[0, 1) \times \partial M$. Note that $\partial U = \partial M$. The desired conclusion follows from the commutative diagram

$$\begin{array}{ccc}
E_m(\mathring{M}) & \xrightarrow{\partial} & E_{m-1}(\partial M) \\
\downarrow & & \| \\
E_m(\mathring{U}) & \xrightarrow{\partial} & E_{m-1}(\partial U) \\
\simeq \downarrow i & & \| \\
E_m(\mathbb{R} \times \partial M) & \xrightarrow[s_{\partial M}]{} & E_{m-1}(\partial M).
\end{array}$$

Indeed, the upper square is commutative due to the naturality of the boundary map associated to the commutative diagram of split exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_0(\mathring{M}) & \longrightarrow & C_0(M) & \longrightarrow & C_0(\partial M) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & C_0(\mathring{U}) & \longrightarrow & C_0(U) & \longrightarrow & C_0(\partial U) & \longrightarrow & 0.
\end{array}$$

In the lower square, $i([\![D_{\mathring{U}}]\!]) = [\![D_{\mathbb{R}\times\partial M}]\!] = [\![D_{\partial M}]\!] \widehat{\boxtimes} [\![D_{\mathbb{R}}]\!]$, the second equality being a consequence of Theorem 4.1. The diagram's commutativity follows from the description $\partial = s_{\partial M}$ given in Lemma 4.8. $\square$

The key results about the K-homology classes that we have considered in this paper are Theorems 3.10, 4.1, 4.4, 4.6, and 4.9. One may want to compare our proofs with the ones given in [Higson and Roe 2000, Chapters 9–11] or [Baum et al. 1989], both using Kasparov cycles. See also [Guentner 1998] for a presentation of some of these properties using also E-theory, but for ungraded C^*-algebras and with other applications in mind.

5. An application: cobordism invariance of the index

Let N^n be a compact Spin-manifold *without* boundary, with D_N the Dirac operator on N. Let $\epsilon = \epsilon^N : N \to \bullet$ be the map that crushes the entire manifold to a point.

Definition 5.1. The *real index of* D_N is

$$\mathrm{Index}(D_N) = \epsilon^N_*([\![D_N]\!]) \in E_n(\bullet).$$

Remark 5.2. Because $E_n(\bullet) = \mathbb{Z}, 0, 0, 0, \mathbb{Z}, 0, \mathbb{Z}/2, \mathbb{Z}/2$ as n ranges from 0 to 7, this real index is a bit subtle. If the manifold is spin^c and n is even, the (complex) index defined in Definition 5.1 equals the familiar Fredholm index

$$\dim\ker D_+ - \dim\ker D_+^* \in \mathbb{Z}.$$

(The notation is that of (2–1), and recall that the ellipticity makes $\overline{D_N}$ a Fredholm operator. One needs also the pairings with the K-theory groups to obtain the claimed equality.) The above identification justifies the terminology of Definition 5.1. The cobordism invariance of the index is contained in the next result.

Theorem 5.3. *Let M^m be a compact Spin-manifold with boundary ∂M. Then*

$$\mathrm{Index}(D_{\partial M}) = 0.$$

Proof. This follows easily after some moments of contemplation of the diagram

$$\begin{tikzcd}
E_m(\mathring{M}) \arrow[r, "\partial"] & E_{m-1}(\partial M) \arrow[r, "i_*"] \arrow[d, "\epsilon^{\partial M}_*"'] & E_{m-1}(M) \arrow[dl, "\epsilon^M_*"] \\
& E_{m-1}(\bullet) &
\end{tikzcd}$$

Indeed,

$$\begin{aligned}
\mathrm{Index}(D_{\partial M}) &\overset{\text{def}}{=} \epsilon^{\partial M}_*([\![D_{\partial M}]\!]) = \epsilon^{\partial M}_*(\partial[\![D_{\mathring{M}}]\!]) \quad \text{(by Theorem 4.9)} \\
&= \left(\epsilon^M_* \circ i_*\right)(\partial[\![D_{\mathring{M}}]\!]) \\
&= \epsilon^M_*\left((i_*\circ\partial)([\![D_{\mathring{M}}]\!])\right) = \epsilon^M_*(0) = 0.
\end{aligned}$$

$\square$

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