Pacific Journal of Mathematics

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VERONICA FELLI AND MOHAMEDEN OULD AHMEDOU

Volume 218 No. 1 January 2005

PACIFIC JOURNAL OF MATHEMATICS
Vol. 218, No. 1, 2005

A GEOMETRIC EQUATION WITH CRITICAL NONLINEARITY ON THE BOUNDARY

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A theorem of Escobar asserts that if a three-dimensional smooth compact Riemannian manifold M with boundary is of positive type and is not conformally equivalent to the standard three-dimensional ball, a necessary and sufficient condition for a C^2 function H on M to be the mean curvature of some conformal scalar flat metric is that H be positive somewhere. We show that, when the boundary is umbilic and the function H is positive everywhere, all such metrics stay in a compact set with respect to the C^2 norm and the total degree of all solutions is -1.

1. Introduction

José F. Escobar [1992a] raised the following question: When is a compact Riemannian manifold with boundary conformally equivalent to one that has zero scalar curvature and whose boundary has constant mean curvature? This problem can be seen as a generalization to higher dimensions of the Riemann Mapping Theorem, which says that an open, simply connected proper subset of the plane is conformally diffeomorphic to the disk. In higher dimensions few regions are conformally diffeomorphic to the ball. However one can still ask whether a domain is conformal to a manifold that resembles the ball in two ways: namely, it has zero scalar curvature and its boundary has constant mean curvature. Escobar's problem is equivalent to seeking a smooth positive solution u to the following nonlinear boundary value problem on an n-dimensional Riemannian manifold with boundary (M^n, g), where $n \geq 3$:

$$\text{(P)} \qquad \begin{cases} -\Delta_g u + \dfrac{(n-2)}{4(n-1)} R_g u = 0, \quad u > 0 & \text{in } \mathring{M} \\ \dfrac{\partial u}{\partial \nu} + \dfrac{n-2}{2} h_g u = c u^{n/(n-2)} & \text{on } \partial M, \end{cases}$$

where R_g is the scalar curvature of M, h_g is the mean curvature of ∂M, ν is the outer normal vector with respect to g, and c is a constant whose sign is uniquely determined by the conformal structure.

MSC2000: 35J60, 53C21.

Felli is supported by MURST under the national project "Variational Methods and Nonlinear Differential Equations". Ould Ahmedou was supported by a postdoctoral fellowship from SISSA.

For almost all manifolds, Escobar [1992a; 1996] established that (P) has a solution. More recently in [Ould Ahmedou 2003] this problem was studied using the tool of critical points at infinity developed by A. Bahri [1989] (see also [Bahri and Coron 1988; Bahri and Brezis 1996]). Going beyond the existence results of [Ould Ahmedou 2003], we proved in [Felli and Ould Ahmedou 2003] that, when (M, g) is locally conformally flat with umbilic boundary but not conformal to the standard ball, all solutions of (P) stay in a compact set with respect to the C^2 norm and the total degree of all solutions is -1.

The heart of the proof of the result above is some fine analysis of the possible blow-up behaviour of solutions to (P). More specifically, we obtained energy-independent estimates of solutions to

$$\begin{cases} L_g u = 0, \quad u > 0 & \text{in } \mathring{M}, \\ B_g u = (n-2)u^q & \text{on } \partial M, \end{cases}$$

where

$$1 < 1 + \varepsilon_0 \le q \le \frac{n}{n-2}, \quad L_g = \Delta_g - \frac{n-2}{4(n-1)} R_g, \quad B_g = \frac{\partial}{\partial \nu_g} + \frac{n-2}{2} h_g.$$

Instead of looking for conformal metrics with zero scalar curvature and constant mean curvature as in (P), one may also look for scalar-flat conformal metrics with boundary mean curvature being a given function H; this problem is equivalent to finding a smooth positive solution u to

$$(\mathrm{P}_H) \qquad \begin{cases} L_g u = 0, \quad u > 0 & \text{in } \mathring{M}, \\ B_g u = H u^{n/(n-2)} & \text{on } \partial M. \end{cases}$$

Such a problem was studied by Escobar [1996], who proved that if a positive three-dimensional smooth compact Riemannian manifold M is not conformally equivalent to the standard three-ball, a necessary and sufficient condition for a C^2 function H on M to be the mean curvature of some conformal flat metric is that H be positive somewhere. Recall that a manifold is called of positive type, or simply positive, if the quadratic part of the Euler functional associated to (P) is positive definite.

In our work we assume that the boundary is umbilic, that is, the traceless part of the second fundamental form vanishes on the boundary. Moreover we assume that the function H is positive.

Consider for $1 < q \le 3$ the problem

$$(\mathrm{P}_{H,q}) \qquad \begin{cases} L_g u = 0, \quad u > 0 & \text{in } \mathring{M}, \\ B_g u = H u^q & \text{on } \partial M. \end{cases}$$

We use $\mathcal{M}_{H,q}$ to denote the set of solutions of $(\mathrm{P}_{H,q})$ in $C^2(M)$. Our first theorem gives a priori estimates of solutions of $(\mathrm{P}_{H,q})$ in $H^1(M)$ norm.

Theorem 1.1. *Let (M, g) be a three-dimensional smooth compact Riemannian manifold with umbilic boundary. Then, for all $\varepsilon_0 > 0$,*

$$\|u\|_{H^1(M)} \le C \qquad \text{for all } u \in \bigcup_{1+\varepsilon_o \le q \le 3} \mathcal{M}_{H,q},$$

where C depends only on M, g, ε_0, $\|H\|_{C^2(\partial M)}$, and the positive lower bound of H.

Our next theorem states that for any positive C^2 function H, all such metrics stay bounded with respect to the C^2 norm and the total Leray–Schauder degree of all the solutions of $(\mathrm{P}_{H,q})$ is -1.

Theorem 1.2. *Let (M, g) be a positive three-dimensional smooth compact Riemannian manifold with umbilic boundary which is not conformally equivalent to the standard three-dimensional ball. Then, for any $1 < q \le 3$ and positive function $H \in C^2(\partial M)$, there exists some constant C (depending only on M, g, $\|H\|_{C^2}$, the positive lower bound of H, and q) such that*

$$\frac{1}{C} \le u \le C \quad \text{and} \quad \|u\|_{C^2(M)} \le C$$

for all solutions u of $(\mathrm{P}_{H,q})$. The total degree of all solutions of $(\mathrm{P}_{H,q})$ is -1. Consequently, equation $(\mathrm{P}_{H,q})$ with $q = 3$ has at least one solution.

The hypothesis that (M, g) is not conformally equivalent to the standard three-dimensional ball is necessary since (P_H) may have no solution in this case due to the Kazdan–Warner-type condition for manifolds with boundary and for the mean curvature proved in [Escobar 1996]. On the ball sufficient conditions on H in dimensions 3 and 4 are given in [Djadli et al. 2004; Escobar and Garcia 2004], and perturbative results were obtained in [Chang et al. 1998].

Recently S. Brendle [2002a; 2002b] obtained on surfaces some results related to ours. He used curvature flow methods, in the spirit of M. Struwe [2002] and X.-X. Chen [2001]. The curvature flow method was introduced in [Hamilton 1988] and used in [Chow 1991; Ye 1994; Bartz et al. 1994].

The remainder of the paper is organized as follows. In Section 2 we provide the main local blow-up analysis giving first sharp pointwise estimates to a sequence of solutions near isolated simple blow-up points, then we prove that an isolated blow-up is in fact an isolated simple blow-up, ruling out the possibility of bubbles on top of bubbles. In Section 3 we rule out the possibility of bubble accumulations and establish Theorem 1.1. In Section 4 we study compactness of solutions of (P_H) and establish Theorem 1.2. In the Appendix we provide some standard descriptions of singular behaviour of positive solutions to some linear boundary value elliptic equations in punctured half balls and collect some useful results.

2. Local blow-up analysis

We may assume without loss of generality that $h_g \equiv 0$. Indeed, let φ_1 be a positive eigenfunction associated to the first eigenvalue λ_1 of the problem

$$\begin{cases} L_g\varphi = \lambda_1\varphi & \text{in } \mathring{M}, \\ B_g\varphi = 0 & \text{on } \partial M. \end{cases}$$

Setting $\tilde{g} = \varphi_1^4 g$ and $\tilde{u} = \varphi_1^{-1}u$, where u is a solution of $(\mathrm{P}_{H,q})$ (with $q = 3$), one can easily check that $R_{\tilde{g}} > 0$, $h_{\tilde{g}} \equiv 0$, and $\tilde{u}$ satisfies

$$\begin{cases} L_{\tilde{g}}\tilde{u} = 0 & \text{in } \mathring{M}, \\ \dfrac{\partial \tilde{u}}{\partial \nu} = H\tilde{u}^3 & \text{on } \partial M. \end{cases}$$

For simplicity, we work with $\tilde{g}$, denoting it by g. Since ∂M is umbilic with respect to g and $h_{\tilde{g}} = 0$, it follows that the second fundamental form vanishes at each point of the boundary, that is, the boundary is a totally geodesic submanifold. Hence we can take conformal normal coordinates around any point of the boundary [Escobar 1992b].

Recall the definitions of isolated and isolated simple blow-ups, first introduced by R. Schoen [1991] and used extensively by Y.-Y. Li [1995; 1996].

Definition 2.1 (isolated blow-up point). Let (M, g) be a smooth compact n-dimensional Riemannian manifold with boundary and take $\bar{r} > 0$, $\bar{c} > 0$, $\bar{x} \in \partial M$. Let $H \in C^0(\overline{B_{\bar{r}}(\bar{x})})$ be a positive function, where $B_{\bar{r}}(\bar{x})$ denotes the geodesic ball in (M, g) of radius $\bar{r}$ centered at $\bar{x}$. Suppose that, for certain sequences $q_i = 3 - \tau_i$, $\tau_i \to 0$, $H_i \to H$ in $C^2(\overline{B_{\bar{r}}(\bar{x})})$, the sequence $\{u_i\}_{i\in\mathbb{N}}$ solves

$$\begin{cases} L_g u_i = 0, \quad u_i > 0 & \text{in } B_{\bar{r}}(\bar{x}), \\ \dfrac{\partial u_i}{\partial \nu} = H_i u_i^{q_i} & \text{on } \partial M \cap B_{\bar{r}}(\bar{x}). \end{cases} \tag{2–1}$$

We say that $\bar{x}$ is an isolated blow-up point of $\{u_i\}_i$ if there exists a sequence of local maximum points x_i of u_i such that $x_i \to \bar{x}$, $u_i(x_i) \to \infty$ and for some $C_1 > 0$,

$$u_i(x) \le C_1 d(x, x_i)^{-1/(q_i-1)}, \qquad \text{for all } x \in B_{\bar{r}}(x_i) \text{ and all } i.$$

To describe the behaviour of blowing-up solutions near an isolated blow-up point, we define spherical averages of u_i centered at x_i as follows

$$\bar{u_i}(r) = \fint_{M\cap\partial B_r(\bar{x})} u_i = \frac{1}{\mathrm{Vol}_g(M \cap \partial B_r(\bar{x}))} \int_{M\cap\partial B_r(\bar{x})} u_i.$$

Definition 2.2 (isolated simple blow-up point). Let $x_i \to \bar{x}$ be an isolated blow-up point of $\{u_i\}_i$ as in Definition 2.1. We say that $x_i \to \bar{x}$ is an isolated simple blow-up point of $\{u_i\}_i$ if, for some positive constants $\tilde{r} \in (0, \bar{r})$ and $C_2 > 1$, the function $\bar{w}_i(r) := r^{1/(q_i-1)}\bar{u}_i(r)$ satisfies, for large i,

$$\bar{w}_i'(r) < 0 \quad \text{for } r \text{ satisfying } C_2 u_i^{1-q_i}(x_i) \le r \le \tilde{r}.$$

Notation. For later use we introduce the following symbols:

$\mathbb{R}^3_+$ is the open upper half-space $\{(x^1, x^2, x^3) \in \mathbb{R}^2 \times \mathbb{R} : x^3 > 0\}$;
$B_r^+(\bar{x})$ is the open upper hemisphere $\{x = (x', x^3) \in \mathbb{R}^3_+ : |x - \bar{x}| < r\}$;
$\overline{B}_r^+(\bar{x})$ is the closure of $B_r^+(\bar{x})$;
$\Gamma_1(B_r^+(\bar{x}))$ is the closed equatorial disk $\partial B_r^+(\bar{x}) \cap \partial\mathbb{R}^3_+$;
$\Gamma_2(B_r^+(\bar{x}))$ is the open upper hemisphere $\partial B_r(\bar{x}) \cap \mathbb{R}^3_+$;
$\overline{\Gamma}_2(B_r^+(\bar{x}))$ is the closure of $\Gamma_2(B_r^+(\bar{x}))$.

When the center of a ball is 0 we omit it from the notation, so $B_r^+ = B_r^+(0)$, etc.

For any $\bar{x} \in \partial M$, by choosing a geodesic normal coordinate system centered at $\bar{x}$, we can assume without loss of generality that

$$\bar{x} = 0, \quad g_{ij}(0) = \delta_{ij}, \quad B_1^+ \subset M,$$
$$\{(x', 0) = (x^1, x^2, 0) : |x'| < 1\} \subset \partial M, \quad \Gamma^k_{ij}(0) = 0,$$

where Γ^k_{ij} is the Christoffel symbol.

Let $H_i \to H$ in $C^2(\Gamma_1(B_3^+))$ be a sequence of positive functions, q_i a sequence of numbers satisfying $2 \le q_i \le 3$ and $q_i \to 3$, and $\{v_i\}_i \subset C^2(\overline{B}_3^+)$ a sequence of solutions to

$$(\mathrm{P}_i) \qquad \begin{cases} -\Delta_g v_i + \frac{1}{8} R_g v_i = 0, \quad v_i > 0 & \text{in } B_3^+, \\ \dfrac{\partial v_i}{\partial \nu} = H_i v_i^{q_i} & \text{on } \Gamma_1(B_3^+). \end{cases}$$

We now give some properties of isolated and isolated simple blow-ups. We will use c to denote positive constants that may vary from formula to formula and may depend only on M, g, and $\bar{r}$. A similar analysis of blow-ups was also carried out in [Escobar and Garcia 2004], where (M, g) was the standard ball endowed with euclidean metric. See also [Felli and Ould Ahmedou 2003].

The following lemma gives a Harnack Inequality, whose proof is contained (apart from minor modifications) in [Felli and Ould Ahmedou 2003, Lemma 2.3] and [Escobar and Garcia 2004].

Lemma 2.3. *Let v_i satisfy* (P_i) *and let $y_i \to \bar{y} \in \Gamma_1(B_3^+)$ be an isolated blow-up of $\{v_i\}_i$. Then, for any* $0 < r < \bar{r}$,

$$\max_{\overline{B}^+_{2r}(y_i)\setminus B^+_{r/2}(y_i)} v_i \le C_3 \min_{\overline{B}^+_{2r}(y_i)\setminus B^+_{r/2}(y_i)} v_i,$$

where C_3 is a positive constant independent of i and r.

Lemma 2.4. *Let v_i satisfy* (P$_i$) *and let $y_i \to \bar{y} \in \Gamma_1(B_1^+)$ be an isolated blow-up point. Then for any $R_i \to +\infty$ and $\varepsilon_i \to 0^+$ we have, after passing to a subsequence,*

$$\left\| v_i^{-1}(y_i)\, v_i\big(\exp_{y_i}(v_i^{1-q_i}(y_i)x)\big) - \sqrt{\frac{1}{(1+h_i x^3)^2 + h_i^2|x'|^2}}\, \right\|_{C^1(B_{2R_i}^+)} + \left\| v_i^{-1}(y_i)\, v_i\big(\exp_{y_i}(v_i^{1-q_i}(y_i)x)\big) - \sqrt{\frac{1}{(1+h_i x^3)^2 + h_i^2|x'|^2}}\, \right\|_{H^1(B_{2R_i}^+)} \le \varepsilon_i \tag{2–2}$$

and

$$\frac{R_i}{\log v_i(y_i)} \underset{i\to+\infty}{\longrightarrow} 0, \tag{2–3}$$

where $x = (x', x^3) \in B_1^+$ and $h_i = H_i(y_i)$.

Proof. Let $g_i = (g_i)_{\alpha\beta}(x)\, dx^\alpha\, dx^\beta = g_{\alpha\beta}\big(v_i^{1-q_i}(y_i)x\big)\, dx^\alpha\, dx^\beta$ denote the scaled metric. Set

$$\xi_i(x) = v_i^{-1}(y_i)\, v_i\big(y_i + v_i^{1-q_i}(y_i)x\big) \quad \text{for } x \in B^{-T_i}_{v_i^{q_i-1}(y_i)},$$

defined on the set

$$B^{-T_i}_{v_i^{q_i-1}(y_i)} := \big\{z \in \mathbb{R}^3 : |z| < v_i^{q_i-1}(y_i) \quad \text{and} \quad z^3 > -T_i\big\},$$

where $T_i = y_i^3 v_i^{q_i-1}(y_i)$. Then the following conditions are satisfied:

(a) in $B^{-T_i}_{v_i^{q_i-1}(y_i)}$,

$$-\Delta_{g_i}\xi_i + \tfrac{1}{8} v_i^{2(1-q_i)}(y_i) R_{g_i}\big(y_i + v_i^{1-q_i}(y_i)x\big)\, \xi_i = 0 \quad \text{and} \quad \xi_i > 0;$$

(b) on $\partial B^{-T_i}_{v_i^{q_i-1}(y_i)} \cap \{z \in \mathbb{R}^3 : z^3 = -T_i\}$,

$$\frac{\partial \xi_i}{\partial \nu_{g_i}} = H_i(y_i + v_i^{1-q_i}(y_i)x)\xi_i^{q_i}$$

(c) $\xi_i(0) = 1$,

(d) 0 is a local maximum point of ξ_i;

(e) for some positive constant $\tilde{c}$,

$$0 < \xi_i(x) \le \tilde{c}|x|^{-1/(q_i-1)}. \tag{2–4}$$

Now we prove that ξ_i is locally bounded. Using Hopf's boundary point lemma and Lemma 2.3, we derive that for $0 < r < 1$

$$1 = \xi_i(0) \ge \min_{\overline{\Gamma}_1(B_r^+)} \xi_i \ge \min_{\overline{\Gamma}_2(B_r^+)} \xi_i \ge c \max_{\overline{\Gamma}_2(B_r^+)} \xi_i,$$

which implies that, for some c independent of r,

$$\max_{\overline{\Gamma}_2(B_r^+)} \xi_i \le c.$$

From this we derive easily that ξ_i is locally bounded. Applying standard elliptic estimates to $\{\xi_i\}$, we conclude, after passing to a subsequence, that $\xi_i \to \xi$ in $C^2_{\mathrm{loc}}(\mathbb{R}^3_+)$ and $H^1_{\mathrm{loc}}(\mathbb{R}^3_+)$ for some ξ satisfying

$$\begin{cases} \Delta\xi = 0, \quad \xi > 0 & \text{in } \mathbb{R}^3_{-T}, \\ \dfrac{\partial \xi}{\partial \nu} = \big(\lim_i H_i(y_i)\big)\xi^3 & \text{on } \partial\mathbb{R}^3_{-T}, \end{cases}$$

where $\mathbb{R}^3_{-T} := \big\{x = (x', x^3) \in \mathbb{R}^3 : x^3 > -T\big\}$ and $T = \lim_i T_i$. By the Liouville Theorem and (2–4) we have $T < +\infty$. By a Liouville-type theorem from [Li and Zhu 1995] and [Chipot et al. 1996] (see Theorem A.3 in the Appendix), we easily deduce that $T = 0$ and

$$\xi(x', x^3) = \left(\frac{1}{(1 + \lim_i H_i(y_i)x^3)^2 + (\lim_i H_i(y_i))^2|x'|^2}\right)^{1/2}. \qquad \square$$

Before stating our next result, we point out that it follows from Lemma A.5 of the Appendix that, for $\delta_0 > 0$ small enough, there exists a unique function $G(\,\cdot\,, \bar{y}) \in C^2\big(\overline{B}^+_{\delta_o}(\bar{y}) \setminus \{\bar{y}\}\big)$ satisfying

$$\begin{cases} -\Delta_g G(\,\cdot\,, \bar{y}) + \frac{1}{8} R_g G(\,\cdot\,, \bar{y}) = 0 & \text{in } B^+_{\delta_o}(\bar{y}), \\ \dfrac{\partial}{\partial \nu} G(\,\cdot\,, \bar{y}) = 0 & \text{on } \Gamma_1(B^+_{\delta_o}(\bar{y})) \setminus \{\bar{y}\}, \\ \lim_{y \to \bar{y}} d(y, \bar{y}) G(y, \bar{y}) = 1. \end{cases}$$

Now we state our main estimate on isolated simple blow-up points.

Proposition 2.5. *Let v_i satisfy* (P$_i$) *and let $y_i \to \bar{y} \in \Gamma_1(B^+_1)$ be an isolated simple blow-up point, with* (2–2) *and* (2–3) *for all i. Then for some positive constant C depending only on C_1, $\tilde{r}$, $\|H_i\|_{C^2(\Gamma_1(B^+_3))}$, and $\inf_{y \in \Gamma_1(B^+_1)} H_i(y)$ we have*

$$v_i(y) \le C v_i^{-1}(y_i) d(y, y_i)^{-1}, \quad \textit{for} \quad d(y, y_i) \le \frac{\tilde{r}}{2} \tag{2–5}$$

where C_1 and $\tilde{r}$ are given in Definitions 2.1 and 2.2. Furthermore, after passing to some subsequence, for some positive constant b,

$$v_i(y_i) v_i \underset{i \to +\infty}{\longrightarrow} b\, G(\,\cdot\,, \bar{y}) + E \quad \textit{in } C^2_{\mathrm{loc}}\big(\overline{B}^+_{\tilde{\rho}}(\bar{y}) \setminus \{\bar{y}\}\big),$$

where $\tilde{\rho} = \min(\delta_0, \tilde{r}/2)$ and $E \in C^2(B^+_{\tilde{\rho}}(\bar{y}))$ satisfies

$$\begin{cases} -\Delta_g E + \frac{1}{8} R_g E = 0 & \textit{in } B^+_{\tilde{\rho}}, \\ \dfrac{\partial E}{\partial \nu} = 0 & \textit{on } \Gamma_1(B^+_{\tilde{\rho}}). \end{cases}$$

Proposition 2.5 will be established through a series of lemmas.

Lemma 2.6. *Let v_i satisfy* (P_i) *and let $y_i \to \bar{y} \in \Gamma_1(B_1^+)$ be an isolated simple blow-up. Assume $R_i \to +\infty$ and $0 < \varepsilon_i < e^{-R_i}$ are sequences for which* (2–2) *and* (2–3) *hold. Then, for any $0 < \delta < \frac{1}{100}$, there exists $\rho_1 \in (0, \tilde{r})$ independent of i* (*but depending on δ*) *such that*

$$v_i(y_i) \le C_4 v_i^{-\lambda_i}(y_i) d(y, y_i)^{-1+\delta} \quad \text{for all } r_i \le d(y, y_i) \le \rho_1, \tag{2–6}$$

$$\nabla_g v_i(y_i) \le C_4 v_i^{-\lambda_i}(y_i) d(y, y_i)^{-2+\delta} \quad \text{for all } r_i \le d(y, y_i) \le \rho_1, \tag{2–7}$$

$$\nabla_g^2 v_i(y_i) \le C_4 v_i^{-\lambda_i}(y_i) d(y, y_i)^{-3+\delta} \quad \text{for all } r_i \le d(y, y_i) \le \rho_1, \tag{2–8}$$

where $r_i = R_i v_i^{1-q_i}(y_i)$, $\lambda_i = (1-\delta)(q_i - 1) - 1$, and C_4 is some positive constant independent of i.

Proof. We assume for simplicity that g is the flat metric. The general case can be derived essentially in the same way. Let $r_i = R_i v_i^{1-q_i}(y_i)$. Lemma 2.4 implies that

$$v_i(y) \le c v_i(y_i) R_i^{-1} \quad \text{for } d(y, y_i) = r_i. \tag{2–9}$$

We then derive from Lemma 2.3, (2–9), and the definition of an isolated simple blow-up that, for $r_i \le d(y, y_i) \le \tilde{r}$, we have

$$v_i^{q_i - 1}(y) \le c R_i^{-1+o(1)} d(y, y_i)^{-1}. \tag{2–10}$$

Set $T_i = y_i^3 v_i^{q_i - 1}(y_i)$. From the proof of Lemma 2.4 we know that $\lim_i T_i = 0$. It is not restrictive to take $y_i = (0, 0, y_i^3)$. Thus we have $d(0, y_i^3) = o(r_i)$. So

$$B_1^+(0) \setminus B_{2r_i}^+(0) \subset \left\{ \tfrac{3}{2} r_i \le d(y, y_i) \le \tfrac{3}{2} \right\}.$$

We now apply the maximum principle stated in Theorem A.1; to this aim we set

$$\varphi_i(y) = M_i \left(|y|^{-\delta} - \varepsilon |y|^{\delta - 1} y^3 \right) + A v_i^{-\lambda_i}(y_i) \left(|y|^{-1+\delta} - \varepsilon |y|^{-2+\delta} y^3 \right)$$

with M_i and A to be chosen later, and let Φ_i be the boundary operator defined by

$$\Phi_i(v) = \frac{\partial v}{\partial \nu} - H_i v_i^{q_i - 1}(y_i) v.$$

A direct computation yields

$$\Delta \varphi_i(y) = M_i |y|^{-\delta} \big(-\delta(1-\delta) + O(\varepsilon) \big) + |y|^{-(3-\delta)} A v_i^{-\lambda_i}(y_i) \big(-\delta(1-\delta) + O(\varepsilon) \big).$$

Thus one can choose $\varepsilon = O(\delta)$ such that $\Delta \varphi_i \le 0$.

Another straightforward computation taking into account (2–10) shows that for $\delta > 0$ there exists $\rho_1(\delta) > 0$ such that

$$\Phi_i \varphi_i > 0 \quad \text{on } \Gamma_1(B_{\rho_1}^+).$$

Setting

$$\Omega = D_i = B_{\rho_1}^+ \setminus B_{2r_i}^+(0),$$

$$\Sigma = \Gamma_1(D_i) := \partial D_i \cap \partial \mathbb{R}_+^3, \qquad \Gamma = \Gamma_2(D_i) := \partial D_i \cap \mathbb{R}_+^3, \qquad V \equiv 0,$$

$$h = H_i v_i^{q_i - 1}, \qquad v = \varphi_i - v_i, \qquad \psi = v_i,$$

and choosing $A = O(\delta)$ such that $\varphi_i \geq 0$ on $\Gamma_2(D_i)$ and $M_i = \max_{\Gamma_1(B^+_{\rho_1})} v_i$, we deduce from Theorem A.1 that

$$v_i(x) \leq \varphi_i(x). \tag{2–11}$$

By the Harnack inequality and the assumption that the blow-up is isolated simple, we derive that

$$M_i \leq c v_i^{-\lambda_i}(y_i). \tag{2–12}$$

Now (2–6) follows from (2–11) and (2–12).

To derive (2–7) from (2–6), we argue as follows. For $r_i \leq |\tilde{y}| \leq \rho_1/2$, we consider

$$w_i(z) = |\tilde{y}|^{1-\delta} v_i^{\lambda_i}(y_i)\, v_i(|\tilde{y}|z) \quad \text{for } \tfrac{1}{2} \leq |z| \leq 2,\ z^3 \geq 0.$$

It follows from (P_i) that w_i satisfies

$$\begin{cases} -\Delta w_i = 0 & \text{in } \left\{\tfrac{1}{2} < |z| < 2 : z^3 > 0\right\}, \\ \dfrac{\partial w_i}{\partial \nu} = H_i(|\tilde{y}|z)\, |\tilde{y}|^{-\lambda_i} v_i^{\lambda_i(1-q_i)}(y_i) w_i^{q_i} & \text{on } \left\{\tfrac{1}{2} < |z| < 2 : z^3 = 0\right\}. \end{cases} \tag{2–13}$$

In view of (2–6), we have $w_i(z) \leq c$ for any z such that $\frac{1}{2} \leq |z| \leq 2$ and $z^3 \geq 0$. We then derive from (2–13) and gradient elliptic estimates that

$$\left|\nabla w_i(z)\right| \leq c \quad \text{for } z \in \Gamma_2(B_1^+),$$

which implies that

$$\left|\nabla v_i(\tilde{y})\right| \leq c\, |\tilde{y}|^{-2+\delta} v_i^{-\lambda_i}(y_i).$$

This establishes (2–7). Estimate (2–8) can be derived in a similar way. We omit the details. □

Later on we will fix δ close to 0, hence fix ρ_1. Our aim is to obtain (2–6) with $\delta = 0$ for $r_i \leq d(y, y_i) \leq \rho_1$, which together with Lemma 2.4 yields Proposition 2.5.

Now we state a Pohozaev-type identity, which is basically contained in [Li and Zhu 1997]. In the following, we will be working in some geodesic normal coordinate $x = (x^1, x^2, x^3)$ with $g_{ij}(0) = \delta_{ij}$ and $\Gamma_{ij}^k(0) = 0$. We use also the notation $\nabla = (\partial_1, \partial_2, \partial_3)$, $dx = dx^1 \wedge dx^2 \wedge dx^3$ and ds to denote the surface area element with respect to the flat metric.

Lemma 2.7. *For $H \in C^2(\Gamma_1(B_1^+))$ and $a \in C^2(\Gamma_1(B_1^+))$, let $u \in C^2(\overline{B}_1^+)$ satisfy, for $q > 0$,*

$$\begin{cases} -\Delta_g u + \frac{1}{8} R_g u = 0, \quad u > 0 & \text{in } B_1^+, \\ \dfrac{\partial u}{\partial \nu} = H u^q & \text{on } \Gamma_1(B_1^+). \end{cases}$$

Then, for any r such that $0 < r \le 1$,

$$\frac{1}{q+1}\int_{\Gamma_1(B_r^+)}(x'\cdot\nabla_{x'}H)u^{q+1}\,ds+\left(\frac{2}{q+1}-\frac{1}{2}\right)\int_{\Gamma_1(B_r^+)}Hu^{q+1}\,ds$$
$$-\frac{1}{16}\int_{B_r^+}(x\cdot\nabla R_g)u^2\,dx-\frac{1}{8}\int_{B_r^+}R_g\,u^2\,dx-\frac{r}{16}\int_{\Gamma_2(B_r^+)}R_g\,u^2\,ds$$
$$-\frac{r}{q+1}\int_{\partial\Gamma_1(B_r^+)}Hu^{q+1}\,ds$$
$$=\int_{\Gamma_2(B_r^+)}B(r,x,u,\nabla u)\,ds+A(g,u),$$

where

$$B(r,x,u,\nabla u)=\tfrac{1}{2}u\frac{\partial u}{\partial\nu}+\tfrac{1}{2}r\left(\frac{\partial u}{\partial\nu}\right)^2-\tfrac{1}{2}r\,|\nabla_T u|^2 \tag{2–14}$$

($\nabla_T u$ *being the component of* ∇u *tangent to* $\Gamma_2(B_r^+)$) *and*

(2–15)

$$A(g,u)=\int_{B_r^+}(x^k\partial_k u)(g_{ij}-\delta_{ij})\partial_{ij}u\,dx-\int_{B_r^+}(x^l\partial_l u)(g_{ij}-\Gamma_{ij}^k\partial_k u)\,dx$$
$$+\frac{1}{2}\int_{B_r^+}u(g^{ij}-\delta^{ij})\partial_{ij}u\,dx-\frac{1}{2}\int_{B_r^+}u\,g^{ij}\Gamma_{ij}^k\partial_k u\,dx$$
$$-\int_{\Gamma_1(B_r^+)}x^i\frac{\partial u}{\partial x_i}(g^{ij}-\delta^{ij})\frac{\partial u}{\partial x_i}\nu_j-\frac{n-2}{2}\int_{\Gamma_1(B_r^+)}(g^{ij}-\delta^{ij})\frac{\partial u}{\partial x_i}\nu_j u.$$

Regarding the term $A(g,u_i)$, where u_i is a solution of (P_i), we have the following estimate, whose proof is a direct consequence of Lemmas 2.4 and 2.6.

Lemma 2.8. *Let* $\{v_i\}_i$ *satisfy* (P_i) *and let* $y_i\to\bar y\in\Gamma_1(B_1^+)$ *be an isolated simple blow-up point. Assume that* $R_i\to+\infty$ *and* $0<\varepsilon_i<e^{-R_i}$ *are sequences for which* (2–2) *and* (2–3) *hold. Then, for* $0<r<\rho_1$, *we have*

$$|A(g,v_i)|\le C_5 r v_i^{-2\lambda_i}(y_i),$$

where C_5 *is some constant independent of i and r.*

Using Lemmas 2.4, 2.6, 2.7 and 2.8, together with standard elliptic estimates, we derive the following estimate about the rate of blow-up of the solutions of (P_i).

Lemma 2.9. *Let* v_i *satisfy* (P_i) *and let* $y_i\to\bar y\in\Gamma_1(B_1^+)$ *be an isolated simple blow-up point. Assume that* $R_i\to+\infty$ *and* $0<\varepsilon_i<e^{-R_i}$ *are sequences for which* (2–2) *and* (2–3) *hold. Then*

$$\tau_i=O\big(v_i^{-2\lambda_i}(y_i)\big).$$

Consequently $v_i^{\tau_i}(y_i)\to 1$ *as* $i\to\infty$.

Lemma 2.10. *Let v_i satisfy* (P$_i$) *and let $y_i \to \bar{y} \in \Gamma_1(B_1^+)$ be an isolated simple blow-up point. Then, for* $0 < r < \tilde{r}/2$, *we have*

$$\limsup_{i\to+\infty} \max_{y\in\Gamma_2(B_r^+(y_i))} v_i(y_i)\, v_i(y) \le C(r).$$

Proof. By Lemma 2.3, it is enough to establish the lemma for $r > 0$ small enough. Without loss of generality we may take $\bar{r} = 1$. Pick any $y_r \in \Gamma_2(B_r^+)$ and set

$$\xi_i(y) = v_i^{-1}(y_r)\, v_i(y).$$

Then ξ_i satisfies

$$\begin{cases} -\Delta_g \xi_i + \frac{1}{8} R_g \xi_i = 0 & \text{in } B_{1/2}^+(\bar{y}), \\ \dfrac{\partial \xi_i}{\partial \nu} = H_i v_i^{q_i-1}(y_r) \xi_i^{q_i} & \text{on } \Gamma_1(B_{1/2}^+(\bar{y})). \end{cases}$$

It follows from Lemma 2.3 that for any compact set $K \subset B_{1/2}^+(\bar{y}) \setminus \{\bar{y}\}$ there exists some constant $c(K)$ such that

$$c(K)^{-1} \le \xi_i \le c(K) \quad \text{on } K.$$

We also know from (2–6) that $v_i(y_r) \to 0$ as $i \to +\infty$. Then by standard elliptic theory, we have, after passing to a subsequence, that $\xi_i \to \xi$ in $C^2_{\text{loc}}(B_{1/2}^+(\bar{y}) \setminus \{\bar{y}\})$, where ξ satisfies

$$\begin{cases} -\Delta_g \xi + \frac{1}{8} R_g \xi = 0 & \text{in } B_{1/2}^+(\bar{y}), \\ \dfrac{\partial \xi}{\partial \nu} = 0 & \text{on } \Gamma_1(B_{1/2}^+) \setminus \{\bar{y}\}. \end{cases}$$

From the assumption that $y_i \to \bar{y}$ is an isolated simple blow-up point of $\{v_i\}_i$, we know that the function $r^{1/2}\bar{\xi}(r)$ is nonincreasing in the interval $(0, \tilde{r})$ and so we deduce that ξ is singular at $\bar{y}$. So it follows from Corollary A.8 that for r small enough there exists some positive constant $m > 0$ independent of i such that for i large we have

$$-\frac{1}{8}\int_{B_r^+} R_g \xi_i = \int_{B_r^+} -\Delta_g \xi_i = -\int_{\Gamma_1(B_r^+)} \frac{\partial \xi_i}{\partial \nu} - \int_{\Gamma_2(B_r^+)} \frac{\partial \xi_i}{\partial \nu} > m - \int_{\Gamma_1(B_r^+)} \frac{\partial \xi_i}{\partial \nu},$$

which implies that

$$-\frac{1}{8}\int_{B_r^+} R_g \xi_i + \int_{\Gamma_1(B_r^+)} \frac{\partial \xi_i}{\partial \nu} > m. \tag{2–16}$$

On the other hand,

$$\int_{\Gamma_1(B_r^+)} \frac{\partial \xi_i}{\partial \nu} = \int_{\Gamma_1(B_r^+)} H_i v_i^{q_i-1}(y_r)\xi_i^{q_i} \le v_i^{-1}(y_r) \int_{\Gamma_1(B_r^+)} H_i v_i^{q_i}. \tag{2–17}$$

Using Lemmas 2.4 and 2.6, we derive that

$$\int_{\Gamma_1(B_r^+)} H_i v_i^{q_i} \le c v_i^{-1}(y_i). \tag{2–18}$$

Hence our lemma follows from (2–16), (2–17), and (2–18). □

Proof of Proposition 2.5. We first establish (2–5) arguing by contradiction. Suppose the contrary; then, possibly passing to a subsequence still denoted by v_i, there exists a sequence $\{\tilde{y}_i\}_i$ such that $d(\tilde{y}_i, y_i) \le \tilde{r}/2$ and

$$v_i(\tilde{y}_i)\, v_i(y_i)\, d(\tilde{y}_i, y_i) \underset{i\to+\infty}{\longrightarrow} +\infty. \tag{2–19}$$

Set $\tilde{r}_i = d(\tilde{y}_i, y_i)$. From Lemma 2.4 it is clear that $\tilde{r}_i \ge r_i = R_i v_i^{1-q_i}(y_i)$. Set

$$\tilde{v}_i(x) = \tilde{r}_i^{1/(q_i-1)} v_i(y_i + \tilde{r}_i x) \quad \text{in } B_2^{-T_i}, \quad T_i = \tilde{r}_i^{-1} y_i^3.$$

Clearly $\tilde{v}_i$ satisfies

$$\begin{cases} -\Delta_{g_i} \tilde{v}_i + \frac{1}{8}\, \widetilde{R}_{g_i} \tilde{v}_i = 0, \quad v_i > 0 & \text{in } B_2^{-T_i}, \\ \dfrac{\partial \tilde{v}_i}{\partial \nu} = \widetilde{H}_i(x) \tilde{v}_i^{q_i}(x) & \text{on } \partial B_2^{-T_i} \cap \{x^3 = -T_i\}, \end{cases}$$

where

$$\begin{aligned} (g_i)_{\alpha\beta} &= g_{\alpha\beta}(\tilde{r}_i x)\, dx^\alpha dx^\beta, \\ \widetilde{R}_{g_i}(x) &= \tilde{r}_i^2 R_{g_i}(y_i + \tilde{r}_i x), \\ \widetilde{H}_i(x) &= H_i(y_i + \tilde{r}_i x). \end{aligned}$$

Lemma 2.10 yields $\max_{x\in\Gamma_2(B^+_{1/2})} \tilde{v}_i(0)\tilde{v}_i(x) \le c$ for some positive constant c, so

$$v_i(\tilde{y}_i)\, v_i(y_i)\, d(y_i, y_i) \le c.$$

This contradicts (2–19). Therefore (2–5) is established. Now take

$$w_i(x) = v_i(y_i)\, v_i(x).$$

From (P_i) it is clear that w_i satisfies

$$\begin{cases} -\Delta_g w_i + \frac{1}{8}\, R_g w_i = 0, & \text{in } B_3^+ \\ \dfrac{\partial w_i}{\partial \nu} = H_i(x)\, v_i^{1-q_i}(y_i) w_i^{q_i} & \text{on } \Gamma_1(B_3^+). \end{cases}$$

Estimate (2–5) implies that $w_i(x) \le c\, d(x, y_i)^{-1}$. Since $y_i \to \bar{y}$, w_i is locally bounded in any compact set not containing $\bar{y}$. Then, up to a subsequence, $w_i \to w$ in $C^2_{\mathrm{loc}}\big(B_{\tilde{\rho}}(\bar{y}) \setminus \{\bar{y}\}\big)$ for some $w > 0$ satisfying

$$\begin{cases} -\Delta_g w + \frac{1}{8}\, R_g w = 0 & \text{in } B^+_{\tilde{\rho}}(\bar{y}), \\ \dfrac{\partial w}{\partial \nu} = 0 & \text{on } \Gamma_1(B^+_{\tilde{\rho}}) \setminus \{\bar{y}\}. \end{cases}$$

From Proposition A.7, we have

$$w = b\, G(\cdot, \bar{y}) + E \quad \text{in } B^+_{\tilde{\rho}} \setminus \{0\},$$

where $b \geq 0$, E is a regular function satisfying

$$\begin{cases} -\Delta_g E + \frac{1}{8} R_g E = 0 & \text{in } B^+_{\tilde{\rho}}, \\ \dfrac{\partial E}{\partial \nu} = 0 & \text{on } \Gamma_1(B^+_{\tilde{\rho}}), \end{cases}$$

and $G \in C^2(B^+_{\tilde{\rho}} \setminus \{\bar{y}\})$ satisfies

$$\begin{cases} -L_g G(\,\cdot\,, \bar{y}) = 0 & \text{in } B^+_{\tilde{\rho}}, \\ \dfrac{\partial G_a}{\partial \nu} = 0 & \text{on } \Gamma_1(B^+_{\tilde{\rho}}) \setminus \{\bar{y}\}, \end{cases}$$

and $\lim_{y \to \bar{y}} d(y, \bar{y}) G(y, \bar{y})$ is a constant. Moreover w is singular at $\bar{y}$. Indeed, from the definition of an isolated simple blow-up we know that $r^{1/2}\bar{w}(r)$ is a nonincreasing function in the interval $(0, \tilde{r})$, which implies that w is singular at the origin and hence $b > 0$. The proof of Proposition 2.5 is thereby complete. □

Using Proposition 2.5, one can strengthen the results of Lemmas 2.6 and 2.8 using just (2–5) instead of (2–6), thus obtaining the following corollary.

Corollary 2.11. *Let $\{v_i\}_i$ satisfy* (P$_i$) *and let $y_i \to \bar{y} \in \Gamma_1(B^+_1)$ be an isolated simple blow-up point. Assume that $R_i \to +\infty$ and $0 < \varepsilon_i < e^{-R_i}$ are sequences for which* (2–2) *and* (2–3) *hold. Then there exists $\rho_1 \in (0, \tilde{r})$ such that*

$$\big|\nabla_g v_i(y)\big| \leq C_4 v_i^{-1}(y_i) d(y, y_i)^{-2} \quad \textit{for all } r_i \leq d(y, y_i) \leq \rho_1 \tag{2–20}$$

and

$$\big|\nabla_g^2 v_i(y)\big| \leq C_4 v_i^{-1}(y_i) d(y, y_i)^{-3} \quad \textit{for all } r_i \leq d(y, y_i) \leq \rho_1, \tag{2–21}$$

where $r_i = R_i v_i^{1-q_i}(y_i)$ and C_4 is some positive constant independent of i. Moreover

$$\big|A(g, v_i)\big| \leq C_5 r v_i^{-2}(y_i),$$

for some positive constant C_5 independent of i.

We prove an upper bound estimate for $\nabla_g H_i(y_i)$.

Lemma 2.12. *Let v_i satisfy* (P$_i$) *and let $y_i \to \bar{y} \in \Gamma_1(B^+_1)$ be an isolated simple blow-up point. Then*

$$\nabla_g H_i(y_i) = O(v_i^{-2}(y_i)).$$

Proof. Let $x = (x^1, x^2, x^3)$ be geodesic normal coordinates centered at y_i and let η be a smooth cut-off function such that $0 \leq \eta \leq 1$ and

$$\eta(x) = \begin{cases} 1 & \text{if } x \in \overline{B}^+_{1/4}, \\ 0 & \text{if } x \notin \overline{B}^+_{1/2}. \end{cases}$$

Multiply (P_i) by $\eta(\partial v_i/\partial x_1)$ and integrate by parts over B_1^+, thus obtaining

$$(2\text{–}22)\quad 0=\int_{B_1^+}\nabla_g v_i\cdot\nabla\Big(\eta\frac{\partial v_i}{\partial x_1}\Big)\,dV+\frac{1}{8}\int_{B_1^+}R_g v_i\eta\frac{\partial v_i}{\partial x_1}-\int_{\Gamma_1(B_{1/2}^+)}\frac{\partial v_i}{\partial \nu}\eta\frac{\partial v_i}{\partial x_1}\,d\sigma.$$

From (P_i), (2–5), and (2–2) we have

(2–23)

$$\begin{aligned}\int_{\Gamma_1(B_{1/2}^+)}&\frac{\partial v_i}{\partial\nu}\eta\frac{\partial v_i}{\partial x_1}\,d\sigma+\frac{1}{8}\int_{B_1^+}R_g v_i\eta\frac{\partial v_i}{\partial x_1}\\ &=\int_{\Gamma_1(B_{1/2}^+)}H_i v_i^{q_i}\eta\frac{\partial v_i}{\partial x_1}\,d\sigma+O(v_i^{-2}(y_i))\\ &=-\frac{1}{q_i+1}\frac{\partial H_i}{\partial x_1}(y_i)\int_{\Gamma_1(B_{1/2}^+)}\eta v_i^{q_i+1}\,d\sigma+O\bigg(\int_{\Gamma_1(B_{1/2}^+)}|x'|v_i^{q_i+1}\bigg)+O(v_i^{-2}(y_i))\\ &=-\frac{1}{q_i+1}\frac{\partial H_i}{\partial x_1}(y_i)\int_{\Gamma_1(B_{1/2}^+)}\eta v_i^{q_i+1}\,d\sigma+O(v_i^{-2}(y_i)).\end{aligned}$$

On the other hand, from (2–20) it follows that

$$\begin{aligned}(2\text{–}24)\quad \int_{\Gamma_1(B_1^+)}&\nabla_g v_i\cdot\nabla_g\Big(\eta\frac{\partial v_i}{\partial x_1}\Big)\,d\sigma\\ &=\int_{B_1^+}(\nabla_g v_i\cdot\nabla_g\eta)\frac{\partial v_i}{\partial x_1}\,dV+\int_{B_1^+}\nabla_g v_i\cdot\eta\nabla_g\Big(\frac{\partial v_i}{\partial x_1}\Big)\,dV\\ &=-\frac{1}{2}\int_{B_{1/2}^+\setminus B_{1/4}^+}\frac{\partial\eta}{\partial x_1}|\nabla_g v_i|^2\,dV+O(v_i^{-2}(y_i))=O(v_i^{-2}(y_i)).\end{aligned}$$

Putting together (2–22), (2–23), and (2–24), we find

$$\frac{\partial H_i}{\partial x_1}(y_i)=O(v_i^{-2}(y_i)).$$

Repeating the same argument for the derivatives with respect to x_2 and x_3, we come to the required estimate. □

Corollary 2.13. *Under the assumptions of Lemma 2.12*,

$$\int_{\Gamma_1(B_r^+)}x'\cdot\nabla_{x'}H_i v_i^{q_i+1}\,d\sigma=O(v_i^{-4}(y_i)).$$

Proof. We have

$$\begin{aligned}\int_{\Gamma_1(B_r^+)}&x'\cdot\nabla_{x'}H_i v_i^{q_i+1}\,d\sigma\\ &=\int_{\Gamma_1(B_r^+)}\nabla_{x'}H_i(y_i)\cdot(x'-y_i)v_i^{q_i+1}\,d\sigma+O\bigg(\int_{\Gamma_1(B_r^+)}|x'|^2v_i^{q_i+1}\,d\sigma\bigg).\end{aligned}$$

From Proposition 2.5 and Lemma 2.4, $\int_{\Gamma_1(B_r^+)}(x'-y_i)v_i^{q_i+1}\,d\sigma=O(v_i^{-2}(y_i))$. The conclusion follows from Lemma 2.12, Corollary 2.11, and (2–2). □

Proposition 2.14. *Let v_i satisfy* (P_i), $y_i \to \bar{y}$ *be an isolated simple blow-up point with, for some* $\tilde{\rho} > 0$,

$$v_i(y_i)\, v_i \underset{i\to+\infty}{\longrightarrow} h \quad \text{in } C^2_{\mathrm{loc}}\big(B^+_{\tilde{\rho}}(\bar{y}) \setminus \{\bar{y}\}\big).$$

Assume, for some $\beta > 0$, *that in some geodesic normal coordinate system* $x = (x^1, x^2, x^3)$ *we have*

$$h(x) = \frac{\beta}{|x|} + A + o(1) \quad \text{as } |x| \to 0.$$

Then $A \le 0$.

Proof. For $r > 0$ small, the Pohozaev-type identity of Lemma 2.7 yields

$$\begin{aligned}(2\text{–}25)\qquad & \frac{1}{q_i+1}\int_{\Gamma_1(B_r^+)} (x'\cdot\nabla_{x'}H_i)\, v_i^{q_i+1}\, ds + \Big(\frac{2}{q_i+1} - \frac{1}{2}\Big)\int_{\Gamma_1(B_r^+)} H_i v_i^{q_i+1}\, ds \\ & - \frac{1}{16}\int_{B_r^+} (x\cdot\nabla R_g)\, v_i^2\, dx - \frac{1}{8}\int_{B_r^+} R_g\, v_i^2\, dx - \frac{r}{16}\int_{\Gamma_2(B_r^+)} R_g\, v_i^2\, ds \\ & - \frac{r}{q_i+1}\int_{\partial\Gamma_1(B_r^+)} H_i v_i^{q_i+1} \\ & = \int_{\Gamma_2(B_r^+)} B(r, x, v_i, \nabla v_i)\, ds + A(g, v_i),\end{aligned}$$

where B and $A(g, v_i)$ are defined in (2–14) and (2–15) respectively. Multiply (2–25) by $v_i^2(y_i)$ and let $i \to \infty$. Using Corollary 2.11, Lemma 2.4, and Corollary 2.13, one has

(2–26)

$$\lim_{r\to 0^+}\int_{\Gamma_2(B_r^+)} B(r, x, h, \nabla h) = \lim_{r\to 0^+}\limsup_{i\to\infty} v_i^2(y_i)\int_{\Gamma_2(B_r^+)} B(r, x, v_i, \nabla v_i) \ge 0.$$

On the other hand, a direct calculation yields

$$(2\text{–}27)\qquad \lim_{r\to 0^+}\int_{\Gamma_2(B_r^+)} B(r, x, h, \nabla h) = -c\, A$$

for some $c > 0$. The conclusion follows from (2–26) and (2–27). □

Now we can prove that an isolated blow-up point is in fact an isolated simple blow-up point.

Proposition 2.15. *Let v_i satisfy* (P_i) *and* $y_i \to \bar{y}$ *be an isolated blow-up point. Then $\bar{y}$ must be an isolated simple blow-up point.*

Proof. The proof is much the same as that of [Felli and Ould Ahmedou 2003, Prop. 2.11]. For the reader's convenience, we include it here. From Lemma 2.4, it follows that

$$(2\text{–}28)\qquad \bar{w}_i{}'(r) < 0 \quad \text{for every } C_2 v_i^{1-q_i}(y_i) \le r \le r_i.$$

Suppose that the blow-up is not simple; then there exist sequences $\tilde{r}_i \to 0^+$ and $\tilde{c}_i \to +\infty$ such that $\tilde{c}_i v_i^{1-q_i}(y_i) \le \tilde{r}_i$ and, after passing to a subsequence,

$$\bar{w}_i'(\tilde{r}_i) \ge 0. \tag{2–29}$$

From (2–28) and (2–29) it is clear that $\tilde{r}_i \ge r_i$ and $\bar{w}_i$ has at least one critical point in the interval $[r_i, \tilde{r}_i]$. Let μ_i be the smallest critical point of $\bar{w}_i$ in this interval. We have

$$\tilde{r}_i \ge \mu_i \ge r_i \quad \text{and} \quad \lim_{i\to\infty} \mu_i = 0.$$

Let $g_i = (g_i)_{\alpha\beta}\, dx^\alpha dx^\beta = g_{\alpha\beta}(\mu_i x)\, dx^\alpha dx^\beta$ be the scaled metric and set

$$\xi_i(x) = \mu_i^{1/(q_i-1)} v_i(y_i + \mu_i x).$$

Then ξ_i satisfies

$$\begin{cases} -\Delta_{g_i}\xi_i + \frac{1}{8} R_{g_i}\xi_i = 0 & \text{in } B^{-T_i}_{1/\mu_i}, \\ \dfrac{\partial \xi_i}{\partial \nu} = \widetilde{H}_i(x)\xi^{q_i}\xi_i^{q_i} & \text{on } \partial B^{-T_i}_{1/\mu_i} \cap \{x^3 = -T_i\}; \end{cases}$$

$\lim_{i\to\infty} \xi_i(0) = \infty$ and 0 is a local maximum point of ξ_i; also $r^{1/(q_i-1)}\bar{\xi}_i(r)$ has negative derivative in $c\,\xi_i(0)^{1-q_i} < r < 1$ and

$$\frac{d}{dr}\big(r^{1/(q_i-1)}\bar{\xi}_i(r)\big)\big|_{r=1} = 0, \tag{2–30}$$

where $T_i = \mu_i^{-1} y_i^3$, $\tilde{a}_i(x) = \mu_i a_i(y_i + \mu_i x)$, and $\widetilde{H}_i(x) = H_i(y_i + \mu_i x)$. Arguing as in the proof of Lemma 2.4, we can easily prove that $T_i \to 0$. Since 0 is an isolated simple blow-up point, by Proposition 2.5 and Lemma 2.3, we have, for some $\beta > 0$,

$$\xi_i(0)\xi_i \underset{i\to+\infty}{\longrightarrow} h = \beta\,|x|^{-1} + E \quad \text{in } C^2_{\text{loc}}(\mathbb{R}^3_+ \setminus \{0\}), \tag{2–31}$$

with E satisfying

$$\begin{cases} -\Delta E = 0, & \text{in } \mathbb{R}^3_+, \\ \dfrac{\partial E}{\partial \nu} = 0, & \text{on } \partial\mathbb{R}^3_+. \end{cases}$$

By the maximum principle we have $E \ge 0$. Reflecting E to be defined on all $\mathbb{R}^3$ and thus using Liouville's Theorem, we deduce that E is a constant. Using (2–30) and (2–31), we deduce that $E \equiv b$. Therefore $h(x) = b(G_a(x, \bar{y}) + 1)$, contradicting Proposition 2.14. □

3. Ruling out bubble accumulations

Now we can proceed as in [Felli and Ould Ahmedou 2003] to obtain the following results, which rule out possible accumulations of bubbles, thus implying that only isolated blow-up points may occur for blowing-up sequences of solutions.

Proposition 3.1. *Let (M, g) be a smooth compact three-dimensional Riemannian manifold with umbilic boundary. For any $R \geq 1$ and $0 < \varepsilon < 1$, there exist positive constants δ_0, c_0, and c_1 depending only on M, g, $\|H\|_{C^2(\partial M)}$, $\inf_{y\in\partial M} H(y)$, R, and ε, such that for all u in*

$$\bigcup_{3-\delta_0\leq q\leq 3} \mathcal{M}_{H,q}$$

with $\max_M u \geq c_0$ there exists $\mathcal{S} = \{p_1, \dots, p_N\} \subset \partial M$ with $N \geq 1$ satisfying the following conditions:

(i) *each p_i is a local maximum point of u in M and*

$$\overline{B}_{\bar r_i}(p_i) \cap \overline{B}_{\bar r_j}(p_j) = \varnothing, \quad \text{for } i \neq j,$$

where $\bar r_i = Ru^{1-q}(p_i)$ and $\overline{B}_{\bar r_i}(p_i)$ denotes the geodesic ball in (M, g) of radius $\bar r_i$ and centered at p_i;

(ii) $$\left\| u^{-1}(p_i)u\big(\exp_{p_i}(yu^{1-q}(p_i))\big) - \sqrt{\frac{1}{(1+hx^3)^2 + h^2|x'|^2}} \right\|_{C^2(B^M_{2R}(0))} < \varepsilon,$$

where

$$B^M_{2R}(0) = \big\{ y \in T_{p_i}M : |y| \leq 2R,\ u^{1-q}(p_i)y \in \exp^{-1}_{p_i}(B_\delta(p_i)) \big\},$$

$y = (y', y^n) \in \mathbb{R}^n$, *and* $h > 0$;

(iii) $d^{1/(q-1)}(p_j, p_i)u(p_j) \geq c_0$, *for $j > i$, while $d(p, \mathcal{S})^{1/(q-1)}u(p) \leq c_1$, for all $p \in M$, where $d(\cdot\,, \cdot)$ denotes the distance function in metric g.*

Proposition 3.2. *Let (M, g) be a smooth compact three-dimensional Riemannian manifold with umbilic boundary. For suitably large R and small $\varepsilon > 0$, there exist δ_1 and d depending only on M, g, $\|a\|_{C^2(\partial M)}$, $\|H\|_{C^2(\partial M)}$, $\inf_{y\in\partial M} H(y)$, R, and ε, such that for all u in $\bigcup_{3-\delta_1\leq q\leq 3} \mathcal{M}_{a,H,q}$ with $\max_M u \geq c_0$, we have*

$$\min\{d(p_i, p_j) : i \neq j,\ 1 \leq i, j \leq N\} \geq d$$

where $c_0, p_1, \dots, p_N$ are given by Proposition 3.1.

The previous two propositions imply that any blow-up point is in fact an isolated blow-up point. Thanks to Proposition 2.15, any blow-up point is in fact an isolated simple blow-up point.

Proof of Theorem 1.1. Arguing by contradiction, suppose that there exist sequences $q_i \to q \in (1, 3]$, $u_i \in \mathcal{M}_{H_i,q_i}$ such that $\|u_i\|_{H^1(M)} \to +\infty$ as $i \to \infty$, which, in view of standard elliptic estimates, implies that $\max_M u_i \to +\infty$.

From [Hu 1994] (see also [Li and Zhang 2003]), we know that $q = 3$. By Proposition 3.2, for some small $\varepsilon > 0$, large $R > 0$, and some $N \geq 1$ there exist $y_i^{(1)}, \dots, y_i^{(N)} \in \partial M$ such that conditions (i)–(iii) Proposition 3.1 hold. The points $\{y_i^{(1)}\}_i, \dots, \{y_i^{(N)}\}_i$ are isolated blow-up points and hence, by Proposition 2.15, isolated simple blow-up points. From (2–2) and Proposition 2.5, the sequence $\big\{\|u_i\|_{H^1(M)}\big\}_i$ is bounded, which gives a contradiction. □

4. Compactness of the solutions

Before proving Theorem 1.2, we state the following result about the compactness of solutions of $(\mathrm{P}_{H,q})$ when q stays strictly below the critical exponent. The proof is basically the same as that of [Felli and Ould Ahmedou 2003, Theorem 3.1].

Theorem 4.1. *Let (M, g) be a smooth compact three-dimensional Riemannian manifold with umbilic boundary. Then for any $\delta_1 > 0$ there exists a constant $C > 0$ depending only on M, g, δ_1, $\|H\|_{C^2(\partial M)}$, and the positive lower bound of H on ∂M such that for all $u \in \bigcup_{1+\delta_1 \leq q \leq 3-\delta_1} \mathcal{M}_{H,q}$ we have*

$$\|u\|_{C^2(M)} \leq C \qquad \text{and} \qquad \frac{1}{C} \leq u(x) \leq C \quad \text{for all } x \in M.$$

Proof of Theorem 1.2. Due to elliptic estimates and Lemma 2.3, we have to prove just the L^∞ bound, i.e., $u \leq C$. Suppose the contrary; then there exists a sequence $q_i \to q \in (1, 3]$ with

$$u_i \in \mathcal{M}_{H,q_i} \quad \text{and} \quad \max_M u_i \to +\infty,$$

where $\bar{c}$ is some positive constant independent of i. From Theorem 4.1, q must be 3. It follows from Propositions 2.15 and 3.2 that, after passing to a subsequence, $\{u_i\}_i$ has N (with $1 \leq N < \infty$) isolated simple blow-up points denoted by $y^{(1)}, \dots, y^{(N)}$. Let $y_i^{(\ell)}$ denote the local maximum points as in Definition 2.1. It follows from Proposition 2.5 that

$$u_i(y_i^{(1)})u_i \underset{i\to+\infty}{\longrightarrow} h(y) = \sum_{j=1}^{N} b_j G(y, y^{(j)}) + E(y) \quad \text{in } C^2_{\text{loc}}\big(M \setminus \{y^{(1)}, \dots, y^{(N)}\}\big),$$

where $b_j > 0$ and $E \in C^2(M)$ satisfies

$$\begin{cases} -L_g E = 0 & \text{in } M, \\ \dfrac{\partial E}{\partial \nu} = 0 & \text{on } \partial M. \end{cases} \tag{4–1}$$

Since the manifold is of positive type, $E \equiv 0$. Therefore,

$$u_i(y_i^{(1)})u_i \underset{i\to+\infty}{\longrightarrow} h(y) = \sum_{j=1}^{N} b_j G_a(y, y^{(j)}) \quad \text{in } C^2_{\text{loc}}\big(M \setminus \{y^{(1)}, \dots, y^{(N)}\}\big).$$

Let $x = (x^1, x^2, x^3)$ be a geodesic normal coordinate system centered at $y_i^{(1)}$. From Lemma A.5, the Positive Mass Theorem, and the assumption that the manifold is not conformally equivalent to the standard ball, we derive that there exists a positive constant A such that

$$h(x) = h(\exp_{y_i^{(1)}}(x)) = c|x|^{-1} + A_i + O(|x|^{-\alpha}) \quad \text{for } |x| \text{ close to } 0$$

and $A_i \geq A > 0$. This contradicts the result of Proposition 2.14. The compactness part of Theorem 1.2 is proved. Once we have compactness, we can proceed as in [Felli and Ould Ahmedou 2003, Section 4] to prove that the total degree of the solutions is -1. □

Appendix

Here we recall some needed results and describe the singular behaviour of positive solutions to certain boundary value elliptic equations in punctured half-balls.

For $n \geq 3$, let $B_r^+ : \{x = (x', x^n) \in \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R} : |x| < r \text{ and } x^n > 0\}$ and set $\Gamma_1(B_r^+) := \partial B_r^+ \cap \partial \mathbb{R}_+^n$, $\Gamma_2(B_r^+) := \partial B_r^+ \cap \mathbb{R}_+^n$. Consider a smooth Riemannian metric $g = g_{ij}\, dx^i dx^j$ in B_1^+, and $a \in C^1(\Gamma_1(B_1^+))$.

We first recall a maximum principle; for the proof see [Han and Li 1999].

Theorem A.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ and let $\partial\Omega = \Gamma \cup \Sigma$, $V \in L^\infty(\Omega)$, and $h \in L^\infty(\Sigma)$ be such that there exists $\psi \in C^2(\Omega) \cap C^1(\overline{\Omega})$ positive in $\overline{\Omega}$ and satisfying*

$$\begin{cases} \Delta_g \psi + V\psi \leq 0 & \text{in } \Omega, \\ \dfrac{\partial \psi}{\partial \nu} \geq h\psi & \text{on } \Sigma. \end{cases}$$

If $v \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies

$$\begin{cases} \Delta_g v + V v \leq 0 & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \nu} \geq h v & \text{on } \Sigma, \\ v \geq 0, & \text{on } \Gamma, \end{cases}$$

then $v \geq 0$ in $\overline{\Omega}$.

Next we state a maximum principle that holds for the operator T defined by

$$Tu = v \quad \text{if and only if} \quad \begin{cases} L_g u = 0 & \text{in } \mathring{M}, \\ \dfrac{\partial u}{\partial \nu} = v & \text{on } \partial M. \end{cases}$$

Proposition A.2 [Escobar 1996]. *Let (M, g) be a Riemannian manifold with boundary of positive type. Then, for any $u \in C^2(\mathring{M}) \cap C^1(M)$ satisfying $L_g u \geq 0$ in $\mathring{M}$ and $\partial u/\partial \nu \leq 0$ on ∂M, we have $u \leq 0$ in M.*

Proof. Let $u^+(x) = \max\{0, u(x)\}$. Then

$$0 \leq \int_M (L_g u)u^+\, dV - \int_{\partial M} \frac{\partial u}{\partial \nu} u^+\, d\sigma = -\int_M \left|\nabla_g u^+\right|^2 dV - \frac{1}{8}\int_M R_g^2 |u^+|^2\, dV.$$

Since M is of positive type $\int |\nabla_g u|^2 + \frac{1}{8}\int R_g u^2$ is an equivalent norm, hence $u^+ \equiv 0$. □

We now recall a Louville-type theorem, from [Li and Zhu 1995]. See also [Escobar 1990; Chipot et al. 1996].

Theorem A.3. *If v is a solution of*

$$\begin{cases} -\Delta v = 0 & \text{in } \mathbb{R}^n_+, \\ \dfrac{\partial v}{\partial x^n} = c v^{n/(n-2)} & \text{on } \partial\mathbb{R}^n_+, \end{cases}$$

and c is a negative constant, then either $v \equiv 0$ or v is of the form

$$v(x', x^n) = \left(\frac{\varepsilon}{(x_0^n + x^n)^2 + |x' - x_0'|^2}\right)^{\frac{n-2}{2}} \quad \text{for } x' \in \mathbb{R}^{n-1},\ x^n \in \mathbb{R},$$

where $x_0^n = -(n-2)\varepsilon/c$, for some $\varepsilon > 0$, and $x_0' \in \mathbb{R}^{n-1}$.

Lemma A.4. *Suppose that $u \in C^2(B_1^+ \setminus \{0\})$ is a solution of*

$$\text{(A–1)} \qquad \begin{cases} -L_g u = 0, & \text{on } B_1^+, \\ \dfrac{\partial u}{\partial \nu} = 0, & \text{on } \Gamma_1(B_1^+ \setminus \{0\}), \end{cases}$$

and $u(x) = o\left(|x|^{2-n}\right)$ as $|x| \to 0$. Then $u \in C^{2,\alpha}(B_{1/2}^+)$ for any $0 < \alpha < 1$.

Proof. We reflect across $\Gamma_1(B_1^+)$ to extend u as a solution of $-L_g u = 0$ on $B_1 \setminus \{0\}$, then use [Gilbarg and Serrin 1955/56] to conclude that 0 is a removable singularity. The result follows from standard elliptic regularity. □

Lemma A.5. *There exists some constant $\delta_0 > 0$ depending only on n, $\|g_{ij}\|_{C^2(B_1^+)}$ and $\|H\|_{L^\infty(B_1^+)}$ such that for all $0 < \delta < \delta_0$ there exists some function G satisfying*

$$\text{(A–2)} \qquad \begin{cases} -L_g G = 0, & \text{in } B_\delta^+, \\ \dfrac{\partial G}{\partial \nu} = 0, & \text{on } \Gamma_1(B_\delta^+) \setminus \{0\}, \\ \lim_{|x| \to 0} |x|^{-1} G(x) = 1 \end{cases}$$

such that, for some constant A and some $\alpha \in (0, 1)$,

$$G(x) = |x|^{-1} + A + O(|x|^\alpha) \quad \text{for all } x \in B_\delta^+.$$

Proof. Reflecting across $\Gamma_1(B_\delta^+)$, the lemma is reduced to [Li and Zhu 1997, Proposition B.1]. □

Reflecting again across $\Gamma_1(B_1^+)$, we derive from [Li and Zhu 1999, Lemma 9.3] the following result:

Lemma A.6. *Assume that $u \in C^2(B_1^+ \setminus \{0\})$ satisfies*

$$\begin{cases} -L_g u = 0 & \textit{in } B_1^+, \\ \dfrac{\partial u}{\partial \nu} = 0 & \textit{on } \Gamma_1(B_1^+) \setminus \{0\}. \end{cases}$$

Then

$$\alpha = \limsup_{r \to 0^+} \max\nolimits_{x \in \Gamma_2(B_r^+)} u(x)|x|^{n-2} < +\infty.$$

Proposition A.7. *Suppose that $u \in C^2(B_1^+ \setminus \{0\})$ satisfies*

$$\text{(A–3)} \qquad \begin{cases} -L_g u = 0 & \textit{in } B_1^+, \\ \dfrac{\partial u}{\partial \nu} = 0 & \textit{on } \Gamma_1(B_1^+) \setminus \{0\}. \end{cases}$$

Then there exists some constant $b \geq 0$ such that

$$u(x) = b\, G(x) + E(x) \quad \textit{in } B_{1/2}^+ \setminus \{0\},$$

where G is defined in Lemma A.5, and $E \in C^2(B_1^+)$ satisfies

$$\text{(A–4)} \qquad \begin{cases} -L_g E = 0 & \textit{in } B_{1/2}^+, \\ \dfrac{\partial E}{\partial \nu} = 0 & \textit{on } \Gamma_1(B_{1/2}^+). \end{cases}$$

Proof. Set

$$\text{(A–5)} \qquad b = b(u) = \sup\bigl\{\lambda \geq 0 : \lambda G \leq u \text{ in } \overline{B}_{\delta_0}^+ \setminus \{0\}\bigr\}.$$

By the previous lemma we know that $0 \leq b \leq \alpha < +\infty$. Two cases may occur.

Case 1: $b = 0$. We claim that for all $\varepsilon > 0$ there exists $r_\varepsilon \in (0, \delta_0)$ such that

$$\min\nolimits_{x \in \Gamma_2(B_r^+)} \bigl\{u(x) - \varepsilon G(x)\bigr\} \leq 0 \quad \text{for any } 0 < r < r_\varepsilon.$$

For suppose otherwise. Then there exist $\varepsilon_0 > 0$ and a sequence $r_j \to 0^+$ such that

$$\min\nolimits_{|x| = r_j} \bigl\{u(x) - \varepsilon_0 G(x)\bigr\} > 0 \quad \text{and} \quad u(x) - \varepsilon_0 G(x) > 0 \text{ on } \Gamma_2(B_{\delta_0}^+).$$

We prove that $\varepsilon_0 \leq b$, contradicting the assumptions. To do this note that

$$\begin{cases} -\Delta_g(u - \varepsilon_0 G) + \frac{1}{8} R_g(u - \varepsilon_0 G) = 0 & \text{in } B_{\delta_0}^+, \\ \dfrac{\partial}{\partial \nu}(u - \varepsilon_0 G) = 0 & \text{on } \Gamma_1(B_{\delta_0}^+) \setminus \{0\}, \end{cases}$$

and apply Theorem A.1 with

$$v = u - \varepsilon_0 G, \quad \Sigma = \Gamma_1(B_{\delta_0}^+) \setminus \Gamma_1(B_{r_j}^+), \quad \Gamma = \Gamma_2(B_{r_j}^+) \cup \Gamma_2(B_{\delta_0}^+);$$

this yields $u-\varepsilon_0 G\ge 0$ in the annulus $B^+_{\delta_0}\setminus B^+_{r_j}$ for any j, and consequently in $B^+_{\delta_0}\setminus\{0\}$. Therefore $\varepsilon_0\le b$ and we have our contradiction.

Hence, for any $\varepsilon>0$ and $0<r<r_\varepsilon$ there exists $x_\varepsilon\in\Gamma_2(B^+_r)$ such that

$$u(x_\varepsilon)\le\varepsilon G(x_\varepsilon).$$

By the Harnack inequality of Lemma 2.3 we have that

$$\max\nolimits_{|x|=r} u(x)\le c\,u(x_\varepsilon)\le c\,\varepsilon G(x_\varepsilon).$$

Since $G(x)\sim|x|^{2-n}$ for $|x|$ small, we conclude that

$$u(x)=o\big(|x|^{2-n}\big)\quad\text{for } |x|\sim 0.$$

Therefore from Lemma A.4 we obtain that u is regular. Setting $E(x)=u(x)$, the conclusion in this case follows.

Case 2. $b>0$.

We consider $v(x)=u(x)-b\,G(x)$ in $\overline{B}^+_{\delta_0}\setminus\{0\}$. By definition of b, it is clear that $v\ge 0$ in $\overline{B}^+_{\delta_0}\setminus\{0\}$. Moreover v satisfies

$$\begin{cases}-\Delta_g v+\dfrac{1}{8}\,R_g v=0, & \text{in } B^+_{\delta_0},\\ \dfrac{\partial v}{\partial\nu}=0, & \text{on } \Gamma_1(B^+_{\delta_0})\setminus\{0\},\end{cases}$$

so that from the maximum principle we know that either $v\equiv 0$ or $v>0$ in $B^+_{\delta_0}\setminus\{0\}$. If $v\equiv 0$, take $E\equiv 0$ and we are done. Otherwise $v>0$ and satisfies the same equation as u. Set

$$\tilde b=b(v)=\sup\big\{\lambda\ge 0:\lambda G\le v \text{ in } \overline{B}^+_{\delta_0}\setminus\{0\}\big\}.$$

If $\lambda\ge 0$ and $\lambda G\le v$ in $B^+_{\delta_0}\setminus\{0\}$, then $\lambda G\le u-b\,G$ with $b>0$, i.e. $(\lambda+b)\,G\le u$. By the definition of b, this implies that $\lambda+b\le b$, so $\lambda=0$. Therefore $\tilde b=0$. Arguing as in Case 1, we can prove that $v(x)=o\big(|x|^{2-n}\big)$ for $|x|\sim 0$. Lemma A.4 ensures that v is regular, so that choosing $E(x)=v(x)$ we are done.

The proof of Proposition A.7 is thereby complete. □

Corollary A.8. *Let u be a solution of* (A–3) *which is singular at* 0. *Then*

$$\lim_{r\to 0^+}\int_{\Gamma_2(B^+_r)}\frac{\partial u}{\partial\nu}\,d\sigma=b\lim_{r\to 0^+}\int_{\Gamma_2(B^+_r)}\frac{\partial G}{\partial\nu}\,d\sigma=-\frac{n-2}{2}\,b\,|\mathbb{S}^{n-1}|,$$

where $\mathbb{S}^{n-1}$ *denotes the standard n-dimensional sphere and* $b>0$ *is given by Proposition A.7.*

Proof. From the previous proposition, we know that

$$u(x)=b\,G(x)+E(x)\quad\text{in } B^+_{1/2}(0)\setminus\{0\},\quad\text{with } b\ge 0.$$

Since u is singular at 0, b must be strictly positive. From (A–4), we have

$$0 = -\int_{B_r^+} \Delta_g E\, dV - \int_{\Gamma_2(B_r^+)} \frac{\partial E}{\partial \nu}\, d\sigma + \frac{1}{8}\int_{B_r^+} R_g E.$$

Hence, since E is regular, we obtain

$$\int_{\Gamma_2(B_r^+)} \frac{\partial E}{\partial \nu}\, d\sigma = \frac{1}{8}\int_{B_r^+} R_g(x) E(x)\, d\sigma \underset{r\to 0^+}{\longrightarrow} 0$$

and so

$$\lim_{r\to 0^+} \int_{\Gamma_2(B_r^+)} \frac{\partial u}{\partial \nu}\, d\sigma = \lim_{r\to 0^+} b \int_{\Gamma_2(B_r^+)} \frac{\partial G}{\partial \nu}\, d\sigma.$$

From Lemma A.5 we know that G is of the form

$$G(x) = |x|^{-1} + \mathcal{R}(x),$$

where $\mathcal{R}$ is regular. Since

$$\int_{\Gamma_2(B_r^+)} \frac{\partial}{\partial \nu} |x|^{-1}\, d\sigma = -\frac{1}{2}\, |\mathbb{S}^{n-1}|$$

and

$$\int_{\Gamma_2(B_r^+)} \frac{\partial \mathcal{R}}{\partial \nu}\, d\sigma \underset{r\to +0^+}{\longrightarrow} 0,$$

we conclude that

$$\lim_{r\to 0^+} \int_{\Gamma_2(B_r^+)} \frac{\partial u}{\partial \nu}\, d\sigma = -\tfrac{1}{2}\, b\, |\mathbb{S}^{n-1}|$$

and the corollary follows. □

Acknowledgements

We thank the referee for a careful reading, and for useful comments and suggestions. We are grateful to Prof. A. Ambrosetti and Prof. Alice Chang for their interest in this work and for their constant support. Ahmedou thanks Prof. Reiner Schätzle for useful discussions when preparing a final version of this paper.

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Received July 26, 2001. Revised September 9, 2002.

VERONICA FELLI
UNIVERSITÀ DI MILANO BICOCCA
DIPARTIMENTO DI MATEMATICA E APPLICAZIONI
VIA COZZI 53, 20125 MILANO
ITALY

felli@matapp.unimib.it

MOHAMEDEN OULD AHMEDOU
EBERHARD-KARLS-UNIVERSITÄT TÜBINGEN
MATHEMATISCHES INSTITUT
AUF DER MORGENSTELLE , D-72076 TÜBINGEN
GERMANY

ahmedou@everest.mathematik.uni-tuebingen.de