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We give an explicit description of the poles of the Igusa local zeta function associated to a polynomial mapping g, in the case in which it is a nondegenerate homogeneous mapping of degree d. The proof uses a generalization of the p-adic stationary phase formula and Néron p-desingularization.

1. Introduction

Let K be nonarchimedean local field, O_K the valuation ring of K, P_K the maximal ideal of O_K, and $\overline{K} = O_K/P_K$ the residue field of K. The cardinality of the residue field of K is denoted by q; thus $\overline{K} = \mathbb{F}_q$. Denote the valuation of $z \in K$ by $v(z) \in \mathbb{Z} \cup \{+\infty\}$, and let $|z|_K = q^{-v(z)}$ and $\operatorname{ac} z = z\pi^{-v(z)}$, where π is a fixed uniformizing parameter for O_K. For $x = (x_1, \dots, x_l) \in K^l$, set $\|x\|_K := \max_{1 \leqslant i \leqslant l} |x_i|_K$.

Let $g_i(x) \in O_K[x_1, \dots, x_n]$ be a nonconstant polynomial for $i = 1, \dots, l$, and $g(x) = (g_1(x), \dots, g_l(x)) : K^n \to K^l$ a polynomial mapping with $l \leqslant n$. Let χ_i be a character of $O_K^\times$, i.e., a homomorphism $\chi_i : O_K^\times \to \mathbb{C}^\times$ with finite image, for $i = 1, \dots, l$. We formally put $\chi_i(0) = 0$, $i = 1, \dots, l$, $\chi := (\chi_1, \dots, \chi_l)$, and define

$$\chi(\operatorname{ac} g(x)) := \textstyle\prod_{i=1}^{l} \chi_i(\operatorname{ac} g_i(x)).$$

The Igusa local zeta function associated to χ and g is then defined as

$$Z(s, \chi, g) := \int_{O_K^n} \chi(\operatorname{ac} g(x)) \, \|g(x)\|_K^s \, |dx|, \quad s \in \mathbb{C} \text{ with } \operatorname{Re} s > 0, \tag{1–1}$$

where $|dx|$ denotes the Haar measure on K^n normalized so O_K^n has measure 1.

Local zeta functions of type (1–1) were introduced by Weil [1965] and their basic properties for general g and $l = 1$ were first studied by Igusa. Using resolution of singularities Igusa proved that $Z(s, \chi, g)$ admits a meromorphic continuation to the complex plane as a rational function of q^{-s} [Igusa 1978; 2000, Theorem 8.2.1]. His proof was generalized by Meuser [1981, Theorem 1] to the case $l \geqslant 1$. Using p-adic cell decomposition, Denef [1984] gave a completely different proof of the rationality of $Z(s, \chi_{\text{triv}}, g)$, with $l \geqslant 1$.

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We write $Z(s, g) = Z(s, \chi_{\mathrm{triv}}, g)$ when $\chi(z) = \chi_{\mathrm{triv}}(z) = (1, \dots, 1)$. Let $N_m(g)$ be the number of solutions of

$$g_i(x) \equiv 0 \mod P_K^m,\ i = 1, \dots, l,\ \text{in } (O_K/P_K^m)^n, \tag{1–2}$$

and set $P(t, g) := \sum_{m=0}^{\infty} N_m(g)(q^{-n}t)^m$, with $N_0(g) = 1$. By [Meuser 1981, Theorem 2], the Poincaré series $P(t, g)$ is related to $Z(s, g)$ by the formula

$$P(t, g) = \frac{1 - tZ(s, g)}{1 - t},\quad t = q^{-s}.$$

Thus the rationality of $Z(s, g)$ implies that of $P(t, g)$.

Igusa's local zeta functions are related to the number of solutions of congruences mod π^m and exponential sums mod π^m. Indeed, in the case $l = 1$, Igusa showed that the largest pole of $Z(s, g)$ controls the asymptotic behavior of number of solutions of (1–2), and the largest pole of $Z(s, \chi, g)$ controls the asymptotic behavior absolute value of $\sum_{x \bmod \pi^m} \Psi(\lambda g(x))$, where Ψ is an standard additive character. See [Denef 1991; Igusa 1978; 1974; 1975; 2000]. Motivated by this work, Igusa asked [1978, p. 32] how one could extend his method and results to the case of polynomial mappings. The theory of local zeta functions for the case $l > 1$ is just starting, and only some partial results have been obtained; see [Denef 1984; 2000; Lichtin 2000; Meuser 1981; Zuniga-Galindo 2004].

The first result of this paper, Theorem 2.3, is a generalization of the p-adic stationary phase formula to polynomial mappings. Igusa has used the p-adic stationary phase formula to compute the local zeta functions of several classes of prehomogeneous vectors spaces (see [Igusa 1994] and the references therein); the author has used this formula and Néron p-desingularization [Artin 1969] in the study and computation of local zeta functions associated to several classes of polynomials [Zuniga-Galindo 2001; 2003a; 2003b]. Section 2 of this paper includes applications of the generalized formula to the calculation of certain integrals are included in.

Our second main result, Theorem 3.1, provides an explicit description of the poles of $Z(s, g, \chi)$, with $1 \leqslant l \leqslant n$, in the case where g is a nondegenerate homogeneous mapping of degree d (Definition 3.1). The proof uses Theorem 2.3 and Néron p-desingularization. Still in Section 3 we prove some consequences of this description of the poles of $Z(s, g, \chi)$.

2. The p-adic Stationary Phase Formula for Mappings

In this section we give a generalization of the p-adic stationary phase formula (abbreviated SPF), and compute some integrals that will be used in the next section. The results about the SPF given here constitute a generalization of some results of [Zuniga-Galindo 2001, Section 2] and [Zuniga-Galindo 2003b, Section 2].

Some p-adic integrals. If ξ is an element of O_K, we denote by $\bar{\xi}$ its image under the canonical homomorphism $O_K \to O_K/\pi P_K = \mathbb{F}_q$, i.e., the reduction of ξ modulo π. If $g = (g_1, \dots, g_l) : K^n \to K^l$, where $l \leqslant n$, is a polynomial mapping with $g_i(x) \in O_K[x_1, \dots, x_n] \setminus P_K[x_1, \dots, x_n]$, $i = 1, \dots, l$, we denote by $\bar{g} := (\bar{g}_1, \dots, \bar{g}_l) : \mathbb{F}_q^n \to \mathbb{F}_q^l$ its reduction modulo π. We fix a lifting R of $\mathbb{F}_q^n$ in O_K^n, and if $\xi \in R$ satisfies $\xi = \bar{\xi} \bmod \pi$, we say that ξ is the lifting of $\bar{\xi}$.

Given $\xi_0 \in O_K$, we define the *dilatation* of $g_i(x)$ at ξ_0 as

$$g_{i,\xi_0}(x) := \pi^{-e_{i,\xi_0}} g_i(\xi_0 + \pi x),$$

where e_{i,ξ_0} is the minimum order of π in the coefficients of $g_i(\xi_0 + \pi x)$.

If $y = (y_1, \dots, y_M) \in K^M$ and $z = (z_1, \dots, z_N) \in K^N$, we set

$$\|(y, z)\|_K := \big\|(y_1, \dots, y_M, z_1, \dots, z_N)\big\|_K.$$

Proposition 2.1. *Let $h(x) = (h_1(x), \dots, h_N(x))$ and $g(x) = (g_1(x), \dots, g_M(x))$ be polynomial mappings with $h_j(x)$, $g_i(x) \in O_K[x_1 \dots, x_n]$ for $j = 1, \dots, N$, $i = 1, \dots, M$, and $N \geqslant 1$, $M \geqslant 1$. Let*

(2–1)

$$I(s, \chi, h, g, c_0) := \int_{O_K^n} \chi(\operatorname{ac} h(x)) \big\|(c_0, g(x))\big\|_K^s |dx| \quad \textit{for } s \in \mathbb{C} \textit{ with } \operatorname{Re} s > 0,$$

where $\chi = (\chi_1, \dots, \chi_N)$ and c_0 a nonzero element of O_K. Then $I(s, \chi, h, g, c_0)$ is a polynomial in q^{-s} with complex coefficients.

Proof. We work by induction on $\alpha(I(s, \chi, h, g, c_0)) = \alpha := \max(v(c_0), M)$. The proof of the case $\alpha = 1$ involves two subcases: (I) $v(c_0) = 0$, $M = 1$, and (II) $v(c_0) = 1$, $c_0 = \pi u$, $u \in O_K^\times$, $M = 1$. The first follows immediately; the second is proved as follows:

$$\begin{aligned} I(s, \chi, h, g, c_0) &= \int_{O_K^n} \chi(\operatorname{ac} h(x)) \big\|(\pi u, g(x))\big\|_K^s |dx| \\ &= \sum_{\xi \in R} q^{-n} \int_{O_K^n} \chi(\operatorname{ac} h(\xi + \pi x)) \big\|(\pi u, g(\xi + \pi x))\big\|_K^s |dx|, \end{aligned}$$

with

$$q^{-n} \int_{O_K^n} \chi(\operatorname{ac} h(\xi + \pi x)) \big\|(\pi u, g(\xi + \pi x))\big\|_K^s |dx| \\ = \begin{cases} q^{-n-s} \int_{O_K^n} \chi(\operatorname{ac} h(\xi + \pi x)) |dx| & \text{if } \bar{g}(\bar{\xi}) = 0, \\ q^{-n} \int_{O_K^n} \chi(\operatorname{ac} h(\xi + \pi x)) |dx| & \text{if } \bar{g}(\bar{\xi}) \neq 0. \end{cases}$$

Suppose that the result is valid for every integral of form (2–1) with $1 \leqslant \alpha \leqslant k$, $k \in \mathbb{N}$. Let $I(s, \chi, h, g, c_0)$ be an integral of form (2–1), with $\alpha = k + 1$. Set

$$V := \big\{\xi \in O_K^n \mid \bar{\xi} \in \bar{g}^{-1}(0)\big\},$$

$I=\{1,\dots,M\}$, $J\subseteq I$, and

$$V_J := \left\{\xi\in O_K^n \mid v(g_i(\xi))=0 \iff i\in J\right\}.$$

Denote by $\overline{V}$, $\overline{V}_J$ the images of V, V_J under the canonical homomorphism $O_K^n\to (O_K/\pi P_K)^n=\mathbb{F}_q^n$. With this notation O_K^n can be partitioned as

$$O_K^n = V \bigcup \bigcup_{\substack{J\subseteq I\\ J\neq\varnothing}} V_J,$$

and from this it follows that

$$\int_{O_K^n} = \int_V + \sum_{\substack{J\subseteq I\\ J\neq\varnothing}} \int_{V_J}.$$

Therefore the proof of the case $\alpha=k+1$ is reduced to showing that

$$\int_V \chi(\mathrm{ac}\, h(x)) \left\|(c_0, g(x))\right\|_K^s |dx| \in \mathbb{C}[q^{-s}], \tag{2–2}$$

and

$$\int_{V_J} \chi(\mathrm{ac}\, h(x)) \left\|(c_0, g(x))\right\|_K^s |dx| \in \mathbb{C}[q^{-s}] \quad \text{for nonempty } J\subseteq I. \tag{2–3}$$

To prove (2–2) we proceed as follows. By decomposing V into equivalence classes modulo π we obtain

(2–4)
$$\int_V \chi(\mathrm{ac}\, h(x)) \left\|(c_0, g(x))\right\|_K^s |dx|$$
$$= \sum_{\xi\in\overline{V}} q^{-n-\gamma s} \int_{O_K^n} \left(\prod_{i=1}^N \chi_i(\mathrm{ac}\, h_{i,\xi}(x))\right) \left\|\left(\pi^{-\gamma}c_0,\ (\pi^{e_{j,\xi}-\gamma} g_{i,\xi}(x))_{i\in I}\right)\right\|_K^s |dx|,$$

where ξ is the lifting of $\bar{\xi}$ and $\gamma=\min\left(e_{1,\xi},\dots,e_{M,\xi}, v(c_0)\right)$. Since $\bar{g}_i(\bar{\xi})=0$ for $i=1,\dots,M$, and $v(c_0)\geqslant 1$, it follows that $\gamma>1$. Equation (2–2) follows from (2–4) by the induction hypothesis because each integral in the right side of (2–4) has an $\alpha\leqslant k$.

The proof of (2–3) is as follows. By decomposing V_J into equivalence classes modulo π we get

$$\int_{V_J} \chi(\mathrm{ac}\, h(x)) \left\|(c_0, g(x))\right\|_K^s |dx| \tag{2–5}$$
$$= \sum_{\xi\in\overline{V_J}} q^{-n} \int_{O_K^n} \left(\prod_{i=1}^N \chi_i(\mathrm{ac}\, h_{i,\xi}(x))\right) \left\|\left(c_0,\ (\pi^{e_{j,\xi}} g_{j,\xi}(x))_{j\in I\setminus J}\right)\right\|_K^s |dx|.$$

Since $J \neq \varnothing$, it follows that the cardinality of $I \setminus J$ is less than M. Now (2–3) follows from (2–5) by the induction hypothesis because each integral in the right-hand side of (2–5) has an $\alpha \leqslant k$. $\square$

Given $c = (c_1, \ldots, c_l)$ a nonzero element of O_K^l, we set

$$I(s) = I(s, \chi, c) := \int_{O_K^l} \prod_{i=1}^{l} \chi(\mathrm{ac}\, x_i) \left\| (c_1 x_1, \ldots, c_l x_l) \right\|_K^s |dx|$$

for $s \in \mathbb{C}$ with $\operatorname{Re} s > 0$.

Proposition 2.2. *With the preceding notation,*

$$I(s, \chi, c) = \left\{ \begin{array}{ll} 0 & \text{if } \chi \neq \chi_{\mathrm{triv}}, \\ \dfrac{L(q^{-s}, \chi)}{(1 - q^{-l-s})} & \text{if } \chi = \chi_{\mathrm{triv}}, \end{array} \right.$$

where $L(q^{-s}, \chi)$ is a polynomial in q^{-s} with complex coefficients.

Proof. If $\chi \neq \chi_{\mathrm{triv}}$ there exists at least one index, say 1, such that $\chi_1 \neq \chi_{\mathrm{triv}}$. Let u be an element of $O_K^\times$ such that $\chi_1(u) \neq 1$. Since $I(s, \chi, c)(1 - \chi_1(u)) = 0$, it follows that $I(s, \chi, c) = 0$.

If $\chi = \chi_{\mathrm{triv}}$, we set $I(s, \chi_{\mathrm{triv}}, c) = I(s, c)$. For a nonempty subset $J \subseteq I = \{1, 2, \ldots, l\}$, we set

(2–6) $$W_J := \{x \in O_K^l \mid v(x_i) = 0 \iff i \in J\}$$

and

$$I_J(s, c) := \int_{W_J} \left\| (c_1 x_1, \ldots, c_l x_l) \right\|_K^s |dx| \quad \text{for } s \in \mathbb{C} \text{ with } \operatorname{Re} s > 0.$$

On the other hand, for each $l \geqslant 1$, O_K^l admits the partition

(2–7) $$O_K^l = (P_K)^l \bigcup \bigcup_{\substack{J \subseteq \{1, \ldots, l\} \\ J \neq \varnothing}} W_J,$$

where W_J is defined in (2–6). From this partition it follows that

$$I(s, c) = \int_{(P_K)^l} \|(c_1 x_1, \ldots, c_l x_l)\|_K^s |dx| + \sum_{\substack{J \subseteq \{1, \ldots, l\} \\ J \neq \varnothing}} I_J(s, c).$$

On the other hand, a direct calculation shows that

$$\int_{(P_K)^l} \|(c_1 x_1, \ldots, c_l x_l)\|_K^s |dx| = q^{-l-s} I(s, c);$$

thus in order to prove the result it is sufficient to show that $I_J(s, c)$ is a polynomial in q^{-s} for every $J \neq \varnothing$. To do so we proceed as follows. By renaming the variables, we may assume that $J = \{1, 2, \ldots, f\}$, with $f \leqslant l$. Since

$$\left\| (c_1 x_1, \ldots, c_l x_l) \right\|_K = \left\| (c_0, c_{f+1} x_{f+1}, \ldots, c_l x_l) \right\|_K \quad \text{for any } (x_1, \ldots, x_l) \in W_J,$$

where c_0 is an element of O_K satisfying $|c_0|_K = \max_{1\leqslant i\leqslant f} |c_i|_K$, we obtain

$$I_J(s,c) = (1-q^{-1})^f \int_{O_K^{l-f}} \left\| (c_0, c_{f+1}x_{f+1}, \ldots, c_l x_l) \right\|_K^s |dx|.$$

Now the result follows by applying Proposition 2.1 to $I_J(s,c)$. $\square$

The p-adic stationary phase formula. For $A \subseteq O_K^n$, denote by $Z_A(s,\chi,F)$ the integral $\int_A \chi(\mathrm{ac}\, F(x))\, \|F(x)\|_K^s\, |dx|$. Let L be an arbitrary field and consider a polynomial mapping $g : L^n \to L^l$, where $l \leqslant n$. Let

$$\mathcal{J}_g(z) := \left(\frac{\partial g_i}{\partial x_j}\right)_{\substack{1\leqslant i\leqslant l\\ 1\leqslant j\leqslant n}}(z)$$

be the Jacobian matrix of g at $z \in L^n$, and $\mathcal{C}_g(L)$ the L-critical set of g:

$$\mathcal{C}_g(L) = \left\{ z \in L^n \mid \mathrm{rank}_L \mathcal{J}_g(z) \leqslant l-1 \right\}.$$

Finally, put $\mathrm{Sing}_g(L) := \mathcal{C}_g(L) \cap g^{-1}(0)$.

The next result is a generalization of the p-adic stationary phase formula (see [Igusa 2000, Theorem 10.2.1], for example) to the current setting.

Theorem 2.3. *Let $f_i(x) \in O_K[x_1,\ldots,x_n] \setminus P_K[x_1,\ldots,x_n]$ be a nonconstant polynomial for $i = 1,\ldots,l$, with $l \leqslant n$, and set $F^*(x) = (f_1(x),\ldots,f_l(x))$. Let c_i, $i = 1,\ldots,l$, be nonzero elements of O_K such that at least one c_i belongs to $O_K^\times$, and set $F(x) = (c_1 f_1(x),\ldots,c_l f_l(x))$. Let $\bar{E}$ be a nonempty subset of $\mathbb{F}_q^n$, $\bar{S} = \mathrm{Sing}_{\bar{F}^*}(\mathbb{F}_q) \cap \bar{E}$, and E, S the preimages of $\bar{E}$ and $\bar{S}$ under $O_K^n \to (O_K/P_K)^n$. Then*

$$Z_E(s,\chi,F) = L_0(q^{-s},\chi,F) + \frac{L_1(q^{-s},\chi,F)}{(1-q^{-l-s})} + Z_S(s,\chi,F),$$

where $L_0(q^{-s},\chi,F)$ and $L_1(q^{-s},\chi,F)$ are polynomials in q^{-s} with complex coefficients and degrees independent of χ. Moreover $L_1(q^{-s},\chi) = 0$ if $\chi \neq \chi_{triv}$.

Proof. Since $E \setminus S$ and S form a partition of E,

$$Z_E(s,\chi,F) = Z_{E\setminus S}(s,\chi,F) + Z_S(s,\chi,F).$$

Thus it is sufficient to show that

$$Z_{E\setminus S}(s,\chi,F) = L_0(q^{-s},\chi,F) + \frac{L_1(q^{-s},\chi,F)}{(1-q^{-l-s})}.$$

By decomposing $E \setminus S$ into equivalence classes modulo π, we check that

$$Z_{E\setminus S}(s,\chi,F) = \sum_{\bar{\xi}\in \bar{E}\setminus\bar{S}} q^{-n} \int_{O_K^n} \chi(\mathrm{ac}\, F(\xi+\pi x))\, \left\|F(\xi+\pi x)\right\|_K^s\, |dx|.$$

We first consider the contribution of the points $\xi \in E \setminus S$ satisfying $\bar{F}^*(\bar{\xi}) \neq 0$. This condition is equivalent to the existence of a nonempty subset $T = T_\xi$ of $I = \{1, \ldots, l\}$ such that

$$\bar{f}_i(\bar{\xi}) \neq 0 \iff i \in T. \tag{2–8}$$

From (2–8) it follows that

$$\left\|(c_1 f_1(\xi + \pi x), \ldots, c_l f_l(\xi + \pi x))\right\|_K = \left\|(c_0, (c_i f_i(\xi + \pi x))_{i \in I \setminus T})\right\|_K$$

for any $x \in O_K^n$, where c_0 is an element of O_K satisfying $|c_0|_K = \max_{i \in T} |c_i|_K$.

Thus the contribution of an $\xi \in E \setminus S$ satisfying (2–8) is

$$q^{-n} \int_{O_K^n} \prod_{i=1}^{l} \chi_i(\operatorname{ac} c_i f_i(\xi + \pi x)) \left\|(c_0, (c_i f_i(\xi + \pi x))_{i \in I \setminus T})\right\|_K^s |dx|.$$

By Proposition 2.1 the preceding integral is a polynomial $L_{\bar{\xi}}(q^{-s}, \chi)$. By adding all these polynomials we obtain

$$L_0(q^{-s}, \chi, F) = \sum_{\substack{\bar{\xi} \in \bar{E} \setminus \bar{S} \\ \bar{F}^*(\bar{\xi}) \neq 0}} L_{\bar{\xi}}(q^{-s}, \chi, F).$$

Now we consider the contribution of points $\xi \in E \setminus S$ satisfying $\bar{F}^*(\bar{\xi}) = 0$. This condition implies that the Jacobian matrix of $\bar{F}^*$ at $\bar{\xi}$ has rank l.

Set

$$y_i = \phi_i(x) = \begin{cases} \dfrac{f_i(\xi + \pi x) - f_i(\xi)}{\pi} & \text{for } i = 1, \ldots, l, \\ x_i & \text{for } i = l+1, \ldots, n. \end{cases}$$

By the nonarchimedean implicit function theorem $y = \phi(x) := (\phi_1(x), \ldots, \phi_n(x))$ gives a measure-preserving mapping from O_K^n to O_K^n. By performing a change of variables from the x_i to the y_i one can check that

$$q^{-n} \int_{O_K^n} \chi(\operatorname{ac} F(\xi + \pi x)) \left\|F(\xi + \pi x)\right\|_K^s |dx| \\ = q^{-n-s} \int_{O_K^l} \prod_{i=1}^{l} \chi(\operatorname{ac} c_i y_i) \left\|(c_i y_i)_{1 \leqslant i \leqslant l}\right\|_K^s |dy|.$$

By Proposition 2.2, the integral in this equation is a rational function of the form

$$\frac{R_{\bar{\xi}}(q^{-s})}{(1 - q^{-l-s})},$$

and the polynomial $R_{\bar{\xi}}(q^{-s})$ vanishes if $\chi \neq \chi_{\text{triv}}$. By adding all these polynomials we get

$$L_1(q^{-s}, \chi) = \sum_{\substack{\bar{\xi} \in \bar{E} \setminus \bar{S} \\ \bar{F}^*(\bar{\xi}) = 0}} R_{\bar{\xi}}(q^{-s}, \chi). \qquad \square$$

Néron p-desingularization. We now use techniques of Néron p-desingularization (see [Artin 1969, Sect. 4], for example) together with the SPF to compute certain p-adic integrals. The results presented here are generalizations of results in [Zuniga-Galindo 2001, Section 2] and [Zuniga-Galindo 2003b, Section 2].

Let $g_i(x) \in O_K[x_1, \dots, x_n] \setminus P_K[x_1, \dots, x_n]$ be a nonconstant polynomial for $i = 1, \dots, l$ and let $g(x) = (g_1(x), \dots, g_l(x)) : K^n \to K^l$ be a polynomial mapping with $l \leqslant n$. We denote by $S(g)$ the subset of R mapped bijectively to the set $\mathrm{Sing}_{\bar{g}}(\mathbb{F}_q)$ by the canonical homomorphism $O_K^n \to (O_K/P_K)^n$. For simplicity we say that $S(g)$ is the lifting of $\mathrm{Sing}_{\bar{g}}(\mathbb{F}_q)$.

Given $\xi_0 \in S(g)$, we define the *dilatation* of $g_i(x)$ at ξ_0 as

$$g_{i,\xi_0}(x) := \pi^{-e_{i,\xi_0}} g_i(\xi_0 + \pi x), \tag{2–9}$$

where e_{i,ξ_0} is the minimum order of π in the coefficients of $g_i(\xi_0 + \pi x)$. We say that $S(g_{i,\xi_0})$, the lifting of $\mathrm{Sing}_{\bar{g}_{i,\xi_0}}(\mathbb{F}_q)$, is *the first generation of descendants of* ξ_0. Given a sequence of points $(\xi_k)_{k\in\mathbb{N}}$ in O_K^n with $\xi_0 \in S(g)$, we define inductively $e_{i,\xi_0,\dots,\xi_k}$, $g_{i,\xi_0,\dots,\xi_k}(x)$, and $S(g_{i,\xi_0,\dots,\xi_k})$ by setting

$$g_{i,\xi_0,\dots,\xi_k}(x) := \pi^{-e_{i,\xi_0,\dots,\xi_k}} g_{i,\xi_0,\dots,\xi_{k-1}}(\xi_k + \pi x) \quad \text{for } k \geqslant 1, \tag{2–10}$$

where ξ_k lies in $S(g_{i,\xi_0,\dots,\xi_{k-1}})$ and $e_{i,\xi_0,\dots,\xi_k}$ is the minimum order of π in the coefficients of $g_{i,\xi_0,\dots,\xi_{k-1}}(\xi_k + \pi x)$. The $(k+1)$*-st generation of descendants of* ξ_0 is by definition the set

$$S(g_{i,\xi_0,\dots,\xi_k}) = \bigcup_{\xi_k \in S(f_{\xi_0,\dots,\xi_{k-1}})} S(g_{i,\xi_0,\dots,\xi_{k-1},\xi_k}).$$

These definitions extend to polynomial mappings as follows. Given a sequence of points $(\xi_k)_{k\in\mathbb{N}}$ in O_K^n, with $\xi_0 \in S(g)$, we define inductively $g_{\xi_0,\dots,\xi_k}(x)$ and $S(g_{\xi_0,\dots,\xi_k})$ by setting

$$g_{\xi_0,\dots,\xi_k}(x) := \big(g_{1,\xi_0,\dots,\xi_k}(x), \dots, g_{l,\xi_0,\dots,\xi_k}(x)\big) \quad \text{for } k \geqslant 0, \tag{2–11}$$

where $\xi_k \in S(g_{\xi_0,\dots,\xi_{k-1}})$ and $g_{i,\xi_0,\dots,\xi_k}(x)$ is the dilatation of $g_{i,\xi_0,\dots,\xi_{k-1}}(x)$ at ξ_k. The set

$$S(g_{i,\xi_0,\dots,\xi_k}) := \bigcup_{\xi_k \in S(g_{\xi_0,\dots,\xi_{k-1}})} S(g_{\xi_1,\dots,\xi_{k-1},\xi_k})$$

is called *the* $(k+1)$*-th generation of descendants* of ξ_0.

Remark 2.1. With the hypothesis of Theorem 2.3, we have

$$S = \bigcup_{\bar{\xi} \in \overline{S(F^*)} \cap \bar{E}} (\xi + \pi O_K^n),$$

and then

$$Z_S(s,\chi,F) = \sum_{\bar{\xi} \in \overline{S(F^*)} \cap \bar{E}} q^{-n-\alpha_\xi} \int_{O_K^n} \prod_{i=1}^{l} \chi_i(\mathrm{ac}\, f_{i,\xi}(x)) \left\| \left(\pi^{e_{i,\xi}-\alpha_\xi} f_{i,\xi}(x)\right)_{1 \leqslant i \leqslant l} \right\|_K^s |dx|,$$

where $\alpha_\xi := \min_i\{e_{i,\xi}\}$. Therefore the stationary phase formula can be rewritten as

$$Z_E(s,\chi,F) = L_0(q^{-s},\chi,F) + \frac{L_1(q^{-s},\chi,F)}{(1-q^{-l-s})} + \sum_{\bar{\xi}\in\overline{S(F^*)}\cap\bar{E}} q^{-n-\alpha_\xi} \int_{O_K^n} \prod_{i=1}^{l} \chi_i(\mathrm{ac}\, f_{i,\xi}(x)) \, \|(\pi^{e_{i,\xi}-\alpha_\xi} f_{i,\xi}(x))_{1\leqslant i\leqslant l}\|_K^s \, |dx|.$$

Let $g_i \in O_K[x_1,\ldots,x_n]\setminus P_K[x_1,\ldots,x_n]$, for $i=1,\ldots,l\leqslant n$, be nonconstant polynomials. Set $g(x) = \big(g_1(x),\ldots,g_l(x)\big) : K^n \to K^l$, and let ξ be a point of O_K^n such that $\xi \notin \mathrm{Sing}_g(K)$. Define

$$\mathcal{L}(g,\xi) = \min_M\{v(M(\xi))\},$$

where M runs over the minors of the Jacobian matrix $\mathcal{J}_g(\xi)$ of g at ξ.

Proposition 2.4. *With the notation just defined*:

(1) *If* $\xi \in O_K^n$ *and* $\xi \in \bar{g}^{-1}(0)$, *then*

$$\mathcal{L}(g,\xi) = 0 \iff \xi \notin \mathrm{Sing}_{\bar{g}}(\mathbb{F}_q).$$

(2) *If* $A \subseteq O_K^n$ *is an open and compact subset such that* $A \cap \mathrm{Sing}_g(K) = \varnothing$, *there exists a constant* $c(g,A) \in \mathbb{N}$ *such that*

$$\sup_{\xi\in A} \mathcal{L}(g,\xi) \leqslant c(g,A).$$

Proof. Part (1) follows from the Jacobian criterion. Part (2) is proved by contradiction. If $\mathcal{L}(g,\xi)$ is not bounded on A, there exists a sequence $(\xi_i)_{i\in\mathbb{N}}$ of points of A such that $\mathcal{L}(g,\xi_i) \to \infty$ when $i \to \infty$. Since A is compact the sequence $(\xi_i)_{i\in\mathbb{N}}$ has a limit point ξ^* in A. Thus, since $\mathrm{Sing}_g(K)$ is closed, ξ^* lies in $A \cap C_g(K)$; that is, $A \cap \mathrm{Sing}_g(K) \neq \varnothing$, contradicting $A \cap \mathrm{Sing}_g(K) = \varnothing$. □

The index $\mathcal{L}(g,\xi)$ appears naturally associated to SPF, as already noted in [Zuniga-Galindo 2001]. It plays a central role in the construction of the Néron p-desingularization of the special fiber of smooth schemes over $\mathrm{Spec}\,(O_K)$; see, for example, [Artin 1969, Section 4].

Lemma 2.5. *Let* $g_i \in O_K[x_1,\ldots,x_n]\setminus P_K[x_1,\ldots,x_n]$, *for* $i=1,\ldots,l\leqslant n$, *be nonconstant polynomials. Set* $g(x) = \big(g_1(x),\ldots,g_l(x)\big) : K^n \to K^l$. *Let* $A \subseteq O_K^n$ *be an open and compact subset such that* $A \cap \mathrm{Sing}_g(K) = \varnothing$. *Take* $\xi_0 \in A$.

(1) *Suppose that* ξ_k, *for some* $k \geqslant 0$, *satisfies*

$$\xi_k \in \begin{cases} S(g) & \text{if } k=0, \\ S(g_{\xi_0,\ldots,\xi_{k-1}}) & \text{if } k\geqslant 1, \end{cases} \tag{2–12}$$

where $\xi_1,\ldots,\xi_{k-1}$ *are descendants of* ξ_0, *and that* ξ_k *has at least one descendant in* $S(g_{\xi_0,\ldots,\xi_k})$. *Then*

$$\mathcal{L}(g_{\xi_0,\ldots,\xi_k},0) \leqslant \mathcal{L}(g,\xi_0+\xi_1\pi+\cdots+\xi_k\pi^k) - (k+1).$$

(2) *For any* $\xi_0 \in A \cap S(g)$, *if* $k \geqslant c(g, A) + 1$, *then* $S(g_{i,\xi_0,\dots,\xi_k}) = \varnothing$.

Proof. (1) For each $j = 0, 1, \dots, k$, we take $\xi_{j+1} \in S(g_{i,\xi_0,\dots,\xi_j})$, $e_{i,\xi_0,\dots,\xi_j}$ as in (2–9), (2–10), and define

$$E_{i,k} := \sum_{j=0}^{k} e_{i,\xi_0,\dots,\xi_j}.$$

Thus $E_{i,k} \geqslant k+1$ for every i. With the above notation one checks that

$$g_i(\xi_0 + \xi_1\pi + \dots + \xi_k\pi^k + \pi^{k+1}x) = \pi^{E_{i,k}} g_{i,\xi_1,\dots,\xi_{k-1},\xi_k}(x).$$

It follows that

$$\mathscr{L}(g, \xi_0 + \xi_1\pi + \dots + \xi_k\pi^k) \geqslant \sum_{i=1}^{l}(E_{i,k} - k - 1) + \mathscr{L}(g_{\xi_0,\dots,\xi_{k-1},\xi_k}, 0).$$

Since $(E_{i,k} - k - 1) \geqslant 0$ for every i, this inequality implies

$$\mathscr{L}(g, \xi_0 + \xi_1\pi + \dots + \xi_k\pi^k) \geqslant E_{1,k} - k - 1 + \mathscr{L}(g_{\xi_0,\dots,\xi_{k-1},\xi_k}, 0).$$

The result follows from this inequality by showing that $E_{1,k} \geqslant 2(k+1)$. This last assertion follows from

$$e_{1,\xi_0,\dots,\xi_j} \geqslant 2 \quad \text{for } j = 0, 1, \dots, k,$$

which we prove as follows. Let $\xi_j \in O_K^n$, for $j = 0, 1, \dots, k$, represent a point satisfying the conditions of part (1) of the lemma. We write

$$g_{i,\xi_1,\dots,\xi_{j-1}}(\xi_j + \pi x) = g_{i,\xi_1,\dots,\xi_{j-1}}(\xi_j) + \pi \sum_{j=1}^{n} \frac{\partial g_{i,\xi_1,\dots,\xi_{j-1}}}{\partial x_j}(\xi_j) x_j + \pi^2 (\text{terms of degree} \geqslant 2)$$

for $i = 1, \dots, l$. From (2–12) it follows that

$$v(g_{i,\xi_1,\dots,\xi_{j-1}}(\xi_j)) \geqslant 1,$$

and there follows also the existence of a minor $M(\xi_j)$ of the Jacobian matrix of $g_{i,\xi_1,\dots,\xi_{j-1}}$ at ξ_j such that

$$\mathscr{L}(g_{\xi_0,\dots,\xi_{j-1}}, \xi_j) = v(M(\xi_j)) \geqslant 1.$$

Without loss of generality we may suppose that

$$M(\xi_j) = \det\Big(\frac{\partial g_i}{\partial x_j}\Big)_{\substack{1 \leqslant i \leqslant l \\ 1 \leqslant j \leqslant l}}(\xi_j).$$

Thus the rank of $\overline{M}(\bar{\xi}_j)$ over $\mathbb{F}_q$ is less than l. By taking an invertible linear transformation $\Omega : O_K^n \to O_K^n$, with entries in O_K, we may assume that, say,

$$\frac{\partial g_{1,\xi_1,\dots,\xi_{j-1}}}{\partial x_j}(\xi_j) = \pi a_{1,j} \quad \text{for all } j.$$

Therefore

$$g_{1,\xi_1,\dots,\xi_{j-1}}(\xi_j+\pi x)=\pi a_{1,0}+\pi^2\sum_j a_{1,j}x_j+\pi^2(\text{terms of degree}\geqslant 2). \tag{2–13}$$

At the same time, the lemma's condition that ξ_k has at least one descendant in $S(g_{\xi_0,\dots,\xi_k})$ implies that the congruence

$$g_{1,\xi_1,\dots,\xi_{j-1}}(\xi_j+\pi x)\equiv 0 \bmod \pi^2$$

has a solution in R. This fact and (2–13) imply that $a_{1,0}\equiv 0 \bmod \pi$; therefore

$$g_{1,\xi_1,\dots,\xi_{j-1}}(\xi_j+\pi x)=\pi^{e_{1,\xi_0,\dots,\xi_j}}g_{1,\xi_1,\dots,\xi_j}(x),$$

with $e_{1,\xi_0,\dots,\xi_j}\geqslant 2$ for $j=0,1,\dots,k$.

Part (2) of the lemma follows immediately from Part (1). □

Lemma 2.6. *Let $E\subseteq O_K^n$ be the preimage under the canonical homomorphism $O_K^n\longrightarrow(O_K/\pi O_K)^n$ of a subset $\overline{E}\subseteq\mathbb{F}_q^n$. Let $g(x)=\big(g_1(x),\dots,g_l(x)\big):K^n\to K^l$ be a polynomial mapping such that $g_i(x)\in O_K[x_1,\dots,x_n]\setminus P_K[x_1,\dots,x_n]$ for $i=1,\dots,l$, $l\leqslant n$, and suppose $\operatorname{Sing}_g(K)\cap E=\varnothing$. Then*

$$Z_E(s,\chi,g)=\frac{L(q^{-s},\chi,g)}{1-q^{-l-s}},$$

where $L(q^{-s},\chi,g)$ is a polynomial in q^{-s} with complex coefficients.

Proof. We define inductively I_k as follows:

$$I_1=S(g)\cap E,$$
$$I_k=\big\{(\xi_1,\dots,\xi_k)\mid(\xi_1,\dots,\xi_{k-1})\in I_{k-1}\text{ and }\xi_k\in S(f_{\xi_1,\dots,\xi_{k-1}})\big\}\quad\text{for }k\geqslant 2.$$

We set $E(\xi_1,\dots,\xi_k):=\alpha_{\xi_1}+\alpha_{\xi_1,\xi_2}+\dots+\alpha_{\xi_1,\xi_2,\dots,\xi_k}$. If $m=c(g,E)+1$, then $I_{m+1}=\varnothing$, because part (2) of Lemma 2.5 implies that $S(f_{\xi_1,\xi_2,\dots,\xi_m})=\varnothing$ for every $(\xi_1,\xi_2,\dots,\xi_m)\in I_m$. The result follows by applying the stationary phase formula $m+1$-times:

$$\begin{aligned}Z_E(s,\chi,g)&=L_0(E,\bar g,\chi)+\frac{L_1(E,\bar g,\chi)}{(1-q^{-l-s})}\\&\quad+\sum_{k=1}^m q^{-kn}\Bigg(\sum_{(\xi_1,\dots,\xi_k)\in I_k}L_0(\bar g_{\xi_1,\dots,\xi_k},O_K^n,\chi)q^{-E(\xi_1,\dots,\xi_k)s}\Bigg)\\&\quad+\frac{1}{(1-q^{-l-s})}\sum_{k=1}^m q^{-kn}\Bigg(\sum_{(\xi_1,\dots,\xi_k)\in I_k}L_1(\bar g_{\xi_1,\dots,\xi_k},O_K^n,\chi)q^{-E(\xi_1,\dots,\xi_k)s}\Bigg).\end{aligned}$$

□

3. Local zeta functions of homogeneous nondegenerate mappings

In this section we state and prove the second main result of this paper (Theorem 3.1) and some of its consequences.

Definition 3.1. A polynomial mapping $g(x) = \big(g_1(x), \ldots, g_l(x)\big) : K^n \to K^l$ is called a nondegenerate homogeneous mapping of degree d if:

(1) $g_i(x) \in O_K[x_1, \ldots, x_n]$ is a nonconstant homogeneous polynomial of degree d, for $i = 1, \ldots, l$;
(2) $l \leqslant n$;
(3) $\mathrm{Sing}_g(K) \bigcap (K^\times)^n = \varnothing$.

Theorem 3.1. *Let $g(x) = (g_1(x), \ldots, g_l(x)) : K^n \to K^l$ be a nondegenerate homogeneous mapping of degree d. Then*

$$Z(s, \chi, g) = \frac{L(q^{-s}, \chi, g)}{(1 - q^{-n-ds})(1 - q^{-l-s})},$$

where $L(q^{-s}, \chi, g)$ is a polynomial in q^{-s} with complex coefficients.

Proof. By partitioning O_K^n as $(P_K)^n \cup W$, with

$$W := \big\{\xi \in O_K^n \mid v(\xi_i) = 0, \text{ for some } i\big\},$$

we check that

$$\begin{aligned} Z(s, \chi, g) &= Z_{(P_K)^n}(s, \chi, g) + Z_W(s, \chi, g) \\ &= q^{-n-ds} Z(s, \chi, g) + Z_W(s, \chi, g). \end{aligned}$$

The result follows by applying Lemma 2.6 to $Z_W(s, \chi, g)$. □

Corollary 3.2. *For $i = 1, \ldots, l$, let $g_i(x) \in O_K[x_1, \ldots, x_n]$ be a nonconstant homogeneous polynomial of degree d, and assume $g(x) = \big(g_1(x), \ldots, g_l(x)\big) : K^n \to K^l$ is a nondegenerate homogeneous mapping of degree d. If*

$$N_m(g) = \mathrm{Card}\big(\{\xi \in (O_K/P_K^m)^n \mid g_i(\xi) \equiv 0 \; mod \; \pi^m, i = 1, \ldots, l\}\big),$$

then

$$\lim\sup_{m\to\infty} \big(N_m(g_1, \ldots, g_l)\big)^{1/m} \leqslant q^{n-\min(l,n/d)}.$$

In particular $N_m(g_1, \ldots, g_l) \leqslant q^{m(n-\min(l,n/d))}$ for m big enough.

Proof. The result follows by estimating the radius of convergence

$$r = \frac{1}{\lim\sup_{m\to\infty}\big(N_m(g)q^{-n}\big)^{1/m}}$$

of the Poincaré series $P(t, g) = \sum_{m=0}^{\infty} N_m(g)(q^{-n}t)$. Since

$$P(t, g) = \frac{1 - tZ(s, g)}{1 - t},$$

with $t = q^{-s}$ (see [Meuser 1981, Theorem 2]), Theorem 3.1 implies that

$$r \geqslant q^{(n-\min(l,n/d))}.$$

The statement in the corollary is a simple reformulation of this equation. □

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