Pacific
Journal of
Mathematics

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Volume 218 No. 2 February 2005

PACIFIC JOURNAL OF MATHEMATICS
Vol. 218, No. 2, 2005

MODULAR DIOPHANTINE INEQUALITIES AND NUMERICAL SEMIGROUPS

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We study the set of integer solutions to the modular diophantine inequality $ax \bmod b \leq x$.

Introduction

Given $x \in \mathbb{Q}$, we set $\lceil x \rceil = \min\{z \in \mathbb{Z} \mid z \geq x\}$ and $\lfloor x \rfloor = \max\{z \in \mathbb{Z} \mid z \leq x\}$, as usual. Given integers m, n with $n > 0$, we set $m \bmod n = m - n\lfloor m/n \rfloor$ and $m \bmod (-n) = m \bmod n$. A *modular diophantine inequality* is an expression of the form $ax \bmod b \leq x$ with a, b integers such that $b \neq 0$. Since $ax \bmod b \geq 0$, the set S of solutions to such an inequality is contained in the set $\mathbb{N}$ of nonnegative integers. S is a *numerical semigroup*, that is, S is closed under addition, $0 \in S$ and $\mathbb{N} \setminus S$ is finite. Not every numerical semigroup arises from a modular diophantine inequality, and Section 2 presents a procedure for testing numerical semigroups for this property. Theorem 12 is crucial for obtaining this algorithm, and thus Section 1 is devoted to it. One of the main consequences of this theorem is that if the inequalities $ax \bmod b \leq x$ and $cx \bmod d \leq x$ have the same solutions, then

$$b - (a, b) - (a-1, b) = d - (c, d) - (c-1, d),$$

where (x, y) denotes the greatest common divisor of the integers x and y.

A numerical semigroup S is said to be *modular with modulus b and factor a* if $S = \{x \in \mathbb{N} \mid ax \bmod b \leq x\}$. The preceding remark ensures that $b-(a, b)-(a-1, b)$ is an invariant of S, which we call the *weight* of S and denote by $\mathrm{w}(S)$.

If S is a numerical semigroup, the largest integer not in S is called the *Frobenius number* of S and is denoted by $\mathrm{g}(S)$. This integer has been widely studied; see for instance [Brauer 1942; Brauer and Shockley 1962; Johnson 1960; Selmer 1977; Sylvester 1884; Curtis 1990; Davison 1994; Djawadi and Hofmeister 1996]. In this direction it is worth highlighting [Ramírez Alfonsín 2000; $\geq$ 2005], where a review of this problem is given, with many references. In the literature one can also find a large number of publications devoted to the study of one-dimensional analytically

Supported by the project BFM2000-1469.
MSC2000: 20M05, 20M14, 20M30, 13H10.

irreducible local domains via their value semigroups, which are numerical semigroups; see, for instance, [Apéry 1946; Barucci et al. 1997; Bertin and Carbonne 1977; Delorme 1976; Fröberg et al. 1987; Kunz 1970; Teissier 1973; Watanabe 1973]. As a consequence of this study, some interesting kinds of numerical semigroups arise, such as symmetric and pseudo-symmetric numerical semigroups. In Section 1 we prove that a modular numerical semigroup S is symmetric if and only if $\mathrm{w}(S) = \mathrm{g}(S)$, and pseudo-symmetric if and only if $\mathrm{g}(S) = \mathrm{w}(S) + 1$. Sections 3 and 4 are devoted to modular numerical semigroups with modulus equal to their weight plus two and three, respectively. We show that those of weight plus two are obtained from a symmetric numerical semigroup by adjoining its Frobenius number to it, and that those with weight plus three arise from a pseudo-symmetric numerical semigroup by adding to it its Frobenius number and this number divided by two.

In Section 5 we study those modular numerical semigroups S such that the factor of S divides the modulus. For these numerical semigroups we can explicitly give formulas for the multiplicity, the minimal generator set, the Apéry set and the Frobenius number, so the case $a \mid b$ is now well understood.

Section 6 addresses the problem of computing the Frobenius number in the complementary case $a \nmid b$, solving it when $(a-1)(a - (b \bmod a)) < b$. We have not been able to solve the general case.

1. Modular numerical semigroups

Let a and b be integers such that $b \neq 0$. Since $ax \bmod b = (a \bmod b)x \bmod b$ and $ax \bmod b = ax \bmod (-b)$, in order to study the solutions of $ax \bmod b \leq x$, we can assume that b is a positive integer and that $0 \leq a < b$.

Proposition 1. *The set of integer solutions of a modular diophantine inequality is a numerical semigroup.*

Proof. Let a and b be two integers such that $0 \leq a < b$ and let $S = \{x \in \mathbb{N} \mid ax \bmod b \leq x\}$. Clearly $0 \in S$, and if x is an integer greater than or equal to b, then $x \in S$. Hence $\mathbb{N} \setminus S$ is finite. For $x, y \in S$, we have $a(x + y) \bmod b \leq ax \bmod b + ay \bmod b \leq x + y$, whence $x + y \in S$, so S is closed under addition. □

A numerical semigroup S arising as in the proposition is said to be *modular*. The modular semigroup with modulus b factor a will be denoted by $\mathrm{S}(a, b)$; thus $\mathrm{S}(a, b) = \{x \in \mathbb{N} \mid ax \bmod b \leq x\}$. When we write $\mathrm{S}(a, b)$ we will generally assume tacitly that a and b are integers with $0 \leq a < b$.

Example 2. $\mathrm{S}(2, 3) = \mathrm{S}(2, 4) = \{0, 2, 3, \rightarrow\}$, where $\rightarrow$ means that all the elements beyond 3 are in the set. Thus a and b don't have to be unique.

The goal of this section is to prove Theorem 12, which counts the natural numbers *absent* from $\mathrm{S}(a, b)$. We prepare the ground with some simple results.

Lemma 3. *Let a and b be integers such that $0 \le a < b$. Then $ax \bmod b \le x$ if and only if $(b+1-a)x \bmod b \le x$.*

Proof. If $ax \bmod b \le x$, there exist $q, r \in \mathbb{N}$ such that $ax = qb + r$ with $0 \le r \le x$. Hence $(b+1-a)x = (b+1)x - ax = bx - qb + x - r$ and $(b+1-a)x \bmod b \le x - r \le x$. The converse follows by interchanging a with $b+1-a$. □

Lemma 4. *Let S be a modular numerical semigroup with modulus $b \ge 2$. Then there exists a positive integer a such that $a \le \frac{1}{2}(b+1)$ and $S = \mathrm{S}(a, b)$.*

Proof. Write $S = \mathrm{S}(a, b)$ with $0 \le a < b$. By Lemma 3, $S = \mathrm{S}(b+1-a, b)$, so if $a > \frac{1}{2}(b+1)$ we can replace a by $b+1-a \le \frac{1}{2}(b+1)$. Also if $a = 0$ we can replace it by $a = 1$, since $S = \mathbb{N}$ for both these values of a. □

Lemma 5. *Let a and b be integers such that $0 \le a < b$ and let $x \in \mathbb{N}$. Then*

$$a(b-x) \bmod b = \begin{cases} 0 & \textit{if } ax \bmod b = 0, \\ b - (ax \bmod b) & \textit{if } ax \bmod b \ne 0, \end{cases}$$

and $ax \bmod b > x$ implies that $a(b-x) \bmod b < b - x$. □

Corollary 6. *If $S = \mathrm{S}(a, b)$ and $x \in \mathbb{N} \setminus S$, then $b - x \in S$.* □

Given a subset A of $\mathbb{N}$, we denote by $\mathrm{H}(A)$ the complement $\mathbb{N} \setminus A$, and by $\langle A \rangle$ the submonoid of $\mathbb{N}$ generated by A (the set of finite sums of elements of A).

Remark 7. If $S = \mathrm{S}(a, b) \ne \mathbb{N}$ for positive a and b, then $b-1 \notin \mathrm{H}(S)$, since otherwise $b - (b-1) = 1$ would be an element of S. Moreover $x \in S$ for all integers $x \ge b$. Therefore the Frobenius number $\mathrm{g}(S)$ is at most $b - 2$.

We now characterize the case $\mathrm{g}(S) = b-2$. If $\mathrm{g}(S) = b-2$, Corollary 6 implies that $b - (b-2) = 2 \in S$. Hence b is odd and $S = \langle 2, b \rangle$. In addition, since $2 \in S$, $2a \bmod b \le 2$ and this leads to $2a > b$, whence $a > \frac{1}{2}b$. But Lemma 4 says we can take $a \le \frac{1}{2}(b+1)$, which then means $a = \frac{1}{2}(b+1)$. Hence $S = \mathrm{S}(\frac{1}{2}(b+1), b)$. Conversely, if $S = \mathrm{S}(\frac{1}{2}(b+1), b)$ with b odd, it is easy to check that $S = \langle 2, b \rangle$ and thus $\mathrm{g}(S) = b - 2$.

Example 8. Suppose $b \ge 2$ and $S = \mathrm{S}(2, b)$. Then $S = \left\{0, \left\lfloor \frac{1}{2}(b+1) \right\rfloor, \rightarrow\right\}$. For clearly $\{b, \rightarrow\} \subseteq S$. Now take $0 < x < b$. Then $x \in S$ if and only if $2x \bmod b \le x$. However, $2x \bmod b = 2x$ if and only if $2x < b$, and thus in this case $x \notin S$. If $2x \ge b$, then $2x \bmod b = 2x - b \le x$, whence $x \in S$.

Lemma 9. *Let $S = \mathrm{S}(a, b)$ and let x be an integer such that $0 \le x \le b$. Then x and $b - x$ are both in S if and only if $ax \bmod b \in \{0, x\}$.*

Proof. If $ax \bmod b \notin \{0, x\}$, Lemma 5 gives $a(b-x) \bmod b = b - (ax \bmod b)$. If $x \in S$, the right-hand side exceeds $b - x$ (since $ax \bmod b < x$). Thus $b - x \notin S$.

Conversely, if $ax \bmod b = 0$, clearly $x \in S$ and also $b - x \in S$ by Lemma 5; whereas if $ax \bmod b = x \neq 0$, again $x \in S$, and Lemma 5 gives $a(b-x) \bmod b = b - (ax \bmod b) = b - x$, so $b - x \in S$. □

Lemma 10. *Let a and b be positive integers and x an integer such that $0 \leq x < b$.*

(1) $ax \bmod b = 0$ *if and only if x is a multiple of $b/(a, b)$.*

(2) $ax \bmod b = x$ *if and only if x is a multiple of $b/(b, a-1)$.* □

Lemma 11. *Let $S = S(a, b)$ with $0 < a < b$. Let $\alpha = (b, a-1)$ and $\beta = (b, a)$, and let x be an integer such that $0 \leq x \leq b$. Then*

$$\{x, b-x\} \subset S \iff x \in \left\{0, \frac{b}{\alpha}, 2\frac{b}{\alpha}, \dots (\alpha-1)\frac{b}{\alpha}, \frac{b}{\beta}, 2\frac{b}{\beta}, \dots, (\beta-1)\frac{b}{\beta}, b\right\} =: X.$$

The cardinality of X is $\alpha + \beta$.

Proof. The equivalence is just Lemmas 9 and 10 put together. To show there is no duplication in the elements of X as written, note that $(\alpha, \beta) = 1$. If $sb/\alpha = tb/\beta$ for some $s, t \in \mathbb{N}$, then $s\beta = t\alpha = k\alpha\beta$ for some $k \in \mathbb{N}$. Hence $s = k\alpha$ and $t = k\beta$. □

Theorem 12. *Let $S = \mathrm{S}(a, b)$ for some integers $0 \leq a < b$. Then*

$$\#\mathrm{H}(S) = \frac{b + 1 - (a, b) - (a-1, b)}{2}.$$

Here as usual # denotes cardinality.

Proof. Let α, β and X be as in Lemma 11. By Corollary 6 and Lemma 11, for $0 \leq x \leq b$, at most one of x, $b-x$ lies in $\mathrm{H}(S)$, and it's exactly one unless $x \in X$. Thus $\#\mathrm{H}(S) = \frac{1}{2}(b + 1 - \# X) = \frac{1}{2}(b + 1 - \alpha - \beta)$. □

Example 13. If p is an odd prime, $\#\mathrm{H}(\mathrm{S}(a, p)) = \frac{1}{2}(p-1)$ for all a with $1 < a < p$.

As an immediate consequence of Theorem 12 we obtain:

Corollary 14. *Suppose* $\mathrm{S}(a, b) = \mathrm{S}(c, d)$. *Then*

$$b - (a, b) - (a-1, b) = d - (c, d) - (c-1, d).$$

Example 15. The converse of Corollary 14 is false. For instance, $\langle 4, 5, 6\rangle = \mathrm{S}(3, 12) \neq \mathrm{S}(2, 10) = \langle 5, 6, 7, 8, 9\rangle$.

Recall that we have defined the *weight* of $S = \mathrm{S}(a, b)$ as $\mathrm{w}(S) := b - (a, b) - (a-1, b)$; by Theorem 12, this number equals $2\#\mathrm{H}(S)-1$, and so is an invariant of S. Note that $\mathrm{w}(\mathbb{N}) = -1$. If $S \neq \mathbb{N}$, we can choose a, b with $2 \leq a < b$; hence $(a, b)+(a-1, b) \leq \frac{1}{2}b+\frac{1}{3}b < b$, so $\mathrm{w}(S) \geq 1$. Thus, like the Frobenius number, the weight of a modular numerical semigroup is at least 1, except for the case $S=\mathbb{N}$, where $\mathrm{w}(S)=\mathrm{g}(S)=-1$.

Theorem 12 and the inequality $\#\,H(S)\geq \frac{1}{2}(\mathrm{g}(S)+1)$, valid for *any* numerical semigroup S (see [Fröberg et al. 1987], for instance), yield:

Corollary 16. *If S is a modular numerical semigroup, then* $\mathrm{w}(S)$ *is odd and greater than or equal to* $\mathrm{g}(S)$. □

In view of this, modular numerical semigroups S with $\mathrm{w}(S)=\mathrm{g}(S)$ and $\mathrm{g}(S)$ odd, or with $\mathrm{w}(S)=\mathrm{g}(S)+1$ and $\mathrm{g}(S)$ even, have minimal possible weight with respect to their Frobenius numbers. The next result characterizes this kind of numerical semigroup, but before proving it we need to recall some concepts.

A numerical semigroup S is *symmetric* if $x\in\mathbb{N}\setminus S$ implies $\mathrm{g}(S)-x\in S$. It is straightforward to prove that a symmetric numerical semigroup has odd Frobenius number. A numerical semigroup is *pseudo-symmetric* if $\mathrm{g}(S)$ is even and $x\in\mathbb{N}\setminus S$ implies that either $x=\mathrm{g}(S)/2$ or $\mathrm{g}(S)-x\in S$. A numerical semigroup S is symmetric if and only if $\#\,\mathrm{H}(S)=\frac{1}{2}(\mathrm{g}(S)+1)$, and pseudo-symmetric if and only if $\#\,\mathrm{H}(S)=\frac{1}{2}(\mathrm{g}(S)+2))$; see [Fröberg et al. 1987], for instance.

A numerical semigroup is *irreducible* if it cannot be expressed as the intersection of two numerical semigroups containing it properly. In [Rosales and Branco 2003] it is shown that S is irreducible if and only if S is symmetric or pseudo-symmetric (depending on the parity of $\mathrm{g}(S)$).

Corollary 17. *Let S be a modular numerical semigroup.*

(1) *S is symmetric if and only if* $\mathrm{w}(S)=\mathrm{g}(S)$.

(2) *S is pseudo-symmetric if and only if* $\mathrm{w}(S)=\mathrm{g}(S)+1$.

Proof. S is symmetric if and only if $\#\,\mathrm{H}(S)=\frac{1}{2}(\mathrm{g}(S)+1)$. By Theorem 12, $\#\,\mathrm{H}(S)=\frac{1}{2}(\mathrm{w}(S)+1)$, whence S is symmetric if and only if $\mathrm{g}(S)=\mathrm{w}(S)$. The proof of (2) is analogous. □

Example 18. If b is an odd integer, there exists a modular numerical semigroup S with $\mathrm{w}(S)=b$. It suffices to take $S=\mathrm{S}(2,b+2)$, since $\mathrm{w}(\mathrm{S}(2,b+2))=b+2-(2,b+2)-(1,b+2)=b+2-1-1=b$.

2. Determining whether a numerical semigroup is modular

In this section we give a procedure for deciding whether a given numerical semigroup is a modular numerical semigroup, and if so to express it in the form $\mathrm{S}(a,b)$.

Lemma 19. *Let S be a modular numerical semigroup with modulus b and $S\neq\mathbb{N}$. Then $b\leq 12\,\#\,\mathrm{H}(S)-6$.*

Proof. As we saw right after Example 15, if $a\geq 2$ we have $(a,b)+(a-1,b)\leq\frac{5}{6}b$. By Theorem 12, $\#\,\mathrm{H}(S)\geq\frac{1}{2}(b+1-\frac{5}{6}b)$ and thus $b\leq 12\,\#\,\mathrm{H}(S)-6$. □

For a numerical semigroup S, the *multiplicity* of S, denoted by $\mathrm{m}(S)$, is the least positive integer in S. Here is an immediate consequence of Lemma 11:

Lemma 20. *For $S = \mathrm{S}(a, b)$,*

$$b - \mathrm{m}(S) \in S \Longleftrightarrow \mathrm{m}(S) = \min\left\{\frac{b}{(a, b)}, \frac{b}{(a-1, b)}\right\}.$$

Lemma 21. *Let S be a modular numerical semigroup with modulus b. Then*

$$b \geq \mathrm{g}(S) + \mathrm{m}(S).$$

Proof. Since $1, 2, \ldots, \mathrm{m}(S)-1$ are not in S, Corollary 6 ensures that $b-\mathrm{m}(S)+1, \ldots, b-1$ are. But $\{b, \mathrm{m}(S)\} \subset S$, so $\{b - \mathrm{m}(S) + 1, \rightarrow\} \subseteq S$. This implies that $\mathrm{g}(S) \leq b - \mathrm{m}(S)$. □

Lemma 22. *For $S = \mathrm{S}(a, b)$,*

$$b = \mathrm{g}(S) + \mathrm{m}(S) \Longleftrightarrow \mathrm{m}(S) \neq \min\left\{\frac{b}{(a, b)}, \frac{b}{(a-1, b)}\right\}.$$

Proof. Follows from Lemmas 20 and 21. □

Now we have all the ingredients to give the algorithm announced at the start of this section, to decide whether a numerical semigroup is of the form $S(a, b)$, and if so, produce such a pair (a, b) (or all such pairs with $a \leq \frac{1}{2}(b+1)$, if the algorithm is not stopped after the first pair is found).

Algorithm 23. Given a numerical semigroup S distinct from $\mathbb{N}$:

(1) Compute $\#\mathrm{H}(S)$, $\mathrm{g}(S)$ and $\mathrm{m}(S)$.

(2) Set $b = \mathrm{g}(S) + \mathrm{m}(S)$.

(3) For every $a \in A := \left\{ a \in \mathbb{N} \;\middle|\; \begin{array}{l} 2 \leq a \leq \frac{1}{2}(b+1), \\ b = 2\#\mathrm{H}(S) + (a, b) + (a-1, b) - 1, \\ \mathrm{m}(S) < \min\{b/(a, b), b/(a-1, b)\} \end{array} \right\}$

compute $\mathrm{S}(a, b)$; if $S = \mathrm{S}(a, b)$, return this answer and stop.

(4) Compute $B = \big\{ b \in \{k \cdot \mathrm{m}(S) \mid k \in \mathbb{N}\} \;\big|\; 2\#\mathrm{H}(S) + 1 \leq b \leq 12\#\mathrm{H}(S) - 6 \big\}$.

(5) For every $b \in B$

for every $a \in A_b := \left\{ a \in \mathbb{N} \;\middle|\; \begin{array}{l} 2 \leq a \leq \frac{1}{2}(b+1), \\ b = 2\#\mathrm{H}(S) + (a, b) + (a-1, b) - 1, \\ \mathrm{m}(S) = \min\{b/(a, b), b/(a-1, b)\} \end{array} \right\}$

compute $\mathrm{S}(a, b)$; if $S = \mathrm{S}(a, b)$, return this answer and stop.

(6) Return "S is not modular".

We briefly justify the correctness of Algorithm 23. In Steps (2) and (3) we check whether S is a modular numerical semigroup with modulus $\mathrm{g}(S)+\mathrm{m}(S)$, and the correct working of these steps relies on Lemmas 4 and 22 and Theorem 12. If S is not a modular numerical semigroup with modulus $\mathrm{g}(S)+\mathrm{m}(S)$, Lemma 22 gives $\mathrm{m}(S)=\min\{b/(a,b), b/(a-1,b)\}$. This implies that $\mathrm{m}(S)$ divides b. Theorem 12 states that $b=2\#\mathrm{H}(S)+(a,b)+(a-1,b)-1$, so $b\geq 2\#\mathrm{H}(S)+1$; at the same time $b\leq 12\#\mathrm{H}(S)-6$ by Lemma 19. Therefore Steps (4) and (5) cover the case $b\neq \mathrm{g}(S)+\mathrm{m}(S)$.

Example 24. Let $S=\langle 3,5\rangle$. Then $\#\mathrm{H}(S)=4$, $\mathrm{g}(S)=7$ and $\mathrm{m}(S)=3$. In Step (2) we get $b=10$. Step (3) yields $A=\{2,3,4\}$, then $\mathrm{S}(2,10)=\langle 5,6,7,8,9\rangle$, $\mathrm{S}(3,10)=\langle 4,5,7\rangle$, and $\mathrm{S}(4,10)=\langle 3,5\rangle=S$, so the algorithm returns $S=S(4,10)$.

Example 25. Let $S=\langle 3,8,10\rangle$. In this case $\#\mathrm{H}(S)=5$, $\mathrm{g}(S)=7$ and $\mathrm{m}(S)=3$. In Step (2) we obtain $b=10$ and in Step (3), $A=\varnothing$. The only nonempty set A_b with $b\in B$ is $A_{15}=\{5\}$. Since $S\neq\mathrm{S}(5,15)=\langle 3,7,11\rangle$, the algorithm returns No.

Example 26. Let $S=\langle 10,11,12\rangle$. Then $\#\mathrm{H}(S)=25$, $\mathrm{g}(S)=49$ and $\mathrm{m}(S)=10$. In Step (2) we obtain $b=59$ and A is empty. Computing B, we obtain

$$B=\{60,70,80,90,100,110,120,130,140,150,160,170,180,\\ 190,200,210,220,230,240,250,260,270,280,290\}.$$

The only nonempty set A_b with $b\in B$ is $A_{60}=\{6\}$. It turns out that $S=\mathrm{S}(6,60)$.

Remark 27. If the input to Algorithm 23 is known to be symmetric, the procedure can be improved, because if $S=\mathrm{S}(a,b)$ is symmetric then b must be equal to $\mathrm{g}(S)+(a,b)+(a-1,b)$ (note that $\mathrm{w}(S)=\mathrm{g}(S)$ by Corollary 17). A similar argument applies to the pseudo-symmetric case.

Remark 28. The intersection $\bigcap_{i=1}^{n}\mathrm{S}(a_i,b_i)$ of $n\geq 1$ modular numerical semigroups is a numerical semigroup; it need not be modular, as can be seen from Example 25, since we can write $\langle 3,8,10\rangle=\langle 3,4\rangle\cap\langle 3,5\rangle=\mathrm{S}(3,8)\cap\mathrm{S}(4,10)$.

Nor can every numerical semigroup be written as such an intersection: for instance, $\langle 7,8,10,13\rangle$ is a symmetric, hence irreducible, numerical semigroup; thus it cannot be an intersection of modular numerical semigroups other than by being itself a modular numerical semigroup. Algorithm 23 says that it is not.

3. Modular numerical semigroups whose modulus is its weight plus two

We now study modular numerical semigroups $S=\mathrm{S}(a,b)$ whose modulus b equals $\mathrm{w}(S)+2$. Since $b=\mathrm{w}(S)+(a,b)+(a-1,b)\geq\mathrm{w}(S)+2$, the condition $b=\mathrm{w}(S)+2$ is equivalent to $(a,b)=(a-1,b)=1$ (so b is odd), and it characterizes modular numerical semigroups whose moduli are minimal with respect to their weights.

Every numerical semigroup S is finitely generated (as an additive monoid). This is easy to see — for instance, start with two relatively prime $r, s \in S$ and then adjoin all elements of $S \cap \{0, 1, \dots, rs-1\}$ as yet unaccounted for. Among all generating sets one can of course choose one that is *minimal*, say $\mathcal{M}(S)$. A minute's thought shows that $\mathcal{M}(S)$ is characterized by containing exactly those nonzero elements of S that cannot be expressed as a sum of two nonzero elements of S:

$$\mathcal{M}(S) = (S \setminus \{0\}) \setminus \big((S \setminus \{0\}) + (S \setminus \{0\})\big).$$

In particular, $\mathcal{M}(S)$ is unique. We set $\mathrm{e}(S) = \#\mathcal{M}(S)$ and call this number the *embedding dimension* of S; the elements of $\mathcal{M}(S)$ are called *minimal generators*.

Proposition 29. *Let* $S = \mathrm{S}(a, b)$ *with* $2 \le a < b$ *and* $(a, b) = (a-1, b) = 1$. *Then*

(1) $b = \mathrm{g}(S) + \mathrm{m}(S)$,

(2) $\#\mathrm{H}(S) = \frac{1}{2}(\mathrm{g}(S) + \mathrm{m}(S) - 1)$,

(3) *b is the largest minimal generator of S.*

Proof. (1) We already know that $b-1 \in S$ when $2 \le a < b$. Hence $\mathrm{m}(S) \ne b$. Using Lemma 22, we get $b = \mathrm{g}(S) + \mathrm{m}(S)$.

(2) Immediate from Theorem 12.

(3) First we prove that b is a minimal generator of S. Assume to the contrary that $b = x + y$ with $x, y \in S \setminus \{0\}$. Then $ax \bmod b \le x$ and $ay \bmod b \le y$, and thus $(ax \bmod b) + (ay \bmod b) \le x + y = b$. Since $a(x + y) \bmod b = ab \bmod b = 0$, we deduce that $(ax \bmod b) + (ay \bmod b) \in \{0, b\}$. Thus either $ax \bmod b = x$ and $ay \bmod b = y$, or $ax \bmod b = 0$ and $ay \bmod b = 0$. These two cases contradict the two halves of Lemma 10.

To see that b is the largest minimal generator, take $x \in S$ with $x > b$. By applying (1) we obtain $x > \mathrm{g}(S) + \mathrm{m}(S)$, which implies that $x - \mathrm{m}(S) > \mathrm{g}(S)$; this forces $x - \mathrm{m}(S) \in S$. Thus $x = \mathrm{m}(S) + (x - \mathrm{m}(S))$ cannot be a minimal generator of S. □

Proposition 29 allows us to relate the modular numerical semigroups in question with *unitary extensions of symmetric numerical semigroups* or *UESY-semigroups* in short. A numerical semigroup S is a UESY-semigroup if there exists a symmetric numerical semigroup S' such that $S' \subset S$ and $\#(S \setminus S') = 1$. In [Rosales ≥ 2005b] this condition is shown to be equivalent to the existence of a symmetric numerical semigroup S' such that $S = S' \cup \{\mathrm{g}(S')\}$. The following result also appears there.

Proposition 30. *Let S be a numerical semigroup*, $S \ne \mathbb{N}$. *The following conditions are equivalent*:

(1) *S is a UESY-semigroup.*

(2) $\#\mathrm{H}(S) = \frac{1}{2}(\mathrm{g}(S) + \mathrm{m}(S) - 1)$ *and* $\mathrm{g}(S) + \mathrm{m}(S)$ *is a minimal generator of S.* □

A *pseudo-Frobenius number* [Rosales and Branco 2002] of a numerical semigroup S is an integer $x \notin S$ such that $x+s \in S$ for all $s \in S \setminus \{0\}$. The set of pseudo-Frobenius numbers of S is denoted by $\mathrm{Pg}(S)$, and its cardinality, called the *type* of S, is denoted by $\mathrm{t}(S)$. Clearly $\mathrm{g}(S) \in \mathrm{Pg}(S)$. Moreover S is symmetric if and only if $\mathrm{Pg}(S) = \{\mathrm{g}(S)\}$, and S is pseudo-symmetric if and only if $\mathrm{Pg}(S) = \{\mathrm{g}(S), \frac{1}{2}\mathrm{g}(S)\}$; see [Barucci et al. 1997; Fröberg et al. 1987], for instance.

In [Rosales $\geq$ 2005b] it is proved that if S is a UESY-semigroup distinct from $\mathbb{N}$, then $\mathrm{t}(S) = \mathrm{e}(S)-1$. This, plus Propositions 29 and 30, gives:

Corollary 31. *Let $S = \mathrm{S}(a, b)$ be such that $2 \leq a < b$ and $(a, b) = (a-1, b) = 1$. Then $\mathrm{t}(S) = \mathrm{e}(S)-1$ and there exists a symmetric numerical semigroup S' such that $S = S' \cup \{\mathrm{g}(S')\}$.* □

Theorem 32. *Let $S = \mathrm{S}(a, b)$. Then $b = \mathrm{w}(S) + 2$ if and only if S is a UESY-semigroup and b is a minimal generator of S.*

Proof. If $b = \mathrm{w}(S)+2 = b-(a, b)-(a-1, b)+2$, we deduce $(a, b) = (a-1, b) = 1$. Corollary 31 then says that S is a UESY-semigroup, and Proposition 29 that b is a minimal generator of S.

Conversely, if b is a minimal generator of S it equals $\mathrm{g}(S) + \mathrm{m}(S)$, by Lemma 21 and the fact, shown in the proof of Proposition 29, that a minimal generator of S cannot exceed $\mathrm{g}(S) + \mathrm{m}(S)$. If S is a UESY, then, $\#\,\mathrm{H}(S) = \frac{1}{2}(\mathrm{g}(S) + m(S)-1)$ by Proposition 30 and $\#\,\mathrm{H}(S) = \frac{1}{2}(\mathrm{w}(S) + 1)$ by Theorem 12. Thus $b = \mathrm{w}(S) + 2$. □

Corollary 33. *Let S be a modular numerical semigroup with modulus b. Then $b = \mathrm{w}(S) + 2$ if and only if $S \setminus \{b\}$ is a symmetric numerical semigroup. Therefore, if b is a prime integer, $S \setminus \{b\}$ is a symmetric numerical semigroup.*

Proof. If $b = \mathrm{w}(S)+2$, Theorem 32 says b is a minimal generator of S, so $S' = S \setminus \{b\}$ is a numerical semigroup with $\mathrm{g}(S') = b$. By Corollary 6, S' is symmetric.

Conversely, if $S \setminus \{b\}$ is a symmetric numerical semigroup, then S is a UESY-semigroup with b as a minimal generator. Now Theorem 32 gives $b = \mathrm{w}(S) + 2$.

Finally, b prime implies $(a, b) = (a-1, b) = 1$, so $\mathrm{w}(S) = b - 2$. □

Corollary 34. *Let $b \geq 3$ be an integer. Then b is prime if and only if b is the largest minimal generator of $\mathrm{S}(a, b)$ for every a such that $2 \leq a \leq \sqrt{b}$.*

Proof. If b is prime Proposition 29 applies; this proves one direction. Conversely, suppose b is not a prime — say $b = ac$ with integers $a, c \geq 2$ and $a \leq \sqrt{b}$. For $S = \mathrm{S}(a, b)$, we have $ac \bmod b = 0$ and thus $c \in S$. But then $b = ac$ cannot be a minimal generator of S. □

4. Modular numerical semigroups whose modulus is its weight plus three

We now study modular numerical semigroups $S = \mathrm{S}(a, b)$ such that $b = \mathrm{w}(S) + 3$; this condition is equivalent to $(a, b) + (a-1, b) = 3$. There are two cases:

- $(a, b) = 1$ and $(a-1, b) = 2$.
- $(a, b) = 2$ and $(a-1, b) = 1$.

In both situations b must be even and by Corollary 6 we deduce that $\frac{1}{2}b \in S$.

Let S be a numerical semigroup with minimal generating set $\{n_1, \ldots, n_p\}$. We say that $x \in S$ has a *unique expression* if the equality $x = a_1 n_1 + \cdots + a_p n_p$, with $a_1, \ldots, a_p \in \mathbb{N}$, determines $a_1, \ldots, a_p$ uniquely.

Proposition 35. *Let* $S = \mathrm{S}(a, b)$ *be such that* $2 \le a < b$ *and* $(a, b) + (a-1, b) = 3$.

(1) $\mathrm{m}(S) \ne \frac{1}{2}b \Leftrightarrow S \ne \{0, \frac{1}{2}b, \rightarrow\} \Leftrightarrow b = \mathrm{g}(S) + \mathrm{m}(S) \Leftrightarrow \#\,\mathrm{H}(S) = \dfrac{\mathrm{g}(S)+\mathrm{m}(S)-2}{2}$.

(2) $\frac{1}{2}b$ *is a minimal generator of* S.

(3) b *has a unique expression in* S.

Proof. (1) Follows easily from Corollary 6, Lemma 22 and Theorem 12.

(2) Suppose $x + y = \frac{1}{2}b$ with $x, y \in S$. Then $ax \bmod b \le x$ and $ay \bmod b \le y$, whence $ax \bmod b + ay \bmod b \le x + y = \frac{1}{2}b$. Thus $\frac{1}{2}ab \bmod b = a(x+y) \bmod b = ax \bmod b + ay \bmod b$. We must show that $x = 0$ or $y = 0$. We distinguish two cases. If $(a, b) = 2$, then $\frac{1}{2}ab \bmod b = 0$, so $ax \bmod b = 0$ and $ay \bmod b = 0$; then Lemma 10 shows that both x and y are multiples of $\frac{1}{2}b$, which leads to the desired conclusion. Similarly, if $(a-1, b) = 2$, then $\frac{1}{2}ab \bmod b = \frac{1}{2}b$, so $ax \bmod b = x$ and $ay \bmod b = y$; Lemma 10 again shows that x and y are multiples of $\frac{1}{2}b$.

(3) We prove that if $x, y \in S \backslash \{0\}$ are such that $x + y = b$, then $x = y = \frac{1}{2}b$. Arguing as in the proof of Proposition 29(3), we see that either $(ax \bmod b, ay \bmod y) = (x, y)$ or $ax \bmod b = ay \bmod y = 0$. Lemma 10 implies that x and y are both multiples of $\frac{1}{2}b$, and since $x \ne 0 \ne y$, we conclude that $x = y = \frac{1}{2}b$. □

Paralleling what we did in Section 3 for the case $b = \mathrm{w}(S) + 2$, we can use Proposition 35 to relate modular numerical semigroups such that $b = \mathrm{w}(S) + 3$ with a previous studied class of numerical semigroups. A numerical semigroup S is called a *PESPY-semigroup* if there exists a pseudo-symmetric numerical semigroup S' such that $S = S' \cup \left\{\frac{1}{2}\mathrm{g}(S'), \mathrm{g}(S')\right\}$ (the two additional elements are the pseudo-Frobenius numbers of S'; see [Barucci et al. 1997; Fröberg et al. 1987]).

Numerical semigroups of the form $\{0, x, \rightarrow\}$ with x a positive integer are called *intervals*. The following result appears in [Rosales ≥ 2005a].

Proposition 36. *Let* S *be a numerical semigroup that is not an interval. The following conditions are equivalent*:

(1) S *is a PEPSY-semigroup.*

(2) $\#\,\mathrm{H}(S) = \frac{1}{2}(\mathrm{g}(S) + \mathrm{m}(S) - 2)$, $\frac{1}{2}(\mathrm{g}(S) + \mathrm{m}(S))$ *is a minimal generator of* S *and* $\mathrm{g}(S) + \mathrm{m}(S)$ *is an element of unique expression of* S. □

The next result is an immediate consequence of Propositions 35 and 36.

Corollary 37. *Let $S = \mathrm{S}(a, b)$ be such that $2 \le a < b$, $(a, b) + (a-1, b) = 3$ and S is not an interval. Then S is a PEPSY-semigroup.*

In [Rosales $\ge$ 2005a] it is proved that if S is a PEPSY-semigroup that is not an interval, then $\mathrm{t}(S) = \mathrm{e}(S)-1$. Thus:

Corollary 38. *Let $S = \mathrm{S}(a, b)$ be such that $2 \le a < b$, $(a, b) + (a-1, b) = 3$ and S is not an interval. Then $\mathrm{t}(S) = \mathrm{e}(S)-1$.*

Remark 39. Among numerical semigroups, interval semigroups have maximal embedding dimension relative to multiplicity: $\mathrm{e}(S) = \mathrm{m}(S)$. For any numerical semigroup with maximal embedding dimension, $\mathrm{t}(S) = \mathrm{m}(S)-1 = \mathrm{e}(S)-1$ (see [Barucci et al. 1997], for instance). Hence the assumption "S is not an interval" can be dropped from Corollary 38.

Theorem 40. *Assume that $S = \mathrm{S}(a, b)$ is not an interval. Then $b = \mathrm{w}(S) + 3$ if and only if S is a PEPSY-semigroup, $\frac{1}{2}b$ is a minimal generator of S and b has a unique expression in S.*

Proof. Necessity follows from Corollary 37 and Proposition 35. Sufficiency: Lemma 21 says that $b \ge \mathrm{g}(S)+\mathrm{m}(S)$. If $b > \mathrm{g}(S)+\mathrm{m}(S)$, then $\mathrm{m}(S)+(b-\mathrm{m}(S))$ and $\frac{1}{2}b + \frac{1}{2}b$ are distinct expressions for b in S ($\mathrm{m}(S) \ne \frac{1}{2}b$ since otherwise S is an interval, by Corollary 6). Therefore $b = \mathrm{g}(S) + \mathrm{m}(S)$. By Proposition 36, we know that $\#\mathrm{H}(S) = \frac{1}{2}(\mathrm{g}(S) + \mathrm{m}(S) - 2)$ and Theorem 12 ensures that $\#\mathrm{H}(S) = \frac{1}{2}(\mathrm{w}(S) + 1)$, whence $b = \mathrm{g}(S) + \mathrm{m}(S) = \mathrm{w}(S) + 3$. □

Corollary 41. *Let S be a modular numerical semigroup with modulus b. Then $b = \mathrm{w}(S)+3$ if and only if $S \setminus \left\{\frac{1}{2}b, b\right\}$ is a pseudo-symmetric numerical semigroup. Therefore, if $b = 2p$ and $a < p$ for some positive prime p, then $S \setminus \left\{\frac{1}{2}b, b\right\}$ is a pseudo-symmetric numerical semigroup.*

Proof. Suppose $b = \mathrm{w}(S)+3$. By Theorem 40, $\frac{1}{2}b$ is a minimal generator of S and b has a unique expression in S. This implies that $S' = S \setminus \left\{\frac{1}{2}b, b\right\}$ is a numerical semigroup, and clearly $\mathrm{g}(S') = b$. Using Corollary 6 we can easily deduce that S' is pseudo-symmetric.

Conversely, if $S \setminus \left\{\frac{1}{2}b, b\right\}$ is a pseudo-symmetric numerical semigroup, then S is a PEPSY-semigroup by definition, $\frac{1}{2}b$ is a minimal generator of S and $b = \frac{1}{2}b + \frac{1}{2}b$ is the unique expression of b in S. Thus $b = \mathrm{w}(S) + 3$ by Theorem 40. □

5. When the factor divides the modulus

We next focus on numerical semigroups of the form $S = \mathrm{S}(a, ab)$, where we may as well assume $a, b > 1$. First a general definition: given a numerical semigroup

S and $n \in S \setminus \{0\}$, the *Apéry set* of n in S [Apéry 1946] is

$$\mathrm{Ap}(S, n) = \{s \in S \mid s - n \notin S\}.$$

This set always has n elements $w(0) = 0$, $w(1)$, ..., $w(n-1)$, where $w(i)$ is the least element congruent to i modulo n. Note also that $x \in \mathbb{Z}$ is an element of S if and only if $x \geq w(x \bmod n)$. Consequently

$$\mathrm{g}(S) = \max(\mathrm{Ap}(S, n)) - n. \tag{$*$}$$

The following result is a consequence of [Rosales 1996, Lemma 3.3] and gives a characterization of Apéry sets which will be useful later.

Lemma 42. *Let $m > 0$ be an integer and let $X = \{0 = w(0), w(1), \dots, w(m-1)\}$ be a subset of $\mathbb{N}$ such that $i < w(i) \equiv i \bmod m$ for all $i \in \{1, \dots, m-1\}$. Let S be the submonoid of $\mathbb{N}$ generated by $X \cup \{m\}$. Then S is a numerical semigroup with multiplicity m. Moreover,* $\mathrm{Ap}(S, m) = X$ *if and only if for all $i, j \in \{1, \dots, m-1\}$ there exist $k \in \{0, \dots, m-1\}$ and $t \in \mathbb{N}$ such that $w(i) + w(j) = w(k) + tm$.* □

Getting back to $S = \mathrm{S}(a, ab)$, with $a, b > 1$, we will give a description of the particular Apéry set $\mathrm{Ap}(S, \mathrm{m}(S))$ in terms of a, b, and this will lead to an explicit formula for the Frobenius number of S. We also show how the minimal generating set for such numerical semigroups can be computed from a and b as well as the corresponding sets of pseudo-Frobenius numbers.

Lemma 43. $\mathrm{m}(\mathrm{S}(a, ab)) = b$.

Proof. Let $S = \mathrm{S}(a, ab)$ and let $x \in \{1, \dots, b-1\}$. Then $ax < ab$ and thus $ax \bmod ab = ax > x$, whence $x \notin S$. Clearly $b \in S$ and consequently $\mathrm{m}(S) = b$. □

Theorem 44. $\mathrm{Ap}(\mathrm{S}(a, ab), b) = \{0, k_1 b + 1, k_2 b + 2, \dots, k_{b-1} b + b-1\}$, *where $k_i = \lceil (a-1)i/b \rceil$ for all $i \in \{1, \dots, b-1\}$.*

Proof. Let S' be the semigroup generated by $\{b, k_1 b+1, \dots, k_{b-1} b + b-1\}$. Since $k_i \geq 1$ for all $i \in \{1, \dots, b-1\}$ we have $\mathrm{m}(S') = b$. Clearly $k_1 \leq \cdots \leq k_{b-1}$ and $k_i + k_j \geq k_{i+j}$ for all $i, j \in \{1, \dots, b-1\}$ with $2 \leq i + j \leq b-1$. Using Lemma 42, we deduce that $\mathrm{Ap}(S', b) = \{0, k_1 b + 1, \dots, k_{b-1} b + b-1\}$. Recall that $x \in \mathbb{Z}$ belongs to S' if and only if $x \geq k_{x \bmod b} b + x \bmod b$, since this latter number is the element in $\mathrm{Ap}(S', b)$ that is congruent to x modulo b. So, for x an integer we have $x \in S' \iff \lfloor x/b \rfloor \geq k_{x \bmod b} \iff \lfloor x/b \rfloor \geq \lceil (a-1)(x \bmod b)/b \rceil \iff \lfloor x/b \rfloor \geq (a-1)(x \bmod b)/b \iff \lfloor x/b \rfloor b \geq (a-1)(x \bmod b) \iff x - (x \bmod b) \geq (a-1)(x \bmod b) \iff a(x \bmod b) \leq x \iff ax \bmod ab \leq x$. Thus $S' = \mathrm{S}(a, ab)$. □

Using this result and equality $(*)$ with $n = \mathrm{m}(S)$, we obtain:

Corollary 45. $\mathrm{g}(\mathrm{S}(a, ab)) = \left\lceil (b-1)(a-1)/b \right\rceil b - 1$.

Particularizing the formula given in Theorem 12 for the case at hand, we get

$$\#\,\mathrm{H}(\mathrm{S}(a, ab)) = \frac{a(b-1) - (a-1, b) + 1}{2}.$$

Minimal generators. We next turn our attention to the minimal generating set $\{n_0 < n_1 < \cdots < n_p\}$ of $\mathrm{S}(a, ab)$. We know that $n_0 = b$, by Lemma 43; our goal is to describe the remaining minimal generators.

Lemma 46. *Let x and y be positive integers. Then $\lceil x/b \rceil + \lceil y/b \rceil = \lceil (x+y)/b \rceil$ if and only if $x \equiv 0 \bmod b$ or $y \equiv 0 \bmod b$ or $(x \bmod b) + (y \bmod b) > b$.* □

Remark 47. If S is any numerical semigroup and $m \in S \setminus \{0\}$, then S is generated by $X = (\mathrm{Ap}(S, m) \setminus \{0\}) \cup \{m\} = \{m, w(1), \ldots, w(m-1)\}$, and the minimal generating set of S is $X \setminus (X + X)$. Now, in the case of $S = \mathrm{S}(a, ab)$, Theorem 44 says that $\mathrm{Ap}(S, b) = \{0, k_1 b + 1, \ldots, ak_{b-1} b + b-1\}$, with $k_i = \lceil (a-1)i/b \rceil$ for all $i \in \{1, \ldots, b-1\}$. Thus $k_t b + t$ is a minimal generator of S if and only if $k_t \neq k_i + k_{t-i}$ for all $i \in \{1, \ldots, t-1\}$.

Lemma 48. *Let $S = \mathrm{S}(a, ab)$ with $a, b > 1$, set $k_i = \lceil (a-1)i/b \rceil$ for all $i \in \{1, \ldots, b-1\}$ and take $t \in \{1, \ldots, b-1\}$.*

(i) *If $t < b/(a-1, b)$, then $k_t b + t$ is a minimal generator of S if and only if $(a-1)i \bmod b < (a-1)t \bmod b$ for all $i \in \{1, \ldots, t-1\}$.*

(ii) *If $t > b/(a-1, b)$, then $k_t b + t$ is not a minimal generator of S.*

(iii) *If $t = b/(a-1, b)$, then $k_t b + t$ is a minimal generator of S.*

Proof. Using Lemma 46 and Remark 47, we see that $k_t b + t$ is a minimal generator of S if and only if $(a-1)i \not\equiv 0 \bmod b$ and $(a-1)i \bmod b + (a-1)(t-i) \bmod b \le b$ for all $i \in \{1, \ldots, t-1\}$. Observe that

$$(\dagger) \qquad \frac{b}{(a-1, b)} = \frac{\mathrm{lcm}(a-1, b)}{a-1} = \min\{i \mid (a-1)i \bmod b = 0\}.$$

(i) From the foregoing we deduce that if $t < b/(a-1, b)$, then $k_t b + t$ is a minimal generator of S if and only if $(a-1)i \bmod b + (a-1)(t-i) \bmod b \le b$ for all $i \in \{1, \ldots, t-1\}$. If $(a-1)i \bmod b + (a-1)(t-i) \bmod b = b$, then $(a-1)t \bmod b = 0$, which is impossible in view of ($\dagger$), since $t < b/(a-1, b)$. Hence $k_t b + t$ is a minimal generator of S if and only if for all $i \in \{1, \ldots, t-1\}$ one has $(a-1)i \bmod b + (a-1)(t-i) \bmod b < b$, which is equivalent to $(a-1)i \bmod b + (a-1)(t-i) \bmod b = (a-1)t \bmod b$. Since $(a-1)(t-i) \bmod b \neq 0$, we conclude that $k_t b + t$ is a minimal generator of S if and only if $(a-1)i \bmod b < (a-1)t \bmod b$ for all $i \in \{1, \ldots, t-1\}$.

(ii) Let $i = b/(a-1, b)$. Then $(a-1)i \equiv 0 \bmod b$ and in view of Lemma 46 we get $k_i + k_{t-i} = k_t$, which implies that $k_t b + b$ is not a minimal generator of S.

(iii) In this setting $(a-1)t \bmod b=0$ and $(a-1)i \bmod b\neq 0$ for all $i\in\{1,\ldots,t-1\}$. Hence for every $i\in\{1,\ldots,t-1\}$ one gets $(a-1)i \bmod b+(a-1)(t-i) \bmod b=b$, and by Lemma 46 we deduce that $k_t\neq k_i+k_{t-i}$ for any $i\in\{1,\ldots,t-1\}$. Therefore $k_t b+t$ is a minimal generator of S. □

Lemma 48 yields an explicit description of the minimal generating set of S:

Theorem 49. *Let $S=\mathrm{S}(a,ab)$ with $a,b>1$, and set $k_i=\lceil (a-1)i/b\rceil$ for $i\in\{1,\ldots,b-1\}$.*

(1) *If $(b,a-1)=1$, the minimal generating set of S is $\{b, k_{t_1}b+t_1,\ldots,k_{t_r}b+t_r\}$, where $\{t_1,\ldots,t_r\}=\big\{t\in\{1,\ldots,b-1\} \bigm| (a-1)i \bmod b<(a-1)t \bmod b$ for all $i\in\{1,\ldots,t-1\}\big\}$.*

(2) *If $(b,a-1)\neq 1$, let $t_{r+1}=b/(b,a-1)$. Then the minimal generating set of S is $\{b,k_{t_1}b+t_1,\ldots,k_{t_r}b+t_r,k_{t_{r+1}}b+t_{r+1}\}$, where $\{t_1,\ldots,t_r\}=\big\{t\in\{1,\ldots,t_{r+1}-1\} \bigm| (a-1)i \bmod b<(a-1)t \bmod b$ for all $i\in\{1,\ldots,t-1\}\big\}$.* □

Example 50. Let $S=\mathrm{S}(5,35)$. Applying Theorem 49(1) with $a=5$ and $b=7$, we see that $\{t_1,\ldots,t_r\}=\{1,3,5\}$ (observe that 1 is always in $\{t_1,\ldots,t_r\}$), and that S is minimally generated by $\{7,8,17,26\}$.

Example 51. Let $S=\mathrm{S}(5,30)$. Applying Theorem 49(2) with $a=5$ and $b=6$, we see that $t_{r+1}=3$, $\{t_1,\ldots,t_r\}=\{1\}$, and S is minimally generated by $\{6,7,15\}$.

Corollary 52. *Let $S=\mathrm{S}(a,ab)$ with $a,b>1$. Set $k_i=\lceil (a-1)i/b\rceil$ for $i\in\{1,\ldots,b-1\}$, and*

$$t=\begin{cases}\min\{x\in\mathbb{N}\mid (a-1)x\equiv b-1 \bmod b\} & \text{if } (b,a-1)=1,\\ b/(b,a-1) & \text{if } (b,a-1)\neq 1.\end{cases}$$

Then $k_t b+t$ is the greatest minimal generator of S. □

Corollary 53. *Let $a\geq 3$ and let b be a positive integer. Then $\mathrm{e}(\mathrm{S}(a,ab))\geq \lfloor b/(a-1)\rfloor+1$.*

Proof. The integer b is always a minimal generator of $\mathrm{S}(a,ab)$. Also, if $(a-1)t\leq b$, then by Lemma 48, $k_t b+t$ is a minimal generator of S. □

Pseudo-Frobenius numbers. For any numerical semigroup S, we define an order $\leq_S$ on S as follows: $a\leq_S b$ if $b-a\in S$. Given a subset A of S, denote by $\mathrm{Max}_{\leq_S}A$ the set of maximal elements of A with respect to $\leq_S$. The following result appears in [Rosales and Branco 2002].

Lemma 54. *Let S be any numerical semigroup with multiplicity m. If*

$$\mathrm{Max}_{\leq_S}(\mathrm{Ap}(S,m))=\{w_{i_1},\ldots,w_{i_t}\},$$

the pseudo-Frobenius numbers of S (page 387) *are precisely $w_{i_1}-m,\ldots,w_{i_t}-m$.*

Note that if $w, w' \in \mathrm{Ap}(S, m)$ and $w - w' \in S$, this forces $w - w'$ to be in $\mathrm{Ap}(S, m)$ as well. Hence

$$\begin{aligned}\mathrm{Max}_{\leq_S}(\mathrm{Ap}(S, m)) \\ &= \{w \in \mathrm{Ap}(S, m) \mid w + w' \notin \mathrm{Ap}(S, m) \text{ for all } 0 \neq w' \in \mathrm{Ap}(S, m)\}.\end{aligned}$$

Let $S = \mathrm{S}(a, ab)$ with $a, b > 1$. Our aim is to compute the set $\mathrm{Max}_{\leq_S}(\mathrm{Ap}(S, b))$ and thus, in view of Lemma 54, the pseudo-Frobenius set $\mathrm{Pg}(S)$.

Remark 55. By Theorem 44, $k_i b + i \notin \mathrm{Max}_{\leq_S}(\mathrm{Ap}(S, b))$ if and only if there exists $j \in \{1, \ldots, b-1\}$ such that $i + j \leq b-1$ and $k_i + k_j = k_{i+j}$. Minimal generators are $\leq_S$-minimal elements of $\mathrm{Ap}(S, b)$, which is why the condition just stated is similar (dual) to the one presented on the previous page for minimal generators.

Theorem 56. *Let a and b be two integers greater than one, and let $S = \mathrm{S}(a, ab)$. Let $k_i = \lceil (a-1)i/b \rceil$ for $i \in \{1, \ldots, b-1\}$. Then $k_i b + i \in \mathrm{Max}_{\leq_S}(\mathrm{Ap}(S, b))$ if and only if one of the following conditions hold*:

(i) $(a-1)i \equiv 0 \bmod b$ *and* $i = b-1$,

(ii) $(a-1)i \not\equiv 0 \bmod b$ *and for all* $t \in \{i + 1, \ldots, b-1\}$, *either* $(a-1)i \bmod b < (a-1)t \bmod b$ *or* $(a-1)t \bmod b = 0$.

Proof. Assume that $(a-1)i \equiv 0 \bmod b$ and $i < b-1$. Then by Lemma 46, we deduce that $k_i + k_1 = k_{i+1}$ and thus $k_i b + i \notin \mathrm{Max}_{\leq_S}(\mathrm{Ap}(S, b))$. If $(a-1)i \not\equiv 0 \bmod b$, then by Lemma 46 we have $k_i b_i + i \in \mathrm{Max}_{\leq_S}(\mathrm{Ap}(S, b))$ if and only if for all $t \in \{i + 1, \ldots, b-1\}$ we have $(a-1)(t - i) \not\equiv 0 \bmod b$ and $(a-1)i \bmod b + (a-1)(t - i) \bmod b \leq b$. If $(a-1)i \bmod b + (a-1)(t - i) \bmod b < b$, then $(a-1)i \bmod b + (a-1)(t - i) \bmod b = (a-1)t \bmod b$ and thus $(a-1)i \bmod b < (a-1)t \bmod b$. If $(a-1)i \bmod b+(a-1)(t-i) \bmod b = b$, then $(a-1)t \bmod b = 0$.

To prove the converse, assume $k_i b + i \notin \mathrm{Max}_{\leq_S}(\mathrm{Ap}(S, b))$. Then there exists $t \in \{1+i, \ldots, b-1\}$ such that $k_i + k_{t-i} = k_t$. By using Lemma 46, we deduce that $(a-1)i \equiv 0 \bmod b$ or $(a-1)(t-i) \equiv 0 \bmod b$ or $(a-1)i \bmod b+(a-1)(t-i) \bmod b > b$. If $(a-1)i \equiv 0 \bmod b$, then i must be equal to $b-1$, but this is impossible since $t \in \{i + 1, \ldots, b-1\}$. If $(a-1)(t - i) \equiv 0 \bmod b$, then $(a-1)i \bmod b = (a-1)t \bmod b$, which is also impossible by hypothesis. Finally if $(a-1)i \bmod b + (a-1)(t-i) \bmod b > b$, then $(a-1)t \bmod b = (a-1)i \bmod b + (a-1)(t-i) \bmod b - b < (a-1)i \bmod b$, leading again to a contradiction. □

Example 57. Let $S = \mathrm{S}(5, 30)$. Applying Theorem 56 we get $\mathrm{Max}_{\leq_S}(\mathrm{Ap}(S, 6)) = \{29\}$, which by Lemma 54 means that $\mathrm{Pg}(S) = \{23\}$. Thus $\mathrm{S}(5, 30)$ is symmetric.

Proposition 58. *Let $S = \mathrm{S}(a, ab)$ with $a, b > 1$.*

(1) *S is symmetric if and only if* $(a-1, b) + (a-1) \bmod b = b$.

(2) *S is pseudo-symmetric if and only if* $(a-1, b) + (a-1) \bmod b = b + 1$.

Proof. (1) Combining Corollaries 45 and 17(1), we see that S is symmetric if and only if $\lceil (b-1)(a-1)/b \rceil b - 1 = ab - a - (a-1, b)$. The left-hand side can be written as $\big(a-1-\lfloor (a-1)/b \rfloor\big) b - 1 = (a-1)b - \lfloor (a-1)/b \rfloor b - 1 = ab - b - (a-1-(a-1) \bmod b) - 1$. Thus S is symmetric if and only if $(a-1) \bmod b + (a-1, b) = b$.

(2) As above, but this time using Corollary 17(2). □

Corollary 59. *Let k be a positive integer and let b be a multiple of k. Then* $\mathrm{S}(b-k+1+bn, (b-k+1+bn)b)$ *is symmetric for all $n \in \mathbb{N}$.* □

The pseudo-symmetric case is completely different:

Corollary 60. $\mathrm{S}(a, ab)$ *is not pseudo-symmetric for any choice of $a, b > 1$.*

Proof. Set $q = \lfloor (a-1)/b \rfloor$ and choose $u, v \in \mathbb{Z}$ such that $(a-1, b) = u(a-1) + vb$. If $\mathrm{S}(a, ab)$ is pseudo-symmetric, we have $(a-1, b) + (a-1) \bmod b = b+1$, hence $u(a-1) + vb + (a-1) - qb = b+1$, or yet $(u+1)(a-1) + (v-q-1)b = 1$. But this implies $(a-1, b) = 1$ and hence $1 + (a-1) \bmod b = b+1$, an impossibility. □

Some families. We now present some families of numerical semigroups of the form $\mathrm{S}(a, ab)$ with $a, b > 1$ such that $(a-1, b) = 1$. For these families we can compute the minimal generating set and pseudo-Frobenius numbers explicitly. As a consequence of Theorems 49 and 56 one gets:

Proposition 61. *Let $S = \mathrm{S}(a, ab)$ with $a, b > 1$ and $(a-1, b) = 1$. Set $k_i = \lceil (a-1)i/b \rceil$ for $i \in \{1, \dots, b-1\}$ and take $t \in \{1, \dots, b-1\}$.*

(1) *$k_t b + t$ is a minimal generator of S if and only if $(a-1)i \bmod b < (a-1)t \bmod b$ for all $i \in \{1, \dots, t-1\}$.*

(2) *$k_t b + t \in \mathrm{Max}_{\leq_S}(\mathrm{Ap}(S, b))$ if and only if $(a-1)t \bmod b < (a-1)i \bmod b$ for all $i \in \{t+1, \dots, b-1\}$.*

Let S_n be the symmetric group in n elements $\{1, \dots, n\}$, and for k relatively prime to $n+1$, define the permutation $\sigma_{k,n+1} \in \mathrm{S}_n$ by $\sigma(i) = ki \bmod (n+1)$ for $i = 1, \dots, n$. Such a permutation is called *modular*. Next, given any permutation $\sigma \in \mathrm{S}_n$, set

$$\mathrm{E}(\sigma) = \{t \in \{1, \dots, n\} \mid \sigma(i) < \sigma(t) \text{ for all } i \in \{1, \dots, t-1\}\},$$
$$\mathrm{T}(\sigma) = \{t \in \{1, \dots, n\} \mid \sigma(t) < \sigma(i) \text{ for all } i \in \{t+1, \dots, n\}\}.$$

With this notation we can rewrite Proposition 61 as follows.

Corollary 62. *Let $S = \mathrm{S}(a, ab)$ with $a, b > 1$ and $(a-1, b) = 1$. Then*

$$\mathrm{e}(S) = \#\mathrm{E}(\sigma_{a-1,b}) + 1 \quad \textit{and} \quad \mathrm{t}(S) = \#\mathrm{T}(\sigma_{a-1,b}).$$

The minimal generating set of S is $\{b\} \cup \big\{ \lceil (a-1)i/b \rceil b + i \;\big|\; i \in \mathrm{E}(\sigma_{a-1,b}) \big\}$, *and*

$$\mathrm{Max}_{\leq_S}(\mathrm{Ap}(S, b)) = \big\{ \lceil (a-1)i/b \rceil b + i \;\big|\; i \in \mathrm{T}(\sigma_{a-1,b}) \big\}.$$

Example 63. Let $S = \mathrm{S}(6, 42)$. Apply Corollary 62 with $a = 6$ and $b = 7$. Clearly $\sigma_{5,7} = (154623)$, $\mathrm{E}(\sigma_{5,7}) = \{1, 4\}$ and $\mathrm{T}(\sigma_{5,7}) = \{3, 6\}$. Hence $\mathrm{e}(S) = 3$ and $\mathrm{t}(S) = 2$. The set $\{7, \lceil(5\times 1)/7\rceil 7+1, \lceil(5\times 4)/7\rceil 7+4\} = \{7, 8, 25\}$ is a minimal generating set of S and $\mathrm{Max}_{\leq_S}(\mathrm{Ap}(S, 7)) = \{\lceil(5\times 3)/7\rceil 7+3, \lceil(5\times 6)/7\rceil 7+6\} = \{24, 41\}$.

Corollary 64. *Let $S = \mathrm{S}((b-1)+bn, ((b-1)+bn)b)$ with $n \in \mathbb{N}$ and $b \geq 5$ odd. Then S is minimally generated by $\left\{b, (n+1)b+1, \left(\frac{b-1}{2}+n\frac{b+1}{2}\right)b+\frac{b+1}{2}\right\}$, and*

$$\mathrm{Max}_{\leq_S}(\mathrm{Ap}(S, b)) = \left\{\left(\tfrac{b-1}{2}+n\tfrac{b-1}{2}\right)b+\tfrac{b-1}{2}, ((b-2)+n(b-1))b+b-1\right\}.$$

Proof. Since $(b-2+bn, b) = (b-2, b) = 1$, we can apply Corollary 62. By inspection we see that $\mathrm{E}(\sigma_{b-2,b}) = \{1, (b+1)/2\}$ and $\mathrm{T}(\sigma_{b-2,b}) = \{(b-1)/2, b-1\}$. We can conclude the proof using Corollary 62, taking into account that

$$\left\lceil\frac{((b-2)+bn)1}{b}\right\rceil = n+1, \quad \left\lceil\frac{((b-2)+bn)(b\pm 1)/2}{b}\right\rceil = \frac{b-1}{2}+n\frac{b\pm 1}{2}, \quad \text{and}$$

$$\left\lceil\frac{((b-2)+bn)(b-1)}{b}\right\rceil = (b-2)+n(b-1). \qquad \square$$

Corollary 65. *Let b be an integer greater than or equal to two and let $n \in \mathbb{N}$. Then $S = \mathrm{S}((n+1)b, (n+1)b^2)$ is minimally generated by $\{b, (n+1)b+1\}$ and* $\mathrm{Max}_{\leq_S}(\mathrm{Ap}(S, b)) = \{(n+1)(b-1)b+b-1\}$.

Proof. Use Corollary 62 and the fact that $\sigma_{(n+1)b-1,b} = \sigma_{b-1,b}$ swaps i and $b-i$. $\square$

Corollary 66. *Let $S = \mathrm{S}(2+nb, (2+nb)b)$ with $n \in \mathbb{N}$ and $b \geq 2$. Then S is minimally generated by*

$$X = \{b, (n+1)b+1, (2n+1)b+2, \dots, ((b-1)n+1)b+b-1\}$$

and $\mathrm{Max}_{\leq_S}(\mathrm{Ap}(S, b)) = X \setminus \{b\}$.

Proof. Use Corollary 62 and the fact that $\sigma_{1+nb,b} = \sigma_{1,b}$ is the identity. $\square$

Corollary 67. *Let $S = \mathrm{S}(3+nb, (3+nb)b)$ with and $n \in \mathbb{N}$ $b \geq 3$ odd. Then S is minimally generated by $\left\{b, (n+1)b+1, (2n+1)b+2, \dots, \left(\frac{b-1}{2}n+1\right)+\frac{b-1}{2}\right\}$ and*

$$\mathrm{Max}_{\leq_S}(\mathrm{Ap}(S, b)) = \left\{\left(\tfrac{b+1}{2}n+2\right)b+\tfrac{b+1}{2}, \dots, ((b-1)n+2)b+b-1\right\}.$$

Proof. By considering $\sigma_{2+bn,b} = \sigma_{2,b}$ we see that $\mathrm{E}(\sigma_{2,b}) = \left\{1, \dots, \frac{1}{2}(b-1)\right\}$ and $\mathrm{T}(\sigma_{2,b}) = \left\{\frac{1}{2}(b+1), \dots, b-1\right\}$. Using Corollary 62, the proof follows easily from

$$\left\lceil\frac{(2+bn)i}{b}\right\rceil b = \begin{cases}(ni+1)b+i & \text{if } i \leq \frac{1}{2}(b-1),\\ (ni+2)b+i & \text{if } i \geq \frac{1}{2}(b+1).\end{cases} \qquad \square$$

6. The Frobenius number in other special cases

In Section 5 we studied $\mathrm{S}(a, b)$ with $a \mid b$. We now give some partial results for the Frobenius number in the complementary case, $a \nmid b$. We are able to find the number when $(a-1)(a - (b \bmod a)) < b$. We use without further comment the fact that, for q a rational number and x a positive integer, $x < \lceil q \rceil$ implies $x < q$.

Lemma 68. *Let $S = \mathrm{S}(a, b)$ with $0 < a < b$ and $b \bmod a \neq 0$. Then*

$$\mathrm{g}(\mathrm{S}(a, b)) \leq b - \lceil b/a \rceil.$$

Proof. Let x be a positive integer. If $x < \lceil b/a \rceil$, then $x < b/a$ and thus $ax \bmod b = ax > x$. Hence $x \notin S$ and in view of Corollary 6, this leads to $b - x \in S$. As $y \in S$ for all $y \geq b$, we conclude that $\mathrm{g}(S) \leq b - \lceil b/a \rceil$. □

Lemma 69. *Let a and b be positive integers such that $a < b$ and $b \bmod a \neq 0$. Then $a \lceil b/a \rceil \bmod b = a - (b \bmod a)$.* □

Proposition 70. *Let a and b be positive integers such that $a < b$ and $b \bmod a \neq 0$. Then $\mathrm{g}(\mathrm{S}(a, b)) = b - \lceil b/a \rceil$ if and only if $(a-1)(a - (b \bmod a)) < b$.*

Proof. Let $S = \mathrm{S}(a, b)$. From Lemma 68 we deduce that $\mathrm{g}(S) = b - \lceil b/a \rceil$ if and only if $b - \lceil b/a \rceil \notin S$, or in other words, $a(b - \lceil b/a \rceil) \bmod b > b - \lceil b/a \rceil$. This by Lemma 69 is equivalent to $((b \bmod a) - a) \bmod b > b - \lceil b/a \rceil$, and this condition holds if and only if $b + (b \bmod a) - a > b - \lfloor b/a \rfloor - 1$. Hence $\mathrm{g}(S) = b - \lceil b/a \rceil$ if and only if $\lfloor b/a \rfloor + 1 + (b \bmod a) > a$, or equivalently $(b - (b \bmod a))/a + 1 + (b \bmod a) > a$, and this holds if and only if $b > (a-1)(a - (b \bmod a))$. □

Corollary 71. *Let a and b be positive integers such that $a < b$,$b \bmod a \neq 0$ and $(a-1)(a - (b \bmod a)) < b$. Then $\mathrm{m}(\mathrm{S}(a, b)) = \lceil b/a \rceil$.*

Proof. Let $S = \mathrm{S}(a, b)$. By Proposition 70, we know that $\mathrm{g}(S) = b - \lceil b/a \rceil$. Thus $b - \lceil b/a \rceil \notin S$ and thus by Corollary 6, $\lceil b/a \rceil = b - (b - \lceil b/a \rceil) \in S$. Besides, if x is a positive integer such that $x < \lceil b/a \rceil$, then $x < b/a$, whence $ax \bmod b = ax > x$ and thus $x \notin S$. Therefore $\mathrm{m}(S) = \lceil b/a \rceil$. □

Though we have given an explicit formula for $\mathrm{g}(\mathrm{S}(a, b))$ for several cases, we have not been able to find such a formula for arbitrary positive integers a and b. We propose this as an open question.

Problem 1. Find a formula for $\mathrm{g}(\mathrm{S}(a, b))$ with a and b positive integers.

Acknowledgments

We thank J. I. García-García and the referee for their comments and suggestions.

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Received March 20, 2003. Revised March 16, 2004.

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