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This note contains a new proof of a theorem of Gang Xiao saying that the bicanonical map of a surface S of general type is generically finite if and only if $p_2(S) > 2$. Such properties are also studied for adjoint linear systems $|K_S + L|$, where L is any divisor with $h^0(S, \mathcal{O}_S(L)) \geq 2$.

Introduction

Let S be a complex minimal surface of general type. Since

$$K_S^2 + 1 - q(S) + p_g(S) \geq 2,$$

the Riemann–Roch Theorem implies that $p_2(S) \geq 2$. If $p_2(S) = 2$, the bicanonical map is composite with a pencil. This note gives an alternative proof of the Theorem of G. Xiao stating the converse:

Theorem 0.1 [Xiao 1985a, Theorem 1]. *Let S be a minimal projective surface of general type. Then the bicanonical map of S is generically finite if and only if $p_2(S) > 2$.*

Xiao's proof depends on his study of genus-2 fibrations over curves and on Horikawa's classification of the possible degenerations. We choose a different approach and deduce the theorem from vanishing theorems for $\mathbb{Q}$-divisors, using in addition just some well known and fundamental properties of surfaces of general type.

We present such a new proof mainly as an interesting application of the $\mathbb{Q}$-divisor method used for similar problems in higher-dimensional birational geometry (see [Chen 2003], for example). Using more involved results on surfaces, there are other, slightly shorter proofs of Xiao's Theorem.

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In the last section, we show that adjoint linear systems $|K_S + L|$ on surfaces of general type can only be composite with a pencil of curves if L is a divisor with $h^0(S, \mathbb{O}_S(L)) \le 2$. We discuss some examples, showing that this bound is sharp. This result may be applied to study on 3-folds (see [Chen 2003], for example).

Notation. For a linear system $|L|$ on a surface S the induced rational map is denoted by φ_L. The linear system is composite with a pencil of curves if $\dim \varphi_L(S) = 1$. The symbol $\equiv$ stands for numerical equivalence of divisors, whereas $\sim$ denotes linear equivalence. K_S denotes the canonical divisor, and if $f : S \to B$ is a surjective morphism, $K_{S/B} = K_S - f^*K_B$. For a a real number $\ulcorner a \urcorner$ denotes the round-up, that is, the integer with $\ulcorner a \urcorner - 1 < a \le \ulcorner a \urcorner$. For a $\mathbb{Q}$-divisor $D = \sum a_i \cdot D_i$ we write $\ulcorner D \urcorner = \sum \ulcorner a_i \urcorner \cdot D_i$ and $\llcorner D \lrcorner = -\ulcorner - D \urcorner$ for the round-down. The base field is $\mathbb{C}$.

1. Proof of Theorem 0.1

Recall the Kawamata–Viehweg vanishing theorem (from [Esnault and Viehweg 1992, p. 49], for example).

Theorem 1.1 [Kawamata 1982; Viehweg 1982]. *Let X be a smooth projective variety and L a divisor on X. Assume that D is an effective $\mathbb{Q}$-divisor with normal crossing supports such that one of the following holds true*:

(i) *$L - D$ is nef and big.*

(ii) *$L - D$ is nef and $\kappa(L - \llcorner D \lrcorner) = \dim X$.*

Then $H^i(X, \mathbb{O}_S(K_X + L - \llcorner D \lrcorner)) = 0$ for all $i > 0$.

Remark 1.2. As is well known, on surfaces, one may apply the vanishing theorems without the assumption of normal crossings. In fact, if $\tau : X' \to X$ is a blowing up, with τ^*D a normal crossing divisor, then

$$R^i \tau_* \mathbb{O}_{X'}(K_{X'} + \tau^*L - \llcorner D' \lrcorner) = 0 \quad \text{for } i > 0,$$

and for $i = 0$ it coincides with $\mathbb{O}_X(K_X + L - \llcorner D \lrcorner)$ in codimension one. If X is a surface, for $i > 0$ we have

$$\begin{aligned} 0 &= H^i\big(X', \mathbb{O}_{X'}(K_{X'} + \tau^*L - \llcorner D' \lrcorner)\big) \\ &= H^i\big(X, \tau_*\mathbb{O}_{X'}(K_{X'} + \tau^*L - \llcorner D' \lrcorner)\big) \\ &= H^i\big(X, \mathbb{O}_X(K_X + L - \llcorner D \lrcorner)\big). \end{aligned}$$

We will use the following simple observation, due to Xiao [1985a, Lemme 8].

Lemma 1.3. *Let S be a minimal surface of general type with $q(S) = 0$ and $K_S^2 \le 2$. Let θ be a nontrivial invertible torsion sheaf on S. Then $H^1(S, \theta) = 0$.*

Proof. There exists an étale cover $\tau : T \to S$ with $\tau^*\theta = \mathbb{O}_T$, hence θ is a direct factor of $\tau_*\mathbb{O}_T$. Since $K_S^2 \leq 2 \leq 2\chi(\mathbb{O}_S)$, it follows from [Beauville 1979, Corollary 5.8] that the fundamental group of S is finite, hence the one of T as well. Then both $H^1(T, \mathbb{O}_T)$ and $H^1(S, \theta)$ are zero. □

As a first step, let us reduce the proof of Theorem 0.1 to the case $p_2(S) = 3$.

Proposition 1.4. *Let S be a minimal smooth surface of general type.*

(1) *The bicanonical map of S is generically finite if $p_2(S) \geq 4$.*

(2) *The linear system $|2K_S|$ is not composite with an irrational pencil of curves for $p_2(S) = 3$.*

Proof. Suppose for some S with $p_2(S) \geq 2$ the linear system $|2K_S|$ is composite with a pencil, or for $p_2(S) = 3$ with an irrational pencil. Let $\pi : S' \longrightarrow S$ be any birational modification such that $|2\pi^*(K_S)|$ defines a morphism ϕ_2' and let B_2' be its image. Consider the Stein factorization

$$\phi_2' : S' \xrightarrow{f} B_2 \longrightarrow B_2'.$$

For some fibres C_i of f and for a general fibre C, we may write

$$\pi^*(2K_S) \sim \sum_{i=1}^{a} C_i + Z_2 \equiv a \cdot C + Z_2,$$

where Z_2 is the fixed part. By assumption on the the smooth curve B_2, the sheaf $f_*(\mathbb{O}_{S'}(2K_{S'}))$ is invertible of degree a and the space of its global sections is of dimension ≥ 4, or of dimension ≥ 3 if $B_2 \neq \mathbb{P}^1$. In both cases one finds $a \geq 3$.

Set $G = \pi^*(K_S) - (1/a)Z_2$. We have $K_{S'} + \ulcorner G \urcorner \leq K_{S'} + \pi^*(K_S)$ and the sheaf

$$G - C \equiv \frac{a-2}{a}\pi^*(K_S)$$

is nef and big. Thus Theorem 1.1 implies that

$$|K_{S'} + \ulcorner G \urcorner|\big|_C = |K_C + D|,$$

for some divisor $D = \ulcorner G \urcorner|_C$ of positive degree on the curve C. The genus of C cannot be zero or one; hence $h^0(C, K_C + D) \geq 2$. This implies that the morphism given by $|K_{S'} + \pi^*(K_S)|$ cannot factor through f, a contradiction. □

Proposition 1.5. *Let S be a smooth minimal surface of general type with $p_2(S) = 3$. Assume that $|2K_S|$ is composite with a pencil of curves.*

(i) $K_S^2 = 2$ *and* $p_g(S) = q(S) \leq 1$.

(ii) $|2K_S|$ *is composite with a rational pencil of curves of genus* 2.

(iii) $|2K_S|$ *defines a morphism on* S, *that is, the movable part of* $|2K_S|$ *is basepoint-free.*

(iv) *Let* E *be a component of the fixed part of* $|2K_S|$. *Then* $E \cdot K_S = 0$ *and* E *is a* (-2) *curve.*

Proof. Since $p_2(S) = 3$ one has $p_g(S) \le 2$. The Riemann–Roch theorem and the positivity of the Euler–Poincaré characteristic imply that

$$0 < K_S^2 = 3 - 1 + q(S) - p_g(S) \le 2.$$

By [Bombieri 1973, Theorems 11 and 12], $q(S) = 0$ if either $K_S^2 = 1$ or if $K_S^2 = p_g(S) = 2$. Hence in order to prove (i), one just has to exclude the case $K_S^2 = 1$, $p_g(S) = 1$ and $q(S) = 0$.

Since $p_2(S) = 3$, Proposition 1.4 implies that $|2K_S|$ is composite with a rational pencil of curves. Let $\pi : S' \to S$ be again a minimal birational modification such that $|2K_{S'}|$ defines a morphism $f : S' \to \mathbb{P}^1$. The sheaf $f_*\mathbb{O}_S(2K_S)$ is invertible of degree two; hence we may write

$$2K_{S'} \sim 2C' + Z_2'$$

for a general fibre C' of f. Set $C = \pi_*(C')$ and $Z_2 = \pi_*(Z_2')$; then $2K_S \sim 2C + Z_2$.

If $K_S^2 = 1$ one has $C^2 \le K_S \cdot C \le 1$. Since the genus of C is at least two, $K_S \cdot C + C^2 \ge 2$, which implies $K_S \cdot C = C^2 = 1$ and $K_S^2 \cdot C^2 = (K_S \cdot C)^2$. By the Index Theorem, $K_S \equiv C$. As shown in [Bombieri 1973] or [Catanese 1979], the condition $K_S^2 = p_g(S) = 1$ implies that on S numerical equivalence coincides with linear equivalence. Hence $K_S \sim C$, a contradiction since $p_g(S) \ne h^0(S, \mathbb{O}_S(C)) = 2$.

So far we have obtained (i). For (iii) suppose that π cannot be chosen to be an isomorphism, hence $C^2 > 0$. Then $2 = K_S^2 \ge K_S \cdot C \ge C^2$. On the other hand, the index theorem gives

$$K_S^2 \cdot C^2 \le (K_S \cdot C)^2.$$

Since $K_S \cdot C + C^2$ is even, one finds $K_S^2 = K_S \cdot C = C^2 = 2$, hence $K_S \equiv C$, and $Z_2 = 0$.

Assume $p_g(S) = 1$. Let $D \in |K_S|$ be the unique effective divisor. Then there are two fibers C_1' and C_2' of f such that, for $C_i = \pi(C_i')$ one has $2D = C_1 + C_2$. If $C_1 \ne C_2$, then the C_i are both 2-divisible for $i = 1,\ 2$ and $D \equiv 2P$, where P is a divisor. This implies $D^2 \ge 4$, a contradiction. If $C_1 = C_2$, then $D = C_1$ and thus $h^0(S, \mathbb{O}_S(D)) = 2$, again a contradiction.

Assume $p_g(S) = 0$, hence $q(S) = 0$. Then the sheaf

$$\theta = \mathbb{O}_S(K_S - C)$$

is a nontrivial invertible torsion sheaf on S. The Riemann–Roch Theorem implies $h^1(S, \theta) = 1$, contradicting Lemma 1.3.

So (iii) holds true and we may choose $S' = S$. Since for a general fibre C of f one has $g(C) \geq 2$ and $K_S \cdot C \leq K_S^2 = 2$, one finds $g(C) = 2$, and $Z_2 \cdot K_S = 0$. $\square$

Proof of Theorem 0.1. By Propositions 1.4 and 1.5 it remains to show that there cannot exist a minimal surface S of general type with $p_2(S) = 3$, with $K_S^2 = 2$ and with $p_g(S) = q(S) \leq 1$, and such that the bicanonical map is a genus-two fibration $f : S \to \mathbb{P}^1$.

Writing again Z_2 for the fixed part of $|2K_S|$ and C for a general fibre of f, one has $2K_S \sim 2C + Z_2$. Let $Z_v \leq Z_2$ be the largest effective divisor contained in fibres of f, and $Z_h = Z_2 - Z_v$ the horizontal part of Z_2. In particular $C \cdot K_S = C \cdot Z_h = 4$. We will study step by step the divisors Z_v and Z_h.

Claim 1.6. *The maximal multiplicity a in Z_2 of an irreducible component is two.*

Proof. Suppose $a > 2$, and denote by Γ the total sum of reduced components of multiplicity a in Z_2. We may write

$$\Gamma = \Gamma_1 + \cdots + \Gamma_s,$$

where the Γ_i are connected pairwise disjoint. Proposition 1.5(iv) implies that each Γ_i is a connected tree of rational curves, thus 1-connected. We may replace $2C$ by the sum of two different general fibres of f, say C_1 and C_2. Then

$$K_S - \frac{1}{a}C_1 - \frac{1}{a}C_2 - \frac{1}{a}Z_2$$

is nef and big, and Theorem 1.1 implies that

$$H^1(2K_S - \Gamma_1 - \cdots - \Gamma_s) = H^1\big(2K_S + \ulcorner -\tfrac{1}{a}C_1 - \tfrac{1}{a}C_2 - \tfrac{1}{a}Z_2 \urcorner\big) = 0.$$

Thus we have a surjective map

$$H^0(S, 2K_S) \longrightarrow H^0(\Gamma_1, \mathcal{O}_{\Gamma_1}) \oplus \cdots \oplus H^0(\Gamma_s, \mathcal{O}_{\Gamma_s}) = \bigoplus^s \mathbb{C},$$

contradicting $\Gamma \leq Z_2$. $\square$

Claim 1.7. *The horizontal part Z_h of Z_2 is either reduced, or $Z_h = 2H$ for an irreducible (-2)-curve H.*

Proof. If not, there is an irreducible curve H_1 with $Z_h - 2H_1 \neq 0$. By Claim 1.6 the multiplicities occurring in Z_2 are at most 2, and $Z_h \cdot C = 4$ implies that either $Z_h - 2H_1 = 2H_2$ for a reduced (-2)-curve H_2, or $Z_h - 2H_1$ is reduced. Write $H_2 = 0$ in the second case, such that in both cases

$$\tfrac{1}{2}Z_h - \llcorner \tfrac{1}{2}Z_h \lrcorner + H_2 \neq 0.$$

Consider the effective $\mathbb{Q}$-divisor $G = \frac{1}{2}Z_2 - H_2$. Obviously

$$K_S - G \equiv C + H_2$$

is nef. On the other hand,

$$2(K_S - \lfloor G \rfloor) \geq 2C + Z_h - 2\left\lceil \tfrac{1}{2} Z_h \right\rceil + 2H_2$$

is big. By the vanishing theorem (Theorem 1.1), we have

$$H^1(S, 2K_S - \lfloor G \rfloor) = 0.$$

The divisor $\lfloor G \rfloor \geq H_1$ is again the sum over reduced connected trees Γ_i of (-2)-curves, say

$$\lceil G \rceil = \Gamma_1 + \cdots + \Gamma_s.$$

Thus we have a surjective map

$$H^0(S, 2K_S) \longrightarrow H^0(\Gamma_1, \mathcal{O}_{\Gamma_1}) \oplus \cdots \oplus H^0(\Gamma_s, \mathcal{O}_{\Gamma_s}) = \bigoplus^{s} \mathbb{C},$$

contradicting $0 < 2\lfloor G \rfloor \leq 2G \leq Z_2$. □

Claim 1.8. *Z_h is either the sum of 4 disjoint sections of f or twice an irreducible curve H. Moreover $Z_v = 0$ in both cases.*

Proof. If $Z_h = 2H$ for an irreducible curve H, one has $Z_h^2 = -8$. Otherwise Claim 1.7 only leaves the possibility $Z_h = H_1 + \cdots + H_t$, for $t \leq 4$. In this case, $Z_h^2 \geq -2t \geq -8$, and $Z_h^2 = -8$ if and only if $t = 4$ and $H_i \cdot H_j = 0$ for $i \neq j$. The inequality

$$0 = 2K_S \cdot Z_h = 8 + Z_v \cdot Z_h + Z_h^2, \tag{1–1}$$

implies $Z_h^2 \leq -8$, and we obtain the first part of Claim 1.8.

In both cases (1–1) is an equality, hence $Z_v \cdot Z_h = 0$. Finally the equality

$$0 = 2K_S \cdot Z_v = 2C \cdot Z_v + Z_v^2 + Z_v \cdot Z_h$$

implies $Z_v^2 = 0$ and by the Index theorem $Z_v \equiv 0$. Since $Z_v \geq 0$ one finds $Z_v = 0$. □

Claim 1.9. *In Claim 1.8 the case $Z_h = 2H$ does not occur, and*

$$Z_h = H_1 + \cdots + H_4$$

implies $p_g(S) = q(S) = 0$.

Proof. Assume that $p_g(S) = 1$, and let D denote the effective canonical divisor. Then $2D = C_1 + C_2 + Z_h$ for fibres C_i of f. First of all this implies that the multiplicity of Z_h is divisible by 2, hence $Z_h = 2H$, and $C_1 + C_2$ must be divisible by 2 as well. Since for any divisor B the intersection number $B^2 + B \cdot K_S$ must be even, and since $C_i \cdot K_S = 2$, the fibres C_i cannot be divisible by two. Hence $C_1 = C_2$ and $D = C_1 + H$, a contradiction since $p_g(S) < h^0(S, \mathcal{O}_S(D)) = 2$.

If $p_g(S) = 0$, Proposition 1.5(i) implies $q(S) = 0$. In case $Z_h = 2H$ one finds that $K_S \equiv C + H$ and $\theta = \mathcal{O}_S(K_S - C - H)$ is a 2-torsion sheaf. The Riemann–Roch Theorem implies that $h^1(S, \theta) = 1$, contradicting Lemma 1.3. $\square$

It remains to exclude the existence of a minimal surface S of general type such that, letting $f : S \to \mathbb{P}^1$ be the bicanonical map, there exist a fibre C of f and pairwise disjoint (-2)-curves $H_1, \dots, H_4$ satisfying

$$2K_{S/\mathbb{P}^1} = 6C + H_1 + \cdots + H_4.$$

Write $H = H_1 + \cdots + H_4$. On some open dense subset $U \subset \mathbb{P}^1$ there is a natural involution $\imath$ on $f^{-1}(U)$ with quotient $f^{-1}(U) \to \mathbb{P}^1 \times U$. Since S is minimal, $\imath$ extends to an involution on S, denoted again by $\imath$. The equality

$$0 = 2K_S \cdot \imath(H_i) = 2C \cdot \imath(H_i) + (H_1 + H_2 + H_3 + H_4) \cdot \imath(H_i)$$

implies that $\imath(H_i) \in \{H_1, H_2, H_3, H_4\}$, hence $\imath(H) = H$. For U small enough, each effective bicanonical divisor of $f^{-1}(U)$ is the pullback of a divisor on $\mathbb{P}^1 \times U$, hence none of the H_i can be fixed under $\imath$. Renumbering we may assume that $\imath(H_1) = H_2$ and $\imath(H_3) = H_4$.

Let E be any (-2)-curve on S not equal to any of the H_i. The equality

$$0 = 2K_S \cdot E = 2C \cdot E + (H_1 + H_2 + H_3 + H_4) \cdot E$$

implies that $H_i \cdot E = 0$ for all i. Hence E is a component of a fibre not meeting the H_i.

Let E be any component of a fibre of f. If E does not meet H, then $E \cdot K_S = 0$, hence E is a (-2)-curve.

The morphism $\delta : S \to S'$ to the relative canonical model contracts exactly the (-2)-curves of the fibres. Hence all fibres of $f' : S' \to \mathbb{P}^1$ are reduced and all their components E' meet $H' = \delta(H)$. Moreover the intersection number $E \cdot K_S = E \cdot H$ on S is even. So the reducible fibres of f' have at most two components E'_1 and E'_2, both meeting H' in two points. The components E'_1 and E'_2 need not be Cartier divisors. However, $E'_1 + E'_2$ is Cartier, as are the images H'_i of the H_i.

We write $\imath'$ for the automorphism of S' induced by $\imath$. Since $p_g(S) = q(S) = 0$, the direct image $f_* \mathcal{O}_S(K_{S/\mathbb{P}^1})$ equals $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$. Consider the restriction map

$$\eta : {f'}_* \mathcal{O}_{S'}(K_{S'/\mathbb{P}^1}) = \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2} \longrightarrow \mathcal{O}_{H'_1}(2) = \mathcal{O}_{H_i}(K_{S'/\mathbb{P}^1} \cdot H'_1).$$

Since $\mathcal{O}_C(K_C)$ is generated by global sections, η is nonzero; hence its kernel is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(\epsilon)$, for $\epsilon = 0$ or 1. Let σ' be a general section of $\mathrm{Ker}(\eta)$, and let σ be the induced section of $\mathcal{O}_{S'}(K_{S'/\mathbb{P}^1})$. By construction, H'_1 lies in the zero locus B of σ. For some open dense $U \subset \mathbb{P}^1$ the divisor $B|_{f'^{-1}(U)}$ is invariant under $\imath'$.

Then the section σ is zero on $H'_1 + H'_2$. Altogether we have found an effective Cartier divisor D' with

$$\epsilon \cdot C + H'_1 + H'_2 + D' \sim K_{S'/\mathbb{P}^1}.$$

By construction, D' does not contain a whole fibre, so it is concentrated in the reducible fibres of f'. Let $f'^{-1}(p) = E'_1 + E'_2$ be one of such fibres, and let $\alpha_1 \cdot E'_1 + \alpha_2 \cdot E'_2$ be the part of D' concentrated in $f'^{-1}(p)$. Then one of the α_i must be zero — say α_1. Hence $\alpha_2 > 0$.

The divisor $\imath'^*(\alpha_2 \cdot E'_2)$ is the part of $\imath'^*(D')$ lying in $f'^{-1}(p)$. If $\imath'^*(E'_2) = E'_1$,

$$\alpha_2 \cdot E'_2 - \imath'^*(\alpha_2 \cdot E'_2) = \alpha_2 \cdot E'_2 - \alpha_2 \cdot E'_1$$

is the part concentrated in $f'^{-1}(p)$ of a divisor, linearly equivalent to zero. Then the same holds true for

$$\alpha_2 \cdot \delta^*(E'_2) - \alpha_2 \cdot \delta^*(E'_1).$$

Obviously this is not possible; hence E'_i is invariant under $\imath'$.

We may assume that $E'_1 \cap H'_1 \neq \varnothing$. The component E'_1 meets exactly one of the other H'_i, and being invariant under $\imath'$, this can only be H'_2. Write $D = \delta^*(D')$ and E_i for the proper transform of E'_i. If D' contains E'_2, it cannot contain E'_1, hence D does not contain E_1. Since

$$\epsilon \cdot C + H_1 + H_2 + D \sim K_{S/\mathbb{P}^1}$$

one finds $1 = E_1 \cdot K_{S/\mathbb{P}^1} \geq E_1 \cdot (H_1 + H_2) = 2$, obviously a contradiction. So D' only contains components of reducible fibres meeting H'_1 and H'_2 but neither H'_3 nor H'_4. So $D \cdot H_3 = 0$ and

$$H_3 \cdot (\epsilon \cdot C + H_1 + H_2 + D) = \epsilon < H_3 \cdot K_{S/\mathbb{P}^1} = 2,$$

a contradiction. □

2. Adjoint linear systems

Let S be a surface of general type, not necessary minimal, and let L be a divisor on S. Few criteria are known that imply that φ_{K_S+L} is generically finite, though the linear system $|K_S + L|$ is well understood (see [Reider 1988; Catanese 1990], for instance).

By [Xiao 1985b], for a surface S of general type with $q(S) \geq 3$ the map φ_{K_S} is generically finite; hence the same holds true for φ_{K_S+L} whenever $L \geq 0$. Moreover:

Proposition 2.1. *Let S be a smooth projective surface of general type and let L be an effective divisor on S with $h^0(S, \mathbb{O}_S(L)) > 2$. Then φ_{K_S+L} is generically finite.*

If $h^0(S, \mathcal{O}_S(L)) = 2$ obviously $|L|$ is composite with a pencil. The method used to prove the proposition will also show:

Addendum 2.2. *Assume in Proposition 2.1 that* $h^0(S, \mathcal{O}_S(L)) = 2$. *Then* φ_{K_S+L} *is generically finite, except possibly in one of the following cases*:

(a) $p_g(S) = 0$ *and* $|L|$ *is composite with a rational pencil of hyperelliptic curves.*

(b) $0 < q(S) \leq 2$ *and* $|L|$ *is composite with a rational pencil of curves of genus* $g = q(S) + 1$.

The next two examples shows that exceptional cases (a) and (b) do occur.

Example 2.3. In [Xiao 1985a, pp. 46–49], one finds an example of a surface S of general type with $p_g(S) = q(S) = 0$ and $K_S^2 = 2$, having a pencil $f : S \to \mathbb{P}^1$ of curves of genus 2. If C denotes a general fibre, then

$$H^0(S, \mathcal{O}_S(K_S + C)) = H^0(C, \mathcal{O}_C(K_C)) = \mathbb{C}^{\oplus 2},$$

and $|K_S + C|$ is composite with a rational pencil of genus-2 curves.

Example 2.4. Let C be a smooth curve of genus 2, and let θ be an invertible 2-torsion sheaf on C, with $\theta \neq \mathcal{O}_C$. For $T = \mathbb{P}^1 \times C$, let $p_1 : T \to \mathbb{P}^1$ and $p_2 : T \to C$ be the projections. For $a \geq 3$ consider

$$\delta = p_1^*(O(a)) \otimes p_2^*(\theta).$$

Since $\delta^2 \cong \mathcal{O}_T(D)$ for a nonsingular divisor D, one obtains a smooth double cover $\pi : S \to T$ with

$$\pi_*\mathcal{O}_S(K_S) = \mathcal{O}_T(K_T) \oplus \mathcal{O}_T(K_T) \otimes \delta.$$

It is easy to see that S is a minimal surface of general type, and that $|K_S|$ is composite with a pencil of curves of genus 3. In fact φ_{K_S} coincides with $f = p_1 \circ \pi$. For a general fiber C of f, choose $L = C$. Then $h^0(S, \mathcal{O}_S(L)) = 2$, but $|K_S + L|$ is composite with the same pencil as $|K_S|$.

Note that f is an isotrivial family of curves of genus 3, that

$$f_*\mathcal{O}_S(K_S) = \mathcal{O}_{\mathbb{P}^1}(a-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)^{\oplus 2},$$

and that $q(S) = 2$.

In Examples 2.3 and 2.4 the divisor L is nef, but not big.

Question 2.5. Does there exists a minimal surfaces S of general type and a nef and big divisor L on S with $h^0(S, \mathcal{O}_S(L)) = 2$, for which $|K_S + L|$ is composite with a pencil of curves?

Such examples exist on surfaces S of smaller Kodaira dimension, or on surfaces S of general type for $h^0(S, \mathcal{O}_S(L)) = 1$:

Example 2.6. Let $f : S \to \mathbb{P}^1$ be a family of elliptic curves admitting a section G, and with S nonsingular and projective. For a general fibre C of f choose $L_m = mF + G$. Then $h^0(S, \mathcal{O}_S(L_m)) = m + 1$ and L_m is nef and big whenever $m > \mathrm{Max}\{0, -\frac{1}{2}G^2\}$. However $|K_S + L_m|$ is always composite with a pencil.

Example 2.7. Let S be a minimal surface of general type with $K_S^2 = 1$ and $p_g(S) = q(S) = 0$. Denote by L a divisor numerically equivalent to K_S. Then $h^0(S, \mathcal{O}_S(L)) \leq 1$ and $h^0(S, \mathcal{O}_S(K_S + L)) = 2$. Thus $|K_S + L|$ is automatically composite with a rational pencil of curves. See [Reid 1978] for a classification of such pairs (S, L).

Proof of Proposition 2.1 and Addendum 2.2. Replacing S by a blowing up, we may assume that the moving part of L has no fixed points, hence that φ_L is a morphism.

Consider first the case that $|L|$ is composite with a pencil of curves. Take the Stein factorization

$$g : S \xrightarrow{f} B \xrightarrow{\rho} \mathbb{P}(H^0(S, \mathcal{O}_S(L))), \tag{2–1}$$

so f is a pencil of curves of genus $g \geq 2$. As in the proof of Proposition 1.4, one easily sees that $h^0(S, \mathcal{O}(L)) > 2$ implies that $L \geq C_1 + C_2$ for two fibres C_i of f. The same holds true for $h^0(S, \mathcal{O}_S(L)) = 2$, if ρ is not an isomorphism. In both cases we may as well assume that $L = C_1 + C_2$.

As explained in [Esnault and Viehweg 1992, 7.18], Kollár's vanishing theorem implies that the locally free sheaf $f_*\mathcal{O}_S(K_{S/B})$ is numerically effective, and that $\mathcal{E} = f_*\mathcal{O}_S(K_S + C_1 + C_2)$ is generated by global sections. Hence the tautological sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ on the projective bundle $\mathbb{P}(\mathcal{E})$ is globally generated.

If the genus $g(B)$ is positive, as a tensor product of a numerically effective vector bundle with an invertible sheaf of positive degree, $\mathcal{E}$ is ample.

If $B \cong \mathbb{P}^1$ the sheaf $\mathcal{E} = f_*\mathcal{O}_S(K_{S/B})$ is a direct sum of line bundles of nonnegative degree, say $\nu_1 \leq \nu_2 \leq \cdots \leq \nu_g$. If $q(S) = 0$, the Leray spectral sequence yields $H^1(\mathbb{P}^1, f_*\mathcal{O}_S(K_S)) = 0$, hence $\nu_1 > 0$. If $q(S) \neq 0$, one has $p_g(S) > 0$, hence $\nu_g \geq 2$.

Altogether, in both cases the sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is globally generated and big. The sheaf φ_{K_S+L} factors as

$$S \xrightarrow{\varphi} \mathbb{P}(\mathcal{E}) \xrightarrow{\varphi'} \mathbb{P}^M, \tag{2–2}$$

where φ is the relative canonical map and φ' the rational map induced by global sections of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. Since the genus of the fibres of f is at least two, φ is generically finite. $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ and its restriction to the closure of the image of φ are globally generated and big; hence φ_{K_S+L} is generically finite.

Before finishing the proof of Proposition 2.1 we look at the case where

$$h^0(S, \mathcal{O}_S(L)) = 2 \quad \text{and} \quad B \xrightarrow{\cong} \mathbb{P}^1$$

in (2–1). Here we may assume that $L = C$ for a general fibre of $f : S \to \mathbb{P}^1$. Write again $f_*\mathcal{O}_S(K_{S/B})$ as a direct sum of line bundles of nonnegative degrees $\nu_1 \le \nu_2 \le \cdots \le \nu_g$. If φ_{K_S+L} is composite with a pencil, [Xiao 1985b] implies that $q(S) < 3$. Note that $\nu_i = 0$ for $i = 1, \ldots, q(S)$.

If $p_g(S) > 0$, one also knows that $\nu_g \ge 2$. Hence if $g > q(S) + 1$, the sheaf $f_*\mathcal{O}_S(K_S+C)$ contains a subbundle $\mathcal{E}$ of rank ≥ 2 which is globally generated and nontrivial, that is, not the direct sum of copies of $\mathcal{O}_{\mathbb{P}^1}$. For this bundle consider again the maps (2–2). The first one, φ, is fibrewise given by at least two independent sections of the canonical linear system, hence it is generically finite. Since $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ and its restriction to the image of φ are again generated by global sections and big, $\varphi' \circ \varphi$ is generically finite and one obtains Addendum 2.2, for $p_g(S) > 0$.

If $p_g(S) = 0$, so that $q(S) = 0$, then $\nu_1 = \cdots = \nu_g = 1$, and $\mathcal{E} = f_*\mathcal{O}_S(K_S+C)$ is trivial. Then $\mathbb{P}(\mathcal{E}) = \mathbb{P}^1 \times \mathbb{P}^{g-1}$ and φ in (2–2) is generically finite, whereas φ' is the projection to the second factor. The restriction of φ_{K_S+L} to a smooth fibre F coincides with $|K_F|$. So for F nonhyperelliptic, the assumption that $|K_S + L|$ is composite with a pencil implies that all smooth fibres F are isomorphic and that $(\varphi_L, \varphi_{K_S+L})$ is a birational map $S \to \mathbb{P}^1 \times F$, a contradiction.

To finish the proof of Proposition 2.1 it remains to consider the case that φ_L is generically finite. If $p_g(S) > 0$, the linear system $|L|$ is a subsystem of $|K_S + L|$; hence the latter cannot be composite with a pencil of curves.

For $p_g(S) = q(S) = 0$, blowing up S if necessary, we assume that both φ_{K_S+L} and φ_L are morphisms, hence that the movable parts M of $K_S + L$ and L^0 of L have no fixed points. Replacing L by L^0 we may assume L to be big and globally generated.

Take the Stein factorization

$$\varphi_{K_S+L} : S \xrightarrow{h} B \longrightarrow \mathbb{P}\big(H^0(S, \mathcal{O}_S((K_S+L)-1))\big).$$

If φ_{K_S+L} is not generically finite, h is a fibration onto a smooth curve B with general fibre C. One may write $M \sim \sum_{i=1}^{a} C_i$ for fibres C_i of h and for $a \ge h^0(S, \mathcal{O}_S(K_S+L)) - 1$. Noting that

$$h^0(S, \mathcal{O}_S(K_S+L)) = \tfrac{1}{2} L \cdot (K_S+L) + \chi(\mathcal{O}_S) = \tfrac{1}{2} L \cdot (K_S+L) + 1,$$

one obtains the inequality

$$L \cdot (K_S+L) \ge L \cdot M \ge \big(\tfrac{1}{2} L \cdot (K_S+L)\big)(L \cdot C);$$

hence $1 \le L \cdot C \le 2$.

Consider next the natural map

$$H^0(S, \mathcal{O}_S(L)) \xrightarrow{\alpha} W \subset H^0(C, \mathcal{O}_C(L|_C)),$$

with W the image of α. Because $|L|$ is not composite with a pencil,

$$h^0(C, \mathcal{O}_C(L|_C)) \geq \dim_{\mathbb{C}} W \geq 2.$$

Noting that the genus $g(C)$ is at least 2, one has $h^0(C, \mathcal{O}_C(\Gamma)) \leq j$ whenever Γ is a divisor with $1 \leq \deg \Gamma \leq j$. Hence

$$h^0(C, \mathcal{O}_C(L|_C)) = \dim_{\mathbb{C}} W = L \cdot C = 2.$$

This implies that $h^0(S, \mathcal{O}_S(L - C)) \geq 1$ and $L - C \geq 0$. Since

$$|K_S + C|\big|_C = |K_C|,$$

one finds $\dim \varphi_{K_S+L}(C) = 1$, contradicting the choice of C as a fibre of h. □

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