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We consider vector bundles with fibre a loop space and structure group a loop group. We determine necessary and sufficient conditions for the structure group to reduce to the group of constant loops in terms of the existence of certain finite-dimensional subbundles of the original vector bundle.

1. Introduction

In this paper we study subbundles of loop bundles. Following [Cohen and Stacey 2004], we define a rank-n loop bundle to be a vector bundle $\xi \to X$ with fibre $L\mathbb{C}^n$ and structure group LGL_n. We will find it convenient to assume a reduction of the structure group to LU_n, which is equivalent to choosing a fibrewise inner product on ξ isomorphic in charts to the standard inner product on $L\mathbb{C}^n$.

The problem we wish to consider is this: suppose $\zeta \to X$ is a finite-dimensional vector bundle over X; what are the implications for the structure groups of ζ and of ξ of the statement that ζ is a subbundle of ξ?

The idea behind this problem was to study a loop bundle ξ by considering an increasing sequence of finite-dimensional subbundles ξ_k that approximate it. The aim is to use this sequence to transfer finite-dimensional constructions to infinite dimensions via a limiting sequence, much in the spirit of the construction of the Witten genus in [Witten 1988]. This idea of approximating an infinite-dimensional vector bundle by a filtration of finite-dimensional subbundles has proved very successful in examining the structure of Hilbert manifolds; see for example [Elworthy 1968] and [Eells and Elworthy 1971]. The vector bundles we consider inject naturally into bundles of Hilbert spaces and thus can be trivialised using the contractibility of the unitary group of a Hilbert space [Kuiper 1965]. One can then impose a filtration, referred to as a layer structure in [Eells and Elworthy 1971], and ask whether this is in fact a filtration of the original loop bundle. Under certain reasonable conditions on the filtration, we show that it can only be so in very special circumstances.

Our reasonable condition imposed on the ξ_k is that there be some k such that the family $\{z^n \xi_k : n \in \mathbb{Z}\}$ is fibrewise dense in ξ. Our main result can be summarised

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thus: if there is a finite-dimensional subbundle ζ for which $\{z^n\zeta : n \in \mathbb{Z}\}$ is fibrewise dense in ξ then the structure group of ξ reduces from LU_n to U_n.

The main example of a loop bundle is the tangent bundle of the free loop space of a smooth finite-dimensional manifold M. In [Morava 2001] and [Cohen and Stacey 2004], the question arose as to the existence of a circle equivariant subbundle of TLM nonequivariantly isomorphic to the pull back of TM via an evaluation map. In [Cohen and Stacey 2004], the existence of such a bundle was found to be a strong condition, one of the implications being that the tangent bundle of the based loop space is parallelisable. In this paper we consider a more general setting but the conclusion stays the same.

To state our result precisely, we need to define a class of subspaces of $L\mathbb{C}^n$:

Definition 1.1. A finite-dimensional subspace W of $L\mathbb{C}^n$ is *almost generating* if there is some dense subset T of S^1 such that the natural map $C^\infty(T, \mathbb{C}) \otimes W \to C^\infty(T, \mathbb{C}^n)$ is surjective. It is *generating* if we can take $T = S^1$.

Here we are using the natural restriction map from $L\mathbb{C}^n = C^\infty(S^1, \mathbb{C}^n)$ to $C^\infty(T, \mathbb{C}^n)$ in order to consider W as a subspace of $C^\infty(T, \mathbb{C}^n)$. We will see that we can take T open in S^1. This concept can be extended fibrewise to finite-dimensional subbundles of loop bundles, leading to:

Theorem 1.2. (1) *A loop bundle $\xi \to X$ admits an almost generating subbundle $\zeta \to X$ if and only if the structure group of ξ reduces to U_n, viewed as the constant loops in LU_n. In this case, there is an isomorphism $\xi \cong L\mathbb{C} \otimes \psi$ where ψ is an n-dimensional vector bundle over X.*

(2) *A loop bundle $\xi \to X$ admits a reduction of its structure group to a compact group if and only if it admits a reduction of its structure group to U_n.*

(3) *If ζ is a finite-dimensional subbundle of a loop bundle ξ, the structure group of ζ reduces to $U_{n_1} \times \cdots \times U_{n_k}$ for $n_1, \ldots, n_k \le n$ with $n_1 + \cdots + n_k = \dim \zeta$.*

We note that we are being very precise with the notion of subbundles. We shall expand on this in Section 2.

Our method of attack for this problem is to reduce it to the question of homomorphisms $\rho : G \to LU_n$, where G is a compact Lie group. The adjoint map is thus a path of homomorphisms $\tilde{\rho} : S^1 \times G \to U_n$. From this point of view, the problem bears a strong resemblance to the study of subgroups of compact Lie groups using homology theory as found in [Dynkin 1952], [Brown 1978], and [Gelbrich 1991]. Our question is closest to that studied by [Dynkin 1952] although we consider a slightly different angle and in fact use [Gelbrich 1991] at a key step.

We work over the field of complex numbers. This is mostly to avoid phrases like “unitary (resp., orthogonal)” that would otherwise pervade this text. We can always complexify a real representation to apply the theorems above to the real case. Thus we shall henceforth assume $\mathbb{C}$ and write, for example, M_n for $M_n(\mathbb{C})$, the algebra of complex $n \times n$ matrices.

It is also convenient to fix a category of differentiability of maps. We choose to work in the smooth category, after our original motivation for the question, although all results hold in the continuous category and any stronger one. For a smooth manifold M, therefore, LM denotes the space of smooth loops in M. We shall often use the natural inclusions $U_n \subseteq LU_n \subseteq LGL_n \subseteq LM_n$ and we shall identify each space with its image in LM_n unless otherwise stated.

The paper is organised as follows: in Section 2 we consider the definition of subbundles and show how this translates our question into one on Lie groups. We also prove Theorem 1.2(3). In Section 3 we prove Theorem 1.2(1) and (2).

2. Subbundles

The aspect of vector bundles and their subbundles that we wish to emphasise is the rôle of the structure group. For finite-dimensional vector bundles this is an unnecessary elaboration because the structure group is usually assumed to be GL_n. In infinite dimensions, the general linear group $GL(E)$ is in general either contractible or not a regular Lie group and therefore one usually has a smaller Lie group as structure group. In this case, not all candidates for subbundles will be compatible with the smaller structure group. We make this precise by considering the standard definitions. For simplicity, we assume that X is a connected smooth manifold.

Definition 2.1. (1) Let V be a vector space and G a group acting linearly on V. A *vector bundle* over X with *typical fibre* V and *structure group* G consists of a smooth manifold ξ, a surjective map $\pi : \xi \to X$, and a smooth atlas $\mathcal{U}$ of X such that

(a) for $U \in \mathcal{U}$ there is a diffeomorphism $\phi_U : \pi^{-1}(U) \to U \times V$ such that $\pi|_{\pi^{-1}(U)} = p_U \circ \phi_U$, and

(b) for $U_1, U_2 \in \mathcal{U}$ with $U_1 \cap U_2 \neq \emptyset$, the adjoint of the map $\phi_1 \circ \phi_2^{-1}$: $U_1 \cap U_2 \times V \to U_1 \cap U_2 \times V$ is a map $U_1 \cap U_2 \to G$.

(2) A vector bundle $\xi \to X$ with structure group G admits a *reduction* to $H \subseteq G$ if there is an atlas $\mathcal{V}$ of X compatible with $\mathcal{U}$ such that condition (b) holds with $\mathcal{V}$ replacing $\mathcal{U}$ and H replacing G. (Recall that $\mathcal{V}$ is *compatible with* $\mathcal{U}$ if it is contained in a maximal atlas $\mathcal{U}' \supseteq \mathcal{U}$, all atlases being assumed to satisfy the conditions in (1).)

(3) To say that ζ is a *subbundle* of ξ is to give a smooth atlas $\mathcal{V}$ of X compatible with the atlases $\mathcal{U}_\zeta$ of ζ and $\mathcal{U}_\xi$ of ξ, an inclusion map $i : \zeta \to \xi$ such

that $\pi_\xi \circ i = \pi_\zeta$, and a linear inclusion $j : V_\zeta \to V_\xi$ such that for $U \in \mathcal{V}$, $(\mathrm{id} \times j) \circ \phi_U^\zeta = \phi_U^\xi \circ i : \pi_\zeta^{-1}(U) \to U \times V_\xi$.

The key ingredient in all of this is the compatibility betwixt the atlases. From these definitions, the following proposition is immediate:

Proposition 2.2. *Suppose that $(\zeta, V_\zeta, G_\zeta)$ is a subbundle of (ξ, V_ξ, G_ξ) over X. Then ξ admits a reduction to the stabiliser group of $V_\zeta \subseteq V_\xi$ in G_ξ and ζ admits a reduction to the image of this group in G_ζ.*

As a consequence of this proposition, to prove Theorem 1.2(1) it is sufficient to show that the stabiliser subgroup of an almost generating subspace of $L\mathbb{C}^n$ is conjugate in LU_n to a subgroup of U_n. To prove (2), it is then sufficient to show that any compact subgroup of LU_n has an invariant, generating subspace. We do these things in Section 3. To prove Theorem 1.2(3) we need to examine the image of the stabiliser group of a general $V \subseteq L\mathbb{C}^n$ in $GL(V)$. This is simpler and we do it here.

Proof of Theorem 1.2(3). Let $V \subseteq L\mathbb{C}^n$ be a finite-dimensional subspace and $G \subseteq LU_n$ its stabiliser subgroup. For any $t \in S^1$, the evaluation map $e_t : L\mathbb{C}^n \to \mathbb{C}^n$ is LU_n-equivariant and thus G-equivariant. Using this, we can decompose V into orthogonal subspaces $V_1 \oplus \cdots \oplus V_k$ with $\dim V_k \leq n$ and where G acts on V_i via an evaluation map $e_{t_i} : G \to U_n$. To do this, we fix some $t_1 \in S^1$ for which $e_{t_1} : V \to \mathbb{C}^n$ is not the zero map. The kernel of this is a G-equivariant subspace of V. As G acts unitarily on V, the orthogonal complement of V is also G-invariant and is carried G-equivariantly into $\mathbb{C}^n$ by e_{t_1}. Since V is finite-dimensional, induction on the dimension of V yields the decomposition above. The image of G thus lies in $U_{n_1} \times \cdots \times U_{n_k}$ lying inside $U(V)$ in the natural way. The integers n_i satisfy $n_i \leq n$ for all i and $n_1 + \cdots + n_k = \dim V$.

This is the best general description, as can be shown by the following case: let $k, m, n_1, \ldots, n_k \in \mathbb{N}$ be such that $n_j \leq n$ for each j and $n_1 + \cdots + n_k = m$. Choose k distinct, nonvoid, closed, connected subsets $T_1, \ldots, T_k$ of S^1 and subordinate bump functions $\rho_1, \ldots, \rho_k : S^1 \to [0, 1]$. Let $V_i = \langle \rho_i e_1, \ldots, \rho_i e_{n_i} \rangle$ and $V = V_1 \oplus \cdots \oplus V_k$ (the sum is direct since the T_i are distinct). As U_{n_i} is connected and the T_i are distinct, for $x \in U_{n_i}$ there is some $g : S^1 \to U_n$ such that $g|_{T_j} = 1$ for $j \neq i$ and $g|_{T_i} = x$. By construction, $g \in G$ and thus the image of G in $U(V)$ is precisely $U_{n_1} \times \cdots \times U_{n_k}$. $\square$

3. Compact Lie groups

In this section we prove Theorem 1.2(1) and (2). Our argument proceeds as follows: We first show that in LM_n, finite-dimensional subalgebras have a very precise description. This gives a similar description for the group of units of such an algebra.

We then show that the stabiliser algebra in LM_n of a finite-dimensional subspace of $L\mathbb{C}^n$ is finite-dimensional if and only if the subspace is almost generating. This leads to the proof of Theorem 1.2(1). Finally, we adapt the Peter–Weyl theorem to show that any compact subgroup of LU_n has an almost generating invariant subspace. This leads to the proof of Theorem 1.2(2). We also give an interpretation of this in terms of compact subgroups of LU_n and ΩU_n, namely:

Theorem 3.1. (1) *The only compact subgroups of LU_n are conjugate to subgroups of U_n.*

(2) *There are no compact subgroups of ΩU_n.*

Although we work with LU_n throughout, parts of our results work for LGL_n as well. We use LU_n because from the topological point of view there is no difference between LU_n and LGL_n and the theory for LU_n is slightly simpler than that for LGL_n.

Finite-dimensional subalgebras of LM_n. The aim of this section is to prove the following theorem:

Theorem 3.2. *Let $\rho : G \to LU_n$ be a Lie group homomorphism such that the image of ρ lies in a finite-dimensional subalgebra of LM_n. Then there is some $\gamma \in LU_n$ and a Lie group homomorphism $\tilde{\rho} : G \to U_n$ such that $\rho = c_\gamma \tilde{\rho}$, where $c_\gamma : LU_n \to LU_n$ is the automorphism corresponding to conjugation by γ.*

Corollary 3.3. *Let $\xi \to X$ be a vector bundle with fibre $L\mathbb{C}^n$ and structure group LU_n. If ξ admits a reduction of structure group to G, where $G \subseteq LU_n$ is contained in a finite-dimensional subalgebra of LM_n, then ξ admits a reduction of structure group to U_n and there is an isomorphism of vector bundles $\xi \cong L\mathbb{C} \otimes \psi$, where $\psi \to X$ is an n-dimensional vector bundle.*

Proof. From Theorem 3.2, G is a subgroup of U_n and is included in LU_n via a conjugation of the natural one. If $P \to X$ is the principal G-bundle then the representations defined by the natural inclusion in LU_n and the conjugated inclusion are isomorphic. Thus $\xi \cong P \times_G L\mathbb{C}^n$, where G acts on $L\mathbb{C}^n$ by constant loops. Within this lies the bundle $\psi \to X$ defined by $P \times_G \mathbb{C}^n$ and, moreover, $P \times_G L\mathbb{C}^n \cong L\mathbb{C} \otimes (P \times_G \mathbb{C}^n)$. Hence $\xi \cong L\mathbb{C} \otimes \psi$, as required. □

In [Cohen and Stacey 2004], the bundle ψ is referred to as the *underlying (finite-dimensional) vector bundle* associated to ξ.

The key to the proof of Theorem 3.2 is to consider the characteristic polynomial of $a(t)$ for $a \in LU_n$.

Lemma 3.4. *Let $a \in LM_n$ have a minimum polynomial. Then the characteristic polynomial of $a(t)$ does not depend on t.*

Proof. Let $p(x)$ be the minimum polynomial of a, so $p(a) = 0$. Evaluation at $t \in S^1$ yields $p(a(t)) = 0$ and so the minimum polynomial of $a(t)$ divides $p(x)$. Thus the characteristic polynomial of $a(t)$ is a degree-n monic divisor of $p(x)^n$. As such divisors are classified by their roots, they form a discrete subset of the space of degree-n monic polynomials in $\mathbb{C}$. Thus the map that sends $t \in S^1$ to the characteristic polynomial of $a(t)$ takes values in a discrete set. Being continuous, this map is therefore constant. □

Corollary 3.5. *Let $a \in LM_n$ have a minimum polynomial. Then $a \in LGL_n$ if and only if $a(t_0) \in GL_n$ for some $t_0 \in S^1$.*

Proof. A matrix is invertible if and only if 0 is not a root of its characteristic polynomial. Since $a(t)$ and $a(t_0)$ have the same characteristic polynomial, $a(t)$ is invertible for all t if and only if $a(t_0)$ is invertible. □

Lemma 3.6. *Let $a \in LU_n$ have a minimum polynomial. Then $a(t)$ is conjugate to $a(0)$ for all t.*

Proof. We have $a(t) \in U_n$ for all t. All unitary operators are diagonalisable, so each $a(t)$ is conjugate to a diagonal matrix, with diagonal entries given by the roots of the characteristic polynomial of $a(t)$. As this is independent of t, each $a(t)$ is conjugate to the same diagonal matrix and hence to $a(0)$. □

Corollary 3.7. *Let $a \in LU_n$ have a minimum polynomial. If $a(t_0) = 1$ for some $t_0 \in S^1$ then $a(t) = 1$ for all $t \in S^1$.*

Proof of Theorem 3.2. Assume without loss of generality that $\rho : G \to LU_n$ is injective. Take $g \in \rho(G)$, considered as an element of LM_n. Since g lies in a finite-dimensional subalgebra of LM_n, the subalgebra generated by it is finite-dimensional and hence g has a minimum polynomial in LM_n.

By Corollary 3.7, $e_t\rho : G \to U_n$ is an injective Lie group homomorphism for all $t \in S^1$. Moreover, by Lemma 3.6, the homomorphisms $e_t\rho$ are *pointwise conjugate*; that is, for each $g \in G$ and $t \in S^1$, there is some $x_{g,t} \in U_n$ such that $e_t\rho(g) = x_{g,t}^{-1} e_0\rho(g) x_{g,t}$.

From [Gelbrich 1991], there is thus a path $\gamma : [0, 1] \to U_n$ with $\gamma(0) = 1$ such that $c_{\gamma(t)} e_0\rho = e_t\rho$, where $c_{\gamma(t)} : U_n \to U_n$ is conjugation by $\gamma(t)$. Since $e_1 = e_0 : LU_n \to U_n$, we have $e_1\rho = e_0\rho$ and thus $\gamma(1)$ lies in the centraliser of $H = e_1\rho(G)$ in U_n.

Let $g \in U_n$ lie in the centraliser of H. Since g commutes with all elements of H, its eigenspaces are H-invariant subspaces of $\mathbb{C}^n$, and so any element of U_n that acts by scalars on the eigenspaces of g will also commute with all elements of H and so lie in the centraliser of H. There is a path from the identity to g through such elements and thus the centraliser of H in U_n is connected.

Thus we can find a path β from $1 = \gamma(0)$ to $\gamma(1)$ lying in the centraliser of H. Hence $c_{\gamma(t)\beta(t)} e_0 \rho = c_{\gamma(t)} e_0 \rho$, and so replacing γ by $\gamma' = \gamma\beta$ yields a loop in U_n such that $c_{\gamma'(t)} e_0 \rho = e_t \rho$.

Therefore $\rho = c_{\gamma'(t)} e_0 \rho = c_{\gamma'(t)} \tilde{\rho}$, where $\tilde{\rho} = e_0 \rho : G \to U_n$ is a Lie group homomorphism. □

Almost generating subspaces of $L\mathbb{C}^n$. We now consider finite-dimensional almost generating subspaces of $L\mathbb{C}^n$. We give an alternative way to characterise them, from which we prove the following proposition.

Proposition 3.8. *Let $V \subseteq L\mathbb{C}^n$ be a finite-dimensional subspace. The stabiliser algebra of V is finite-dimensional if and only if V is almost generating.*

Combined with Theorem 3.2, this yields Theorem 1.2(1), because if $\rho : G \to LU_n$ is such that there is a G-invariant almost generating subspace of $L\mathbb{C}^n$ then $\rho(G)$ lies in the stabiliser subalgebra of that subspace. From Theorem 3.2, this implies that ρ is conjugate in LU_n to a map that factors through the inclusion $U_n \to LU_n$.

Proposition 3.9. *A finite-dimensional subspace V of $L\mathbb{C}^n$ is almost generating if and only if there is a dense subset T of S^1 such that the maps $e_t|_V \colon V \to \mathbb{C}^n$ are surjective for $t \in T$.*

As part of the proof, we will see that the set T can be taken to be open.

Proof. Let V be a finite-dimensional subspace of $L\mathbb{C}^n$ of dimension m. Consider the function $r : S^1 \to \mathbb{N}$ given by $r(t) = \dim \operatorname{Im} e_t|_V$. Choose a basis $\{v_i : 1 \le i \le m\}$ for V.

Let $t \in S^1$. As $\{v_i(t) : 1 \le i \le m\}$ spans $e_t(V)$, which is of dimension $r(t)$, there is some subset N_t of $\{1, \dots, m\}$ of size $r(t)$ such that $\{v_i(t) : i \in N_t\}$ is a basis for $e_t(V)$. In particular, $\{v_i(t) : i \in N_t\}$ is a linearly independent set. Because the v_i are continuous, there is some neighbourhood U_t of t for which $\{v_i(s) : i \in N_t\}$ is linearly independent for $s \in U_t$.

One direct consequence of this is that $\dim e_s(V) \ge r(t)$ for $s \in U_t$. Hence the function r has a local minimum at t. As t was arbitrary, r has local minima at all points of S^1. In particular, the set $T := r^{-1}(n)$ is open in S^1.

Using the method prescribed, there is an open covering $\mathcal{U}$ of T such that for each $U \in \mathcal{U}$, there is an n element subset N_U of $\{1, \dots, m\}$ with the property that $\{v_i(t) : i \in N_U\}$ is a basis for $\mathbb{C}^n$ for $t \in U$. Choose a smooth partition of unity $\{\rho_\lambda : \lambda \in \Lambda\}$ subordinate to $\mathcal{U}$. For $\lambda \in \Lambda$, let $U_\lambda \in \mathcal{U}$ be such that the support of ρ_λ is contained in U_λ. Let N_λ be the corresponding n-element subset of $\{1, \dots, m\}$.

Let $w \in C^\infty(T, \mathbb{C}^n)$. For $\lambda \in \Lambda$, as $\{v_i(t) : i \in N_\lambda\}$ is a basis for $\mathbb{C}^n$ for $t \in U_\lambda$, there are smooth functions $f_i^\lambda : U_\lambda \to \mathbb{C}$ such that $\sum_{i \in N_\lambda} f_i^\lambda(t) v_i(t) = w(t)$ for $t \in U_\lambda$. For $i \notin N_\lambda$, define $f_i^\lambda = 0$ on U_λ. Let $f_i : T \to \mathbb{C}$ be the smooth function

$f_i = \sum_{\lambda \in \Lambda} \rho_\lambda f_i^\lambda$. Since the ρ_λ form a partition of unity, $\sum_{i=1}^m f_i(t) v_i(t)$ equals $w(t)$ for $t \in T$ and w lies in the image of the map $C^\infty(T, \mathbb{C}) \otimes V \to C^\infty(T, \mathbb{C}^n)$, which is thus demonstrated to be surjective.

Conversely, suppose that $C^\infty(T, \mathbb{C}) \otimes V \to C^\infty(T, \mathbb{C}^n)$ is surjective for some $T \subseteq S^1$. Choose $w \in \mathbb{C}^n$ and $t \in T$. Denote by w_c the constant map $T \to \mathbb{C}^n$, $t \mapsto w$. From the surjectivity and using the basis $\{v_1, \dots, v_m\}$ of V, there are $f_1, \dots, f_m : T \to \mathbb{C}$ such that $\sum f_i v_i = w_c$. In particular, $\sum f_i(t) v_i(t) = w$. Set $\nu_i = f_i(t)$ and consider the element $\sum \nu_i v_i \in V$. At t, this evaluates to $\sum \nu_i v_i(t) = \sum f_i(t) v_i(t) = w$. Hence $e_t : V \to \mathbb{C}^n$ is surjective for $t \in T$.

Thus $C^\infty(T, \mathbb{C}) \otimes V \to C^\infty(T, \mathbb{C}^n)$ is surjective if and only if $e_t : V \to \mathbb{C}^n$ is surjective for all $t \in T$. Hence V is almost generating if and only if there is a dense set $T \subseteq S^1$, which we may assume to be open, such that $e_t : V \to \mathbb{C}^n$ is surjective for all $t \in T$. V is generating if and only if we are able to take $T = S^1$. □

Proposition 3.10. *Let $V \subseteq L\mathbb{C}^n$ be a finite-dimensional subspace. Its stabiliser algebra A_V is finite-dimensional if and only if V is almost generating.*

Proof. Because A_V is the stabiliser algebra of V, there is an algebra homomorphism $A_V \to \operatorname{End} V$. If this is injective, this demonstrates that A_V is finite-dimensional. If it is not injective, we shall show that its kernel contains a subalgebra isomorphic to $\Omega_\flat \mathbb{C}$, the algebra of basepoint-preserving smooth functions on S^1 that are infinitely flat at 1.

Suppose that V is almost generating and let $a \in A_V$ lie in the kernel of $A_V \to \operatorname{End} V$. Thus $av = 0$ for all $v \in V$. Let $T \subseteq S^1$ be as in the definition of almost generating for V. Let $t \in T$ and $w \in \mathbb{C}^n$. By Proposition 3.9, there is some $v \in V$ such that $v(t) = w$. Hence $0 = a(t)v(t) = a(t)w$. As w is arbitrary in $\mathbb{C}^n$, $a(t) = 0$. The continuity of a combined with the density of T in S^1 now show that $a = 0$. Hence $A_V \to \operatorname{End} V$ is injective.

Suppose that V is not almost generating. Let $T \subseteq S^1$ be the open set on which $\dim e_t(V) = n$. As V is not almost generating, $\overline{T} \neq S^1$. Let l be the maximum value of $\dim e_t(V)$ on $S^1 \setminus \overline{T}$; this value is attained. From the proof of Proposition 3.9, the set of $t \in S^1 \setminus \overline{T}$ where $\dim e_t(V) = l$ is open. Choose $t_0 \in S^1 \setminus \overline{T}$ such that $\dim e_{t_0}(v) = l$.

There are vectors $v_1, \dots, v_l \in V$ such that $\{v_1(t_0), \dots, v_l(t_0)\}$ is a linearly independent set at t_0. Extend this to a basis for $\mathbb{C}^n$ by adding $w_{n-l}, \dots, w_n$. There is an open contractible neighbourhood U of t_0 for which the set

$$\{v_1(t), \dots, v_l(t), w_{n-l}, \dots, w_n\}$$

is a basis for $\mathbb{C}^n$ for $t \in U$ with $e_t(V)$ spanned by $\{v_1(t), \dots, v_l(t)\}$.

Let $\Omega_U\mathbb{C}$ be the space of smooth $\mathbb{C}$-valued functions on S^1 with support in U. As U is a nonempty open contractible subset of S^1, it is diffeomorphic to $(0, 1)$ and such a diffeomorphism in turn defines a diffeomorphism $\Omega_U\mathbb{C} \cong \Omega_\flat\mathbb{C}$.

For $t \in U$, let $P_t : \mathbb{C}^n \to \mathbb{C}^n$ be the projection onto $\langle w_n\rangle$ with kernel spanned by $\{v_1(t), \dots, v_l(t), w_{n-l}, w_{n-1}\}$. Define an action of $\Omega_U\mathbb{C}$ on $L\mathbb{C}^n$ by $f \cdot w(t) = f(t)P_t w(t)$ for $t \in U$ and zero elsewhere. That f is zero outside U shows that this is well defined.

By construction, for $v \in V$ and $t \in U$, $P_t v(t) = 0$ and hence $f \cdot v = 0$. Thus $f \in A_V$ and, moreover, $f \in \ker A_V \to \operatorname{End} V$. Thus $\ker A_V \to \operatorname{End} V$ contains a subalgebra isomorphic to $\Omega_\flat\mathbb{C}$. (It is possible to extend this argument to determine $\ker A_V \to \operatorname{End} V$ precisely, but the proposition is already proved.) □

Compact Lie groups. We now show that every compact Lie group acting on $L\mathbb{C}^n$ through a representation $\rho : G \to LU_n$ has an invariant generating subspace. This implies Theorem 1.2(2) directly. Rather than considering the action of G on $L\mathbb{C}^n$, we consider the induced action of G on LM_n. We wish to find vectors in $L\mathbb{C}^n$ that generate a finite-dimensional G-invariant generating subspace of $L\mathbb{C}^n$, and considering LM_n allows us to find these vectors all in one go.

We shall use aspects of the Peter–Weyl theorem for representations of compact Lie groups. As usually stated, the Peter–Weyl theorem deals with unitary representations on Hilbert spaces. This is not readily applicable to our situation. However, the proof of the Peter–Weyl theorem actually applies in a stronger case, sufficient for our purposes. We refer to [Knapp 1988] for the necessary machinery. In that reference, the proof of the Peter–Weyl theorem is only given for unitary groups. Thus to derive the necessary extension for all compact Lie groups, we first use the Peter–Weyl theorem to deduce that all compact Lie groups are isomorphic to some subgroup of a unitary group and then use the proof of the Peter–Weyl theorem given by Knapp to deduce the strengthening we require.

The following definitions are from [Knapp 1988, Chapter III, Section I]:

Definition 3.11. (1) A *matrix coefficient* of G is a function $G \to \mathbb{C}$ defined by $g \mapsto \langle \psi(g)u, v\rangle$, where $\psi : G \to U(V)$ is a unitary representation of G on a finite-dimensional vector space V and $u, v \in V$ are fixed elements.

(2) Let $\psi : G \to GL(X)$ be a representation of G on a Banach space X. For $f : G \to \mathbb{C}$ continuous, define $\psi(f) : X \to X$ by

$$\psi(f)x = \int_G f(g)\,\psi(g)x\,d\mu.$$

The map $C(G) \times X \to X$ taking (f, x) to $\psi(f)x$ is linear in both variables and is continuous for the product topology. In fact, it is continuous as a map

$L^1(G) \times X \to X$, as the following estimate shows:

$$\begin{aligned}\|\psi(f)x\| &= \left\| \int_G f(g)\,\psi(g)\,x\,d\mu \right\| \\ &\le \int_G |f(g)|\,\|\psi(g)\|\,\|x\|\,d\mu \\ &\le \|x\| \int_G |f(g)|\,d\mu \sup_{g\in G} \|\psi(g)\| \\ &\le \|x\|\,\|f\|_1\,\|G\|,\end{aligned}$$

where $\|G\| := \sup_{g\in G} \|\psi(g)\|$. This exists by virtue of the facts that G is compact and the map $G \to GL(X) \to \mathbb{R}$ given by $g \mapsto \|\psi(g)\|$ is continuous.

[Knapp 1988, Theorem 1.14] states that every finite-dimensional representation of a compact Lie group is smooth. As a corollary of this we obtain:

Corollary 3.12. *All matrix coefficients of G are smooth functions on G.*

Using the technique illustrated in the proof of the Peter–Weyl theorem in [Knapp 1988, Theorem 3.7(a)], we find:

Proposition 3.13. *The linear span of the matrix coefficients of G is a uniformly dense subspace of $C(G)$, the space of continuous $\mathbb{C}$-valued functions on G.*

Proof. If $\phi : G \to H$ is a Lie group homomorphism, any matrix coefficient of H defines a matrix coefficient on G. Therefore, considering G as a subgroup of some U_m, the linear span of the matrix coefficients of U_m is contained within the linear span of the matrix coefficients of G. The restriction map $C(U_m) \to C(G)$ is a continuous open map, and therefore, as the linear span of matrix coefficients of U_m is uniformly dense in $C(U_m)$, its image is uniformly dense in $C(G)$. Hence the linear span of the matrix coefficients of G is uniformly dense in $C(G)$. □

Let $\rho : G \to LGL_n$ be a Lie group homomorphism and consider the induced action of G on LM_n. The inclusion $LGL_n \to L_{\text{cts}}GL_n$ of smooth loops in continuous loops defines a representation of G on $L_{\text{cts}}M_n$ and thus defines $C(G) \times L_{\text{cts}}M_n \to L_{\text{cts}}M_n$ as above. Moreover, the restriction to the subspaces of smooth maps on the left defines a continuous, bilinear map $C^\infty(G) \times LM_n \to LM_n$. In other words, if it so happens that $f : G \to \mathbb{C}$ and $\alpha : S^1 \to M_n$ are smooth then

$$t \mapsto \int_G f(g)\,\rho(g)\,\alpha(t)\,d\mu$$

defines a smooth map $S^1 \to M_n$.

The proof of [Knapp 1988, Theorem 3.7(d)], together with the preceding arguments, will yield:

Proposition 3.14. *For any $\alpha \in LM_n$ and $\epsilon > 0$ there is some $\beta \in LM_n$ for which $\|\alpha - \beta\|_\infty < \epsilon$ and β lies in a G-invariant, finite-dimensional subspace of LM_n.*

We are regarding M_n as the (finite-dimensional) Banach algebra of operators on $\mathbb{C}^n$ and equip it with the operator norm, corresponding to the standard norm on $\mathbb{C}^n$. This in turn defines the norm $\|\cdot\|_\infty$ on $L_{\mathrm{cts}} M_n$.

Proof. The method of the proof is to express β as $\rho(h)\alpha$ for a suitable choice of h. We start in the continuous case and then use the remark above to deduce that if α and h were originally smooth, β is smooth. Let $\|G\|_\infty = \sup_{g\in G}\{\|\rho(g)\|_\infty\}$.

As in the proof of [Knapp 1988, Theorem 3.7(d)], we note that if $h \in C(G)$ is a (finite) linear combination of matrix coefficients, $\rho(h)\alpha$ lies in a G-invariant, finite-dimensional subspace of LM_n. The case of $\alpha = 0$ being trivial, we assume that $\alpha \neq 0$.

For an open neighbourhood U of $1 \in G$, let $f_U : G \to \mathbb{C}$ be a positive, continuous function with support in U, $\|f_U\|_1 = 1$. Then

$$\begin{aligned}
\rho(f_U)\alpha - \alpha &= \int_G f_U(g)\rho(g)\alpha\, d\mu - \|f_U\|_1\, \alpha \\
&= \int_G f_U(g)\rho(g)\alpha\, d\mu - \int_G |f_U(g)|\, d\mu\, \alpha \\
&= \int_G f_U(g)\rho(g)\alpha\, d\mu - \int_G f_U(g)\alpha\, d\mu \\
&= \int_G f_U(g)\,(\rho(g)-1)\,\alpha\, d\mu,
\end{aligned}$$

so

$$\begin{aligned}
\|\rho(f_U)\alpha - \alpha\|_\infty &\le \int_G |f_U(g)|\, \|\rho(g)-1\|_\infty\, \|\alpha\|_\infty\, d\mu \\
&= \int_U |f_U(g)|\, \|\rho(g)-1\|_\infty\, \|\alpha\|_\infty\, d\mu \\
&\le \|\alpha\|_\infty \sup_{g\in U} \|\rho(g)-1\|_\infty .
\end{aligned}$$

Thus we can find U_0 sufficiently small for which $f_0 := f_{U_0}$ satisfies the inequality $\|\rho(f_0)\alpha - \alpha\|_\infty < \epsilon/2$. As $f_0 \in C(G)$, there is some linear combination of matrix coefficients, h, such that $\|f_0 - h\|_\infty < \epsilon/(2\,\|G\|_\infty\,\|\alpha\|_\infty)$. Thus

$$\begin{aligned}
\|\rho(h)\alpha - \alpha\|_\infty &= \|\rho(h)\alpha - \rho(f_0)\alpha + \rho(f_0)\alpha - \alpha\|_\infty \\
&\le \|\rho(h)\alpha - \rho(f_0)\alpha\|_\infty + \|\rho(f_0)\alpha - \alpha\|_\infty \\
&< \|h - f_0\|_\infty\, \|\alpha\|_\infty\, \|G\|_\infty + \epsilon/2 \\
&< \epsilon.
\end{aligned}$$

Set $\beta = \rho(h)\alpha$; then β satisfies the conditions of the proposition for the continuous case. Being a linear combination of matrix coefficients, h is smooth. Hence if α is smooth, β is also smooth. □

Theorem 3.15. *Let $\rho : G \to LGL_n$ be a Lie group homomorphism with G compact. Then there exists a G-invariant, generating subspace of $L\mathbb{C}^n$.*

Proof. From Proposition 3.14 we know that there exists a $\beta \in LM_n$, lying in a G-invariant, finite-dimensional subspace of LM_n, such that $\|1-\beta\|_\infty < 1$. The condition $\|1-\beta\|_\infty < 1$ implies that $\beta(t)$ is invertible for all t and hence the columns of $\beta(t)$ form a basis for $\mathbb{C}^n$ for each t.

Because β lies in a G-invariant, finite-dimensional subspace of LM_n, the loops $t \mapsto \beta(t)x_i$ each lie in a G-invariant, finite-dimensional subspace of $L\mathbb{C}^n$, where $\{x_1, \dots, x_n\}$ is the standard basis for $\mathbb{C}^n$. Thus there is a G-invariant, finite-dimensional subspace V of $L\mathbb{C}^n$ containing the loops defined by the columns of β. As evaluation at t maps these vectors to a basis for $\mathbb{C}^n$, the evaluation map $e_t : V \to \mathbb{C}^n$ is surjective for all t. Hence V is generating. □

Corollary 3.16. *Let $\rho : G \to LU_n$ be a homomorphism of Lie groups with G compact. Then there is a homomorphism $\tilde{\rho} : G \to U_n$ and an element $\gamma \in LU_n$ such that $\rho = c_\gamma \tilde{\rho}$.*

This, together with Corollary 3.3, yields a proof of Theorem 1.2(2). It also proves Theorem 3.1, as follows:

Proof of Theorem 3.1. If G is a compact subgroup of either LU_n or ΩU_n, there is a homomorphism of Lie groups $\rho : G \to LU_n$. Thus there is a conjugation in LU_n taking G to a subgroup of U_n. If G were originally in ΩU_n, conjugation would not change that, so G is conjugate to a subgroup of $\Omega U_n \cap U_n = \{1\}$. Hence ΩU_n has no compact subgroups and those of LU_n are conjugate to subgroups of U_n. □

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References

[Brown 1978] R. F. Brown, "A homological characterization of inner automorphisms of compact Lie groups", pp. 38–42 in *Algebraic and geometric topology* (Santa Barbara, 1977), edited by K. C. Millett, Lecture Notes in Math. **664**, Springer, Berlin, 1978. MR 80a:57021 Zbl 0407.57012

[Cohen and Stacey 2004] R. L. Cohen and A. Stacey, "Fourier decompositions of loop bundles", pp. 85–95 in *Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory* (Evanston, IL), edited by P. Goerss and S. Priddy, Contemp. Math. **346**, Amer. Math. Soc., Providence, RI, 2004. MR 2066497

[Dynkin 1952] E. B. Dynkin, "A connection between homologies of a compact Lie group and its subgroups", *Doklady Akad. Nauk SSSR* (*N.S.*) **87** (1952), 333–336. In Russian. MR 14,620b Zbl 0048.02101

[Eells and Elworthy 1971] J. Eells and K. D. Elworthy, "On Fredholm manifolds", pp. 215–219 in *Actes du Congrès International des Mathématiciens* (Nice, 1970), vol. 2, Gauthier-Villars, Paris, 1971. MR 55 #6463 Zbl 0225.58002

[Elworthy 1968] K. D. Elworthy, "Fredholm maps and $\mathrm{GL}_c(E)$-structures", *Bull. Amer. Math. Soc.* **74** (1968), 582–586. MR 36 #7160 Zbl 0159.25102

[Gelbrich 1991] G. Gelbrich, "On pointwise conjugate homomorphisms of compact Lie groups", *Sem. Sophus Lie* **1**:2 (1991), 217–223. MR 93c:55017 Zbl 0765.55006

[Knapp 1988] A. W. Knapp, *Lie groups, Lie algebras, and cohomology*, Mathematical Notes **34**, Princeton University Press, Princeton, NJ, 1988. MR 89j:22034 Zbl 0648.22010

[Kuiper 1965] N. H. Kuiper, "The homotopy type of the unitary group of Hilbert space", *Topology* **3** (1965), 19–30. MR 31 #4034 Zbl 0129.38901

[Morava 2001] J. Morava, "The tangent bundle of an almost-complex free loopspace", *Homology Homotopy Appl.* **3**:2 (2001), 407–415. Erratum: *ibid.* **5**:1 (2003), 71. MR 2004d:58006a Zbl 1035.58009

[Witten 1988] E. Witten, "The index of the Dirac operator in loop space", pp. 161–181 in *Elliptic curves and modular forms in algebraic topology* (Princeton, 1986), edited by P. S. Landweber, Lecture Notes in Math. **1326**, Springer, Berlin, 1988. MR 970288 Zbl 0679.58045

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