WEYL TRANSFORMS ASSOCIATED WITH A SINGULAR SECOND-ORDER DIFFERENTIAL OPERATOR

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For a class of singular second-order differential operators $\Delta$, we define and study the Weyl transforms $W_\sigma$ associated with $\Delta$, where $\sigma$ is a symbol in $S^m$, for $m \in \mathbb{R}$. We give criteria in terms of $\sigma$ for boundedness and compactness of the transform $W_\sigma$.

Introduction

Herman Weyl [1931] studied extensively the properties of pseudodifferential operators arising in quantum mechanics, regarding them as bounded linear operators on $L^2(\mathbb{R}^n)$, the space of square-integrable functions on $\mathbb{R}^n$ with respect to Lebesgue measure. M. W. Wong calls these operators, which are the subject of his book [Wong 1998], Weyl transforms.

Here we consider the second-order differential operator defined on $]0, +\infty[$ by

$$\Delta u = u'' + \frac{A'}{A}u' + \rho^2 u,$$

where $A$ is a nonnegative function satisfying certain conditions and $\rho$ is a nonnegative real number.

This operator plays an important role in analysis. For example, many special functions (orthogonal polynomials) are eigenfunctions of an operator of $\Delta$ type. The radial part of the Beltrami–Laplacian in a symmetric space is also of $\Delta$ type. Many aspects of such operators have been studied; we mention, in chronological order, [Chebli 1979; Trimèche 1981; Zeuner 1989; Xu 1994; Trimèche 1997; Nessibi et al. 1998]. In particular, the first two of these references investigate standard constructions of harmonic analysis, such as translation operators, convolution product, and Fourier transform, in connection with $\Delta$.

Building on these results, we define and study the Weyl transforms associated with $\Delta$, giving criteria for boundedness and compactness of these transforms. To obtain these results we first define the Fourier–Wigner transform associated with $\Delta$, and establish an inversion formula.

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More precisely, in Section 1 we recall some properties of harmonic analysis for the operator $\Delta$. In Section 2 we define the Fourier–Wigner transform associated with $\Delta$, study some of its properties, and prove an inversion formula.

In Section 3 we introduce the Weyl transform $W_\sigma$ associated with $\Delta$, with $\sigma$ a symbol in class $S^m$, for $m \in \mathbb{R}$, and we give its connection with the Fourier–Wigner transform. We prove that, for $\sigma$ sufficiently smooth, $W_\sigma$ is a compact operator from $L^2(d\nu)$ (the space of square-integrable functions with respect to the measure $d\nu(x) = A(x) \, dx$) into itself.

In Section 4 we define $W_\sigma$ for $\sigma$ in a certain space $L^p(d\nu \otimes d\gamma)$, with $p \in [1, 2]$, and we establish that $W_\sigma$ is again a compact operator.

In Section 5 we define $W_\sigma$ for $\sigma$ in another function space, and use this to prove in Section 6 that for $p > 2$ there exists a function $\sigma$ in the $L^p$ space corresponding to that of Section 4, with the property that the Weyl transform $W_\sigma$ is not bounded on $L^2(d\nu)$.

1. The operator $\Delta$

We consider the second-order differential operator $\Delta$ defined on $]0, +\infty[$ by

$$\Delta u = u'' + \frac{A'}{A}u' + \rho^2 u,$$

where $\rho$ is a nonnegative real number and

$$A(x) = x^{2\alpha+1}B(x), \quad \alpha > -\frac{1}{2},$$

for $B$ a positive, even, infinitely differentiable function on $\mathbb{R}$ such that $B(0) = 1$. Moreover we assume that $A$ and $B$ satisfy the following conditions:

(i) $A$ is increasing and $\lim_{x \to +\infty} A(x) = +\infty$.

(ii) $\frac{A'}{A}$ is decreasing and $\lim_{x \to +\infty} \frac{A'(x)}{A(x)} = 2\rho$.

(iii) There exists a constant $\delta > 0$ such that

$$\frac{B'(x)}{B(x)} = D(x) \exp(-\delta x) \quad \text{if } \rho = 0,$$

$$\frac{A'(x)}{A(x)} = 2\rho + D(x) \exp(-\delta x) \quad \text{if } \rho > 0,$$

where $D$ is an infinitely differentiable function on $]0, +\infty[$, bounded and with bounded derivatives on all intervals $[x_0, +\infty[$, for $x_0 > 0$.

This operator was studied in [Chebli 1979; Nessibi et al. 1998; Trimèche 1981], and the following results have been established:
(I) For all $\lambda \in \mathbb{C}$, the equation

\[
\begin{cases}
\Delta u = -\lambda^2 u \\
u(0) = 1, \ u'(0) = 0
\end{cases}
\]

admits a unique solution, denoted by $\varphi_\lambda$, with the following properties:

• $\varphi_\lambda$ satisfies the product formula

\[
\varphi_\lambda(x)\varphi_\lambda(y) = \int_0^\infty \varphi_\lambda(z)w(x, y, z)A(z)\,dz \quad \text{for } x, y \geq 0;
\]

where $w(x, y, \cdot)$ is a measurable positive function on $[0, +\infty[$, with support in $[|x-y|, x+y]$, satisfying

\[
\int_0^\infty w(x, y, z)A(z)\,dz = 1,
\]

$w(x, y, z) = w(y, x, z)$ for $z \geq 0$,

$w(x, y, z) = w(x, z, y)$ for $z > 0$;

• for $x \geq 0$, the function $\lambda \mapsto \varphi_\lambda(x)$ is analytic on $\mathbb{C}$;

• for $\lambda \in \mathbb{C}$, the function $x \mapsto \varphi_\lambda(x)$ is even and infinitely differentiable on $\mathbb{R}$;

• $|\varphi_\lambda(x)| \leq 1$ for all $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}$;

• for $x > 0$, and $\lambda > 0$ we have

\[
\varphi_\lambda(x) = \frac{1}{\sqrt{B(x)}}j_\alpha(\lambda x) + A^{-1/2}(x)\theta_\lambda(x),
\]

where $j_\alpha$ is defined by $j_\alpha(0) = 1$ and $j_\alpha(s) = 2^\alpha \Gamma(\alpha + 1)s^{-\alpha} J_\alpha(s)$ if $s \neq 0$ (with $J_\alpha$ the Bessel function of first kind), and the function $\theta_\lambda$ satisfies

\[
|\theta_\lambda(x)| \leq \frac{c_1}{x^\alpha + 2} \left( \int_0^x |Q(s)|\,ds \right) \exp \left( \frac{c_2}{\lambda} \int_0^x |Q(s)|\,ds \right)
\]

with $c_1, c_2$ positive constants and $Q$ the function defined on $]0, +\infty[$ by

\[
Q(x) = \frac{1}{4} - \frac{\alpha^2}{x^2} + \frac{1}{4} \left( \frac{A'(x)}{A(x)} \right)^2 + \frac{1}{2} \left( \frac{A'(x)}{A(x)} \right)' - \rho^2.
\]

(II) For nonzero $\lambda \in \mathbb{C}$, the equation $\Delta u = -\lambda^2 u$ has a solution $\Phi_\lambda$ satisfying

\[
\Phi_\lambda(x) = A^{-1/2}(x) \exp(i\lambda x) V(x, \lambda),
\]

with $\lim_{x \to +\infty} V(x, \lambda) = 1$. Consequently there exists a function (spectral function)

\[
\lambda \mapsto c(\lambda),
\]
such that
\[ \varphi_{\lambda} = c(\lambda)\Phi_{\lambda} + c(-\lambda)\Phi_{-\lambda} \quad \text{for nonzero } \lambda \in \mathbb{C}. \]

Moreover there exist positive constants \( k_1, k_2, k_3 \) such that
\[ k_1 |\lambda|^\alpha + 1/2 \leq |c(\lambda)|^{-1} \leq k_2 |\lambda|^\alpha + 1/2 \]
for all \( \lambda \) such that \( \text{Im}\lambda \leq 0 \) and \( |\lambda| \geq k_3 \).

**Notation.** We denote by
- \( d\nu(x) \) the measure defined on \([0, +\infty[\) by
  \[ d\nu(x) = A(x) \, dx; \]
- \( L^p(d\nu) \), for \( 1 \leq p \leq +\infty \), the space of measurable functions on \([0, +\infty[\) satisfying
  \[ \|f\|_{p,\nu} := \left( \int_0^{+\infty} |f(x)|^p \, d\nu(x) \right)^{1/p} < +\infty \quad \text{for } 1 \leq p < +\infty, \]
  \[ \|f\|_{\infty,\nu} := \text{ess sup}_{x \in [0, +\infty[} |f(x)| < +\infty; \]
- \( d\gamma(\lambda) \) the measure defined on \([0, +\infty[\) by
  \[ d\gamma(\lambda) = \frac{d\lambda}{2\pi |c(\lambda)|^2}; \]
- \( L^p(d\gamma) \), for \( 1 \leq p \leq +\infty \), the space of measurable functions on \([0, +\infty[\) satisfying \( \|f\|_{p,\gamma} < +\infty; \)
- \( D_\alpha(\mathbb{R}) \) the space of even, infinitely differentiable functions on \( \mathbb{R} \), with compact support;
- \( \mathbb{H}_\alpha(\mathbb{C}) \) the space of even analytic functions on \( \mathbb{C} \), rapidly decreasing of exponential type.

**Definition 1.1.** The translation operator associated with \( \Delta \) is defined on \( L^1(d\nu) \) by
\[ \mathcal{T}_x f(y) = \int_0^{+\infty} f(z) w(x, y, z) \, d\nu(z) \quad \text{for } x, y \geq 0, \]
where \( w \) is defined in (1–3). The convolution product associated with \( \Delta \) is defined by
\[ (f * g)(x) = \int_0^{+\infty} \mathcal{T}_x f(y) g(y) \, d\nu(y) \quad \text{for } f, g \in L^1(d\nu). \]
Properties of translation and convolution.

- The translation operator satisfies
  \[ T_x \varphi_\lambda(y) = \varphi_\lambda(x) \varphi_\lambda(y). \]

- Let \( f \in L^1(d\nu) \). Then
  \[ \int_0^{+\infty} T_x f(y) d\nu(y) = \int_0^{+\infty} f(y) d\nu(y) \quad \text{for } x \in [0, +\infty[ \]
  and
  \[ \| T_x f \|_{1,\nu} \leq \| f \|_{1,\nu}. \]

- Let \( f \in L^p(d\nu) \) with \( 1 \leq p \leq +\infty \). For all \( x \in [0, +\infty[ \), the function \( T_x f \) belongs to \( L^p(d\nu) \) and
  \[ \| T_x f \|_{p,\nu} \leq \| f \|_{p,\nu}. \]

- For \( f, g \in L^1(d\nu) \) the function \( f \ast g \) also lies in \( L^1(d\nu) \). The convolution product is commutative and associative.

- For \( f \in L^1(d\nu) \) and \( g \in L^p(d\nu) \), with \( 1 \leq p < +\infty \), the function \( f \ast g \) lies in \( L^p(d\nu) \) and we have
  \[ \| f \ast g \|_{p,\nu} \leq \| f \|_{1,\nu} \| g \|_{p,\nu}. \]

- For \( f, g \) even and continuous on \( \mathbb{R} \), with supports
  \[ \text{supp } f \subset [-a, a] \quad \text{and} \quad \text{supp } g \subset [-b, b], \]
  the function \( f \ast g \) is continuous on \( \mathbb{R} \) and
  \[ \text{supp}(f \ast g) \subset [-a-b, a+b]. \]

**Definition 1.2.** The Fourier transform associated with the operator \( \Delta \) is defined on \( L^1(d\nu) \) by
\[
\mathcal{F} f(\lambda) = \int_0^{+\infty} f(x) \varphi_\lambda(x) d\nu(x) \quad \text{for } \lambda \in \mathbb{R}.
\]

Properties of the Fourier transform.

- For \( f \in L^1(d\nu) \) such that \( \mathcal{F} f \in L^1(d\gamma) \), we have the inversion formula
  \[ f(x) = \int_0^{+\infty} \mathcal{F} f(\lambda) \varphi_\lambda(x) d\gamma(\lambda) \quad \text{for a.e. } x \in [0, +\infty[. \]

- For \( f \in L^1(d\nu) \),
  \[ \mathcal{F}(T_x f)(\lambda) = \varphi_\lambda(x) \mathcal{F} f(\lambda) \quad \text{for all } x \in [0, +\infty[ \text{ and } \lambda \in \mathbb{R}. \]
For $f, g \in L^1(d\nu)$,
\[ \mathcal{F}(f * g)(\lambda) = \mathcal{F}f(\lambda) \mathcal{F}g(\lambda), \quad \text{for all } \lambda \in [0, +\infty[. \]

- $\mathcal{F}$ can be extended to an isometric isomorphism from $L^2(d\nu)$ onto $L^2(d\gamma)$.

This means that
\[
\begin{align*}
(1-9) & \quad \|\mathcal{F}f\|_{2,\gamma} = \|f\|_{2,\nu}, \\
(1-10) & \quad \|\mathcal{F}^{-1}f\|_{2,\nu} = \|f\|_{2,\gamma},
\end{align*}
\]

Proposition 1.3. Let $f$ be in $L^p(d\nu)$, with $p \in [1, 2]$. Then $\mathcal{F}f$ belongs to $L^{p'}(d\gamma)$, with
\[
\frac{1}{p} + \frac{1}{p'} = 1,
\]
and
\[
(1-11) \quad \|\mathcal{F}f\|_{p',\gamma} \leq \|f\|_{p,\nu}.
\]

Proof. Since $|\varphi_\lambda(x)| \leq 1$ for $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}$, we get $\|\mathcal{F}f\|_{\infty,\gamma} \leq \|f\|_{1,\nu}$. This, together with (1–9) and the Riesz–Thorin Theorem [Stein 1956; Stein and Weiss 1971], shows that for under the conditions of the proposition $\mathcal{F}f$ belongs to $L^{p'}(d\gamma)$ and satisfies (1–11). □

From [Chebli 1979], the Fourier transform $\mathcal{F}$ is a topological isomorphism from $D_s(\mathbb{R})$ onto $\mathcal{H}_s(\mathbb{C})$ (see page 204 for notation). The inverse mapping is given by
\[
(1-12) \quad \mathcal{F}^{-1}f(x) = \int_{0}^{+\infty} f(\lambda)\varphi_\lambda(x) d\gamma(\lambda) \quad \text{for } x \in \mathbb{R}.
\]

2. Fourier–Wigner transform associated with $\Delta$

Definition 2.1. The Fourier–Wigner transform associated with the operator $\Delta$ is the mapping $V$ defined on $D_s(\mathbb{R}) \times D_s(\mathbb{R})$ by
\[
V(f, g)(x, \lambda) = \int_{0}^{+\infty} f(y)\mathcal{F}g(y)\varphi_\lambda(y) d\nu(y) \quad \text{for } (x, \lambda) \in \mathbb{R} \times \mathbb{R}.
\]

Remark. The transform $V$ can also be written in the forms
\[
(2-1) \quad V(f, g)(x, \lambda) = \mathcal{F}(f \mathcal{F}^{-1}g)(\lambda) = \varphi_\lambda f * g(x).
\]

Notation. We denote by
- $D_s(\mathbb{R}^2)$ the space of infinitely differentiable functions on $\mathbb{R}^2$, even with respect to each variable, with compact support;
- $S_s(\mathbb{R}^2)$ the space of infinitely differentiable functions on $\mathbb{R}^2$, even with respect to each variable, rapidly decreasing together with all their derivatives;
The Fourier–Wigner transform $V$ is a bilinear mapping from $D_a(\mathbb{R}) \times D_a(\mathbb{R})$ into $S_a(\mathbb{R}^2)$.

(ii) For $p \in [1, 2]$ and $p'$ such that $1/p + 1/p' = 1$, we have
\[
\|V(f, g)\|_{p', v \otimes y} \leq \|f\|_{p, v} \|g\|_{p', v}.
\]

The transform $V$ can be extended to a continuous bilinear operator, denoted also by $V$, from $L^p(dv) \times L^p(dv)$ into $L^{p'}(dv \otimes dy)$.

Proof. (i) Let $F$ be the function defined on $\mathbb{R}^2$ by $F(x, y) = f(y) \mathcal{F}_x g(y)$. It’s clear that $F \in D_a(\mathbb{R}^2)$, and we have
\[
V(f, g)(x, \lambda) = I \otimes \mathcal{F}(F)(x, \lambda),
\]
where $I$ is the identity operator. This and the fact that $\mathcal{F}$ is a topological isomorphism from $D_a(\mathbb{R})$ onto $H_s(\mathbb{C})$ imply (i).

(ii) This follows from the first equality in (2–1) together with Proposition 1.3, Minkowski’s inequality for integrals [Folland 1984, p.186], and the fact that
\[
\|\mathcal{F}_x g\|_{p', v} \leq \|g\|_{p', v} \quad \text{for } x \in \mathbb{R}. \quad \square
\]

Theorem 2.3. For $f, g \in D_a(\mathbb{R})$, we have
\[
\mathcal{F} \otimes \mathcal{F}^{-1} (V(f, g)) (\mu, \lambda) = \varphi_\mu(\lambda) f(\lambda) \mathcal{F} g(\mu) \quad \text{for } \mu, \lambda \in \mathbb{R}.
\]

Proof. Using Definition 2.1 and Fubini’s Theorem we have, for all $\mu, \lambda \in \mathbb{R},$
\[
\mathcal{F} \otimes \mathcal{F}^{-1} (V(f, g)) (\mu, \lambda) = \int_0^{+\infty} \int_0^{+\infty} V(f, g)(x, y) \varphi_\mu(x) \varphi_\lambda(y) dv(x) dy(y)
\]
\[
= \int_0^{+\infty} \int_0^{+\infty} \mathcal{F}(f \mathcal{F}_x g)(y) \varphi_\mu(x) \varphi_\lambda(y) dv(x) dy(y)
\]
\[
= \int_0^{+\infty} \varphi_\mu(x) \left( \int_0^{+\infty} \mathcal{F}(f \mathcal{F}_x g)(y) \varphi_\lambda(y) dy(y) \right) dv(x).
\]
From (1–8) we deduce
\[
\mathcal{F} \otimes \mathcal{F}^{-1} (V(f, g)) (\mu, \lambda) = \int_0^{+\infty} \varphi_{\mu}(x) f(\lambda) \mathcal{F}_x g(\lambda) \, d\nu(x)
\]
\[
= f(\lambda) \mathcal{F}(\mathcal{F}_x g)(\mu) = f(\lambda) \varphi_{\mu}(\lambda) \mathcal{F}g(\mu).
\]
□

**Corollary 2.4.** For all \(f, g \in D_*(\mathbb{R})\), we have
\[
\int_0^{+\infty} \mathcal{F} \otimes \mathcal{F}^{-1} (V(f, g)) (\mu, \lambda) \, d\nu(\lambda) = \mathcal{F}f(\mu) \mathcal{F}g(\mu) \quad \text{for} \ \mu \in \mathbb{R},
\]
\[
\int_0^{+\infty} \mathcal{F} \otimes \mathcal{F}^{-1} (V(f, g)) (\mu, \lambda) \, d\gamma(\mu) = f(\lambda) g(\lambda) \quad \text{for} \ \lambda \in \mathbb{R}.
\]

**Proof.** Theorem 2.3 gives
\[
\int_0^{+\infty} \mathcal{F} \otimes \mathcal{F}^{-1} (V(f, g)) (\mu, \lambda) \, d\nu(\lambda) = \int_0^{+\infty} \varphi_{\mu}(\lambda) f(\lambda) \mathcal{F}g(\mu) \, d\nu(\lambda)
\]
\[
= \mathcal{F}f(\mu) \mathcal{F}g(\mu) \quad \text{for} \ \mu \in \mathbb{R},
\]
\[
\int_0^{+\infty} \mathcal{F} \otimes \mathcal{F}^{-1} (V(f, g)) (\mu, \lambda) \, d\gamma(\mu) = \int_0^{+\infty} \varphi_{\mu}(\lambda) f(\lambda) \mathcal{F}g(\mu) \, d\gamma(\mu)
\]
\[
= f(\lambda) \int_0^{+\infty} \varphi_{\mu}(\lambda) \mathcal{F}g(\mu) \, d\gamma(\mu)
\]
\[
= f(\lambda) g(\lambda) \quad \text{for} \ \lambda \in \mathbb{R}. \quad \square
\]

**Theorem 2.5.** Let \(f, g \in L^1(d\nu) \cap L^2(d\nu)\) be such that \(c = \int_0^{+\infty} g(x) \, d\nu(x) \neq 0\). Then
\[
\mathcal{F}f(\lambda) = \frac{1}{c} \int_0^{+\infty} V(f, g)(x, \lambda) \, d\nu(x) \quad \text{for} \ \lambda \in \mathbb{R}.
\]

**Proof.** From Definition 2.1, we have
\[
\int_0^{+\infty} V(f, g)(x, \lambda) \, d\nu(x) = \int_0^{+\infty} \left( \int_0^{+\infty} f(y) \mathcal{F}_x g(y) \varphi_{\lambda}(y) \, dy \right) \, d\nu(x)
\]
for all \(\lambda \in \mathbb{R}\). The result follows from Fubini’s Theorem and the equality
\[
\int_0^{+\infty} \mathcal{F}_x g(y) \, dy = \int_0^{+\infty} g(x) \, dx = c. \quad \square
\]

**Corollary 2.6.** With the hypothesis of Theorem 2.5, if \(\mathcal{F}f \in L^1(d\gamma)\), we have the following inversion formula for the Fourier–Wigner transform \(V\):
\[
f(x) = \frac{1}{c} \int_0^{+\infty} \varphi_{\mu}(x) \left( \int_0^{+\infty} V(f, g)(y, \mu) \, d\gamma(y) \right) \, d\gamma(\mu) \quad \text{for a.e.} \ x \in \mathbb{R}.
\]
3. The Weyl transform associated with $\Delta$

We now introduce the Weyl transform and relate it to the Fourier–Wigner transform. To do this, we must define the class of pseudodifferential operators [Wong 1998].

**Definition 3.1.** Let $m \in \mathbb{R}$. We define $S^m$ to be the set of all infinitely differentiable functions $\sigma$ on $\mathbb{R} \times \mathbb{R}$, even with respect to each variable, and such that for all $p, q \in \mathbb{N}$, there exists a positive constant $C_{p,q,m}$ satisfying

$$\left| \left( \frac{\partial}{\partial x} \right)^p \left( \frac{\partial}{\partial y} \right)^q \sigma(x, y) \right| \leq C_{p,q,m} (1 + y^2)^{m-q}.$$ 

**Definition 3.2.** For $m \in \mathbb{R}$ and $\sigma \in S^m$, we define the operator $H_\sigma$ on $D_\ast(\mathbb{R}) \times D_\ast(\mathbb{R})$ by

$$(3.1) \quad H_\sigma(f, g)(\lambda) = \int_0^{+\infty} \sigma(x, y) \varphi_y(\lambda) V(f, g)(x, y) \, dv(x) \, dy(y),$$

for all $\lambda \in \mathbb{R}$, and we put

$$(3.2) \quad H_\sigma(f, g) = H_\sigma(f, g)(0).$$

**Proposition 3.3.** Define $\sigma \in S^m$ by $\sigma(x, y) = -y^2$ for $x, y \in \mathbb{R}$. Then, for all $f, g \in D_\ast(\mathbb{R})$, we have

$$H_\sigma(f, g)(\lambda) = c \Delta f(\lambda) \quad \text{for } \lambda \in \mathbb{R},$$

where $c = \int_0^{+\infty} g(x) \, dv(x)$.

**Proof.** From (3.1), we have

$$H_\sigma(f, g)(\lambda) = \int_0^{+\infty} \sigma(x, y) \varphi_y(\lambda) V(f, g)(x, y) \, dv(x) \, dy(y) \text{ for } \lambda \in \mathbb{R}.$$ 

Using Definition 2.1 we obtain

$$H_\sigma(f, g)(\lambda) = \int_0^{+\infty} \varphi_y(\lambda) \left( \int_0^{+\infty} f(z) \varphi(z) \, dv(z) \right) \, dx(x) \, dy(y)$$

for $\lambda \in \mathbb{R}$. From Fubini’s Theorem, we get

$$H_\sigma(f, g)(\lambda) = \int_0^{+\infty} \varphi_y(\lambda) \left( \int_0^{+\infty} f(z) \varphi(z) \, dv(z) \right) \, dx(x) \, dy(y)$$

$$= c \int_0^{+\infty} \varphi_y(\lambda) \left( \int_0^{+\infty} f(z) \varphi(z) \, dv(z) \right) \, dy(y)$$

$$= c \int_0^{+\infty} \varphi_y(\lambda) \, dy(y).$$
But, for all \( y \in \mathbb{R} \), \(-y^2 \mathcal{F} f(y) = \mathcal{F}(\Delta f)(y)\). We complete the proof using the inversion formula (1–8).

\[ \square \]

**Definition 3.4.** Let \( \sigma \in S^m; \ m < -\alpha - 1 \). The Weyl transform associated with \( \Delta \) is the mapping \( W_\sigma \) defined on \( D_x(\mathbb{R}) \) by

\[
W_\sigma(f)(\lambda) = \int_0^{+\infty} \left( \int_0^{+\infty} \varphi_\gamma(\lambda)\sigma(x, y) \mathcal{F}_x f(x) \, dv(x) \right) \, d\gamma(y) \quad \text{for } \lambda \in \mathbb{R}.
\]

**Notation.** We denote by

- \( S_x(\mathbb{R}) \) the space of even, infinitely differentiable functions on \( \mathbb{R} \), rapidly decreasing together with all their derivatives.
- \( S^2_x(\mathbb{R}) = \varphi_0 S_x(\mathbb{R}) \), where \( \varphi_0 \) is the solution of (1–2) with \( \lambda = 0 \).

For \( p = 0 \) these two spaces coincide [Trimèche 1997]. The Fourier transform \( \mathcal{F} \) is a topological isomorphism from \( S^2_x(\mathbb{R}) \) onto \( S_x(\mathbb{R}) \), whose inverse is given by (1–12).

**Lemma 3.5.** For \( \sigma \in D_x(\mathbb{R}^2) \), the function \( k \) defined by

\[
k(x, y) = \int_0^{+\infty} \varphi_\lambda(x) \mathcal{F}_x(\sigma(\cdot, \lambda))(y) \, d\gamma(\lambda) \quad \text{for } x, y \in \mathbb{R}
\]

belongs to \( L^p(dv \otimes dv) \), for all \( p \in [2, +\infty[ \).

**Proof.** The defining equation of \( k \) can be rewritten \( k(x, y) = \mathcal{F}_x(G(\cdot, x))(y) \), where

\[
G(x, y) = I \otimes \mathcal{F}^{-1}(\sigma)(x, y) \quad \text{for } x, y \in \mathbb{R},
\]

for \( I \) the identity operator. It follows that, for all \( p \in [2, +\infty[ \),

\[
\int_0^{+\infty} \int_0^{+\infty} |k(x, y)|^p dv(x) \, dv(y) = \int_0^{+\infty} \left( \int_0^{+\infty} |\mathcal{F}_x(G(\cdot, x))(y)|^p dv(y) \right) dv(x)
\]

\[
\leq \int_0^{+\infty} \left( \int_0^{+\infty} |G(y, x)|^p dv(y) \right) dv(x)
\]

\[
\leq \int_0^{+\infty} \left( \int_0^{+\infty} |I \otimes \mathcal{F}^{-1}(\sigma)(y, x)|^p dv(y) \right) dv(x).
\]

We distinguish two cases, \( p = 2 \) and \( p \in ]2, +\infty[, \) the case \( p = +\infty \) being trivial. For \( p = 2 \),

\[
\int_0^{+\infty} \int_0^{+\infty} |k(x, y)|^2 dv(x) \, dv(y) \leq \int_0^{+\infty} \left( \int_0^{+\infty} |\mathcal{F}^{-1}(\sigma(x, \cdot))(y)|^2 dv(x) \right) dv(y).
\]

From (1–10) we deduce that

\[
\int_0^{+\infty} \int_0^{+\infty} |k(x, y)|^2 dv(x) \, dv(y) \leq \int_0^{+\infty} \left( \int_0^{+\infty} |\sigma(y, x)|^2 d\gamma(y) \right) dv(y) < +\infty,
\]
because $\sigma$ belongs to $D_+(\mathbb{R}^2)$. The case $p \in [2, +\infty]$ is more complex. From the hypotheses on $\Delta$, we deduce that, as $x \to +\infty$,

$$
(3–3) \quad A(x) \sim \begin{cases} \frac{x^{2\alpha+1}}{\rho} & \text{if } \rho = 0, \\ \exp(2\rho x) & \text{if } \rho > 0. \end{cases}
$$

- For $\rho = 0$, recall that $\mathcal{F}$ is an isomorphism from $S_+(\mathbb{R}^2)$ onto itself. Thus $I \otimes \mathcal{F}^{-1}(\sigma)$ belongs to $S_+(\mathbb{R}^2)$, and the asymptotics (3–3) implies

$$
(3–4) \quad \int_0^{+\infty} \int_0^{+\infty} |k(x, y)|^p d\nu(x) d\nu(y) \\
\leq \int_0^{+\infty} \left( \int_0^{+\infty} |I \otimes \mathcal{F}^{-1}(\sigma)(y, x)|^p d\nu(x) \right) d\nu(y) < +\infty.
$$

- For $\rho > 0$, we have from [Trimèche 1997, p. 99]

$$
|\varphi_{\lambda}(x)| \leq \varphi_0(x) \leq m(1 + x) \exp(-\rho x) \quad \text{for all } \lambda \in \mathbb{R} \text{ and } x \geq 0,
$$

where $m$ is a positive constant. Then

$$
|I \otimes \mathcal{F}^{-1}(\sigma)(y, x)| \leq m(1 + x) \exp(-\rho x) \int_0^{+\infty} |\sigma(y, z)| d\nu(z).
$$

Since $\sigma$ belongs to $D_+(\mathbb{R}^2)$, there exists a positive constant $M$ such that

$$
\int_0^{+\infty} |\sigma(y, z)| d\nu(z) \leq M \quad \text{for } y \geq 0,
$$

which implies that

$$
|I \otimes \mathcal{F}^{-1}(\sigma)(y, x)| \leq mM(1 + x) \exp(-\rho x).
$$

This, together with the asymptotics (3–3), implies the validity of the same bound (3–4) as in the previous case.

\begin{proof}

\end{proof}

**Theorem 3.6.** Let $\sigma \in D_+(\mathbb{R}^2)$ and $f \in D_+(\mathbb{R})$.

(i) $W_\sigma(f)(x) = \int_0^{+\infty} k(x, y) f(y) d\nu(y)$ for all $x \in \mathbb{R}$.

(ii) $\|W_\sigma(f)\|_{p', \nu} \leq \|k\|_{p', \nu} \otimes \|f\|_{p, \nu}$ for $p \in [1, 2]$ and $p'$ such that $1/p + 1/p' = 1$.

(iii) $W_\sigma$ can be extended to a bounded operator from $L^p(d\nu)$ into $L^{p'}(d\nu)$. In particular, $W_\sigma : L^2(d\nu) \to L^2(d\nu)$ is a Hilbert–Schmidt operator, hence compact.
Proof. (i) From Definition 3.4, we have, for all \( x \in H \):
\[
W_\sigma(f)(x) = \int_0^{+\infty} \varphi_y(x) \left( \int_0^{+\infty} \sigma(z, y) T_x f(z) \, d\nu(z) \right) \, d\gamma(y).
\]
From Fubini’s Theorem, we get, for all \( x \in H \),
\[
W_\sigma(f)(x) = \int_0^{+\infty} f(z) \left( \int_0^{+\infty} \varphi_y(x) T_x [\sigma(., y)](z) \, d\gamma(y) \right) \, d\nu(z).
\]
(ii) Follows from (i), Hölder’s inequality, and Lemma 3.5.

(iii) Since \( k \in L^2(d\nu \otimes d\gamma) \), the mapping
\[
W_\sigma : L^2(d\nu) \rightarrow L^2(d\nu)
\]
is a Hilbert–Schmidt operator, and so compact. □

Theorem 3.7. Let \( m < -\alpha - 1 \) and \( \sigma \in S^m \). For all \( f, g \in D_+(\mathbb{R}) \),
\[
(3–5) \quad \mathbb{H}_\sigma(f, g) = \int_0^{+\infty} f(x) W_\sigma g(x) \, d\nu(x).
\]
Proof. Using (3–2) and Definition 2.1 we obtain
\[
\mathbb{H}_\sigma(f, g) = \int_0^{+\infty} \sigma(x, y) V(f, g)(x, y) \, d\nu(x) \, d\gamma(y)
\]
\[
= \int_0^{+\infty} \left( \int_0^{+\infty} \sigma(x, y) T_x g(\lambda) \varphi_y(\lambda) \, d\nu(\lambda) \right) \, d\gamma(y).
\]
From Fubini’s theorem, we get
\[
\mathbb{H}_\sigma(f, g) = \int_0^{+\infty} f(\lambda) \left( \int_0^{+\infty} \varphi_y(\lambda) \left( \int_0^{+\infty} \sigma(x, y) T_x g(\lambda) \, d\nu(x) \right) \, d\gamma(y) \right) \, d\nu(\lambda)
\]
\[
= \int_0^{+\infty} f(\lambda) W_\sigma(g)(\lambda) \, d\nu(\lambda).\]

4. The Weyl transform with symbol in \( L^p(d\nu \otimes d\gamma) \), for \( 1 \leq p \leq 2 \)

In this section we show using (3–5) that, if \( 1 \leq p \leq 2 \), the Weyl transform with symbol in \( L^p(d\nu \otimes d\gamma) \) is a compact operator.
Notation. We denote by $\mathcal{B}(L^2(dv))$ the $\mathbb{C}^*$-algebra of bounded operators $\Psi$ from $L^2(dv)$ into itself, equipped with the norm

$$\|\Psi\|_* = \sup_{\|f\|_{L^2} = 1} \|\Psi(f)\|_{L^2}.$$  

**Theorem 4.1.** Let $\langle \cdot, \cdot \rangle$ denote the inner product in $L^2(dv)$. There exists a unique operator $Q : L^2(dv \otimes d\gamma) \to \mathcal{B}(L^2(dv))$, whose action we denote by $\sigma \mapsto Q_\sigma$, such that

$$\langle Q_\sigma(g)/\tilde{f} \rangle = \int_0^{+\infty} \left( \int_0^{+\infty} \sigma(x,y) V(f,g)(x,y) \, dv(x) \right) \, d\gamma(y) \quad \text{for } f, g \in L^2(dv).$$

Furthermore, $\|Q_\sigma\|_* \leq \|\sigma\|_{2,\otimes \gamma}$.

**Proof.** Let $\sigma \in D_\times(\mathbb{R}^2)$. For $g \in D_\times(\mathbb{R})$, put $Q_\sigma(g) = W_\sigma(g)$. From Theorems 3.6 and 3.7, we obtain

$$(Q_\sigma(g)/\tilde{f}) = (W_\sigma(g)/\tilde{f}) = H_\sigma(f,g)$$

$$= \int_0^{+\infty} \left( \int_0^{+\infty} \sigma(x,y) V(f,g)(x,y) \, dv(x) \right) \, d\gamma(y).$$

On the other hand, from Proposition 2.2(ii), we have

$$|\langle Q_\sigma(g)/\tilde{f} \rangle| \leq \|\sigma\|_{2,\otimes \gamma} \|f\|_{L^2} \|g\|_{L^2}.$$  

Thus $Q_\sigma \in \mathcal{B}(L^2(dv))$ and

$$(4-1) \quad \|Q_\sigma\|_* \leq \|\sigma\|_{2,\otimes \gamma}.$$  

Now consider $\sigma \in L^2(dv \otimes d\gamma)$. Let $\sigma_k \in D_\times(\mathbb{R}^2)$ be a sequence in $D_\times(\mathbb{R}^2)$ such that $\|\sigma_k - \sigma\|_{2,\otimes \gamma}$ approaches 0 as $k \to +\infty$. From (4-1) we have, for all $k, l \in \mathbb{N}$,

$$\|Q_{\sigma_k} - Q_{\sigma_l}\|_* \leq \|\sigma_k - \sigma_l\|_{2,\otimes \gamma} \leq \|\sigma_k - \sigma\|_{2,\otimes \gamma} + \|\sigma_l - \sigma\|_{2,\otimes \gamma}.$$

Thus $(Q_{\sigma_k})_{k \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{B}(L^2(dv))$. Let it converge to $Q_\sigma$. Clearly $Q_\sigma$ is independent from the choice of $(\sigma_k)_{k \in \mathbb{N}}$, and we have

$$\|Q_\sigma\|_* = \lim_{k \to +\infty} \|Q_{\sigma_k}\|_* \leq \lim_{k \to +\infty} \|\sigma_k\|_{2,\otimes \gamma} = \|\sigma\|_{2,\otimes \gamma}.$$  

We consider first $f, g \in D_\times(\mathbb{R})$. Then

$$\langle Q_\sigma(g)/\tilde{f} \rangle = \lim_{k \to +\infty} \langle Q_{\sigma_k}(g)/\tilde{f} \rangle$$

$$= \lim_{k \to +\infty} \int_0^{+\infty} \left( \int_0^{+\infty} \sigma_k(x,y) V(f,g)(x,y) \, dv(x) \right) \, d\gamma(y)$$

$$= \int_0^{+\infty} \left( \int_0^{+\infty} \sigma(x,y) V(f,g)(x,y) \, dv(x) \right) \, d\gamma(y).$$
Now let $f, g$ be in $L^2(d\nu)$. Pick sequences $(f_k)_{k \in \mathbb{N}}$ and $(g_k)_{k \in \mathbb{N}}$ in $D_s(\mathbb{R})$ converging to $f$ and $g$, respectively, in the $\| \cdot \|_{2,v}$-norm. Then

$$
\langle Q_\sigma(g)/\tilde{f} \rangle = \lim_{k \to +\infty} \langle Q_\sigma(g_k)/\tilde{f_k} \rangle = \lim_{k \to +\infty} \int_0^{+\infty} \left( \int_0^{+\infty} \sigma(x, y)V(f_k, g_k)(x, y) d\nu(x) \right) d\gamma(y) = \int_0^{+\infty} \left( \int_0^{+\infty} \sigma(x, y)V(f, g)(x, y) d\nu(x) \right) d\gamma(y). \tag{\text{\small{\text{□}}}}
$$

We now give an extension of Theorem 4.1 that will allow us to prove that for $1 \leq p \leq 2$ the Weyl transform with symbol in $L^p(d\nu \otimes d\gamma)$, is a compact operator.

**Theorem 4.2.** Let $p \in [1, 2]$. There exists a unique bounded operator

$$Q : L^p(d\nu \otimes d\gamma) \to \mathcal{B}(L^2(d\nu)),$$

whose action is denoted by $\sigma \to Q_\sigma$, such that

$$\langle Q_\sigma(g)/\tilde{f} \rangle = \int_0^{+\infty} \left( \int_0^{+\infty} \sigma(x, y)V(f, g)(x, y) d\nu(x) \right) d\gamma(y) \quad \text{for } f, g \in D_s(\mathbb{R}).$$

Moreover, $\|Q_\sigma\|_* \leq \|\sigma\|_{p,v \otimes \gamma}$.

**Proof.** The case $p = 2$ is given by Theorem 4.1. We turn to the case $p = 1$. For $\sigma \in D_s(\mathbb{R}^2)$, we define $Q_\sigma$ by

$$Q_\sigma(g) = W_\sigma(g) \quad \text{for } g \in D_s(\mathbb{R}).$$

From Theorems 3.6 and 3.7, we have, for $f \in D_s(\mathbb{R})$,

$$\langle Q_\sigma(g)/\tilde{f} \rangle = \mathbb{H}_\sigma(f, g) = \int_0^{+\infty} \left( \int_0^{+\infty} \sigma(x, y)V(f, g)(x, y) d\nu(x) \right) d\gamma(y).$$

From Hölder’s inequality we then obtain

$$\|\langle Q_\sigma(g)/\tilde{f} \rangle\| \leq \|\sigma\|_{1,v \otimes \gamma} \|V(f, g)\|_{1,v \otimes \gamma} \leq \|\sigma\|_{1,v \otimes \gamma} \|f\|_{2,v} \|g\|_{2,v}.$$

This shows that $Q_\sigma \in \mathcal{B}(L^2(d\nu))$ and $\|Q_\sigma\|_* \leq \|\sigma\|_{1,v \otimes \gamma}$.

We extend the definition of $Q_\sigma$ and the two facts just proved to the case of $\sigma \in L^1(d\nu \otimes d\gamma)$, working as in the proof of Theorem 4.1.

Finally, the Riesz–Thorin Theorem [Stein 1956; Stein and Weiss 1971], allows us to generalize the same results from the cases $p = 1$ and $p = 2$ to all $p \in [1, 2]$. \tag{\text{\small{\text{□}}}}

**Theorem 4.3.** Let $p \in [1, 2]$. For $\sigma \in L^p(d\nu \otimes d\gamma)$, the operator $Q_\sigma$ from $L^2(d\nu)$ into itself is compact.
Proof. Given $\sigma \in L^p(d\nu \otimes d\gamma)$, choose a sequence $(\sigma_k)_{k \in \mathbb{N}}$ in $D_s(\mathbb{R}^2)$ approximating $\sigma$ in the $\| \cdot \|_{L^p(\mathbb{R}^2)}$-norm. The last assertion of Theorem 4.2 says that
\[
\| Q_{\sigma_k} - Q_{\sigma} \|_* \leq \| \sigma_k - \sigma \|_{L^p(\mathbb{R}^2)},
\]
so $Q_{\sigma_k}$ approaches $Q_{\sigma}$ in $\mathcal{B}(L^2(d\nu))$. From Theorem 3.6 we know that $W_{\sigma_k} = Q_{\sigma_k}$ is compact for all $k \in \mathbb{N}$. The theorem then follows from the fact that the subspace $\mathcal{H}(L^2(d\nu))$ of $\mathcal{B}(L^2(d\nu))$ consisting of compact operators is a closed ideal of $\mathcal{B}(L^2(d\nu))$. □

5. The Weyl transform with symbol in $S'_{s,0}(\mathbb{R}^2)$

Notation. We denote by

- $S_{s,0}(\mathbb{R}^2)$ the subspace of $S_s(\mathbb{R}^2)$ consisting of functions with compact support with respect to the first variable;
- $S'_{s,0}(\mathbb{R}^2)$ the topological dual of $S_{s,0}(\mathbb{R}^2)$;
- $D'_s(\mathbb{R})$ the space of even distribution on $\mathbb{R}$. It is the topological dual of $D_s(\mathbb{R})$.

Definition 5.1. For $\sigma \in S'_{s,0}(\mathbb{R}^2)$ and $g \in D'_s(\mathbb{R})$, we define the operator $W_{\sigma}(g)$ on $D'_s(\mathbb{R})$ by
\[
(W_{\sigma}(g))(f) = \sigma(V(f, g)) \quad \text{for } f \in D'_s(\mathbb{R}),
\]
where $V$ is the mapping from Definition 2.1. Clearly $W_{\sigma}(g)$ belongs to $D'_s(\mathbb{R})$.

Proposition 5.2. Consider the distribution $\sigma$ of $S'_{s,0}(\mathbb{R}^2)$ given by the constant function $1$. For all $g \in D'_s(\mathbb{R})$, we have
\[
W_{\sigma}(g) = c\delta,
\]
where $c = \int_0^{+\infty} g(x) \, d\nu(x)$ and $\delta$ is the Dirac distribution at $0$.

Proof. For $f, g \in D'_s(\mathbb{R})$, we get
\[
(W_{\sigma}(g))(f) = \sigma(V(f, g)) = \int_0^{+\infty} \left( \int_0^{+\infty} V(f, g)(x, y) \, d\nu(x) \right) \, d\gamma(y).
\]
But from the proof of Theorem 2.5, we have
\[
\int_0^{+\infty} V(f, g)(x, y) \, d\nu(x) = c\mathcal{F}f(y) \quad \text{for } y \in \mathbb{R}.
\]
Integrating both sides over $[0, +\infty[$ with respect to the measure $d\gamma$ and using (1–8), we obtain
\[
\sigma(V(f, g)) = (W_{\sigma}(g))(f) = c \int_0^{+\infty} \mathcal{F}f(y) \, d\gamma(y) = cf(0) = (c\delta, f). \quad \square
\]
Note that by Proposition 5.2, there exists \( \sigma \in S'_* (\mathbb{R}^2) \), given by a function in \( L^\infty (dv \otimes d\gamma) \), such that for all \( g \in D_a (\mathbb{R}) \) satisfying \( c = \int_0^{+\infty} g(x) \, dv(x) \neq 0 \), the distribution \( W_\sigma (g) \) is not given by a function in \( L^2 (dv) \).

6. The Weyl transform with symbol in \( L^p (dv \otimes d\gamma) \), for \( 2 < p < \infty \)

**Theorem 6.1.** Let \( p \in [2, \infty[ \). There exists a function \( \sigma \in L^p (dv \otimes d\gamma) \) such that the Weyl transform \( W_\sigma \) defined by (5–1) is not a bounded linear operator on \( L^2 (dv) \).

We break down the proof into two lemmas, of which the theorem is an immediate consequence.

**Lemma 6.2.** Let \( p \in [2, \infty[ \). Suppose that for all \( \sigma \in L^p (dv \otimes d\gamma) \), the Weyl transform \( W_\sigma \) given by (5–1) is a bounded linear operator on \( L^2 (dv) \). Then there exists a positive constant \( M \) such that

\[
\| W_\sigma \|_\sigma \leq M \| \sigma \|_{p,v \otimes \gamma} \quad \text{for all } \sigma \in L^p (dv \otimes d\gamma).
\]

**Proof.** Under the assumption of the lemma, there exists for each \( \sigma \in L^p (dv \otimes d\gamma) \) a positive constant \( C_\sigma \) such that

\[
\| W_\sigma (g) \|_{2,v} \leq C_\sigma \| g \|_{2,v} \quad \text{for } g \in L^2 (dv).
\]

Let \( f, g \in D_a (\mathbb{R}) \) be such that \( \| f \|_{2,v} = \| g \|_{2,v} = 1 \) and define a linear operator

\[
Q_{f,g} : L^p (dv \otimes d\gamma) \to \mathbb{C}
\]

by

\[
Q_{f,g} (\sigma) = \langle W_\sigma (g)/f \rangle.
\]

Then

\[
\sup_{\| f \|_{2,v} = \| g \|_{2,v} = 1} | Q_{f,g} (\sigma) | \leq C_\sigma.
\]

By the Banach–Steinhaus theorem, the operator \( Q_{f,g} \) is bounded on \( L^p (dv \otimes d\gamma) \), so there exists \( M > 0 \) such that

\[
\| Q_{f,g} \| = \sup_{\| \sigma \|_{p,v \otimes \gamma} = 1} | Q_{f,g} (\sigma) | \leq M.
\]

From this we deduce that for all \( f, g \in D_a (\mathbb{R}) \) and \( \sigma \in L^p (dv \otimes d\gamma) \),

\[
\left| \langle W_\sigma (g)/f \rangle \right| \leq M \| \sigma \|_{p,v \otimes \gamma} \| f \|_{2,v} \| g \|_{2,v},
\]

which implies (6–1).

**Lemma 6.3.** For \( 2 < p < \infty \), there is no positive constant \( M \) satisfying (6–1).
Proof. Suppose there exists such an $M$. Let $p'$ be such that $1/p + 1/p' = 1$. Then $p' \in ]1, 2[.$ We consider, for $f, g \in D_r(\mathbb{R})$, the function $V(f, g)$ of Definition 2.1. We have

$$
\|V(f, g)\|_{p', v \otimes y} = \sup_{\|\sigma\|_{p, v \otimes y} = 1} \left| \int_0^{+\infty} \int_0^{+\infty} \sigma(x, y) V(f, g)(x, y) \, dv(x) \, d\gamma(y) \right|
$$

and consequently

$$
= \sup_{\|\sigma\|_{p, v \otimes y} = 1} |\langle \tilde{W}_\sigma(g), \tilde{f} \rangle| \leq \sup_{\|\sigma\|_{p, v \otimes y} = 1} \|W_\sigma(g)\|_{2,v} \|f\|_{2,v},
$$

and consequently

$$
(6-2) \quad \|V(f, g)\|_{p', v \otimes y} \leq M \|f\|_{2,v} \|g\|_{2,v}.
$$

Now consider $f, g \in L^2(dv)$. Choose sequences $(f_k)_{k \in \mathbb{N}}$ and $(g_k)_{k \in \mathbb{N}}$ in $D_r(\mathbb{R})$ approximating $f$ and $g$ in the $\|\cdot\|_{2,v}$-norm. By Proposition 2.2, the sequence $(V(f_k, g_k))_{k \in \mathbb{N}}$ converges to $V(f, g)$ in $L^{p'}(dv \otimes d\gamma)$, and thus we have extended (6-2) to all $f, g \in L^2(dv)$. We will exhibit an example where this leads to a contradiction.

Let $f$ be an even, measurable function on $\mathbb{R}$, supported in $[-1, 1]$. We have

$$
|V(f, f)(x, y)| \leq |f| * |f|(x),
$$

where $*$ is the convolution product (Definition 1.1). From (1-7), we deduce that for all $y \in \mathbb{R}$, the function $x \mapsto V(f, f)(x, y)$ is supported in $[-2, 2]$. Hölder’s inequality gives

$$
\left( \int_0^{+\infty} \left( \int_0^2 |V(f, f)(x, y)\, dv(x) \right)^{p'} \, d\gamma(y) \right)^{1/p'} \leq \left( \int_0^2 dv(x) \right)^{1/p} \left( \int_0^{+\infty} \left( \int_0^2 |V(f, f)(x, y)|^{p'} \, dv(x) \right) \, d\gamma(y) \right)^{1/p'}
$$

$$
\leq \left( \int_0^2 dv(x) \right)^{1/p} \|V(f, f)\|_{p', v \otimes y} \leq M \left( \int_0^2 dv(x) \right)^{1/p} \|f\|_{2,v}^2,
$$

the last inequality following from (6-2). This proves that the function

$$
y \mapsto \int_0^{+\infty} V(f, f)(x, y) \, dv(x) = c \overline{\mathcal{F}} f(y)
$$

belongs to $L^{p'}(d\gamma)$; here $c = \int_0^{+\infty} f(x) \, dv(x)$. and we have used the proof of Theorem 2.5 for the equality on the right-hand side. Putting this together with the preceding inequality we see that, if $c \neq 0$, the function $\overline{\mathcal{F}} f$ belongs to $L^{p'}(d\gamma)$ and

$$
(6-3) \quad \|\overline{\mathcal{F}} f\|_{p', y} \leq \frac{M}{|c|} \left( \int_0^2 dv(x) \right)^{1/p} \|f\|_{2,v}^2.
$$
Now consider the particular function $f$ given by
\[ f(x) = \frac{|x|^r}{\sqrt{B(x)}} \mathbf{1}_{[-1,1]}(x) \]
where $B$ is the function defined by (1–1) and $\mathbf{1}_{[-1,1]}$ is the characteristic function of the interval $[-1, 1]$. If $r > -(\alpha + 1)$, this function belongs to $L^1(d\nu) \cap L^2(d\nu)$. From (1–4) we get
\[ \mathcal{F}f(\lambda) = \int_0^1 x^{r+2\alpha+1} j_\lambda(x) dx + \int_0^1 x^{r+\alpha+1/2}\theta_\lambda(x) dx \]
\[ = \frac{1}{\lambda^{r+2\alpha+2}} \int_0^\lambda x^{r+2\alpha+1} j_\lambda(x) dx + \int_0^1 x^{r+\alpha+1/2}\theta_\lambda(x) dx. \]
Using the asymptotic expansion of the function $j_\lambda$ [Lebedev 1972; Watson 1944], given by
\[ j_\lambda(x) = \frac{2^{\alpha+1/2} \Gamma(\alpha + 1)}{\sqrt{\pi x^{\alpha+1/2}}} \left( \cos \left( x - \alpha \frac{\pi}{2} - \frac{\pi}{4} \right) + O \left( \frac{1}{\lambda^2} \right) \right) \quad \text{as} \quad x \to +\infty, \]
we deduce that for $-(\alpha + 1) < r < -(\alpha + \frac{1}{2})$, the integral
\[ a := \int_0^{+\infty} x^{r+2\alpha+1} j_\lambda(x) dx \]
exists and is finite, so
\[ \frac{1}{\lambda^{r+2\alpha+2}} \int_0^\lambda x^{r+2\alpha+1} j_\lambda(x) dx \sim \frac{a}{\lambda^{r+2\alpha+2}} \quad \text{as} \quad \lambda \to +\infty. \]
On the other hand, for $\lambda > 1$,
\[ \left| \int_0^1 x^{r+\alpha+1/2}\theta_\lambda(x) dx \right| \leq \frac{c_1}{\lambda^{\alpha+3/2}} \int_0^1 x^{r+\alpha+1/2}\Psi(x) dx, \]
where
\[ \Psi(x) = \left( \int_0^x |Q(s)| ds \right) \exp \left( c_2 \int_0^x |Q(s)| ds \right) \quad \text{for all} \quad x > 0 \]
and $Q$ is given by (1–5). Since $-(\alpha + 1) < r < -(\alpha + \frac{1}{2})$, we deduce that
\[ \mathcal{F}f(\lambda) \sim \frac{a}{\lambda^{r+2\alpha+2}} \quad \text{as} \quad \lambda \to +\infty. \]
Using this and (1–6), it follows that there exist $K, R > 0$ such that
\[ |\mathcal{F}f(\lambda)|' \frac{1}{2\pi |c(\lambda)|^2} \geq \frac{K}{\lambda^{\rho(r+2\alpha+2)-2\alpha-1}} \quad \text{for} \quad \lambda > R; \]
so for \( r \) such that \( p'(r + 2\alpha + 2) < 2\alpha + 2 \), we get

\[
\| \mathcal{F}f \|^{p'}_{\nu} \geq \int_{R}^{
} |\mathcal{F}f(\lambda)|^{p'} \frac{d\lambda}{2\pi |c(\lambda)|^{2}} \geq \int_{R}^{
} K \frac{1}{\lambda^{p'(r+2\alpha+2)+2\alpha+1}} d\lambda = +\infty.
\]

This shows that the relation (6–3) is false if we choose \( r \) so as to satisfy simultaneously the conditions \( r > -(\alpha + 1) \), \( r < -(\alpha + \frac{1}{2}) \) and

\[
r < -(2\alpha + 2) + \frac{2\alpha + 2}{p'}.
\]

This contradiction proves the lemma and Theorem 6.1.

\[\square\]

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