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CYRINE BACCAR AND LAKHDAR TANNECH RACHDI

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WEYL TRANSFORMS ASSOCIATED WITH A SINGULAR SECOND-ORDER DIFFERENTIAL OPERATOR

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For a class of singular second-order differential operators Δ, we define and study the Weyl transforms W_σ associated with Δ, where σ is a symbol in S^m, for $m \in \mathbb{R}$. We give criteria in terms of σ for boundedness and compactness of the transform W_σ.

Introduction

Herman Weyl [1931] studied extensively the properties of pseudodifferential operators arising in quantum mechanics, regarding them as bounded linear operators on $L^2(\mathbb{R}^n)$, the space of square-integrable functions on $\mathbb{R}^n$ with respect to Lebesgue measure). M. W. Wong calls these operators, which are the subject of his book [Wong 1998], Weyl transforms.

Here we consider the second-order differential operator defined on $]0, +\infty[$ by

$$\Delta u = u'' + \frac{A'}{A}u' + \rho^2 u,$$

where A is a nonnegative function satisfying certain conditions and ρ is a nonnegative real number.

This operator plays an important role in analysis. For example, many special functions (orthogonal polynomials) are eigenfunctions of an operator of Δ type. The radial part of the Beltrami–Laplacian in a symmetric space is also of Δ type. Many aspects of such operators have been studied; we mention, in chronological order, [Chebli 1979; Trimèche 1981; Zeuner 1989; Xu 1994; Trimèche 1997; Nessibi et al. 1998]. In particular, the first two of these references investigate standard constructions of harmonic analysis, such as translation operators, convolution product, and Fourier transform, in connection with Δ.

Building on these results, we define and study the Weyl transforms associated with Δ, giving criteria for boundedness and compactness of these transforms. To obtain these results we first define the Fourier–Wigner transform associated with Δ, and establish an inversion formula.

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More precisely, in Section 1 we recall some properties of harmonic analysis for the operator Δ. In Section 2 we define the Fourier–Wigner transform associated with Δ, study some of its properties, and prove an inversion formula.

In Section 3 we introduce the Weyl transform W_σ associated with Δ, with σ a symbol in class S^m, for $m \in \mathbb{R}$, and we give its connection with the Fourier–Wigner transform. We prove that, for σ sufficiently smooth, W_σ is a compact operator from $L^2(d\nu)$ (the space of square-integrable functions with respect to the measure $d\nu(x) = A(x)\,dx$) into itself.

In Section 4 we define W_σ for σ in a certain space $L^p(d\nu \otimes d\gamma)$, with $p \in [1, 2]$, and we establish that W_σ is again a compact operator.

In Section 5 we define W_σ for σ in another function space, and use this to prove in Section 6 that for $p > 2$ there exists a function σ in the L^p space corresponding to that of Section 4, with the property that the Weyl transform W_σ is not bounded on $L^2(d\nu)$.

1. The operator Δ

We consider the second-order differential operator Δ defined on $]0, +\infty[$ by

$$\Delta u = u'' + \frac{A'}{A} u' + \rho^2 u,$$

where ρ is a nonnegative real number and

$$A(x) = x^{2\alpha+1} B(x), \qquad \alpha > -\tfrac{1}{2}, \tag{1–1}$$

for B a positive, even, infinitely differentiable function on $\mathbb{R}$ such that $B(0) = 1$. Moreover we assume that A and B satisfy the following conditions:

(i) A is increasing and $\lim_{x \to +\infty} A(x) = +\infty$.

(ii) $\dfrac{A'}{A}$ is decreasing and $\lim_{x \to +\infty} \dfrac{A'(x)}{A(x)} = 2\rho$.

(iii) There exists a constant $\delta > 0$ such that

$$\frac{B'(x)}{B(x)} = D(x) \exp(-\delta x) \quad \text{if } \rho = 0,$$
$$\frac{A'(x)}{A(x)} = 2\rho + D(x) \exp(-\delta x) \quad \text{if } \rho > 0,$$

where D is an infinitely differentiable function on $]0, +\infty[$, bounded and with bounded derivatives on all intervals $[x_0, +\infty[$, for $x_0 > 0$.

This operator was studied in [Chebli 1979; Nessibi et al. 1998; Trimèche 1981], and the following results have been established:

(I) For all $\lambda \in \mathbb{C}$, the equation

$$\begin{cases} \Delta u = -\lambda^2 u \\ u(0) = 1,\ u'(0) = 0 \end{cases} \tag{1–2}$$

admits a unique solution, denoted by φ_λ, with the following properties:

- φ_λ satisfies the *product formula*

$$\varphi_\lambda(x)\varphi_\lambda(y) = \int_0^\infty \varphi_\lambda(z)\, w(x, y, z) A(z)\, dz \quad \text{for } x, y \geq 0; \tag{1–3}$$

  where $w(x, y, \cdot)$ is a measurable positive function on $[0, +\infty[$, with support in $[|x-y|, x+y]$, satisfying

$$\int_0^\infty w(x, y, z) A(z)\, dz = 1,$$
$$w(x, y, z) = w(y, x, z) \quad \text{for } z \geq 0,$$
$$w(x, y, z) = w(x, z, y) \quad \text{for } z > 0;$$

- for $x \geq 0$, the function $\lambda \mapsto \varphi_\lambda(x)$ is analytic on $\mathbb{C}$;
- for $\lambda \in \mathbb{C}$, the function $x \mapsto \varphi_\lambda(x)$ is even and infinitely differentiable on $\mathbb{R}$;
- $|\varphi_\lambda(x)| \leq 1$ for all $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}$;
- for $x > 0$, and $\lambda > 0$ we have

$$\varphi_\lambda(x) = \frac{1}{\sqrt{B(x)}} j_\alpha(\lambda x) + A^{-1/2}(x)\theta_\lambda(x), \tag{1–4}$$

  where j_α is defined by $j_\alpha(0) = 1$ and $j_\alpha(s) = 2^\alpha \Gamma(\alpha+1) s^{-\alpha} J_\alpha(s)$ if $s \neq 0$ (with J_α the Bessel function of first kind), and the function θ_λ satisfies

$$|\theta_\lambda(x)| \leq \frac{c_1}{\lambda^{\alpha+\frac{3}{2}}} \left(\int_0^x |Q(s)|\, ds \right) \exp\left(\frac{c_2}{\lambda} \int_0^x |Q(s)|\, ds \right)$$

  with c_1, c_2 positive constants and Q the function defined on $]0, +\infty[$ by

$$Q(x) = \frac{\frac{1}{4} - \alpha^2}{x^2} + \frac{1}{4}\left(\frac{A'(x)}{A(x)}\right)^2 + \frac{1}{2}\left(\frac{A'(x)}{A(x)}\right)' - \rho^2. \tag{1–5}$$

(II) For nonzero $\lambda \in \mathbb{C}$, the equation $\Delta u = -\lambda^2 u$ has a solution Φ_λ satisfying

$$\Phi_\lambda(x) = A^{-1/2}(x) \exp(i\lambda x) V(x, \lambda),$$

with $\lim_{x\to+\infty} V(x, \lambda) = 1$. Consequently there exists a function (spectral function)

$$\lambda \mapsto c(\lambda),$$

such that

$$\varphi_\lambda = c(\lambda)\Phi_\lambda + c(-\lambda)\Phi_{-\lambda} \quad \text{for nonzero } \lambda \in \mathbb{C}.$$

Moreover there exist positive constants k_1, k_2, k_3 such that

$$k_1\,|\lambda|^{\alpha+1/2} \le |c(\lambda)|^{-1} \le k_2\,|\lambda|^{\alpha+1/2} \tag{1–6}$$

for all λ such that $\operatorname{Im}\lambda \le 0$ and $|\lambda| \ge k_3$.

Notation. We denote by

- $d\nu(x)$ the measure defined on $[0, +\infty[$ by
$$d\nu(x) = A(x)\,dx;$$
- $L^p(d\nu)$, for $1 \le p \le +\infty$, the space of measurable functions on $[0, +\infty[$ satisfying
$$\|f\|_{p,\nu} := \left(\int_0^{+\infty} |f(x)|^p d\nu(x)\right)^{1/p} < +\infty \quad \text{for } 1 \le p < +\infty,$$
$$\|f\|_{\infty,\nu} := \operatorname*{ess\,sup}_{x\in[0,+\infty[} |f(x)| < +\infty;$$
- $d\gamma(\lambda)$ the measure defined on $[0, +\infty[$ by
$$d\gamma(\lambda) = \frac{d\lambda}{2\pi\,|c(\lambda)|^2};$$
- $L^p(d\gamma)$, for $1 \le p \le +\infty$, the space of measurable functions on $[0, +\infty[$ satisfying $\|f\|_{p,\gamma} < +\infty$;
- $D_*(\mathbb{R})$ the space of even, infinitely differentiable functions on $\mathbb{R}$, with compact support;
- $\mathbb{H}_*(\mathbb{C})$ the space of even analytic functions on $\mathbb{C}$, rapidly decreasing of exponential type.

Definition 1.1. The *translation operator* associated with Δ is defined on $L^1(d\nu)$ by

$$\mathscr{T}_x f(y) = \int_0^{+\infty} f(z)w(x, y, z)\,d\nu(z) \quad \text{for } x, y \ge 0,$$

where w is defined in (1–3). The *convolution product* associated with Δ is defined by

$$(f * g)(x) = \int_0^{+\infty} \mathscr{T}_x f(y)g(y)\,d\nu(y) \quad \text{for } f, g \in L^1(d\nu).$$

Properties of translation and convolution.

- The translation operator satisfies

$$\mathcal{T}_x \varphi_\lambda(y) = \varphi_\lambda(x)\varphi_\lambda(y).$$

- Let $f \in L^1(d\nu)$. Then

$$\int_0^{+\infty} \mathcal{T}_x f(y)\, d\nu(y) = \int_0^{+\infty} f(y)\, d\nu(y) \quad \text{for } x \in [0, +\infty[$$

  and

$$\|\mathcal{T}_x f\|_{1,\nu} \le \|f\|_{1,\nu}.$$

- Let $f \in L^p(d\nu)$ with $1 \le p \le +\infty$. For all $x \in [0, +\infty[$, the function $\mathcal{T}_x f$ belongs to $L^p(d\nu)$ and

$$\|\mathcal{T}_x f\|_{p,\nu} \le \|f\|_{p,\nu}.$$

- For $f, g \in L^1(d\nu)$ the function $f * g$ also lies in $L^1(d\nu)$. The convolution product is commutative and associative.
- For $f \in L^1(d\nu)$ and $g \in L^p(d\nu)$, with $1 \le p < +\infty$, the function $f * g$ lies in $L^p(d\nu)$ and we have

$$\|f * g\|_{p,\nu} \le \|f\|_{1,\nu} \|g\|_{p,\nu}.$$

- For f, g even and continuous on $\mathbb{R}$, with supports

$$\operatorname{supp} f \subset [-a, a] \quad \text{and} \quad \operatorname{supp} g \subset [-b, b],$$

  the function $f * g$ is continuous on $\mathbb{R}$ and

$$\operatorname{supp}(f * g) \subset [-a-b, a+b]. \tag{1–7}$$

Definition 1.2. The *Fourier transform* associated with the operator Δ is defined on $L^1(d\nu)$ by

$$\mathcal{F} f(\lambda) = \int_0^{+\infty} f(x)\varphi_\lambda(x)\, d\nu(x) \quad \text{for } \lambda \in \mathbb{R}.$$

Properties of the Fourier transform.

- For $f \in L^1(d\nu)$ such that $\mathcal{F} f \in L^1(d\gamma)$, we have the inversion formula

$$f(x) = \int_0^{+\infty} \mathcal{F} f(\lambda)\varphi_\lambda(x)\, d\gamma(\lambda) \quad \text{for a.e. } x \in [0, +\infty[. \tag{1–8}$$

- For $f \in L^1(d\nu)$,

$$\mathcal{F}(\mathcal{T}_x f)(\lambda) = \varphi_\lambda(x)\mathcal{F} f(\lambda) \quad \text{for all } x \in [0, +\infty[\text{ and } \lambda \in \mathbb{R}.$$

- For $f, g \in L^1(d\nu)$,

$$\mathcal{F}(f * g)(\lambda) = \mathcal{F}f(\lambda)\mathcal{F}g(\lambda). \quad \text{for all } \lambda \in [0, +\infty[.$$

- $\mathcal{F}$ can be extended to an isometric isomorphism from $L^2(d\nu)$ onto $L^2(d\gamma)$. This means that

$$\|\mathcal{F}f\|_{2,\gamma} = \|f\|_{2,\nu} \quad \text{for } f \in L^2(d\nu), \tag{1–9}$$

$$\|\mathcal{F}^{-1}f\|_{2,\nu} = \|f\|_{2,\gamma} \quad \text{for } f \in L^2(d\gamma). \tag{1–10}$$

Proposition 1.3. *Let f be in $L^p(d\nu)$, with $p \in [1, 2]$. Then $\mathcal{F}f$ belongs to $L^{p'}(d\gamma)$, with $1/p + 1/p' = 1$, and*

$$\|\mathcal{F}f\|_{p',\gamma} \leq \|f\|_{p,\nu}. \tag{1–11}$$

Proof. Since $|\varphi_\lambda(x)| \leq 1$ for $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}$, we get $\|\mathcal{F}f\|_{\infty,\gamma} \leq \|f\|_{1,\nu}$. This, together with (1–9) and the Riesz–Thorin Theorem [Stein 1956; Stein and Weiss 1971], shows that for under the conditions of the proposition $\mathcal{F}f$ belongs to $L^{p'}(d\gamma)$ and satisfies (1–11). □

From [Chebli 1979], the Fourier transform $\mathcal{F}$ is a topological isomorphism from $D_*(\mathbb{R})$ onto $\mathbb{H}_*(\mathbb{C})$ (see page 204 for notation). The inverse mapping is given by

$$\mathcal{F}^{-1}f(x) = \int_0^{+\infty} f(\lambda)\varphi_\lambda(x)\,d\gamma(\lambda) \quad \text{for } x \in \mathbb{R}. \tag{1–12}$$

2. Fourier–Wigner transform associated with Δ

Definition 2.1. The Fourier–Wigner transform associated with the operator Δ is the mapping V defined on $D_*(\mathbb{R}) \times D_*(\mathbb{R})$ by

$$V(f, g)(x, \lambda) = \int_0^{+\infty} f(y)\mathcal{T}_x g(y)\varphi_\lambda(y)\,d\nu(y) \quad \text{for } (x, \lambda) \in \mathbb{R} \times \mathbb{R}.$$

Remark. The transform V can also be written in the forms

$$V(f, g)(x, \lambda) = \mathcal{F}(f\mathcal{T}_x g)(\lambda) = \varphi_\lambda f * g(x). \tag{2–1}$$

Notation. We denote by

- $D_*(\mathbb{R}^2)$ the space of infinitely differentiable functions on $\mathbb{R}^2$, even with respect to each variable, with compact support;
- $S_*(\mathbb{R}^2)$ the space of infinitely differentiable functions on $\mathbb{R}^2$, even with respect to each variable, rapidly decreasing together with all their derivatives;

- $L^p(d\nu \otimes d\nu)$, for $1 \le p \le +\infty$, the space of measurable functions on the product $[0, +\infty[\times [0, +\infty[$ satisfying

$$\| f \|_{p,\nu\otimes\nu} := \left(\int_0^{+\infty} \int_0^{+\infty} | f(x, y)|^p \, d\nu(x) \, d\nu(y) \right)^{1/p} < +\infty \quad \text{for } 1 \le p < +\infty,$$

$$\| f \|_{\infty,\nu\otimes\nu} := \operatorname*{ess\,sup}_{x,y\in[0,+\infty[} | f(x, y)| < +\infty;$$

- $L^p(d\nu \otimes d\gamma)$, for $1 \le p \le +\infty$, the space similarly defined (with $d\nu(x)\, d\gamma(y)$ in the integrand).

Proposition 2.2. (i) *The Fourier–Wigner transform V is a bilinear mapping from $D_*(\mathbb{R}) \times D_*(\mathbb{R})$ into $S_*(\mathbb{R}^2)$.*

(ii) *For $p \in]1, 2]$ and p' such that $1/p + 1/p' = 1$, we have*

$$\|V(f, g)\|_{p',\nu\otimes\gamma} \le \| f \|_{p,\nu} \, \|g\|_{p',\nu}.$$

The transform V can be extended to a continuous bilinear operator, denoted also by V, from $L^p(d\nu) \times L^{p'}(d\nu)$ into $L^{p'}(d\nu \otimes d\gamma)$.

Proof. (i) Let F be the function defined on $\mathbb{R}^2$ by $F(x, y) = f(y)\mathcal{T}_x g(y)$. It's clear that $F \in D_*(\mathbb{R}^2)$, and we have

$$V(f, g)(x, \lambda) = I \otimes \mathcal{F}(F)(x, \lambda),$$

where I is the identity operator. This and the fact that $\mathcal{F}$ is a topological isomorphism from $D_*(\mathbb{R})$ onto $\mathbb{H}_*(\mathbb{C})$ imply (i).

(ii) This follows from the first equality in (2–1) together with Proposition 1.3, Minkowski's inequality for integrals [Folland 1984, p.186], and the fact that

$$\|\mathcal{T}_x g\|_{p',\nu} \le \|g\|_{p',\nu} \quad \text{for } x \in \mathbb{R}. \qquad \square$$

Theorem 2.3. *For $f, g \in D_*(\mathbb{R})$, we have*

$$\mathcal{F} \otimes \mathcal{F}^{-1} \, (V(f, g)) \, (\mu, \lambda) = \varphi_\mu(\lambda) \, f(\lambda) \mathcal{F}g(\mu) \quad \textit{for } \mu, \lambda \in \mathbb{R}.$$

Proof. Using Definition 2.1 and Fubini's Theorem we have, for all $\mu, \lambda \in \mathbb{R}$,

$$\begin{aligned}
\mathcal{F} \otimes \mathcal{F}^{-1} \, (V(f, g)) \, (\mu, \lambda) &= \int_0^{+\infty} \int_0^{+\infty} V(f, g)(x, y) \, \varphi_\mu(x) \, \varphi_y(\lambda) \, d\nu(x) \, d\gamma(y) \\
&= \int_0^{+\infty} \int_0^{+\infty} \mathcal{F}(f \mathcal{T}_x g)(y) \, \varphi_\mu(x) \, \varphi_y(\lambda) \, d\nu(x) \, d\gamma(y) \\
&= \int_0^{+\infty} \varphi_\mu(x) \left(\int_0^{+\infty} \mathcal{F}(f \mathcal{T}_x g)(y) \varphi_y(\lambda) \, d\gamma(y) \right) d\nu(x).
\end{aligned}$$

From (1–8) we deduce

$$\begin{aligned}\mathscr{F}\otimes\mathscr{F}^{-1}\,(V(f,g))\,(\mu,\lambda) &= \int_0^{+\infty}\varphi_\mu(x)\,f(\lambda)\,\mathscr{T}_x g(\lambda)\,d\nu(x)\\ &= f(\lambda)\,\mathscr{F}(\mathscr{T}_\lambda g)(\mu) = f(\lambda)\,\varphi_\mu(\lambda)\,\mathscr{F}g(\mu). \qquad\square\end{aligned}$$

Corollary 2.4. *For all* $f, g \in D_*(\mathbb{R})$, *we have*

$$\int_0^{+\infty}\mathscr{F}\otimes\mathscr{F}^{-1}\,(V(f,g))\,(\mu,\lambda)\,d\nu(\lambda) = \mathscr{F}f(\mu)\,\mathscr{F}g(\mu) \quad \text{for } \mu\in\mathbb{R},$$

$$\int_0^{+\infty}\mathscr{F}\otimes\mathscr{F}^{-1}\,(V(f,g))\,(\mu,\lambda)\,d\gamma(\mu) = f(\lambda)\,g(\lambda) \quad \text{for } \lambda\in\mathbb{R}.$$

Proof. Theorem 2.3 gives

$$\begin{aligned}\int_0^{+\infty}\mathscr{F}\otimes\mathscr{F}^{-1}\,(V(f,g))\,(\mu,\lambda)\,d\nu(\lambda) &= \int_0^{+\infty}\varphi_\mu(\lambda)\,f(\lambda)\mathscr{F}g(\mu)\,d\nu(\lambda)\\ &= \mathscr{F}f(\mu)\,\mathscr{F}g(\mu) \quad \text{for } \mu\in\mathbb{R},\\ \int_0^{+\infty}\mathscr{F}\otimes\mathscr{F}^{-1}\,(V(f,g))\,(\mu,\lambda)\,d\gamma(\mu) &= \int_0^{+\infty}\varphi_\mu(\lambda)\,f(\lambda)\mathscr{F}g(\mu)\,d\gamma(\mu)\\ &= f(\lambda)\int_0^{+\infty}\varphi_\mu(\lambda)\mathscr{F}g(\mu)\,d\gamma(\mu)\\ &= f(\lambda)\,g(\lambda) \quad \text{for } \lambda\in\mathbb{R}. \qquad\square\end{aligned}$$

Theorem 2.5. *Let* $f, g \in L^1(d\nu)\cap L^2(d\nu)$ *be such that* $c = \int_0^{+\infty} g(x)\,d\nu(x) \neq 0$. *Then*

$$\mathscr{F}f(\lambda) = \frac{1}{c}\int_0^{+\infty} V(f,g)(x,\lambda)\,d\nu(x) \quad \text{for } \lambda\in\mathbb{R}.$$

Proof. From Definition 2.1, we have

$$\int_0^{+\infty} V(f,g)(x,\lambda)\,d\nu(x) = \int_0^{+\infty}\left(\int_0^{+\infty} f(y)\,\mathscr{T}_x g(y)\,\varphi_\lambda(y)\,d\nu(y)\right)d\nu(x)$$

for all $\lambda\in\mathbb{R}$. The result follows from Fubini's Theorem and the equality

$$\int_0^{+\infty}\mathscr{T}_x g(y)\,d\nu(y) = \int_0^{+\infty} g(x)\,d\nu(x) = c. \qquad\square$$

Corollary 2.6. *With the hypothesis of Theorem 2.5, if* $\mathscr{F}f \in L^1(d\gamma)$, *we have the following inversion formula for the Fourier–Wigner transform* V:

$$f(x) = \frac{1}{c}\int_0^{+\infty}\varphi_\mu(x)\left(\int_0^{+\infty} V(f,g)(y,\mu)\,d\nu(y)\right)d\gamma(\mu) \quad \text{for a.e. } x\in\mathbb{R}.$$

3. The Weyl transform associated with Δ

We now introduce the Weyl transform and relate it to the Fourier–Wigner transform. To do this, we must define the class of pseudodifferential operators [Wong 1998].

Definition 3.1. Let $m \in \mathbb{R}$. We define S^m to be the set of all infinitely differentiable functions σ on $\mathbb{R} \times \mathbb{R}$, even with respect to each variable, and such that for all $p, q \in \mathbb{N}$, there exists a positive constant $C_{p,q,m}$ satisfying

$$\left| \left(\frac{\partial}{\partial x}\right)^p \left(\frac{\partial}{\partial y}\right)^q \sigma(x, y) \right| \le C_{p,q,m} (1 + y^2)^{m-q}.$$

Definition 3.2. For $m \in \mathbb{R}$ and $\sigma \in S^m$, we define the operator H_σ on $D_*(\mathbb{R}) \times D_*(\mathbb{R})$ by

$$H_\sigma(f, g)(\lambda) = \int_0^{+\infty} \left(\int_0^{+\infty} \sigma(x, y) \varphi_y(\lambda) V(f, g)(x, y)\, d\nu(x) \right) d\gamma(y), \tag{3–1}$$

for all $\lambda \in \mathbb{R}$, and we put

$$\mathbb{H}_\sigma(f, g) = H_\sigma(f, g)(0). \tag{3–2}$$

Proposition 3.3. *Define $\sigma \in S^m$ by $\sigma(x, y) = -y^2$ for $x, y \in \mathbb{R}$. Then, for all $f, g \in D_*(\mathbb{R})$, we have*

$$H_\sigma(f, g)(\lambda) = c\, \Delta f(\lambda) \quad \textit{for } \lambda \in \mathbb{R},$$

where $c = \int_0^{+\infty} g(x)\, d\nu(x)$.

Proof. From (3–1), we have

$$H_\sigma(f, g)(\lambda) = \int_0^{+\infty} \left(\int_0^{+\infty} -y^2 \varphi_y(\lambda) V(f, g)(x, y)\, d\nu(x) \right) d\gamma(y) \text{for } \lambda \in \mathbb{R}.$$

Using Definition 2.1 we obtain

$$H_\sigma(f, g)(\lambda) = \int_0^{+\infty} \left(\int_0^{+\infty} -y^2 \varphi_y(\lambda) \left(\int_0^{+\infty} f(z) \mathcal{T}_x g(z) \varphi_y(z)\, d\nu(z) \right) d\nu(x) \right) d\gamma(y)$$

for $\lambda \in \mathbb{R}$. From Fubini's Theorem, we get

$$\begin{aligned} H_\sigma(f, g)(\lambda) &= \int_0^{+\infty} -y^2 \varphi_y(\lambda) \left(\int_0^{+\infty} f(z) \varphi_y(z) \left(\int_0^{+\infty} \mathcal{T}_z g(x)\, d\nu(x) \right) d\nu(z) \right) d\gamma(y) \\ &= c \int_0^{+\infty} -y^2 \varphi_y(\lambda) \left(\int_0^{+\infty} f(z) \varphi_y(z)\, d\nu(z) \right) d\gamma(y) \\ &= c \int_0^{+\infty} -y^2 \varphi_y(\lambda) \mathcal{F} f(y)\, d\gamma(y). \end{aligned}$$

But, for all $y \in \mathbb{R}$, $-y^2 \mathscr{F} f(y) = \mathscr{F}(\Delta f)(y)$. We complete the proof using the inversion formula (1–8). $\square$

Definition 3.4. Let $\sigma \in S^m$; $m < -\alpha - 1$. The *Weyl transform* associated with Δ is the mapping W_σ defined on $D_*(\mathbb{R})$ by

$$W_\sigma(f)(\lambda) = \int_0^{+\infty} \left(\int_0^{+\infty} \varphi_y(\lambda) \sigma(x, y) \mathscr{T}_\lambda f(x)\, d\nu(x) \right) d\gamma(y) \quad \text{for } \lambda \in \mathbb{R}.$$

Notation. We denote by

- $S_*(\mathbb{R})$ the space of even, infinitely differentiable functions on $\mathbb{R}$, rapidly decreasing together with all their derivatives.
- $S_*^2(\mathbb{R}) = \varphi_0 S_*(\mathbb{R})$, where φ_0 is the solution of (1–2) with $\lambda = 0$.

For $\rho = 0$ these two spaces coincide [Trimèche 1997]. The Fourier transform $\mathscr{F}$ is a topological isomorphism from $S_*^2(\mathbb{R})$ onto $S_*(\mathbb{R})$, whose inverse is given by (1–12).

Lemma 3.5. *For $\sigma \in D_*(\mathbb{R}^2)$, the function k defined by*

$$k(x, y) = \int_0^{+\infty} \varphi_\lambda(x) \mathscr{T}_x\big(\sigma(\,\cdot\,, \lambda)\big)(y)\, d\gamma(\lambda) \quad \textit{for } x, y \in \mathbb{R}$$

belongs to $L^p(d\nu \otimes d\nu)$, for all $p \in [2, +\infty[$.

Proof. The defining equation of k can be rewritten $k(x, y) = \mathscr{T}_x(G(\,\cdot\,, x))(y)$, where

$$G(x, y) = I \otimes \mathscr{F}^{-1}(\sigma)(x, y) \quad \text{for } x, y \in \mathbb{R},$$

for I the identity operator. It follows that, for all $p \in [2, +\infty[$,

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty} |k(x, y)|^p d\nu(x)\, d\nu(y) &= \int_0^{+\infty}\left(\int_0^{+\infty} \big|\mathscr{T}_x(G(\,\cdot\,, x)(y))\big|^p d\nu(y)\right) d\nu(x)\\ &\le \int_0^{+\infty}\left(\int_0^{+\infty} |G(y, x)|^p d\nu(y)\right) d\nu(x)\\ &\le \int_0^{+\infty}\left(\int_0^{+\infty} \big|I \otimes \mathscr{F}^{-1}(\sigma)(y, x)\big|^p d\nu(y)\right) d\nu(x).\end{aligned}$$

We distinguish two cases, $p = 2$ and $p \in \,]2, +\infty[$, the case $p = +\infty$ being trivial. For $p = 2$,

$$\int_0^{+\infty}\int_0^{+\infty} |k(x, y)|^2\, d\nu(x)\, d\nu(y) \le \int_0^{+\infty}\left(\int_0^{+\infty} \big|\mathscr{F}^{-1}(\sigma(x, \cdot\,)(y))\big|^2\, d\nu(x)\right) d\nu(y).$$

From (1–10) we deduce that

$$\int_0^{+\infty}\int_0^{+\infty} |k(x, y)|^2\, d\nu(x)\, d\nu(y) \le \int_0^{+\infty}\left(\int_0^{+\infty} |\sigma(y, x)|^2\, d\gamma(y)\right) d\nu(y) < +\infty,$$

because σ belongs to $D_*(\mathbb{R}^2)$. The case $p \in]2, +\infty[$ is more complex. From the hypotheses on Δ, we deduce that, as $x \to +\infty$,

$$A(x) \sim \begin{cases} x^{2\alpha+1} & \text{if } \rho = 0, \\ \exp(2\rho x) & \text{if } \rho > 0. \end{cases} \tag{3–3}$$

- For $\rho = 0$, recall that $\mathscr{F}$ is an isomorphism from $S_*(\mathbb{R})$ onto itself. Thus $I \otimes \mathscr{F}^{-1}(\sigma)$ belongs to $S_*(\mathbb{R}^2)$, and the asymptotics (3–3) implies

$$\int_0^{+\infty}\int_0^{+\infty} |k(x, y)|^p d\nu(x)\, d\nu(y) \leq \int_0^{+\infty}\left(\int_0^{+\infty} \left|I \otimes \mathscr{F}^{-1}(\sigma)(y, x)\right|^p d\nu(x)\right) d\nu(y) < +\infty. \tag{3–4}$$

- For $\rho > 0$, we have from [Trimèche 1997, p. 99]

  $$|\varphi_\lambda(x)| \leq \varphi_0(x) \leq m(1 + x)\exp(-\rho x) \quad \text{for all } \lambda \in \mathbb{R} \text{ and } x \geq 0,$$

  where m is a positive constant. Then

  $$\left|I \otimes \mathscr{F}^{-1}(\sigma)(y, x)\right| \leq m(1 + x)\exp(-\rho x) \int_0^{+\infty} |\sigma(y, z)|\, d\nu(z).$$

  Since σ belongs to $D_*(\mathbb{R}^2)$, there exists a positive constant M such that

  $$\int_0^{+\infty} |\sigma(y, z)|\, d\nu(z) \leq M \quad \text{for } \ y \geq 0,$$

  which implies that

  $$\left|I \otimes \mathscr{F}^{-1}(\sigma)(y, x)\right| \leq mM(1 + x)\exp(-\rho x).$$

  This, together with the asymptotics (3–3), implies the validity of the same bound (3–4) as in the previous case. □

Theorem 3.6. *Let $\sigma \in D_*(\mathbb{R}^2)$ and $f \in D_*(\mathbb{R})$.*

(i) $W_\sigma(f)(x) = \int_0^{+\infty} k(x, y) f(y)\, d\nu(y)$ *for all* $x \in \mathbb{R}$.

(ii) $\|W_\sigma(f)\|_{p',\nu} \leq \|k\|_{p',\nu\otimes\nu} \|f\|_{p,\nu}$ *for* $p \in [1, 2]$ *and* p' *such that* $1/p + 1/p' = 1$.

(iii) *W_σ can be extended to a bounded operator from $L^p(d\nu)$ into $L^{p'}(d\nu)$. In particular, $W_\sigma : L^2(d\nu) \to L^2(d\nu)$ is a Hilbert–Schmidt operator, hence compact.*

Proof. (i) From Definition 3.4, we have, for all $x \in \mathbb{R}$;

$$\begin{aligned} W_\sigma(f)(x) &= \int_0^{+\infty} \varphi_y(x)\left(\int_0^{+\infty} \sigma(z,y)\mathcal{T}_x f(z)\,d\nu(z)\right) d\gamma(y) \\ &= \int_0^{+\infty} \varphi_y(x)\left(\int_0^{+\infty} f(z)\mathcal{T}_x[\sigma(.,y)](z)\,d\nu(z)\right) d\gamma(y) \end{aligned}.$$

From Fubini's Theorem, we get, for all $x \in \mathbb{R}$,

$$\begin{aligned} W_\sigma(f)(x) &= \int_0^{+\infty} f(z)\left(\int_0^{+\infty} \varphi_y(x)\mathcal{T}_x[\sigma(.,y)](z)\,d\gamma(y)\right) d\nu(z) \\ &= \int_0^{+\infty} f(z)k(x,z)\,d\nu(z). \end{aligned}$$

(ii) Follows from (i), Hölder's inequality, and Lemma 3.5.

(iii) Since $k \in L^2(d\nu \otimes d\nu)$, the mapping

$$W_\sigma : L^2(d\nu) \longrightarrow L^2(d\nu)$$

is a Hilbert–Schmidt operator, and so compact. □

Theorem 3.7. *Let $m < -\alpha - 1$ and $\sigma \in S^m$. For all $f, g \in D_*(\mathbb{R})$,*

$$\mathbb{H}_\sigma(f,g) = \int_0^{+\infty} f(x)W_\sigma g(x)\,d\nu(x). \tag{3–5}$$

Proof. Using (3–2) and Definition 2.1 we obtain

$$\begin{aligned} \mathbb{H}_\sigma(f,g) &= \int_0^{+\infty}\left(\int_0^{+\infty} \sigma(x,y)V(f,g)(x,y)\,d\nu(x)\right) d\gamma(y) \\ &= \int_0^{+\infty}\left(\int_0^{+\infty} \sigma(x,y)\left(\int_0^{+\infty} f(\lambda)\mathcal{T}_x g(\lambda)\varphi_y(\lambda)\,d\nu(\lambda)\right) d\nu(x)\right) d\gamma(y). \end{aligned}$$

From Fubini's theorem, we get

$$\begin{aligned} \mathbb{H}_\sigma(f,g) &= \int_0^{+\infty} f(\lambda)\left(\int_0^{+\infty} \varphi_y(\lambda)\left(\int_0^{+\infty} \sigma(x,y)\mathcal{T}_x g(\lambda)\,d\nu(x)\right) d\gamma(y)\right) d\nu(\lambda) \\ &= \int_0^{+\infty} f(\lambda)W_\sigma(g)(\lambda)\,d\nu(\lambda). \end{aligned}$$ □

4. The Weyl transform with symbol in $L^p(d\nu \otimes d\gamma)$, for $1 \le p \le 2$

In this section we show using (3–5) that, if $1 \le p \le 2$, the Weyl transform with symbol in $L^p(d\nu \otimes d\gamma)$ is a compact operator.

Notation. We denote by $\mathcal{B}(L^2(d\nu))$ the $\mathbb{C}^*$-algebra of bounded operators Ψ from $L^2(d\nu)$ into itself, equipped with the norm

$$\|\Psi\|_* = \sup_{\|f\|_{2,\nu}=1} \|\Psi(f)\|_{2,\nu}\,.$$

Theorem 4.1. *Let $\langle\cdot/\cdot\rangle$ denote the inner product in $L^2(d\nu)$. There exists a unique operator $Q : L^2(d\nu\otimes d\gamma)\to\mathcal{B}(L^2(d\nu))$, whose action we denote by $\sigma\mapsto Q_\sigma$, such that*

$$\langle Q_\sigma(g)/\bar f\rangle=\int_0^{+\infty}\biggl(\int_0^{+\infty}\sigma(x,y)V(f,g)(x,y)\,d\nu(x)\biggr)\,d\gamma(y)\quad \textit{for } f,g\in L^2(d\nu).$$

Furthermore, $\|Q_\sigma\|_*\le\|\sigma\|_{2,\nu\otimes\gamma}$.

Proof. Let $\sigma\in D_*(\mathbb{R}^2)$. For $g\in D_*(\mathbb{R})$, put $Q_\sigma(g)=W_\sigma(g)$. From Theorems 3.6 and 3.7, we obtain

$$\begin{aligned}\langle Q_\sigma(g)/\bar f\rangle &= \langle W_\sigma(g)/\bar f\rangle=\mathbb{H}_\sigma(f,g)\\ &=\int_0^{+\infty}\biggl(\int_0^{+\infty}\sigma(x,y)V(f,g)(x,y)\,d\nu(x)\biggr)\,d\gamma(y).\end{aligned}$$

On the other hand, from Proposition 2.2(ii), we have

$$\bigl|\langle Q_\sigma(g)/\bar f\rangle\bigr|\le\|\sigma\|_{2,\nu\otimes\gamma}\,\|f\|_{2,\nu}\,\|g\|_{2,\nu}\,.$$

Thus $Q_\sigma\in\mathcal{B}(L^2(d\nu))$ and

$$\|Q_\sigma\|_*\le\|\sigma\|_{2,\nu\otimes\gamma}\,. \tag{4–1}$$

Now consider $\sigma\in L^2(d\nu\otimes d\gamma)$. Let $(\sigma_k)_{k\in\mathbb{N}}$ be a sequence in $D_*(\mathbb{R}^2)$ such that $\|\sigma_k-\sigma\|_{2,\nu\otimes\gamma}$ approaches 0 as $k\to+\infty$. From (4–1) we have, for all $k,l\in\mathbb{N}$,

$$\|Q_{\sigma_k}-Q_{\sigma_l}\|_*\le\|\sigma_k-\sigma_l\|_{2,\nu\otimes\gamma}\le\|\sigma_k-\sigma\|_{2,\nu\otimes\gamma}+\|\sigma_l-\sigma\|_{2,\nu\otimes\gamma}.$$

Thus $(Q_{\sigma_k})_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{B}(L^2(d\nu))$. Let it converge to Q_σ. Clearly Q_σ is independent from the choice of $(\sigma_k)_{k\in\mathbb{N}}$, and we have

$$\|Q_\sigma\|_*=\lim_{k\to+\infty}\|Q_{\sigma_k}\|_*\le\lim_{k\to+\infty}\|\sigma_k\|_{2,\nu\otimes\gamma}=\|\sigma\|_{2,\nu\otimes\gamma}.$$

We consider first $f,g\in D_*(\mathbb{R})$. Then

$$\begin{aligned}\langle Q_\sigma(g)/\bar f\rangle &=\lim_{k\to+\infty}\langle Q_{\sigma_k}(g)/\bar f\rangle\\ &=\lim_{k\to+\infty}\int_0^{+\infty}\biggl(\int_0^{+\infty}\sigma_k(x,y)V(f,g)(x,y)\,d\nu(x)\biggr)\,d\gamma(y)\\ &=\int_0^{+\infty}\biggl(\int_0^{+\infty}\sigma(x,y)V(f,g)(x,y)\,d\nu(x)\biggr)\,d\gamma(y).\end{aligned}$$

Now let f, g be in $L^2(d\nu)$. Pick sequences $(f_k)_{k\in\mathbb{N}}$, and $(g_k)_{k\in\mathbb{N}}$ in $D_*(\mathbb{R})$ converging to f and g, respectively, in the $\|\cdot\|_{2,\nu}$-norm. Then

$$\begin{aligned}\langle Q_\sigma(g)/\bar{f}\rangle &= \lim_{k\to+\infty}\langle Q_\sigma(g_k)/\bar{f}_k\rangle \\ &= \lim_{k\to+\infty}\int_0^{+\infty}\left(\int_0^{+\infty}\sigma(x,y)V(f_k,g_k)(x,y)\,d\nu(x)\right)d\gamma(y) \\ &= \int_0^{+\infty}\left(\int_0^{+\infty}\sigma(x,y)V(f,g)(x,y)\,d\nu(x)\right)d\gamma(y). \qquad \square\end{aligned}$$

We now give an extension of Theorem 4.1 that will allow us to prove that for $1\le p\le 2$ the Weyl transform with symbol in $L^p(d\nu\otimes d\gamma)$, is a compact operator.

Theorem 4.2. *Let $p\in[1,2]$. There exists a unique bounded operator*

$$Q: L^p(d\nu\otimes d\gamma)\to \mathcal{B}(L^2(d\nu)),$$

whose action is denoted by $\sigma\to Q_\sigma$, such that

$$\langle Q_\sigma(g)/\bar{f}\rangle = \int_0^{+\infty}\left(\int_0^{+\infty}\sigma(x,y)V(f,g)(x,y)\,d\nu(x)\right)d\gamma(y) \quad \text{for } f, g\in D_*(\mathbb{R}).$$

Moreover, $\|Q_\sigma\|_*\le\|\sigma\|_{p,\nu\otimes\gamma}$.

Proof. The case $p=2$ is given by Theorem 4.1. We turn to the case $p=1$. For $\sigma\in D_*(\mathbb{R}^2)$, we define Q_σ by

$$Q_\sigma(g)=W_\sigma(g) \quad \text{for } g\in D_*(\mathbb{R}).$$

From Theorems 3.6 and 3.7, we have, for $f\in D_*(\mathbb{R})$,

$$\langle Q_\sigma(g)/\bar{f}\rangle = \mathbb{H}_\sigma(f,g) = \int_0^{+\infty}\left(\int_0^{+\infty}\sigma(x,y)V(f,g)(x,y)\,d\nu(x)\right)d\gamma(y).$$

From Hölder's inequality we then obtain

$$\left|\langle Q_\sigma(g)/\bar{f}\rangle\right| \le \|\sigma\|_{1,\nu\otimes\gamma}\,\|V(f,g)\|_{\infty,\nu\otimes\gamma} \le \|\sigma\|_{1,\nu\otimes\gamma}\,\|f\|_{2,\nu}\,\|g\|_{2,\nu}\,.$$

This shows that $Q_\sigma\in\mathcal{B}(L^2(d\nu))$ and $\|Q_\sigma\|_*\le\|\sigma\|_{1,\nu\otimes\gamma}$.

We extend the definition of Q_σ and the two facts just proved to the case of $\sigma\in L^1(d\nu\otimes d\gamma)$, working as in the proof of Theorem 4.1.

Finally, the Riesz–Thorin Theorem [Stein 1956; Stein and Weiss 1971], allows us to generalize the same results from the cases $p=1$ and $p=2$ to all $p\in[1,2]$. $\square$

Theorem 4.3. *Let $p\in[1,2]$. For $\sigma\in L^p(d\nu\otimes d\gamma)$, the operator Q_σ from $L^2(d\nu)$ into itself is compact.*

Proof. Given $\sigma \in L^p(d\nu \otimes d\gamma)$, choose a sequence $(\sigma_k)_{k\in\mathbb{N}}$ in $D_*(\mathbb{R}^2)$ approximating σ in the $\|\cdot\|_{p,\nu\otimes\gamma}$-norm. The last assertion of Theorem 4.2 says that

$$\|Q_{\sigma_k} - Q_\sigma\|_* \le \|\sigma_k - \sigma\|_{p,\nu\otimes\gamma},$$

so Q_{σ_k} approaches Q_σ in $\mathcal{B}(L^2(d\nu))$. From Theorem 3.6 we know that $W_{\sigma_k} = Q_{\sigma_k}$ is compact for all $k \in \mathbb{N}$. The theorem then follows from the fact that the subspace $\mathcal{K}(L^2(d\nu))$ of $\mathcal{B}(L^2(d\nu))$ consisting of compact operators is a closed ideal of $\mathcal{B}(L^2(d\nu))$. □

5. The Weyl transform with symbol in $S'_{*,0}(\mathbb{R}^2)$

Notation. We denote by

- $S_{*,0}(\mathbb{R}^2)$ the subspace of $S_*(\mathbb{R}^2)$ consisting of functions with compact support with respect to the first variable;
- $S'_{*,0}(\mathbb{R}^2)$ the topological dual of $S_{*,0}(\mathbb{R}^2)$;
- $D'_*(\mathbb{R})$ the space of even distribution on $\mathbb{R}$. It is the topological dual of $D_*(\mathbb{R})$.

Definition 5.1. For $\sigma \in S'_{*,0}(\mathbb{R}^2)$ and $g \in D_*(\mathbb{R})$, we define the operator $W_\sigma(g)$ on $D_*(\mathbb{R})$ by

$$\big(W_\sigma(g)\big)(f) = \sigma(V(f,g)) \quad \text{for } f \in D_*(\mathbb{R}), \tag{5–1}$$

where V is the mapping from Definition 2.1. Clearly $W_\sigma(g)$ belongs to $D'_*(\mathbb{R})$.

Proposition 5.2. *Consider the distribution σ of $S'_{*,0}(\mathbb{R}^2)$ given by the constant function* 1. *For all $g \in D_*(\mathbb{R})$, we have*

$$W_\sigma(g) = c\delta,$$

where $c = \int_0^{+\infty} g(x)\,d\nu(x)$ and δ is the Dirac distribution at 0.

Proof. For $f, g \in D_*(\mathbb{R})$, we get

$$\big(W_\sigma(g)\big)(f) = \sigma(V(f,g)) = \int_0^{+\infty}\bigg(\int_0^{+\infty} V(f,g)(x,y)\,d\nu(x)\bigg)\,d\gamma(y).$$

But from the proof of Theorem 2.5, we have

$$\int_0^{+\infty} V(f,g)(x,y)\,d\nu(x) = c\mathcal{F}f(y) \quad \text{for } y \in \mathbb{R}.$$

Integrating both sides over $[0,+\infty[$ with respect to the measure $d\gamma$ and using (1–8), we obtain

$$\sigma(V(f,g)) = \big(W_\sigma(g)\big)(f) = c\int_0^{+\infty} \mathcal{F}f(y)\,d\gamma(y) = cf(0) = (c\delta,\ f). \qquad \square$$

Note that by Proposition 5.2, there exists $\sigma \in S'_{*,0}(\mathbb{R}^2)$, given by a function in $L^\infty(d\nu \otimes d\gamma)$, such that for all $g \in D_*(\mathbb{R})$ satisfying $c = \int_0^{+\infty} g(x)\, d\nu(x) \neq 0$, the distribution $W_\sigma(g)$ is *not* given by a function in $L^2(d\nu)$.

6. The Weyl transform with symbol in $L^p(d\nu \otimes d\gamma)$, for $2 < p < \infty$

Theorem 6.1. *Let $p \in\,]2, \infty[$. There exists a function $\sigma \in L^p(d\nu \otimes d\gamma)$ such that the Weyl transform W_σ defined by (5–1) is not a bounded linear operator on $L^2(d\nu)$.*

We break down the proof into two lemmas, of which the theorem is an immediate consequence.

Lemma 6.2. *Let $p \in\,]2, \infty[$. Suppose that for all $\sigma \in L^p(d\nu \otimes d\gamma)$, the Weyl transform W_σ given by (5–1) is a bounded linear operator on $L^2(d\nu)$. Then there exists a positive constant M such that*

$$\|W_\sigma\|_* \leq M\, \|\sigma\|_{p,\nu\otimes\gamma} \quad \textit{for all } \sigma \in L^p(d\nu \otimes d\gamma). \tag{6–1}$$

Proof. Under the assumption of the lemma, there exists for each $\sigma \in L^p(d\nu \otimes d\gamma)$ a positive constant C_σ such that

$$\|W_\sigma(g)\|_{2,\nu} \leq C_\sigma \|g\|_{2,\nu} \quad \text{for } g \in L^2(d\nu).$$

Let $f, g \in D_*(\mathbb{R})$ be such that $\|f\|_{2,\nu} = \|g\|_{2,\nu} = 1$ and define a linear operator $Q_{f,g} : L^p(d\nu \otimes d\gamma) \to \mathbb{C}$ by

$$Q_{f,g}(\sigma) = \big\langle W_\sigma(g) / \bar{f} \big\rangle.$$

Then

$$\sup_{|f|_{2,\nu} = |g|_{2,\nu} = 1} |Q_{f,g}(\sigma)| \leq C_\sigma.$$

By the Banach–Steinhaus theorem, the operator $Q_{f,g}$ is bounded on $L^p(d\nu \otimes d\gamma)$, so there exists $M > 0$ such that

$$\|Q_{f,g}\| = \sup_{\|\sigma\|_{p,\nu\otimes\gamma} = 1} |Q_{f,g}(\sigma)| \leq M.$$

From this we deduce that for all $f, g \in D_*(\mathbb{R})$ and $\sigma \in L^p(d\nu \otimes d\gamma)$,

$$\big|\langle W_\sigma(g) / \bar{f} \rangle\big| \leq M \|\sigma\|_{p,\nu\otimes\gamma} \|f\|_{2,\nu} \|g\|_{2,\nu},$$

which implies (6–1). □

Lemma 6.3. *For $2 < p < \infty$, there is no positive constant M satisfying (6–1).*

Proof. Suppose there exists such an M. Let p' be such that $1/p + 1/p' = 1$. Then $p' \in]1, 2[$. We consider, for $f, g \in D_*(\mathbb{R})$, the function $V(f, g)$ of Definition 2.1. We have

$$\|V(f,g)\|_{p',\nu\otimes\gamma} = \sup_{\|\sigma\|_{p,\nu\otimes\gamma}=1} \left| \int_0^{+\infty}\int_0^{+\infty} \sigma(x,y)V(f,g)(x,y)\,d\nu(x)\,d\gamma(y) \right|$$
$$= \sup_{\|\sigma\|_{p,\nu\otimes\gamma}=1} \left| \langle W_\sigma(g)/\bar{f}\rangle \right| \le \sup_{\|\sigma\|_{p,\nu\otimes\gamma}=1} \|W_\sigma(g)\|_{2,\nu}\|f\|_{2,\nu},$$

and consequently

$$\|V(f,g)\|_{p',\nu\otimes\gamma} \le M\,\|f\|_{2,\nu}\,\|g\|_{2,\nu}. \tag{6–2}$$

Now consider f, g in $L^2(d\nu)$. Choose sequences $(f_k)_{k\in\mathbb{N}}$ and $(g_k)_{k\in\mathbb{N}}$ in $D_*(\mathbb{R})$ approximating f and g in the $\|\cdot\|_{2,\nu}$-norm. By Proposition 2.2, the sequence $(V(f_k, g_k))_{k\in\mathbb{N}}$ converges to $V(f, g)$ in $L^{p'}(d\nu\otimes d\gamma)$, and thus we have extended (6–2) to all $f, g \in L^2(d\nu)$. We will exhibit an example where this leads to a contradiction.

Let f be an even, measurable function on $\mathbb{R}$, supported in $[-1, 1]$. We have

$$\left|V(f,f)(x,y)\right| \le |f| * |f|(x),$$

where $*$ is the convolution product (Definition 1.1). From (1–7), we deduce that for all $y \in \mathbb{R}$, the function $x \mapsto V(f, f)(x, y)$ is supported in $[-2, 2]$. Hölder's inequality gives

$$\left(\int_0^{+\infty}\left|\int_0^2 V(f,f)(x,y)\,d\nu(x)\right|^{p'} d\gamma(y)\right)^{1/p'}$$
$$\le \left(\int_0^2 d\nu(x)\right)^{1/p}\left(\int_0^{+\infty}\left(\int_0^2 |V(f,f)(x,y)|^{p'}\,d\nu(x)\right)d\gamma(y)\right)^{1/p'}$$
$$= \left(\int_0^2 d\nu(x)\right)^{1/p} \|V(f,f)\|_{p',\nu\otimes\gamma} \le M\left(\int_0^2 d\nu(x)\right)^{1/p} \|f\|^2_{2,\nu},$$

the last inequality following from (6–2). This proves that the function

$$y \mapsto \int_0^{+\infty} V(f,f)(x,y)\,d\nu(x) = c\mathcal{F}f(y)$$

belongs to $L^{p'}(d\gamma)$; here $c = \int_0^{+\infty} f(x)\,d\nu(x)$. and we have used the proof of Theorem 2.5 for the equality on the right-hand side. Putting this together with the preceding inequality we see that, if $c \ne 0$, the function $\mathcal{F}f$ belongs to $L^{p'}(d\gamma)$ and

$$\|\mathcal{F}f\|_{p',\gamma} \le \frac{M}{|c|}\left(\int_0^2 d\nu(x)\right)^{1/p} \|f\|^2_{2,\nu}. \tag{6–3}$$

Now consider the particular function f given by

$$f(x) = \frac{|x|^r}{\sqrt{B(x)}}\,\mathbf{1}_{[-1,1]}(x)$$

where B is the function defined by (1–1) and $\mathbf{1}_{[-1,1]}$ is the characteristic function of the interval $[-1,1]$. If $r > -(\alpha+1)$, this function belongs to $L^1(d\nu)\cap L^2(d\nu)$. From (1–4) we get

$$\begin{aligned}\mathscr{F}f(\lambda) &= \int_0^1 x^{r+2\alpha+1} j_\alpha(\lambda x)dx + \int_0^1 x^{r+\alpha+1/2}\theta_\lambda(x)dx\\ &= \frac{1}{\lambda^{r+2\alpha+2}}\int_0^\lambda x^{r+2\alpha+1} j_\alpha(x)dx + \int_0^1 x^{r+\alpha+1/2}\theta_\lambda(x)\,dx.\end{aligned}$$

Using the asymptotic expansion of the function j_α [Lebedev 1972; Watson 1944], given by

$$j_\alpha(x) = \frac{2^{\alpha+1/2}\Gamma(\alpha+1)}{\sqrt{\pi}x^{\alpha+1/2}}\left(\cos\left(x-\alpha\frac{\pi}{2}-\frac{\pi}{4}\right)+O\left(\frac{1}{x}\right)\right) \quad \text{as } x\to+\infty,$$

we deduce that for $-(\alpha+1) < r < -(\alpha+\frac{1}{2})$, the integral

$$a := \int_0^{+\infty} x^{r+2\alpha+1} j_\alpha(x)\,dx$$

exists and is finite, so

$$\frac{1}{\lambda^{r+2\alpha+2}}\int_0^\lambda x^{r+2\alpha+1} j_\alpha(x)\,dx \sim \frac{a}{\lambda^{r+2\alpha+2}} \quad \text{as } \lambda\to+\infty.$$

On the other hand, for $\lambda > 1$,

$$\left|\int_0^1 x^{r+\alpha+1/2}\theta_\lambda(x)dx\right| \le \frac{c_1}{\lambda^{\alpha+3/2}}\int_0^1 x^{r+\alpha+1/2}\Psi(x)\,dx,$$

where

$$\Psi(x) = \left(\int_0^x |Q(s)|\,ds\right)\exp\left(c_2\int_0^x |Q(s)|\,ds\right) \quad \text{for all } x>0$$

and Q is given by (1–5). Since $-(\alpha+1) < r < -(\alpha+\frac{1}{2})$, we deduce that

$$\mathscr{F}f(\lambda) \sim \frac{a}{\lambda^{r+2\alpha+2}} \quad \text{as } \lambda\to+\infty.$$

Using this and (1–6), it follows that there exist $K, R > 0$ such that

$$|\mathscr{F}f(\lambda)|^{p'}\frac{1}{2\pi|c(\lambda)|^2} \ge \frac{K}{\lambda^{p'(r+2\alpha+2)-2\alpha-1}} \quad \text{for } \lambda > R;$$

so for r such that $p'(r+2\alpha+2) < 2\alpha+2$, we get

$$\|\mathscr{F}f\|_{p',\gamma}^{p'} \geq \int_R^{+\infty} |\mathscr{F}f(\lambda)|^{p'} \frac{d\lambda}{2\pi\,|c(\lambda)|^2} \geq \int_R^{+\infty} \frac{K}{\lambda^{p'(r+2\alpha+2)-2\alpha-1}} d\lambda = +\infty.$$

This shows that the relation (6–3) is false if we choose r so as to satisfy simultaneously the conditions $r > -(\alpha+1)$, $r < -(\alpha+\frac{1}{2})$ and

$$r < -(2\alpha+2) + \frac{2\alpha+2}{p'}.$$

This contradiction proves the lemma and Theorem 6.1. □

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Cyrine Baccar
Department of Mathematics
Faculty of Sciences of Tunis
1060 Tunis
Tunisia

Lakhdar Tannech Rachdi
Department of Mathematics
Faculty of Sciences of Tunis
1060 Tunis
Tunisia
lakhdartannech.rachdi@fst.rnu.tn