TRANVERSAL HOLOMORPHIC SECTIONS AND LOCALIZATION OF ANALYTIC TORSIONS

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We prove a Bott-type residue formula twisted by $\wedge (V^*)$ with a holomorphic vector bundle $V$, and relate certain analytic torsions on the total manifold to the analytic torsions on the zero set of a holomorphic section of $V$.

Introduction

Beasley and Witten [2003], studying half-linear models, have described a compactification on any Calabi–Yau threefold $Y$ that is a complete intersection in a compact toric variety $X$. In particular, a remarkable cancellation involving the instanton effect [Beasley and Witten 2003, (1.3)], involving certain determinants of the $\bar{\delta}$-operator, was derived directly from a residue theorem. One would like to understand its implications in mathematics, for example in Gromov–Witten theory. Bershadsky, Cecotti, Ooguri and Vafa [Bershadsky et al. 1993; 1994] predicted that the analytic torsion of Ray–Singer will play a role regarding the genus-1 Gromov–Witten invariant. Thus we naturally try to understand the results about analytic torsion first.

As an application of [Bismut and Lebeau 1991] and the localization formula (1–3) in this paper, we were able to relate certain analytic torsions on the total manifold with the zero set of a holomorphic transversal section of $V$, generalizing [Bismut 2004, Theorem 6.6] and [Zhang n.d.] with $V = TX$ therein. We expect our formula will be useful for understanding [Beasley and Witten 2003, (1.3)] from a mathematical point of view.

This paper is organized as follows. In Section 1 we prove a Bott-type residue formula. In Section 2 we get a localization formula for Quillen metrics. In Section 3 we get a localization formula for analytic torsions under extra conditions. In Section 4, for the reader’s convenience, we write down six intermediate results, corresponding to [Bismut and Lebeau 1991, Theorems 6.4–6.9].

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1. A Bott-type residue formula

In this section, along the lines of [Bismut 1986, §1], we give a Bott-type residue formula (1–3) by assuming that the holomorphic section is transversal; compare to [Beasley and Witten 2003, (2.32), (2.34)].

Let \( X \) be a compact complex manifold with \( \dim X = n \) and let \( \mathcal{V} \) be a holomorphic vector bundle on \( X \) with \( \dim \mathcal{V} = l \). We assume that the line bundles \( \det T_X \) and \( \det \mathcal{V} \) are holomorphically isomorphic. We fix a holomorphic isomorphism \( \phi: \det \mathcal{V} \simeq \det T_X \), which is clearly unique up to a constant. Thus \( \phi \) defines a map from the \( \mathbb{Z}_2 \)-graded tensor product \( \wedge (\bar{T}^* X) \otimes \wedge (\mathcal{V}^*) \) to \( \wedge (\bar{T}^* X) \otimes \wedge^{\text{max}} (T^* X) \subset \wedge (T^*_R X) \otimes_R \mathbb{C} \). We can define the integral of an element \( \alpha \) of \( \Omega(X, \wedge (\mathcal{V}^*)) \), the set of smooth sections of \( \wedge (\bar{T}^* X) \otimes \wedge (\mathcal{V}^*) \) on \( X \), by

\[
\int_X \alpha = \int_X \phi(\alpha).
\]

Let \( v \) be a holomorphic section of \( \mathcal{V} \) on \( X \). Assume that \( v \) vanishes on a complex manifold \( Y \subset X \). Then \( \mathcal{V}|_Y: TX|_Y \to \mathcal{V}|_Y \) mapping \( U \) to \( \mathcal{V}|_U v \) does not depend on the choice of a connection \( \nabla \) on \( \mathcal{V} \), and \( \mathcal{V}|_U \mathcal{V}|_Y = 0 \) for \( U \in TY \). Let \( N \) be the normal bundle to \( Y \) in \( X \). Assume also that \( \nabla v|_Y: N \to \mathcal{V}|_Y \) is injective, and there is a holomorphic vector subbundle \( \mathcal{V}_1 \) on \( Y \) such that

\[
(1-1) \quad \mathcal{V}|_Y = \mathcal{V}_1 \oplus \text{Im} \mathcal{V}|_Y.
\]

Let \( P^v \) and \( P^{\text{Im} \mathcal{V}} \) be the natural projections from \( \mathcal{V} \) onto \( \mathcal{V}_1 \) and \( \text{Im} \mathcal{V}|_Y \).

Let \( i(v) \) be the standard contraction operator acting on \( \wedge (\mathcal{V}^*) \). A natural question, posed in [Beasley and Witten 2003, §2], is how to express \( \int_X \alpha \) using the local data near the zero set \( Y \) of \( v \) for a \( (\bar{\partial} X + i(v)) \)-closed form \( \alpha \), that is, a form satisfying \( (\bar{\partial} X + i(v))\alpha = 0 \).

First we recall an idea due to Bismut [Bismut 1986]; see also [Zhang 1990].

**Proposition 1.1.** Let \( \alpha \in \Omega(X, \wedge (\mathcal{V}^*)) \) be a \( (\bar{\partial} X + i(v)) \)-closed form. Then

\[
\int_X \alpha = \int_X e^{-((\bar{\partial} X + i(v))\omega)/t} \alpha \quad \text{for any } \omega \in \Omega(X, \wedge (\mathcal{V}^*)) \text{ and } t > 0.
\]

**Proof.** For any \( \omega \in \Omega(X, \wedge (\mathcal{V}^*)) \)

\[
(1-2) \quad \int_X \bar{\partial} X \omega = \int_X \phi(\bar{\partial} X \omega) = \int_X \bar{\partial} X \phi(\omega) = \int_X d\phi(\omega) = 0.
\]

From \((\bar{\partial} X + i(v))^2 = 0\) and \((\bar{\partial} X + i(v))\alpha = 0\), we have

\[
\frac{\partial}{\partial s} \int_X e^{-s((\bar{\partial} X + i(v))\omega)} \alpha = - \int_X (\bar{\partial} X + i(v))(\omega e^{-s((\bar{\partial} X + i(v))\omega)} \alpha) = 0,
\]

and the desired equality follows. \( \square \)
Recall that $\nabla v|_Y : N \rightarrow \text{Im} \nabla v|_Y$ is an isomorphism that induces isomorphisms of holomorphic line bundles $\phi_N = (\det \nabla v|_Y)^*: \text{det}(\text{Im} \nabla v|_Y)^* \rightarrow \text{det} N^*$ and $\phi_Y = \phi|_Y / ((\det \nabla v|_Y)^*) : \text{det} \mathcal{N}_1 \rightarrow \text{det} T^* Y$. These two isomorphisms make the integral $\int_N$ along the normal bundle $N$ and $\int_Y$ well defined.

Let $h^\mathcal{V}$ be a Hermitian metric on $\mathcal{V}$ such that $\mathcal{V}_1$ and $\text{Im} \nabla v|_Y$ are orthogonal on $Y$. Let $g^N_1$ be a Hermitian metric on $N$ such that $\nabla v|_Y : N \rightarrow \text{Im} \nabla v|_Y$ is an isometry. Let $\nabla^\mathcal{V}$ be the curvature of the holomorphic Hermitian connection $\nabla^\mathcal{V}$ on $(\mathcal{V}, h^\mathcal{V})$. Let $j : Y \rightarrow X$ be the natural embedding, and $\{Y_j\}_j$ the connected components of $Y$. On $Y$, define

$$R^\mathcal{V}_j = -(\nabla v)^{-1} P \text{Im} \nabla v R^\mathcal{V}(\cdot, j_\ast \cdot) P^{\mathcal{V}_j} \in T^* Y \otimes \mathcal{V}^* \otimes \text{End } N.$$

$R^\mathcal{V}_j$ is well defined since $P \text{Im} \nabla v R^\mathcal{V}(j_\ast \cdot, j_\ast \cdot) P^{\mathcal{V}_j} = 0$. Thus, for $U \in TY$, $W \in \mathcal{V}_1$, $u_1, u_2 \in N$,

$$\{R^\mathcal{V}_j(U, W) u_1, u_2 \}_{g^N_1} = - \{R^\mathcal{V}_j(U_1, \overline{U}) W, \nabla u_2 v \} = \{W, R^\mathcal{V}((\overline{u_1}, U) \nabla u_2 v)\}.$$

Certainly $\text{det}_N((1 + R^\mathcal{V}_j)/2\pi i)$ is $\overline{\mathcal{V}}$-closed.

The following result verifies a formula of Beasley and Whitney [2003, (2.32), (2.34)] and generalizes corresponding results in [Zhang 1990], [Liu 1995] and [Bott 1967].

**Theorem 1.2.** For any $(\overline{\partial}^X + i(v))$-closed form $\alpha \in \Omega(X, \wedge (\mathcal{V}^*))$,

$$\int_X \alpha = \sum_j \int_{Y_j} \frac{(-1)^{(j-n)(n-\dim Y_j)} \alpha}{\text{det}(1 + R^\mathcal{V}_j)/(2\pi i)}.$$  

**Proof.** Set

$$S = \{\cdot, v\}_h^\mathcal{V} \in C^\infty(X, \mathcal{V}^*).$$

By Proposition 1.1, for any $t \in ]0, +\infty[$,

$$\int_X \alpha = \int_X e^{-\frac{1}{2}(\overline{\partial}^X + i(v)) S} \alpha = \int_X e^{-\frac{1}{2}(\overline{\partial}^X S + |v|^2)} \alpha.$$

Thus, as $t \rightarrow 0$, the integral $\int_X \alpha$ is asymptotically equal to $\int_{\Omega} e^{-\frac{1}{2}(\overline{\partial}^X S + |v|^2)} \alpha$ for any neighborhood $\Omega$ of $Y$.

Take $y \in Y$. Since $Y$ is a complex submanifold, we can find holomorphic coordinates $\{z_i\}_{i=1}^n$ of a neighborhood $U$ of $y$ such that $y$ corresponds to 0 and $\{(\partial/\partial z_i)(0)\}_{i=m+1}^n$ is an orthonormal basis of $(N, g^N_1)$, and, moreover,

$$U \cap Y = \{p \in U, z_{m+1}(p) = \cdots = z_n(p) = 0\}.$$

Let $\{\mu_k\}_{k=1}^{l'}$ and $\{\mu_k\}_{k=l'+1}^l$ be holomorphic frames for $\mathcal{V}_1$ and $\text{Im} \nabla v|_Y$ on $U \cap Y$, with

$$\nabla_{\partial/\partial z_i (0)}^\mathcal{V} \mu_k(0) = \mu_k(0) \quad \text{for } l'+1 \leq k \leq l.$$
and for \( z' = (z_1, \ldots, z_m), \ z'' = (z_{m+1}, \ldots, z_n), \ z = (z', z'') \), define \( \mu_k(z) \) by parallel transport of \( \mu_k(z') \) with respect to \( \nabla^\nu \) along the curve \( u \mapsto (z', uz'') \). Identify \( \nabla^\nu_z \) with \( \nabla^\nu_{(z',0)} \) by identifying \( \mu_k(z) \) with \( \mu_k(z',0) \). Denote by \( W_y(\varepsilon) \) the \( \varepsilon \)-neighborhood of \( y \) in the normal space \( N \). Then

\[
(1-5) \quad \int_{Y \cap U} \int_{W_y(\varepsilon)} e^{-\frac{1}{t} \left( \partial^y S + |v|^2 \right)} \alpha = \int_{Y \cap U} \int_{z \in W_y(\varepsilon/\sqrt{t})} e^{-\frac{1}{t} \left( |v(\sqrt{t}z)|^2 + (\partial^y S)(\sqrt{t}z) \right)} t^{n-m} \alpha(y, \sqrt{t}z).
\]

Define \( z = \sum_j z_j (\partial / \partial z_j) \) and \( \bar{z} = \sum_j \bar{z}_j (\partial / \partial \bar{z}_j) \). The tautological vector field is \( Z = z + \bar{z} \). Then, for \( z \in N_y \),

\[
\frac{1}{2t} |v(\sqrt{t}z)|^2 = \frac{1}{2} |\nabla^\nu_z v|^2 + O(\sqrt{t}) = \frac{1}{2} |z|^2 + O(\sqrt{t})
\]

and

\[
\bar{\partial}^y S = \sum_{k=1}^l \langle \mu_k, \nabla^\nu v \rangle \mu^k.
\]

From now on, set \( z = (0, z') \) and \( Z = z + \bar{z} \). Since \( \nabla^\nu_Z \mu_k(0) = 0 \), we know that

\[
(1-6) \quad \frac{1}{2t} \bar{\partial}^y S(\sqrt{t}z)
\]

\[
= \frac{1}{2t} \sum_{k=1}^l \langle \mu_k, \nabla^\nu v \rangle (\sqrt{t}z) \mu^k(0)
\]

\[
= \frac{1}{2t} \sum_{k=1}^l \left( \langle \mu_k, \nabla^\nu v \rangle (0) + \sqrt{t} \langle \mu_k, \nabla^\nu_Z \nabla^\nu v \rangle (0)
\right.
\]

\[
+ \frac{t}{2} \left( \langle \nabla^\nu_Z \mu_k, \nabla^\nu v \rangle + \langle \mu_k, \nabla^\nu_Z \nabla^\nu v \rangle \right) (0) + O(t^{3/2}) \mu^k(0).
\]

Because of the factor \( t^{n-m} \) in (1–5), it should be clear that in the limit, only those monomials in the vertical form

\[
d\bar{z}_{m+1} \wedge \cdots \wedge d\bar{z}_n \bar{\otimes} \mu^{l+1} \wedge \cdots \wedge \mu^l
\]

whose weight is exactly \( t^{m-n} \) should be kept. Now,

\[
\nabla^\nu_Z \nabla^\nu_{a/\partial z_j} v = R^\nu(Z, \frac{\partial}{\partial z_j}) v + \nabla^\nu_Z \nabla^\nu_{a/\partial z_j} v - \frac{1}{2} [m,n] (j) \nabla^\nu_{a/\partial z_j} v,
\]

\[
\nabla^\nu_Z \nabla^\nu_{a/\partial z_j} v(0) = R^\nu(\bar{z}, \frac{\partial}{\partial z_j}) v + \nabla^\nu_{a/\partial z_j} v = 0,
\]
where $1_{[m,n]}$ is the characteristic function of the interval $[m, n]$. Note that $\nabla^\psi = \nabla^{\psi_1} \oplus \nabla^{\Im \psi}$ on $Y$ and that

$$\left\{ \mu_k, \nabla_\psi^{\psi_1} \nabla_\psi^{\psi_1} \psi_j \right\}(0) = 0 \quad \text{for} \ 1 \leq j \leq m, \ 1 \leq k \leq l'. $$

It follows that in the expression

$$\frac{1}{2\sqrt{t}} \left\{ \mu_k, \nabla_\psi^{\psi_1} \nabla_\psi^{\psi_1} \psi_j \right\}(0) \mu^k(0)$$

a nonzero contribution can only appear in the term

$$\left(1-7\right) \quad \frac{1}{2\sqrt{t}} \left( \sum_{j=1}^m \sum_{k=1}^l \sum_{j=m+1}^n \sum_{k=1}^{l'} \right) \left\{ \mu_k, \nabla_\psi^{\psi_1} \nabla_\psi^{\psi_1} \psi_j \right\}(0) d\tilde{z}^j \otimes \mu^k(0).$$

Similarly, in the last term of (1–6), the only term with a nonzero contribution is

$$\frac{1}{4} \sum_{j=1}^m \sum_{k=1}^{l'} \left( \nabla_\psi^{\psi_1} \nabla_\psi^{\psi_1} \psi_j \right)(0) \left( \mu_k, \nabla_\psi^{\psi_1} \nabla_\psi^{\psi_1} \psi_j \right)(0) d\tilde{z}^j \otimes \mu^k(0).$$

But for $1 \leq j \leq m$, both $\nabla_\psi^{\psi_1} \nabla_\psi^{\psi_1} \psi_j(0)$ and $\nabla_\psi^{\psi_1} \nabla_\psi^{\psi_1} \nabla_\psi^{\psi_1} \psi_j(0) = \nabla_\psi^{\psi_1} \nabla_\psi^{\psi_1} \nabla_\psi^{\psi_1} \psi_j(0)$ vanish, since $v = 0$ on $Y$. Thus, for $1 \leq j \leq m$,

$$\nabla_\psi^{\psi_1} \nabla_\psi^{\psi_1} \nabla_\psi^{\psi_1} \psi_j(0) = 2R^{\psi}(\tilde{z}, \nabla_\psi^{\psi_1} \nabla_\psi^{\psi_1} \psi_j(0) + \nabla_\psi^{\psi_1} \nabla_\psi^{\psi_1} \nabla_\psi^{\psi_1} \psi_j(0).$$

By the preceding discussion, as $t \to 0$, in (1–5), we should replace $\frac{1}{2\pi} \sqrt{\eta} S(y, \sqrt{t})$ by the 2-form

$$\frac{1}{2} \sum_{k=1}^l \left\{ \mu_k, \nabla_\psi^{\psi_1} \psi_j \right\}(0) \mu^k(0) + \sqrt{t} \times \text{expression } \left(1-7\right)$$

$$+ \frac{1}{2} \sum_{j=1}^m \sum_{k=1}^l \left( \mu_k, R^{\psi}(\tilde{z}, \nabla_\psi^{\psi_1} \nabla_\psi^{\psi_1} \psi_j(0) d\tilde{z}^j \otimes \mu^k(0).$$

Set $\beta_Y = d\tilde{z}_1 \cdots d\tilde{z}_m \wedge \mu^1(0) \cdots \mu^l(0), \beta_N = d\tilde{z}_{m+1} \cdots d\tilde{z}_n \wedge \mu^{l+1}(0) \cdots \mu^l(0), \phi(\mu^1(0) \cdots \mu^l(0)) = f dz_1 \cdots dz_n$. Then

$$\phi_Y (\mu^1(0) \cdots \mu^l(0)) \phi_N (\mu^{l+1}(0) \cdots \mu^l(0)) = f dz_1 \cdots dz_n.$$ 

Thus

$$\phi(\beta_Y \wedge \beta_N) = (-1)^{(l-m)} f dz_1 \cdots dz_n \wedge dz_1 \cdots dz_n$$

$$= (-1)^{(l-m)} \phi_Y (\beta_Y) \phi_N (\beta_N).$$

Now, observing that $\int_C \tilde{z}^i e^{-|\tilde{z}|^2} d\tilde{z} d\tilde{z} = 0$ for $i > 0$ and that $\nabla_\psi^{\psi_1} \psi_j : (N, g_Y^\psi) \to (\Im \nabla_\psi, h^\psi_{\Im \nabla_\psi})$ is an isometry and $l - l' = n - m$, we find that the limit of (1–4)
as $t \to 0$ is the sum over $j$ of

\[ (1-8) \int_{Y_j} (-1)^{(l-n)(n-m)} j^* \alpha \int_N \exp \left( -\frac{1}{2} \sum_{k=1}^l \langle \mu_k, \nabla^\vee v \rangle (0) \mu_k (0) \right. \]

\[ \left. -\frac{1}{2} \langle \cdot, P^\vee R^\vee (z, j_v) \nabla^\vee v \rangle (0) - \frac{1}{2} |\nabla^\vee v|^2 \right). \]

The second integrand in this expression can be rewritten as

\[
\exp \left( -\frac{1}{2} \sum_{i=1}^{n-m} d\bar{z}_{m+i} \wedge \mu^{l+i}(0) + \frac{1}{2} \left( R^\vee (z, j_v) P^\vee, \nabla^\vee v \right) (0) - \frac{1}{2} |z|^2 \right)
\]

\[ = \exp \left( \frac{1}{2} \left( (\nabla^\vee v)^{-1} R^\vee (z, j_v) P^\vee, z \right) - \frac{1}{2} |z|^2 \right) \left( \frac{1}{2} \right)^l d\bar{z}_{m+1} d\bar{z}_{m+1} \cdots d\bar{z}_0 d\bar{z}_n. \]

Thus the expression in (1-8) is equal to

\[
\int_{Y_j} \det_N \left( (1 + R^\vee v) / (-2\pi i) \right),
\]

which leads to (1-3).

\[ \square \]

2. Localization of Quillen metrics via a transversal section

Let $X$ be a compact complex manifold of dimension $n$. Let $\mathcal{V}$ and $\xi$ be holomorphic vector bundles on $X$ with dim $\mathcal{V} = m$, and let $v$ be a holomorphic section of $\mathcal{V}$. Assume that $v$ vanishes on a complex manifold $Y \subset X$ and satisfies (1–1). Then we have a complex of holomorphic vector bundles on $X$,

\[ (2-1) \quad 0 \rightarrow \Lambda^m (\mathcal{V}^*) \xrightarrow{i(v)} \Lambda^{m-1} (\mathcal{V}^*) \xrightarrow{i(v)} \cdots \xrightarrow{i(v)} \Lambda^0 (\mathcal{V}^*) \xrightarrow{i(v)} 0. \]

Let $(\Omega (X, \Lambda (\mathcal{V}^*) \otimes \xi), \bar{\partial}^X)$ be the Dolbeault complex associated to the holomorphic vector bundle $\Lambda (\mathcal{V}^*) \otimes \xi$. Let $\mathcal{H}_v (X, \Lambda (\mathcal{V}^*) \otimes \xi)$ be the hypercohomologies of the bicomplex $(\Omega (X, \Lambda (\mathcal{V}^*) \otimes \xi), \bar{\partial}^X, i(v))$. Let $j : Y \rightarrow X$ be the obvious embedding. Now the pullback map $j^*$ induces naturally a map of complexes

\[ (2-2) \quad j^* : (\Omega (X, \Lambda (\mathcal{V}^*) \otimes \xi), \bar{\partial}^X + i(v)) \rightarrow (\Omega (Y, \Lambda (\mathcal{V}_1^*) \otimes \xi), \bar{\partial}^Y). \]

**Theorem 2.1.** The map $j^*$ is a quasi-isomorphism of complexes. In particular, $j^*$ induces an isomorphism

\[ (2-3) \quad \mathcal{H}_v (X, \Lambda (\mathcal{V}^*) \otimes \xi) \simeq H (Y, \Lambda (\mathcal{V}_1^*) \otimes \xi). \]

**Proof.** In [Feng 2003] there is an analytic proof of this theorem when $\mathcal{V} = TX$. There we used the twisted vector bundle $\Lambda (T^* X)$ and here $\Lambda (\mathcal{V}^*)$ takes its place; the proof works just the same. For an algebraic proof, we can modify the proof of [Bismut 2004, Theorem 5.1]. \[ \square \]
Let \( N^{X}_H \) and \( N^{Y}_H \) be the number operators on \( \Lambda(T^*X), \Lambda(\wedge^p) \) corresponding to multiplication by \( p \) on \( \Lambda^p(T^*X), \Lambda^p(\wedge^p) \); do the same replacing \( X \) by \( Y \) and \( \wedge^p \) by \( \wedge^p_1 \). Then \( N^{X}_H - N^{X}_H \) and \( N^{Y}_H - N^{Y}_H \) define \( \mathbb{Z} \)-gradings on \( \Omega(X, \Lambda(\wedge^p) \otimes \xi) \) and \( \Omega(Y, \Lambda(\wedge^p_1) \otimes \xi) \), which in turn induce \( \mathbb{Z} \)-gradings on \( \mathcal{H}_v(X, \Lambda(\wedge^p) \otimes \xi) \) and \( H(Y, \Lambda(\wedge^p_1) \otimes \xi) \), respectively. The isomorphism \( j^* \) preserves these \( \mathbb{Z} \)-gradings.

From [Bismut and Lebeau 1991, (1.24)], we define the complex lines \( \lambda_v(\wedge^p) \) and \( \lambda(\wedge^p_1) \) by

\[
\lambda_v(\wedge^p) = \bigotimes_{p=-m}^n \left( \det \mathcal{H}_v^p(X, \Lambda(\wedge^p) \otimes \xi) \right)^{(-1)^p+1},
\]

\[
\lambda(\wedge^p_1) = \bigotimes_{p=0}^n \bigotimes_{q=0}^m \left( \det H^p(Y, \Lambda^q(\wedge^p_1) \otimes \xi) \right)^{(-1)^p+q+1}.
\]

By (2–3), we have a canonical isomorphism of complex lines

\[
\lambda_v(\wedge^p) \simeq \lambda(\wedge^p_1).
\]

Let \( \rho \) be the nonzero section of \( \lambda(\wedge^p_1)^{-1} \otimes \lambda_v(\wedge^p) \) associated with this canonical isomorphism.

Let \( g^{TX} \) be a Kähler metric on \( TX \). We identify \( N \) with the bundle orthogonal to \( TY \) in \( TX|_Y \). Let \( g^{TY} \) and \( g^N \) be the metrics on \( TY \) and \( N \) induced by \( g^{TX} \). Let \( h^\xi \) be a Hermitian metric on \( \xi \). Let \( h^v \) be a metric on \( v \) such that \( v_1 \) and \( \text{Im} \nabla v |_Y \) are orthogonal on \( Y \) and \( \nabla v |_Y : N \to \text{Im} \nabla v |_Y \) is an isometry.

Let \( dv_X \) be the Riemannian volume form on \( (X, g^{TX}) \). Let \( \langle \cdot, \cdot \rangle_0 \) be the metric on \( \Lambda(T^*X) \otimes \Lambda(\wedge^p) \otimes \xi \) induced by \( g^{TX}, h^v, h^\xi \). The Hermitian product on \( \Omega(X, \Lambda(\wedge^p) \otimes \xi) \) is defined by

\[
(\alpha, \alpha') = \frac{1}{(2\pi)^n} \int_X \langle \alpha, \alpha' \rangle_0 \, dv_X \quad \text{for } \alpha, \alpha' \in \Omega(X, \Lambda(\wedge^p) \otimes \xi).
\]

Let \( \bar{\partial}^X \) and \( v^\wedge = i(v)^* \) be the adjoint of \( \bar{\partial}^X \) and \( i(v) \) with respect to \( \langle \cdot, \cdot \rangle \). Set

\[
V = i(v) + i(v)^*, \quad D^X = \bar{\partial}^X + \bar{\partial}^X^*.
\]

By Hodge theory,

\[
\mathcal{H}_v(X, \Lambda(\wedge^p) \otimes \xi) \simeq \text{Ker}(D^X + V).
\]

Denote by \( P \) be the operator of orthogonal projection from \( \Omega(X, \Lambda(\wedge^p) \otimes \xi) \) onto \( \ker(D^X + V) \) and set \( P^\perp = 1 - P \). Let \( h^{\mathcal{H}_v} \) be the \( L^2 \)-metric on \( \mathcal{H}_v(X, \Lambda(\wedge^p) \otimes \xi) \) induced by the \( L^2 \)-product (2–4) via the isomorphism (2–5). Define in the same way a Hermitian product on \( \Omega(Y, \Lambda(\wedge^p_1) \otimes \xi) \) associated to \( g^{TY}, h^{\wedge^1}, h^\xi \). Let \( \bar{\partial}^Y \) be the adjoint of \( \bar{\partial}^Y \), and \( h^H(Y, \Lambda(\wedge^p_1) \otimes \xi) \) the corresponding \( L^2 \)-metric on
\(H(Y, \wedge (\mathcal{V}_1^* \otimes \xi))\). Set
\[
D^Y = \overline{\partial}^Y + \overline{\partial}^Y.*
\]

Let \(Q\) be the orthogonal projection operator from \(\Omega(Y, \wedge (\mathcal{V}_1^* \otimes \xi))\) on Ker \(D^Y\), and \(Q^\perp = 1 - Q\). Let \(\cdot | \lambda\nu(\mathcal{V}^n)\) and \(\cdot | \lambda\nu(\mathcal{V}^n)\) be the \(L^2\)-metrics on \(\lambda\nu(\mathcal{V}^n)\) and \(\lambda(\mathcal{V}^n)\) induced by \(h\nu\) and \(h^H(Y, \wedge (\mathcal{V}_1^* \otimes \xi))\). Following [Bismut and Lebeau 1991, (1.49)], let
\[
\theta^X_v(s) = -\text{Tr}_s((N^X - N^X_H)((D^X + V)^{-s} P^\perp)).
\]
Then \(\theta^X_v(s)\) extends to a meromorphic function of \(s \in \mathbb{C}\), which is holomorphic at \(s = 0\).

The Quillen metric \(\| \cdot \|_{\lambda\nu(\mathcal{V}^n)}\) on the line \(\lambda\nu(\mathcal{V}^n)\) is defined by
\[
\| \cdot \|_{\lambda\nu(\mathcal{V}^n)} = | \cdot | \lambda\nu(\mathcal{V}^n) \exp \left( -\frac{1}{2} \frac{\partial \theta^X_v}{\partial s}(0) \right).
\]

In the same way, the function
\[
\theta^Y(s) = -\text{Tr}_s((N^Y - N^Y_H)(D^Y)^{-s} Q^\perp)
\]
extends to a meromorphic function of \(s \in \mathbb{C}\), holomorphic at \(s = 0\). The Quillen metric \(\| \cdot \|_{\lambda(\mathcal{V}_1^*)}\) on the line \(\lambda(\mathcal{V}_1^*)\) is defined by
\[
\| \cdot \|_{\lambda(\mathcal{V}_1^*)} = | \cdot | \lambda(\mathcal{V}_1^*) \exp \left( -\frac{1}{2} \frac{\partial \theta^Y}{\partial s}(0) \right).
\]

Let \(\| \cdot \|_{\lambda(\mathcal{V}_1^*)^{-1} \otimes \lambda\nu(\mathcal{V}^n)}\) be the Quillen metric on \(\lambda(\mathcal{V}_1^*)^{-1} \otimes \lambda\nu(\mathcal{V}^n)\) induced by \(\| \cdot \|_{\lambda\nu(\mathcal{V}^n)}\) and \(\| \cdot \|_{\lambda(\mathcal{V}_1^*)}\) as in [Bismut and Lebeau 1991, §1e].

The purpose of this section is to give a formula for \(\| \rho \|^2_{\lambda(\mathcal{V}_1^*)^{-1} \otimes \lambda\nu(\mathcal{V}^n)}\). Now we introduce some notations.

For a holomorphic Hermitian vector bundle \((E, h^E)\) on \(X\), we denote by \(\text{Td}(E)\), \(\text{ch}(E)\), \(\text{c}_{\text{max}}(E)\) the Todd class, Chern character, and top Chern class of \(E\), and by \(\text{Td}(E, h^E)\), \(\text{ch}(E, h^E)\), \(\text{c}_{\text{max}}(E, h^E)\) the Chern–Weil representatives of \(\text{Td}(E)\), \(\text{ch}(E)\), \(\text{c}_{\text{max}}(E)\) with respect to the holomorphic Hermitian connection \(\nabla^E\) on \((E, h^E)\).

Let \(\delta_Y\) be the current of integration on \(Y\). By [Bismut 1992, Theorem 3.6], a current \(\tilde{c}_{\text{max}}(\mathcal{V}, h^{\mathcal{V}})\) on \(X\) is well defined by the holomorphic section \(v\) (which induces an embedding \(v : X \rightarrow \mathcal{V}\)), and this current satisfies
\[
\tilde{\partial} \tilde{\partial} \tilde{c}_{\text{max}}(\mathcal{V}, h^{\mathcal{V}}) = c_{\text{max}}(\mathcal{V}_1, h^{\mathcal{V}_1}) \delta_Y - c_{\text{max}}(\mathcal{V}, h^{\mathcal{V}}) \cdot (2-6)
\]

Let \(\tilde{\text{Td}}(TY, TX, G^{TX|r})\) be the Bott–Chern current on \(Y\) associated to the exact sequence
\[
(2-7) \quad 0 \rightarrow TY \rightarrow TX|_Y \rightarrow N \rightarrow 0
\]
constructed in [Bismut et al. 1988a, §1f], which satisfies
\[ \frac{\bar{\partial} \partial}{2\pi i} \tilde{\text{Td}}(TY, TX, g^{TX|_Y}) = \text{Td}(TX|_Y, g^{TX|_Y}) - \text{Td}(TY, g^{TY}) \text{Td}(N, g^N). \]

Finally, let \( R(x) \) be the power series introduced in [Gillet and Soulé 1991], which is such that if \( \zeta(s) \) is the Riemann zeta function, then
\[ R(x) = \sum_{n \geq 1} \sum_{j=1}^{n} \frac{\zeta(-n) + 2 \frac{\partial \zeta}{\partial s}(-n)}{n!} x^n. \]

We identify \( R \) with the corresponding additive genus. We also set
\[ \text{ch}(\Lambda^*(\mathbb{V}^*_1)) = \sum_i (-1)^i \text{ch}(\Lambda^i(\mathbb{V}^*_1)), \]
and denote by \( \text{ch}(\Lambda^*(\mathbb{V}^*_1), h^{\Lambda^*(\mathbb{V}^*_1)}) \) its Chern–Weil representative.

**Theorem 2.2.** The Quillen metric \( \| \rho \|_{\lambda(\mathbb{V}^*_1)^{-1} \otimes \lambda_i(\mathbb{V}')} \) is given by the exponential of
\[ (2–8) \quad - \int_X \text{Td}(TX, g^{TX}) \text{Td}^{-1}(\mathbb{V}, h^\mathbb{V}) \tilde{c}_{\text{max}}(\mathbb{V}, h^\mathbb{V}) \text{ch}(\xi, h^\xi) 
+ \int_Y \text{Td}^{-1}(N, g^N) \tilde{\text{Td}}(TY, TX|_Y, g^{TX|_Y}) \text{ch}(\Lambda^*(\mathbb{V}^*_1), h^{\Lambda^*(\mathbb{V}^*_1)}) \text{ch}(\xi, h^\xi) 
- \int_Y \text{Td}(TY) R(N) \text{ch}(\Lambda^*(\mathbb{V}^*_1)) \text{ch}(\xi). \]

**Proof.** Set
\[ (2–9) \quad T(\Lambda(\mathbb{V}^*), h^{\Lambda(\mathbb{V}^*)}) = \text{Td}^{-1}(\mathbb{V}, h^\mathbb{V}) \tilde{c}_{\text{max}}(\mathbb{V}, h^\mathbb{V}). \]

By the same argument as in [Bismut et al. 1990, Theorem 3.17], the current
\[ T(\Lambda(\mathbb{V}^*), h^{\Lambda(\mathbb{V}^*)}) \]
is exactly the current on \( X \) associated to (2–1) (evaluated modulo irrelevant \( \partial \) or \( \bar{\partial} \) coboundaries).

Now, from the choice of our metric \( h^\mathbb{V} \), the analogue of [Bismut and Lebeau 1991, Definition 1.21, assumption (A)] is satisfied for the complex (2–1). Then we verify that as far as local index theoretic computations are concerned, the situation is exactly the same as in [Bismut and Lebeau 1991]. Because of the quasi-isomorphism of Theorem 2.1, there are no “small” eigenvalues of the operator \( D + TV \) when \( T \to +\infty \). In Section 3, we write down the intermediate results corresponding to [Bismut and Lebeau 1991, §6c]. Comparing to [Bismut and Lebeau 1991, §§6c–6e], the proof of Theorem 2.2 is complete. □
Remark 2.3. Assume that $Y$ consists only discrete points; then $l \geq n$ and the last two terms of (2–8) are zero. In this case, if $n = l$, then (2–1) is a resolution of $J_\ast (\mathcal{O}_Y)$ and Theorem 2.2 is a direct consequence of [Bismut and Lebeau 1991, Theorem 0.1]. By [Bismut 1992, Theorem 3.2, Definition 3.5], $\tilde{c}_{\text{max}} (\mathcal{V}, h^\mathcal{V})$ is zero if $l > n + 1$.

3. $L^2$ metrics on $H_\ast (X, \wedge (\mathcal{V}^\ast))$ and localization

We keep the assumptions and notations of Section 2.

Let $g^{TX}$ be a Kähler metric on $TX$, and let $g^{TY}, g^N$ be the metrics on $TY, N$ induced by $g^{TX}$. Let $h^\mathcal{V}$ be a metric on $\mathcal{V}$ such that $\mathcal{V}_1$ and $\Im \nabla v |_Y$ are orthogonal on $Y$ and $\nabla v |_Y : (\mathcal{N}, g^N) \to \Im \nabla v |_Y$ is an isometry.

Let $\phi_1 : \det \mathcal{V}_1^\ast \to \det T^* Y$ be a nonzero holomorphic section. Let $h^\mathcal{V}_1$ be a metric on $\mathcal{V}$ such that on $Y, \mathcal{V}_1$ and $\Im \nabla v |_Y$ are orthogonal and

$$| \phi |_{\det \mathcal{V} \otimes \det T^* X, 1} = | \phi_1 |_{\det \mathcal{V}_1 \otimes \det T^* Y, 1} = 1,$$

where $| \cdot |_{\det \mathcal{V} \otimes \det T^* X, 1}$ and $| \cdot |_{\det \mathcal{V}_1 \otimes \det T^* Y, 1}$ are the norms on the holomorphic line bundles $\det \mathcal{V} \otimes \det T^* X$ and $\det \mathcal{V}_1 \otimes \det T^* Y$ induced by $h^\mathcal{V}_1$ and $g^{TX}$.

We will add a subscript 1 to denote the objects induced by $h^\mathcal{V}_1$. For

$$\beta \in \wedge^n (\mathcal{V}^\ast) \otimes \wedge^q (\mathcal{V}_1^\ast),$$

we define $\ast_{\mathcal{V}, 1} \beta \in \wedge^{n-p} (\mathcal{T}^* X) \otimes \wedge^{l-q} (\mathcal{V}_1^\ast)$ by

$$\langle \alpha, \beta \rangle_1 \phi^{-1}(dv_X) = \alpha \wedge \ast_{\mathcal{V}, 1} \beta.$$

It’s useful to write down a local expression for $\ast_{\mathcal{V}, 1}$. If $\{ w^i \}_i=1^n$ and $\{ \mu^i \}_i=1^l$, are orthonormal bases of $T^* X$ and $(\mathcal{V}_1^\ast, h^\mathcal{V}_1)$, then

$$dv_X = (-1)^{n(n+1)/2} (\sqrt{-1})^n \bar{w}^1 \wedge \cdots \wedge \bar{w}^n \otimes w^1 \wedge \cdots \wedge w^n$$

and $\phi^{-1}(w^1 \wedge \cdots \wedge w^n) = f \mu^1 \wedge \cdots \wedge \mu^l$ with $|f| = 1$. If

$$\beta = \bar{w}^1 \wedge \cdots \wedge \bar{w}^p \otimes \mu^1 \wedge \cdots \wedge \mu^q,$$

then

$$\ast_{\mathcal{V}, 1} \beta = (-1)^{(n-p)q+n(n+1)/2} (\sqrt{-1})^n \bar{w}^{p+1} \wedge \cdots \wedge \bar{w}^n \otimes \mu^{q+1} \wedge \cdots \wedge \mu^l.$$

Thus $\ast_{\mathcal{V}, 1} \ast_{\mathcal{V}, 1} \beta = (-1)^{(p+q)(n+l+1)} \beta$, for any $\beta \in \wedge^n (\mathcal{T}^* X) \otimes \wedge^q (\mathcal{V}_1^\ast)$. Combining this with (1–2), we find that

$$\tilde{c}_X \ast_1 \beta = (-1)^{p+q+1} \ast_{\mathcal{V}, 1} \tilde{c}_X \ast_{\mathcal{V}, 1} \beta, \quad (i(v))^* \beta = (-1)^{p+q+1} \ast_{\mathcal{V}, 1} i(v) \ast_{\mathcal{V}, 1} \beta.$$

Thus the antilinear map $\ast_{\mathcal{V}, 1}$ is an isometry from $(\mathcal{H}_s (X, \wedge (\mathcal{V}^\ast)), h^\mathcal{H}_s)$ to itself.
The bilinear form

\[ \alpha, \beta \in \mathcal{H}_v(X, \bigwedge (\mathcal{V}^*)^n) \mapsto \frac{1}{(2\pi)^n} \int_X \alpha \wedge \beta \]

is nondegenerate; indeed, \( \alpha \in \mathcal{H}_v(X, \bigwedge (\mathcal{V}^*)) \) implies \( \ast_{\mathcal{V}, 1} \alpha \in \mathcal{H}_v(X, \bigwedge (\mathcal{V}^*)) \), so \( \alpha \neq 0 \) implies

\[ \int_X \alpha \wedge \ast_{\mathcal{V}, 1} \alpha > 0. \]

Thus the metric \( | \cdot |_{\lambda_v(\mathcal{V}^*), 1} \) on \( \lambda_v(\mathcal{V}^*) \) only depends on the nondegenerate bilinear form (3–1) on \( \mathcal{H}_v(X, \bigwedge (\mathcal{V}^*)) \), which is metric-independent.

Recall the definition of \( \det \nabla v|_Y \) from Section 1. Now, \( \phi|_Y/((\det \nabla v|_Y)^*) \phi_1 \) is a holomorphic function on \( Y \). Since \( Y \) is compact, this function is locally constant. Then we have the following extension of [Bismut 2004, Theorem 5.7].

**Theorem 3.1.**

\[ \log(|\rho|_{\lambda(\mathcal{V}^*), 1}^{-1} \otimes \lambda_v(\mathcal{V}^*))^2 \] = \[ \int_Y \text{Td}(TY) \text{ch}((\bigwedge^m(\mathcal{V}^*)) \log \left| \frac{\phi|_Y/((\det \nabla v|_Y)^*)}{\phi_1} \right| \]

**Proof.** We use \( \phi_1 \) to define the integral \( \int_Y \gamma \) for \( \gamma \in H(Y, \bigwedge (\mathcal{V}^*_1)) \). Since

\[ |\phi_1|_{\det \mathcal{V}_1 \otimes \det T^* Y, 1} = 1, \]

following the same considerations as above, we find that the antilinear operator \( \ast_{\mathcal{V}, 1} \) maps \( H(Y, \bigwedge (\mathcal{V}^*_1)) \) into itself isometrically. Therefore, to evaluate the left-hand side of (3–2), we only need to compare the bilinear forms (3–1) with

\[ a, b \in H(Y, \bigwedge (\mathcal{V}^*_1)) \mapsto \frac{1}{(2\pi)^m} \int_Y a \wedge b. \]

Let \( A_v \in \text{End}^{\text{even}} H(Y, \bigwedge (\mathcal{V}^*_1)) \) be given by

\[ a \mapsto \frac{(-1)^{l-n}(n-m)a}{(2\pi)^{n-m} \det_n((1+R_v^\gamma)/(2\pi i))} \frac{\phi|_Y/((\det \nabla v|_Y)^*)}{\phi_1}. \]

Set

\[ \det A_v = \frac{\det A_v|_{H^{\text{even}}(Y, \bigwedge (\mathcal{V}^*_1))}}{\det A_v|_{H^{\text{odd}}(Y, \bigwedge (\mathcal{V}^*_1))}}; \]

then

\[ (|\rho|_{\lambda(\mathcal{V}^*), 1}^{-1} \otimes \lambda_v(\mathcal{V}^*))^2 = |\det A_v|. \]

Now, \( A_v \) is a degree-increasing operator in \( H(Y, \bigwedge (\mathcal{V}^*_1)) \). Therefore it acts like a triangular matrix whose diagonal part is just multiplication by the locally constant
function \( \frac{\phi|_Y / ((\det \nabla v)|_Y)^*}{\phi_1} \). Using (3–3), we get
\[
\det A_v = \left( \frac{\phi|_Y / ((\det \nabla v)|_Y)^*}{\phi_1} \right) \chi(Y, \wedge(\nabla^*_Y)).
\]

But \( \chi(Y, \wedge(\nabla^*_Y)) = \int_Y Td(TY) \chi(\wedge(\nabla^*_Y)) \); thus we get (3–2).

Let \( g^N \) be the metric on \( N \) such that \( \nabla v|_Y : (N, g^N) \to (\text{Im}(\nabla v), h_1^{\text{Im}(\nabla v)}) \) is an isometry. Let \( \tilde{T}_d^{-1}(N, g^N, g_1^N) \) be the Bott–Chern class constructed in [Bismut et al. 1988a, §1f] such that
\[
\frac{\delta \partial}{2\pi i} \tilde{T}_d^{-1}(N, g^N, g_1^N) = Td^{-1}(N, g^N) - Td^{-1}(N, g^N).
\]
Finally, we can compute the analytic torsion on the total manifold via the zero set of a transversal section \( v \).

**Theorem 3.2.** If \( h_1^N = h^N \) on \( Y \), then
\[
(3–4) \quad -\frac{\delta \partial}{\delta s} X_{v, 1} (0) + \frac{\delta \partial}{\delta s} Y (0) = -\int_X Td(TX, g^{TX}) Td^{-1}(\nabla, h_1^Y) \tilde{\sigma}_{\max} (\nabla, h_1^Y) + \int_Y \left( Td(TX, g^{TX}) \tilde{T}_d(TY, TX|_Y, g^{TX|_Y}) \right. \\
+ Td(TX, g^{TX}) \tilde{T}_d^{-1}(N, g^N, g_1^N) \chi(\wedge^* (\nabla^*_Y), h^* (\nabla^*_Y)) \\
- \left. \int_Y Td(TY) \chi(\wedge^* (\nabla^*_Y)) \left( R(N) + \log \left| \frac{\phi|_Y / ((\det \nabla v)|_Y)^*}{\phi_1} \right| \right) \right).
\]

**Proof.** Since \( h_1^N = h^N \), we have \( | \cdot |_{\lambda(\nabla^*_Y)} = | \cdot |_{\lambda(\nabla^*_Y), 1} \) and \( \| \cdot \|_{\lambda(\nabla^*_Y)} = \| \cdot \|_{\lambda(\nabla^*_Y), 1} \). Let \( \tilde{\chi}(\wedge^* (\nabla^*_Y), h_1^*(\nabla^*_Y), h^*(\nabla^*_Y)) \) be the Bott–Chern class constructed in [Bismut et al. 1988a, §1f], so that
\[
\frac{\delta \partial}{2\pi i} \tilde{\chi}(\wedge^* (\nabla^*_Y), h_1^*(\nabla^*_Y), h^*(\nabla^*_Y)) = \chi(\wedge^*(\nabla^*_Y), h^*(\nabla^*_Y)) - \chi(\wedge^*(\nabla^*_Y), h_1^*(\nabla^*_Y)).
\]
Then by the anomaly formula [Bismut et al. 1988b, Theorem 1.23],
\[
\log \left( \frac{\| \cdot \|_{\lambda(\nabla^*_Y)}^2}{\| \cdot \|_{\lambda(\nabla^*_Y), 1}^2} \right) = \int_X Td(TX, g^{TX}) \tilde{\chi}(\wedge^* (\nabla^*_Y), h_1^*(\nabla^*_Y), h^*(\nabla^*_Y)).
\]

By [Bismut et al. 1990, Theorem 2.5],
\[
(3–5) \quad T(\wedge^* (\nabla^*_Y), h^*(\nabla^*_Y)) = T(\wedge^* (\nabla^*_Y), h_1^*(\nabla^*_Y)) \\
= \chi(\wedge^* (\nabla^*_Y), h^*(\nabla^*_Y)) \tilde{T}_d^{-1}(N, g^N, g_1^N) \delta_Y - \tilde{\chi}(\wedge^*(\nabla^*_Y), h_1^*(\nabla^*_Y), h^*(\nabla^*_Y)).
\]
By (2–9), Theorems 2.2 and 3.1, and the preceding equations, the proof of Theorem 3.2 is complete. \( \square \)
Remark 3.3. If $Y$ consists only of discrete points and $n = l$, then $\phi_1 = \text{Id}$. In this case let $g_{\det}^N$ and $g_{1}^N_{\det}$ be the metrics on $N = \det TX$ induced by $g^N$ and $g_1^N$. By Remark 2.3 and Theorem 3.2,

$$-\frac{\partial \theta^X}{\partial s}(0) = - \int_X Td(TX, g^{TX}) Td^{-1}(\nabla, h_1^N) c_{\text{max}}(\nabla, h_1^N) + \sum_{p \in Y} \left( \frac{1}{2} \log \left( \frac{g_{\det}^N}{g_{1}^N_{\det}} \right) - \log |\phi/(\det \nabla v)|^* \right).$$

Remark 3.4. If $\nabla = TX$ and $v$ is a holomorphic Killing vector field, (3–4) is a special case of [Bismut 1992, Theorems 6.2 and 7.7]. In this case, $h_1^N_{\det} = g^{TX}$, and on $Y$, we have a holomorphic and orthogonal splitting $TX|_Y = TY \oplus N$. Thus $\tilde{T}d(TY, TX|_Y, g^{TX}_{\det}) = 0$. To compute $\tilde{T}d^{-1}(N, g^N, g_1^N)$, note that $g_1^N = g^N((\nabla v), (\nabla v), \cdot)$, as $A = (\nabla v)^*(\nabla v)$ is positive and self-adjoint; thus $(A)^s$ is well defined for $s \in [0, 1]$. Taking $g_s^N = g^N((A)^s, \cdot)$, we obtain by [Bismut et al. 1988a, Theorem 1.30]

$$\tilde{T}d^{-1}(N, g^N, g_1^N) = \int_0^1 \langle (Td^{-1})'(N, g^N_s), \log A \rangle \, ds.$$

But $\nabla v$ is holomorphic, so the curvature $R^N_s$ associated to the holomorphic connection on $(N, g^N_s)$ is $R^N_s = R^N$ for $s \in [0, 1]$. Thus

(3–6) \[ \tilde{T}d^{-1}(N, g^N, g_1^N) = \langle (Td^{-1})'(N, g^N), \log A \rangle. \]

Now

(3–7) \[ Td(TX, g^{TX}) T(\wedge(T^*X), h^{\wedge(T^*X)}) = c_{\text{max}}(TX, g^{TX}) \]

is an $(n-1, n-1)$-form on $X$.

In this case, we get easily the special case of [Bismut 2004, Theorem 4.15] directly from [Ray and Singer 1973] by using Poincaré duality:

(3–8) \[ \frac{\partial \theta^Y}{\partial s}(0) = 0. \]

From (3–4), (3–6), (3–7), and the vanishing of the constant terms of $R(N)$ and $\tilde{T}d(N, g^N) - \frac{1}{2}$, we get

(3–9) \[ -\frac{\partial \theta^X}{\partial s}(0) = \int_Y c_{\text{max}}(TY) \left( R(N) - \frac{\tilde{T}d'}{\tilde{T}d}(N, g^N) - \frac{1}{2}, \log A \right) = 0. \]
4. Appendix: six intermediate results

In this section, to help readers understand how to obtain Theorem 2.2, we write down the corresponding intermediate results from [Bismut and Lebeau 1991, Theorems 6.4-6.9].

Let $\nabla^{\wedge}(\mathcal{V}^s)$ be the connection on $\wedge(\mathcal{V}^s)$ induced by $\nabla^{\mathcal{V}^s}$. Set $C_u = \nabla^{\wedge}(\mathcal{V}^s) + \sqrt{u} \nabla$. Let $\mathcal{B}_{T_2}$ and $\text{Tr}_s(N_H^Y \exp(-\mathcal{B}_{T_2}^2))$ be the operator and the generalized trace associated to the complex (2–7) as in [Bismut and Lebeau 1991, §5]. Let $\Phi$ be the homomorphism from $\wedge^\text{even}(T^*_R X)$ into itself which to $\alpha \in \wedge^p(T^*_R X)$ associates $(2\pi i)^{-p} \alpha$.

**Theorem 4.1.** For any $u_0 > 0$, there exists $C > 0$ such that for $u \geq u_0$, $T \geq 1$,

$$\left| \text{Tr}_s\left(N_H^X e^{-u(D^X+TV)^2}\right) - \text{Tr}_s\left(\left(\frac{1}{2} \dim N + N_H^Y\right)e^{-uD^{Y,2}}\right) \right| \leq \frac{C}{\sqrt{T}}.$$  

$$\left| \text{Tr}_s\left((N^X - N_H^X)e^{-u(D^X+TV)^2}\right) - \text{Tr}_s\left((N^Y - N_H^Y)e^{-uD^{Y,2}}\right) \right| \leq \frac{C}{\sqrt{T}}.$$  

**Theorem 4.2.** Let $\tilde{P}_T$ be the orthogonal projection operator from $\Omega(X, \wedge(\mathcal{V}^s) \otimes \xi)$ to $\ker(D^X + TV)$. There exist $c > 0$ and $C > 0$ such that, for any $u \geq 1$ and $T \geq 1$,

$$\left| \text{Tr}_s\left((N^X - N_H^X)e^{-u(D^X+TV)^2}\right) - \text{Tr}_s\left((N^X - N_H^X)\tilde{P}_T\right) \right| \leq c e^{-C u},$$  

**Theorem 4.3.** There exist $C > 0$ and $\gamma \in [0, 1]$ such that, for any $u \in [0, 1]$ and $0 \leq T \leq 1/u$,

$$\left| \text{Tr}_s\left(N_H^X e^{-u(D^X+TV)^2}\right) - \int_X \text{Td}(TX, g^{TX}) \Phi \text{Tr}_s(N_H^X e^{-C^2}) \right| \leq C(u(1+T))^{\gamma}.$$  

There exists a constant $C' > 0$ such that for $u \in [0, 1]$ and $0 \leq T \leq 1$,

$$\left| \text{Tr}_s\left(N_H^X e^{-u(D^X+TV)^2}\right) - \text{Tr}_s\left(N_H^X e^{-u(D^X)^2}\right) \right| \leq C'T.$$  

**Theorem 4.4.** For any $T > 0$,

$$\lim_{u \to 0} \text{Tr}_s\left(N_H^X e^{-u(D^X+(T/u)V)^2}\right) = \int_Y \Phi \text{Tr}_s\left(N_H^Y e^{-h^2} \right) \text{ch}(\wedge(\mathcal{V}^s), h^\wedge(\mathcal{V}^s)) \chi(\xi, h^\xi).$$  

**Theorem 4.5.** There exist $C > 0$ and $\delta \in [0, 1]$ such that, for any $u \in [0, 1]$ and $T \geq 1$,

$$\left| \text{Tr}_s\left(N_H^X e^{-u(D^X+(T/u)V)^2}\right) - \text{Tr}_s\left(\left(\frac{1}{2} \dim N + N_H^Y\right)e^{-uD^{Y,2}}\right) \right| \leq \frac{C}{T^\delta}.$$  

Let $| \cdot |_{\lambda_+(\mathcal{V}^s), T}^2$ be the $L^2$-metric on $\lambda_+(\mathcal{V}^s)$ induced by $g^{TX}$, $T^2 h^{\mathcal{V}}$ as in (2–5).
Theorem 4.6. As $T \to +\infty$,
\[
\log \left( \frac{|\cdot|_\lambda^2(\mathcal{V}^*)_T}{|\cdot|_\lambda^2(\mathcal{V}^*)} \right) = -\log |\rho|^{2(\gamma_T)}_{\lambda(\mathcal{V}^*)} + \text{Tr}_T((\dim N + 2N^Y_H)Q) \log T + O\left(\frac{1}{T}\right).
\]

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