TRANSVERSAL HOLOMORPHIC SECTIONS AND LOCALIZATION OF ANALYTIC TORSIONS

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We prove a Bott-type residue formula twisted by $\wedge(V^\ast)$ with a holomorphic vector bundle $V$, and relate certain analytic torsions on the total manifold to the analytic torsions on the zero set of a holomorphic section of $V$.

Introduction

Beasley and Witten [2003], studying half-linear models, have described a compactification on any Calabi–Yau threefold $Y$ that is a complete intersection in a compact toric variety $X$. In particular, a remarkable cancellation involving the instanton effect [Beasley and Witten 2003, (1.3)], involving certain determinants of the $\bar{\partial}$-operator, was derived directly from a residue theorem. One would like to understand its implications in mathematics, for example in Gromov–Witten theory. Bershadsky, Cecotti, Ooguri and Vafa [Bershadsky et al. 1993; 1994] predicted that the analytic torsion of Ray–Singer will play a role regarding the genus-1 Gromov–Witten invariant. Thus we naturally try to understand the results about analytic torsion first.

As an application of [Bismut and Lebeau 1991] and the localization formula (1–3) in this paper, we were able to relate certain analytic torsions on the total manifold with the zero set of a holomorphic transversal section of $V$, generalizing [Bismut 2004, Theorem 6.6] and [Zhang n.d.] with $V = TX$ therein. We expect our formula will be useful for understanding [Beasley and Witten 2003, (1.3)] from a mathematical point of view.

This paper is organized as follows. In Section 1 we prove a Bott-type residue formula. In Section 2 we get a localization formula for Quillen metrics. In Section 3 we get a localization formula for analytic torsions under extra conditions. In Section 4, for the reader’s convenience, we write down six intermediate results, corresponding to [Bismut and Lebeau 1991, Theorems 6.4–6.9].
1. A Bott-type residue formula

In this section, along the lines of [Bismut 1986, §1], we give a Bott-type residue formula (1–3) by assuming that the holomorphic section is transversal; compare to [Beasley and Witten 2003, (2.32), (2.34)].

Let $X$ be a compact complex manifold with dim $X = n$ and let $\mathbb{V}$ be a holomorphic vector bundle on $X$ with dim $\mathbb{V} = l$. We assume that the line bundles $\det T X$ and $\det \mathbb{V}$ are holomorphically isomorphic. We fix a holomorphic isomorphism $\phi : \det \mathbb{V} \simeq \det T^* X$, which is clearly unique up to a constant. Thus $\phi$ defines a map from the $\mathbb{Z}_2$-graded tensor product $\wedge (T^* X) \otimes (\mathbb{V}^*)$ to $\wedge (T^* X) \otimes \wedge^{\max} (T^* X) \subset \wedge (T^*_R X) \otimes_R \mathbb{C}$. We can define the integral of an element $\alpha$ of $\Omega(X, \wedge (\mathbb{V}^*))$, the set of smooth sections of $\wedge (T^* X) \otimes (\mathbb{V}^*)$ on $X$, by

$$\int_X \phi(\alpha).$$

Let $v$ be a holomorphic section of $\mathbb{V}$ on $X$. Assume that $v$ vanishes on a complex manifold $Y \subset X$. Then $\nabla u|_Y : TX|_Y \rightarrow \mathbb{V}|_Y$ mapping $U$ to $\nabla_U v$ does not depend on the choice of a connection $\nabla$ on $\mathbb{V}$, and $\nabla_U v|_Y = 0$ for $U \in TY$. Let $N$ be the normal bundle to $Y$ in $X$. Assume also that $\nabla v|_Y : N \rightarrow \mathbb{V}|_Y$ is injective, and there is a holomorphic vector subbundle $\mathbb{V}_1$ on $Y$ such that

$$\mathbb{V}|_Y = \mathbb{V}_1 \oplus \text{Im} \nabla v|_Y.$$  

Let $P^\mathbb{V}_1$ and $P^\text{Im} \nabla v$ be the natural projections from $\mathbb{V}$ onto $\mathbb{V}_1$ and $\text{Im} \nabla v|_Y$. Let $i(v)$ be the standard contraction operator acting on $\wedge (\mathbb{V}^*)$. A natural question, posed in [Beasley and Witten 2003, §2], is how to express $\int_X \phi(\alpha)$ using the local data near the zero set $Y$ of $v$ for a $(\overline{\partial}^X + i(v))$-closed form $\alpha$, that is, a form satisfying $(\overline{\partial}^X + i(v))\alpha = 0$.

First we recall an idea due to Bismut [Bismut 1986]; see also [Zhang 1990].

**Proposition 1.1.** Let $\alpha \in \Omega(X, \wedge (\mathbb{V}^*))$ be a $(\overline{\partial}^X + i(v))$-closed form. Then

$$\int_X \alpha = \int_X e^{-t(\overline{\partial}^X + i(v))\alpha/t} \alpha \quad \text{for any} \ \omega \in \Omega(X, \wedge (\mathbb{V}^*)) \text{and} \ t > 0.$$

**Proof.** For any $\omega \in \Omega(X, \wedge (\mathbb{V}^*))$

$$\int_X \overline{\partial}^X \omega = \int_X \phi(\overline{\partial}^X \omega) = \int_X \overline{\partial}^X \phi(\omega) = \int_X d\phi(\omega) = 0.$$

From $(\overline{\partial}^X + i(v))^2 = 0$ and $(\overline{\partial}^X + i(v))\alpha = 0$, we have

$$\frac{\partial}{\partial s} \int_X e^{-s(\overline{\partial}^X + i(v))\alpha} \alpha = -\int_X (\overline{\partial}^X + i(v))(\omega e^{-s(\overline{\partial}^X + i(v))\omega} \alpha) = 0,$$

and the desired equality follows. \qed
Recall that $\nabla v|_Y : N \to \operatorname{Im \nabla v}|_Y$ is an isomorphism that induces isomorphisms of holomorphic line bundles $\phi_N = (\det \nabla v|_Y)^* : \det(\operatorname{Im \nabla v}|_Y)^* \to \det N^*$ and $\phi_Y = \phi_Y/((\det \nabla v|_Y)^*) : \det \nu^*_Y \to \det T^* Y$. These two isomorphisms make the integral $\int_N$ along the normal bundle $N$ and $\int_Y$ well defined.

Let $h^V$ be a Hermitian metric on $V$ such that $\nabla v|_Y$ and $\operatorname{Im \nabla v}|_Y$ are orthogonal on $Y$. Let $g^N$ be a Hermitian metric on $N$ such that $\nabla v|_Y : N \to \operatorname{Im \nabla v}|_Y$ is an isometry. Let $R^V$ be the curvature of the holomorphic Hermitian connection $\nabla^V$ on $(V, h^V)$. Let $j : Y \to X$ be the natural embedding, and $\{Y_j\}_j$ the connected components of $Y$. On $Y$, define

$$R^V = - (\nabla v)^{-1} P^\operatorname{Im\nabla v} R^V (\cdot, j_\ast \cdot) P^\nu_Y \in \mathcal{T}^\otimes \mathcal{V}^*_Y \otimes \text{End } N.$$  

$R^V$ is well defined since $P^\operatorname{Im\nabla v} R^V (j_\ast \cdot, j_\ast \cdot) P^\nu_Y = 0$. Thus, for $U \in TY$, $W \in \nu_Y$, $u_1, u_2 \in N$,

$$\{ R^V (U, W) u_1, u_2 \}_g^N = - \{ R^V (u_1, U) W, \nabla u_2 v \} = \{ W, R^V (u_1, U) \nabla u_2 v \}.$$  

Certainly $\det_Y(1 + R^V)/2\pi i$ is $\bar{\partial}^Y$-closed.

The following result verifies a formula of Beasley and Whitney [2003, (2.32), (2.34)] and generalizes corresponding results in [Zhang 1990], [Liu 1995] and [Bott 1967].

**Theorem 1.2.** For any $(\bar{\partial}^X + i(v))$-closed form $\alpha \in \Omega(X, \wedge (\nu^*))$,

$$\int_X \alpha = \sum_j \int_{Y_j} \frac{(-1)^{(l-n)(n-\dim Y)}}{\det_N((1 + R^V)^/(2\pi i))} \alpha.$$  

**Proof.** Set

$$S = \langle \cdot, v \rangle_{h^V} \in \mathcal{C}^\infty(X, \nu^*).$$

By **Proposition 1.1,** for any $t \in [0, +\infty[$,

$$\int_X \alpha = \int_X e^{-\frac{i}{2}(\bar{\partial}^X + i(v))s} \alpha = \int_X e^{-\frac{i}{2}(\bar{\partial}^X + i|v|^2)} \alpha.$$  

Thus, as $t \to 0$, the integral $\int_X \alpha$ is asymptotically equal to $\int_U e^{-\frac{i}{2}(\bar{\partial}^X + i|v|^2)} \alpha$ for any neighborhood $U$ of $Y$.

Take $y \in Y$. Since $Y$ is a complex submanifold, we can find holomorphic coordinates $\{z_i\}_{i=1}^n$ of a neighborhood $U$ of $y$ such that $y$ corresponds to 0 and $\{(\partial/\partial z_i)(0)\}_{i=m+1}^n$ is an orthonormal basis of $(N, g^N_{1})$, and, moreover,

$$U \cap Y = \{ p \in U, z_{m+1}(p) = \cdots = z_n(p) = 0 \}.$$  

Let $\{\mu_k\}_{k=1}^{l'}$ and $\{\mu_k\}_{k=l'+1}^i$ be holomorphic frames for $\nu_1$ and $\operatorname{Im \nabla v}|_Y$ on $U \cap Y$, with

$$\nabla^V_{\partial/\partial z_k(0)} v = \mu_k(0) \quad \text{for } l' + 1 \leq k \leq l,$$
and for $z' = (z_1, \ldots, z_m)$, $z'' = (z_{m+1}, \ldots, z_n)$, $z = (z', z'')$, define $\mu_k(z)$ by parallel transport of $\mu_k(z', 0)$ with respect to $\nabla^y$ along the curve $u \mapsto (z', uz'')$. Identify $\nabla_z$ with $\nabla_{(z', 0)}$ by identifying $\mu_k(z)$ with $\mu_k(z', 0)$. Denote by $W_y(\varepsilon)$ the $\varepsilon$-neighborhood of $y$ in the normal space $N$. Then

$$\int_{Y \cap U} \int_{\nabla_y(\varepsilon)} e^{-\frac{1}{2}(\bar{\Delta}^X + |v|^2)} \alpha = \int_{Y \cap U} \int_{z \in W_y(\varepsilon)/\sqrt{t}} e^{-\frac{1}{2}((v(\sqrt{t}z)) + (\bar{\Delta}^X(\sqrt{t}z))_Y^{n-m} \alpha(y, \sqrt{t}z).}
$$

Define $z = \sum_j z_j(\partial/\partial z_j)$ and $\bar{z} = \sum_j \bar{z}_j(\partial/\partial \bar{z}_j)$. The tautological vector field is $Z = z + \bar{z}$. Then, for $z \in N_y$,

$$\frac{1}{2t}|v(\sqrt{t}z)|^2 = \frac{1}{2}|\nabla^y v|^2 + O(\sqrt{t}) = \frac{1}{2}|z|^2 + O(\sqrt{t})$$

and

$$\bar{\Delta}^X S = \sum_{k=1}^{l} (\mu_k, \nabla^y v) \mu^k.$$

From now on, set $z = (0, z'')$ and $Z = z + \bar{z}$. Since $\nabla^y_Z \mu_k(0) = 0$, we know that

$$\frac{1}{2t} \bar{\Delta}^X S(\sqrt{t}z) = \frac{1}{2t} \sum_{k=1}^{l} \left( (\mu_k, \nabla^y v)(0) + \sqrt{t} (\mu_k, \nabla^y_Z \nabla^y v)(0) + \frac{t}{2} (\nabla^y_Z \nabla^y_Z \mu_k, \nabla^y v)(0) + O(t^{3/2}) \right) \mu^k(0).$$

Because of the factor $t^{n-m}$ in (1–5), it should be clear that in the limit, only those monomials in the vertical form

$$d\bar{z}_{m+1} \wedge \cdots \wedge d\bar{z}_n \otimes \mu_l^{l+1} \wedge \cdots \wedge \mu^l$$

whose weight is exactly $t^{m-n}$ should be kept. Now,

$$\nabla^y_Z \frac{\partial}{\partial z_j} v = R^y \left( Z, \frac{\partial}{\partial z_j} \right) v + \nabla^y_{\partial/\partial z_j} \nabla^y_Z v - l_{[m,n]}(j) \nabla^y_{\partial/\partial z_j} v,$$

$$\nabla^y_Z \frac{\partial}{\partial z_j} v(0) = R^y \left( Z, \frac{\partial}{\partial z_j} \right) v + \nabla^y_{\partial/\partial z_j} \nabla^y_Z v = 0,$$
where \(1_{[m,n]}\) is the characteristic function of the interval \([m,n]\). Note that \(\nabla^V = \nabla^V_1 \oplus \nabla^\text{Im}_v\) on \(Y\) and that

\[
\{\mu_k, \nabla^V_z \nabla^V_{\partial/\partial z_j} v\}(0) = 0 \quad \text{for } 1 \leq j \leq m, \ 1 \leq k \leq l'.
\]

It follows that in the expression

\[
\frac{1}{2\sqrt{t}} \{\mu_k, \nabla^V_Z \nabla^V_j v\}(0) \mu^k(0)
\]

a nonzero contribution can only appear in the term

\[
(1-7) \quad \frac{1}{2\sqrt{t}} \left( \sum_{j=1}^m \sum_{k=1}^l + \sum_{j=m+1}^n \sum_{k=1}^l \right) \{\mu_k, \nabla^V_Z \nabla^V_{\partial/\partial z_j} v\}(0) \, \tilde{d}z^j \otimes \mu^k(0).
\]

Similarly, in the last term of \((1-6)\), the only term with a nonzero contribution is

\[
\frac{1}{4} \sum_{j=1}^m \sum_{k=1}^l \left( \{\nabla^V_Z \nabla^V_{\partial/\partial z_j} \mu_k, \nabla^V_{\partial/\partial z_j} v\}(0) + \{\mu_k, \nabla^V_Z \nabla^V_{\partial/\partial z_j} v\}(0) \right) \, \tilde{d}z^j \otimes \mu^k(0).
\]

But for \(1 \leq j \leq m\), both \(\nabla^V_{\partial/\partial z_j} v(0)\) and \(\nabla^V_{\partial/\partial z_j} \nabla^V_z v(0) = \nabla^V_{\partial/\partial z_j} (R^V(\bar{z}, z)v)(0)\) vanish, since \(v = 0\) on \(Y\). Thus, for \(1 \leq j \leq m\),

\[
\nabla^V_Z \nabla^V_{\partial/\partial z_j} v(0) = 2R^V(\bar{z}, \partial/\partial z_j) \nabla^V_z v(0) + \nabla^V_{\partial/\partial z_j} \nabla^V_z \nabla^V_z v(0).
\]

By the preceding discussion, as \(t \to 0\), in \((1-5)\), we should replace \(\frac{1}{2\sqrt{t}} \bar{g}^Y S(y, \sqrt{t}z)\) by the 2-form

\[
\frac{1}{2} \sum_{k=1}^l \{\mu_k, \nabla^V_j v\}(0) \mu^k(0) + \sqrt{t} \times \text{expression } (1-7)
\]

\[
+ \frac{1}{2} \sum_{j=1}^m \sum_{k=1}^l \left( \{\mu_k, R^V(\bar{z}, \partial/\partial z_j) \nabla^V_z v + \nabla^V_{\partial/\partial z_j} \nabla^V_z \nabla^V_z v\}(0) \right) \, \tilde{d}z^j \otimes \mu^k(0).
\]

Set \(\beta_Y = d\bar{z}_1 \cdots d\bar{z}_m \wedge \mu^1(0) \cdots \mu^l(0), \ \beta_N = d\bar{z}_m+1 \cdots d\bar{z}_n \wedge \mu^{l'+1}(0) \cdots \mu^l(0)\), \(\phi(\mu^1(0) \cdots \mu^l(0)) = f d\bar{z}_1 \cdots d\bar{z}_n\). Then

\[
\phi_Y(\mu^1(0) \cdots \mu^l(0)) \phi_N(\mu^{l'+1}(0) \cdots \mu^l(0)) = f d\bar{z}_1 \cdots d\bar{z}_n.
\]

Thus

\[
\phi(\beta_Y \wedge \beta_N) = (-1)^{l'(n-m)} f d\bar{z}_1 \cdots d\bar{z}_n \wedge d\bar{z}_1 \cdots d\bar{z}_n = (-1)^{(l'-m)(n-m)} \phi_Y(\beta_Y) \phi_N(\beta_N).
\]

Now, observing that \(\int_{\mathbb{C}} e^{i|z|^2} d\bar{z} \bar{z} = 0\) for \(i > 0\) and that \(\nabla^V_j v : (N, g^N) \to (\text{Im } \nabla, h^\text{Im } \nabla)\) is an isometry and \(l - l' = n - m\), we find that the limit of \((1-4)\)
as $t \to 0$ is the sum over $j$ of

$$
(1-8) \quad \int_{Y_j} (-1)^{(l-n)(n-m)} j^* \alpha \int_N \exp \left( -\frac{1}{2} \sum_{k=1}^{l} \{ \mu_k, \nabla Y^j v \} (0) \mu_k (0) - \frac{1}{2} \{ \cdot, P^Y (z, j_\ast \cdot) \nabla Y^j v \} (0) - \frac{1}{2} | \nabla Y^j v |^2 \right).
$$

The second integrand in this expression can be rewritten as

$$
\exp \left( -\frac{1}{2} \sum_{i=1}^{n-m} d\bar{z}_{m+i} \wedge \mu^{l+i} (0) + \frac{1}{2} \{ R^Y (z, j_\ast \cdot) P^Y \cdot, \nabla Y^j v \} (0) - \frac{1}{2} | z |^2 \right)
$$

$$
= \exp \left( \frac{1}{2} \{(\nabla Y)^{-1} R^Y (z, j_\ast \cdot) P^Y \cdot, z \} - \frac{1}{2} | z |^2 \right) \left( \frac{1}{2} \right)^{l-\ell'} d\bar{z}_{m+1} d\bar{z}_{m+2} \cdots d\bar{z}_n.
$$

Thus the expression in (1–8) is equal to

$$
\int_{Y_j} \frac{(-1)^{(l-n)(n-m)} \alpha}{\det_N ((1 + R^Y) / (-2\pi i))},
$$

which leads to (1–3).

2. Localization of Quillen metrics via a transversal section

Let $X$ be a compact complex manifold of dimension $n$. Let $\nabla$ and $\xi$ be holomorphic vector bundles on $X$ with $\dim \nabla = m$, and let $v$ be a holomorphic section of $\nabla$. Assume that $v$ vanishes on a complex manifold $Y \subset X$ and satisfies (1–1). Then we have a complex of holomorphic vector bundles on $X$,

$$
(2-1) \quad 0 \to \bigwedge^m (\nabla^*) \xrightarrow{i(v)} \bigwedge^{m-1} (\nabla^*) \xrightarrow{i(v)} \cdots \xrightarrow{i(v)} \bigwedge^1 (\nabla^*) \xrightarrow{i(v)} \bigwedge^0 (\nabla^*) \to 0.
$$

Let $(\Omega (X, \bigwedge (\nabla^*) \otimes \bar{\xi}), \bar{\partial}^X)$ be the Dolbeault complex associated to the holomorphic vector bundle $\bigwedge (\nabla^*) \otimes \bar{\xi}$. Let $\mathcal{H}_v (X, \bigwedge (\nabla^*) \otimes \bar{\xi})$ be the hypercohomologies of the bicomplex $(\Omega (X, \bigwedge (\nabla^*) \otimes \bar{\xi}), \bar{\partial}^X, i(v))$. Let $j : Y \to X$ be the obvious embedding. Now the pullback map $j^*$ induces naturally a map of complexes

$$
(2-2) \quad j^* : (\Omega (X, \bigwedge (\nabla^*) \otimes \bar{\xi}), \bar{\partial}^X + i(v)) \to (\Omega (Y, \bigwedge (\nabla^*) \otimes \bar{\xi}), \bar{\partial}^Y).
$$

**Theorem 2.1.** The map $j^*$ is a quasi-isomorphism of complexes. In particular, $j^*$ induces an isomorphism

$$
(2-3) \quad \mathcal{H}_v (X, \bigwedge (\nabla^*) \otimes \bar{\xi}) \simeq H (Y, \bigwedge (\nabla^*) \otimes \bar{\xi}).
$$

**Proof.** In [Feng 2003] there is an analytic proof of this theorem when $\nabla = TX$. There we used the twisted vector bundle $\bigwedge (T^* X)$ and here $\bigwedge (\nabla^*)$ takes its place; the proof works just the same. For an algebraic proof, we can modify the proof of [Bismut 2004, Theorem 5.1].

□
Let $N^X, N^X_H$ be the number operators on $\bigwedge(T^*X), \bigwedge(V^*)$ corresponding to multiplication by $p$ on $\bigwedge^p(T^*X), \bigwedge^p(V^*)$; do the same replacing $X$ by $Y$ and $V^*$ by $V_1^*$. Then $N^X - N^X_H$ and $N^Y - N^Y_H$ define $\mathbb{Z}$-gradings on $\Omega(X, \bigwedge(V^*) \otimes \xi)$ and $\Omega(Y, \bigwedge(V_1^*) \otimes \xi)$, which in turn induce $\mathbb{Z}$-gradings on $\mathcal{H}_v(X, \bigwedge(V^*) \otimes \xi)$ and $H(Y, \bigwedge(V_1^*) \otimes \xi)$, respectively. The isomorphism $j^*$ preserves these $\mathbb{Z}$-gradings.

From [Bismut and Lebeau 1991, (1.24)], we define the complex lines $\lambda_v(V^*)$ and $\lambda(V_1^*)$ by

$$\lambda_v(V^*) = \bigotimes_{p=-m}^{n} \left( \det \mathcal{H}_v(X, \bigwedge(V^*) \otimes \xi) \right)^{(-1)^{p+1}},$$

$$\lambda(V_1^*) = \bigotimes_{p=0}^{n} \bigotimes_{q=0}^{m} \left( \det H^p(Y, \bigwedge^q(V_1^*) \otimes \xi) \right)^{(-1)^{p+q+1}}.$$

By (2–3), we have a canonical isomorphism of complex lines

$$\lambda_v(V^*) \simeq \lambda(V_1^*).$$

Let $\rho$ be the nonzero section of $\lambda(V_1^*)^{-1} \otimes \lambda_v(V^*)$ associated with this canonical isomorphism.

Let $g^{TX}$ be a Kähler metric on $TX$. We identify $N$ with the bundle orthogonal to $TY$ in $TX|_Y$. Let $g^{TY}$ and $g^N$ be the metrics on $TY$ and $N$ induced by $g^{TX}$. Let $h^\xi$ be a Hermitian metric on $\xi$. Let $h^V$ be a metric on $V$ such that $h_1$ and $\text{Im} \nabla v|_Y$ are orthogonal on $Y$ and $\nabla v|_Y : N \to \text{Im} \nabla v|_Y$ is an isometry.

Let $dv_X$ be the Riemannian volume form on $(X, g^{TX})$. Let $\langle \cdot, \cdot \rangle_0$ be the metric on $\bigwedge(T^*X) \otimes \bigwedge(V^*) \otimes \xi$ induced by $g^{TX}, h^V, h^\xi$. The Hermitian product on $\Omega(X, \bigwedge(V^*) \otimes \xi)$ is defined by

$$(2-4) \quad \langle \alpha, \alpha' \rangle = \frac{1}{(2\pi)^n} \int_X \langle \alpha, \alpha' \rangle_0 dv_X \quad \text{for} \ \alpha, \alpha' \in \Omega(X, \bigwedge(V^*) \otimes \xi).$$

Let $\bar{\partial}^{X*}$ and $v^* \wedge = i(v)^*$ be the adjoint of $\bar{\partial}^X$ and $i(v)$ with respect to $\langle \cdot, \cdot \rangle$. Set

$$V = i(v) + i(v)^*, \quad D^X = \bar{\partial}^X + \bar{\partial}^{X*}.$$

By Hodge theory,

$$(2-5) \quad \mathcal{H}_v(X, \bigwedge(V^*) \otimes \xi) \simeq \text{Ker}(D^X + V).$$

Denote by $P$ be the operator of orthogonal projection from $\Omega(X, \bigwedge(V^*) \otimes \xi)$ onto $\text{ker}(D^X + V)$ and set $P_\perp = 1 - P$. Let $h^{\mathcal{H}_v}$ be the $L^2$-metric on $\mathcal{H}_v(X, \bigwedge(V^*) \otimes \xi)$ induced by the $L^2$-product (2–4) via the isomorphism (2–5). Define in the same way a Hermitian product on $\Omega(Y, \bigwedge(V_1^*) \otimes \xi)$ associated to $g^{TY}, h^{V_1}, h^\xi$. Let $\bar{\partial}^{Y*}$ be the adjoint of $\bar{\partial}^Y$, and $h^{H(Y, \bigwedge(V_1^*) \otimes \xi)}$ the corresponding $L^2$-metric on
Let \( Q \) be the orthogonal projection operator from \( \Omega(Y, \wedge (\mathcal{V}_1^* \otimes \xi)) \) on Ker \( D^Y \), and \( Q^\perp = 1 - Q \). Let \( | \cdot |_{\lambda_v^s(\mathcal{V}^s)} \) and \( | \cdot |_{\lambda(\mathcal{V}^s)} \) be the \( L^2 \)-metrics on \( \lambda_v^s(\mathcal{V}^s) \) and \( \lambda(\mathcal{V}^s) \) induced by \( h_{\mathcal{V}^s}^* \) and \( h^H(Y, \wedge (\mathcal{V}_1^* \otimes \xi)) \). Following [Bismut and Lebeau 1991, (1.49)], let
\[
\theta^*_v(s) = -\text{Tr}_s((N^X - N^X_H)(D^X + V)^{-s} P^\perp).
\]
Then \( \theta^*_v(s) \) extends to a meromorphic function of \( s \in \mathbb{C} \), which is holomorphic at \( s = 0 \).

The Quillen metric \( \| \cdot \|_{\lambda_v^s(\mathcal{V}^s)} \) on the line \( \lambda_v^s(\mathcal{V}^s) \) is defined by
\[
\| \cdot \|_{\lambda_v^s(\mathcal{V}^s)} = | \cdot |_{\lambda_v^s(\mathcal{V}^s)} \exp \left( -\frac{1}{2} \frac{\partial \theta^*_v}{\partial s}(0) \right).
\]
In the same way, the function
\[
\theta^v(s) = -\text{Tr}_s((N^Y - N^Y_H)(D^Y)^{-s} Q^\perp)
\]
extends to a meromorphic function of \( s \in \mathbb{C} \), holomorphic at \( s = 0 \). The Quillen metric \( \| \cdot \|_{\lambda(\mathcal{V}_1^*)} \) on the line \( \lambda(\mathcal{V}_1^*) \) is defined by
\[
\| \cdot \|_{\lambda(\mathcal{V}_1^*)} = | \cdot |_{\lambda(\mathcal{V}_1^*)} \exp \left( -\frac{1}{2} \frac{\partial \theta^v}{\partial s}(0) \right).
\]
Let \( \| \cdot \|_{\lambda(\mathcal{V}_1^*)^{-1} \otimes \lambda_v^s(\mathcal{V}^s)} \) be the Quillen metric on \( \lambda(\mathcal{V}_1^*)^{-1} \otimes \lambda_v^s(\mathcal{V}^s) \) induced by \( \| \cdot \|_{\lambda_v^s(\mathcal{V}^s)} \) and \( \| \cdot \|_{\lambda(\mathcal{V}_1^*)} \) as in [Bismut and Lebeau 1991, §1e].

The purpose of this section is to give a formula for \( \| \rho \|_{\lambda(\mathcal{V}_1^*)^{-1} \otimes \lambda_v^s(\mathcal{V}^s)}^2 \). Now we introduce some notations.

For a holomorphic Hermitian vector bundle \((E, h^E)\) on \( X \), we denote by \( \text{Td}(E) \), \( \text{ch}(E) \), \( c_{\max}(E) \) the Todd class, Chern character, and top Chern class of \( E \), and by \( \text{Td}(E, h^E) \), \( \text{ch}(E, h^E) \), \( c_{\max}(E, h^E) \) the Chern–Weil representatives of \( \text{Td}(E) \), \( \text{ch}(E) \), \( c_{\max}(E) \) with respect to the holomorphic Hermitian connection \( \nabla^E \) on \((E, h^E)\).

Let \( \delta_Y \) be the current of integration on \( Y \). By [Bismut 1992, Theorem 3.6], a current \( \tilde{c}_{\max}(\mathcal{V}, h^\mathcal{V}) \) on \( X \) is well defined by the holomorphic section \( \nu \) (which induces an embedding \( \nu : X \to \mathcal{V} \)), and this current satisfies
\[
(2-6) \quad \frac{\partial \tilde{\theta}}{2\pi i} \tilde{c}_{\max}(\mathcal{V}, h^\mathcal{V}) = c_{\max}(\mathcal{V}_1, h^{\mathcal{V}_1}) \delta_Y - c_{\max}(\mathcal{V}, h^\mathcal{V}).
\]

Let \( \text{Bd}(TY, TX, g^{TX|Y}) \) be the Bott–Chern current on \( Y \) associated to the exact sequence
\[
(2-7) \quad 0 \to TY \to TX|_Y \to N \to 0
\]
constructed in [Bismut et al. 1988a, §1f], which satisfies
\[ \frac{\bar{\partial} \partial}{2\pi i} \tilde{T}_d(TY, TX, g^{TX|Y}) = Td(TX|Y, g^{TX|Y}) - Td(TY, g^{TY}) Td(N, g^N). \]

Finally, let \( R(x) \) be the power series introduced in [Gillet and Soulé 1991], which is such that if \( \zeta(s) \) is the Riemann zeta function, then
\[ R(x) = \sum_{n \geq 1} \left( \sum_{j=1}^{n} \frac{\xi(-n)}{j} + 2 \frac{\partial \xi}{\partial s}(-n) \right) x^n n!, \]

We identify \( R \) with the corresponding additive genus. We also set
\[ \text{ch}(\wedge^*(\nabla^*_i)) = \sum_i (-1)^i \text{ch}(\wedge^i(\nabla^*_i)), \]
and denote by \( \text{ch}(\wedge^*(\nabla^*_i), h^{\wedge^*(\nabla^*_i)}) \) its Chern–Weil representative.

**Theorem 2.2.** The Quillen metric \( \| \rho \|_{\lambda(\nabla^*_i)^{-1} \otimes \lambda_*(\nabla^*_i)}^2 \) is given by the exponential of
\[ (2-8) \quad - \int_X Td(TX, g^{TX}) T^{-1}(\nabla, h^V) \tilde{c}_{\text{max}}(\nabla, h^V) \text{ch}(\xi, h^\xi) \]
\[ + \int_Y T^{-1}(N, g^N) \tilde{T}_d(TY, TX|Y, g^{TX|Y}) \text{ch}(\wedge^*(\nabla^*_i), h^{\wedge^*(\nabla^*_i)}) \text{ch}(\xi, h^\xi) \]
\[ - \int_Y Td(TY) R(N) \text{ch}(\wedge^*(\nabla^*_i)) \text{ch}(\xi). \]

**Proof.** Set
\[ (2-9) \quad T(\nabla^*, h^{\wedge^*(\nabla^*_i)}) = T^{-1}(\nabla, h^V) \tilde{c}_{\text{max}}(\nabla, h^V). \]

By the same argument as in [Bismut et al. 1990, Theorem 3.17], the current
\[ T(\nabla^*, h^{\wedge^*(\nabla^*_i)}) \]
is exactly the current on \( X \) associated to \( (2-1) \) (evaluated modulo irrelevant \( \partial \) or \( \bar{\partial} \) coboundaries).

Now, from the choice of our metric \( h^V \), the analogue of [Bismut and Lebeau 1991, Definition 1.21, assumption (A)] is satisfied for the complex \( (2-1) \). Then we verify that as far as local index theoretic computations are concerned, the situation is exactly the same as in [Bismut and Lebeau 1991]. Because of the quasi-isomorphism of **Theorem 2.1**, there are no “small” eigenvalues of the operator \( D + TV \) when \( T \to +\infty \). In Section 3, we write down the intermediate results corresponding to [Bismut and Lebeau 1991, §6c]. Comparing to [Bismut and Lebeau 1991, §§6c–6e], the proof of **Theorem 2.2** is complete. \( \square \)
3. **L^2 metrics on H^r(X, \wedge(\mathcal{V}^*)) and localization**

We keep the assumptions and notations of Section 2.

Let $g^TX$ be a Kähler metric on $TX$, and let $g^TY, g^N$ be the metrics on $TY, N$ induced by $g^TX$. Let $h^\mathcal{V}$ be a metric on $\mathcal{V}$ such that $\mathcal{V}_1$ and $\text{Im} \nabla v|_Y$ are orthogonal on $Y$ and $\nabla v|_Y : (N, g^N) \to \text{Im} \nabla v|_Y$ is an isometry.

Let $\phi_1: \det \mathcal{V}_1^* \to \det T^*Y$ be a nonzero holomorphic section. Let $h^\mathcal{V}_1$ be a metric on $\mathcal{V}$ such that on $Y$, $\mathcal{V}_1$ and $\text{Im} \nabla v|_Y$ are orthogonal and

$$|\phi|_{\det \mathcal{V} \otimes \det T^*X, 1} = |\phi|_{\det \mathcal{V}_1 \otimes \det T^*Y, 1} = 1,$$

where $|\cdot|_{\det \mathcal{V} \otimes \det T^*X, 1}$ and $|\cdot|_{\det \mathcal{V}_1 \otimes \det T^*Y, 1}$ are the norms on the holomorphic line bundles $\det \mathcal{V} \otimes \det T^*X$ and $\det \mathcal{V}_1 \otimes \det T^*Y$ induced by $h^\mathcal{V}_1$ and $g^TX$.

We will add a subscript 1 to denote the objects induced by $h^\mathcal{V}_1$. For

$$\beta \in \wedge^p (\mathcal{T}^*X) \widehat{\otimes} \wedge^q (\mathcal{V}^*),$$

we define $*_{\mathcal{V}, 1} \beta \in \wedge^{n-p} (\mathcal{T}^*X) \widehat{\otimes} \wedge^{l-q} (\mathcal{V}^*)$ by

$$\langle \alpha, \beta \rangle_1 \phi^{-1}(dv_X) = \alpha \wedge *_{\mathcal{V}, 1} \beta.$$

It’s useful to write down a local expression for $*_{\mathcal{V}, 1}$. If $\{w^i\}_{i=1}^n$ and $\{\mu^j\}_{j=1}^l$ are orthonormal bases of $T^*X$ and $(\mathcal{V}^*, h^\mathcal{V}_1)$, then

$$dv_X = (-1)^{n(n+1)/2}(\sqrt{-1})^n \bar{w}^1 \wedge \cdots \wedge \bar{w}^n \widehat{\otimes} w^1 \wedge \cdots \wedge w^n$$

and $\phi^{-1}(w^1 \wedge \cdots \wedge w^n) = f \mu^1 \wedge \cdots \wedge \mu^l$ with $|f| = 1$. If

$$\beta = \bar{w}^1 \wedge \cdots \wedge \bar{w}^p \widehat{\otimes} \mu^1 \wedge \cdots \wedge \mu^q,$$

then

$$*_{\mathcal{V}, 1} \beta = (-1)^{(n-p)q+n(n+1)/2}(\sqrt{-1})^n f \bar{w}^{p+1} \wedge \cdots \wedge \bar{w}^n \widehat{\otimes} \mu^{q+1} \wedge \cdots \wedge \mu^l.$$

Thus $*_{\mathcal{V}, 1} *_{\mathcal{V}, 1} \beta = (-1)^{(p+q)l} *_{\mathcal{V}, 1} \beta$, for any $\beta \in \wedge^p (\mathcal{T}^*X) \widehat{\otimes} \wedge^q (\mathcal{V}^*)$. Combining this with (1.2), we find that

$$\bar{\partial}^X * \beta = (-1)^{p+q+1} *_{\mathcal{V}, 1} \bar{\partial}^X *_{\mathcal{V}, 1} \beta, \quad (i(v))^a * \beta = (-1)^{p+q+1} *_{\mathcal{V}, 1} i(v) *_{\mathcal{V}, 1} \beta.$$

Thus the antilinear map $*_{\mathcal{V}, 1}$ is an isometry from $(\mathcal{H}_v(X, \wedge(\mathcal{V}^*)), h^\mathcal{V}_1)$ to itself.
The bilinear form
\[(3-1) \quad \alpha, \beta \in \mathcal{H}_v(X, \wedge (\mathcal{V}^*)) \mapsto \frac{1}{(2\pi)^n} \int_X \alpha \wedge \beta \]
is nondegenerate; indeed, \(\alpha \in \mathcal{H}_v(X, \wedge (\mathcal{V}^*))\) implies \(*_{\mathcal{V}_v,1} \alpha \in \mathcal{H}_v(X, \wedge (\mathcal{V}^*))\), so \(\alpha \neq 0\) implies
\[
\int_X \alpha \wedge *_{\mathcal{V}_v,1} \alpha > 0.
\]
Thus the metric \(\| \cdot \|_{\lambda_v(\mathcal{V}^*)}\) on \(\lambda_v(\mathcal{V}^*)\) only depends on the nondegenerate bilinear form \((3-1)\) on \(\mathcal{H}_v(X, \wedge (\mathcal{V}^*))\), which is metric-independent.

Recall the definition of \(\det \nabla v|_Y\) from Section 1. Now,
\[
\phi|_Y/((\det \nabla v|_Y)^*)
\]
is a holomorphic function on \(Y\). Since \(Y\) is compact, this function is locally constant. Then we have the following extension of [Bismut 2004, Theorem 5.7].

**Theorem 3.1.**
\[(3-2) \quad \log (|\rho|_{\lambda(\mathcal{V}_v^\ast)^{-1} \otimes \lambda_v(\mathcal{V}^*)} - 1) = \int_Y \text{Td}(TY) \text{ch}(\wedge (\mathcal{V}_v^*)) \log \left| \frac{\phi|_Y/((\det \nabla v|_Y)^*)}{\phi_1} \right|.\]

**Proof.** We use \(\phi_1\) to define the integral \(\int_Y \gamma\) for \(\gamma \in H(Y, \wedge (\mathcal{V}_v^*))\). Since
\[
|\phi_1|_{\det \nabla v|_{\mathcal{V}_v \otimes \nabla \cdot Y,1}} = 1,
\]
following the same considerations as above, we find that the antilinear operator \(*_{\mathcal{V}_v,1}\) maps \(H(Y, \wedge (\mathcal{V}_v^*))\) into itself isometrically. Therefore, to evaluate the left-hand side of \((3-2)\), we only need to compare the bilinear forms \((3-1)\) with
\[
a, b \in H(Y, \wedge (\mathcal{V}_v^*)) \mapsto \frac{1}{(2\pi)^m} \int_Y a \wedge b.
\]
Let \(A_v \in \text{End}^{\text{even}} H(Y, \wedge (\mathcal{V}_v^*))\) be given by
\[(3-3) \quad a \mapsto \frac{(-1)^{(l-n)(a-m)} a}{(2\pi)^{n-m} \det_N((1 + R^\gamma)/(2\pi i))} \frac{\phi|_Y/((\det \nabla v|_Y)^*)}{\phi_1}.
\]
Set
\[
\det A_v = \frac{\det A_v|_{H^{\text{even}}(Y, \wedge (\mathcal{V}_v^*))}}{\det A_v|_{H^{\text{odd}}(Y, \wedge (\mathcal{V}_v^*))}};
\]
then
\[
(|\rho|_{\lambda(\mathcal{V}_v^\ast)^{-1} \otimes \lambda_v(\mathcal{V}^*)})^2 = |\det A_v|.
\]
Now, \(A_v\) is a degree-increasing operator in \(H(Y, \wedge (\mathcal{V}_v^*))\). Therefore it acts like a triangular matrix whose diagonal part is just multiplication by the locally constant
function $\frac{\phi|_Y}{((\det \nabla v|_Y)^*)}$. Using (3–3), we get
$$\det A_i = \left(\frac{\phi|_Y}{((\det \nabla v|_Y)^*)}\right)^{\chi(Y, \wedge(V_i^o))}.$$ But $\chi(Y, \wedge(V_i^o)) = \int_Y \TD(TY) \ch(\wedge(V_i^o))$; thus we get (3–2).

Let $g^N_1$ be the metric on $N$ such that $\nabla v|_Y : (N, g^N_1) \to (\Im(\nabla v), h^\Im(\nabla v))$ is an isometry. Let $\nabla^{\wedge-1}(N, g^N, g^N_1)$ be the Bott–Chern class constructed in [Bismut et al. 1988a, §1f] such that
$$\nabla^{\wedge-1}(N, g^N, g^N_1) = \TD(N, g^N_1) - \TD(N, g^N).$$ Finally, we can compute the analytic torsion on the total manifold via the zero set of a transversal section $v$.

**Theorem 3.2.** If $h^\wedge_1 = h^\wedge_i$ on $Y$, then

(3–4) $$\frac{\partial \theta^X_{e_1}}{\partial s}(0) + \frac{\partial \theta^Y}{\partial s}(0) = -\int_Y \TD(TX, g^{TX}) \TD^{-1}(N, g^N_1) \tilde{c}_{\max}(V, h^\wedge_1)$$
$$+ \int_Y \left(\TD^{-1}(N, g^N_1) \tilde{\TD}(TY, TX|_Y, g^{TX|_Y})
+ \TD(TX, g^{TX}) \TD^{-1}(N, g^N_1) \ch(\wedge^*(V_i^o), h^\wedge(V_i^o))\right)$$
$$- \int_Y \TD(TY) \ch(\wedge^*(V_i^o))(R(N) + \log \left| \frac{\phi|_Y}{((\det \nabla v|_Y)^*)} \right|).$$

**Proof.** Since $h^\wedge_1 = h^\wedge_i$, we have $| \cdot |_{\lambda(V_i^o)} = | \cdot |_{\lambda(V_i^o), 1}$ and $\| \cdot \|_{\lambda(V_i^o)} = \| \cdot \|_{\lambda(V_i^o), 1}$. Let
$$\ch(\wedge(V^*), h^\wedge(V^*), h^\wedge(V^*))$$
be the Bott–Chern class constructed in [Bismut et al. 1988a, §1f], so that
$$\frac{\partial \theta^X_{e_1}}{\partial s}(0) = \ch(\wedge(V^*), h^\wedge(V^*), h^\wedge(V^*)) = \ch(\wedge(V^*), h^\wedge(V^*)) - \ch(\wedge(V^*), h^\wedge(V^*)).$$

Then by the anomaly formula [Bismut et al. 1988b, Theorem 1.23],
$$\log \left(\frac{\| \cdot \|_{\lambda(V_i^o)}^2}{\| \cdot \|_{\lambda(V_i^o), 1}^2}\right) = \int_X \TD(TX, g^{TX}) \ch(\wedge(V^*), h^\wedge(V^*), h^\wedge(V^*)).$$

By [Bismut et al. 1990, Theorem 2.5],

(3–5) $$\nabla^{\wedge-1}(N, g^N_1, g^N_1) = \ch(\wedge^*(V_i^o), h^\wedge(V_i^o)) \TD^{-1}(N, g^N_1, g^N_1) \delta_Y - \ch(\wedge(V^*), h^\wedge(V^*), h^\wedge(V^*)).$$

By (2–9), Theorems 2.2 and 3.1, and the preceding equations, the proof of Theorem 3.2 is complete.
**Remark 3.3.** If $Y$ consists only of discrete points and $n = l$, then $\phi_1 = \text{Id}$. In this case let $g^N_{\det}$ and $g_{1\det}^N$ be the metrics on $\det N = \det TX$ induced by $g^N$ and $g_1^N$. By Remark 2.3 and Theorem 3.2,

$$
-\frac{\partial \theta_X^{Y,1}}{\partial s}(0) = -\int_X \text{Td}(TX, g^{TX}) \text{Td}^{-1}(\nabla, h^Y_1) \tilde{c}_{\max}(\nabla, h^Y_1) + \sum_{p \in Y} \left( \frac{1}{2} \log \langle g_{\det}^N / g_{1\det}^N \rangle - \log |\phi/(\det v|_Y)^*| \right).
$$

**Remark 3.4.** If $\nabla = TX$ and $v$ is a holomorphic Killing vector field, (3–4) is a special case of [Bismut 1992, Theorems 6.2 and 7.7]. In this case, $h^Y_1 = g^{TX}$, and on $Y$, we have a holomorphic and orthogonal splitting $TX|_Y = TY \oplus N$. Thus $\text{Td}(TY, TX|_Y, g^{TX|_y}) = 0$. To compute $\text{Td}^{-1}(N, g^N, g_1^N)$, note that $g_1^N = g^N((\nabla v)\cdot (\nabla v))$, as $A = (\nabla v)^*(\nabla v)$ is positive and self-adjoint; thus $(A)^s$ is well defined for $s \in [0, 1]$. Taking $g_s^N = g^N((A)^s \cdot \cdot)$, we obtain by [Bismut et al. 1988a, Theorem 1.30]

$$
\text{Td}^{-1}(N, g^N, g_1^N) = \int_0^1 \langle (\text{Td}^{-1})'(N, g_s^N), \log A \rangle ds.
$$

But $\nabla v$ is holomorphic, so the curvature $R^N_s$ associated to the holomorphic connection on $(N, g^N_s)$ is $R^N_s = R^N$ for $s \in [0, 1]$. Thus

$$
(3–6) \quad \text{Td}^{-1}(N, g^N, g_1^N) = \langle (\text{Td}^{-1})'(N, g^N), \log A \rangle.
$$

Now

$$
(3–7) \quad \text{Td}(TX, g^{TX}) T \left( \wedge (T^*X), h^{N(T^*X)} \right) = \tilde{c}_{\max}(TX, g^{TX})
$$

is an $(n-1, n-1)$-form on $X$.

In this case, we get easily the special case of [Bismut 2004, Theorem 4.15] directly from [Ray and Singer 1973] by using Poincaré duality:

$$
(3–8) \quad \frac{\partial \theta_Y^{Y,1}}{\partial s}(0) = 0.
$$

From (3–4), (3–6), (3–7), and the vanishing of the constant terms of $R(N)$ and $\frac{\text{Td}'}{\text{Td}} (N, g^N) - \frac{1}{2}$, we get

$$
(3–9) \quad -\frac{\partial \theta_X^{Y,1}}{\partial s}(0) = \int_Y c_{\max}(TY) \left( R(N) - \left( \frac{\text{Td}'}{\text{Td}} (N, g^N) - \frac{1}{2}, \log A \right) \right) = 0.
$$
4. Appendix: six intermediate results

In this section, to help readers understand how to obtain Theorem 2.2, we write down the corresponding intermediate results from [Bismut and Lebeau 1991, Theorems 6.4-6.9].

Let $\nabla^{\wedge}(\nabla^*)$ be the connection on $\wedge(\nabla^*)$ induced by $\nabla^\nabla$. Set $C_u = \nabla^{\wedge}(\nabla^*) + \sqrt{u}V$. Let $R^2_{T^2}$ and $\mathrm{Tr}_3(N_H^Y \exp(-R^2_{T^2}))$ be the operator and the generalized trace associated to the complex (2–7) as in [Bismut and Lebeau 1991, §5]. Let $\Phi$ be the homomorphism from $\wedge^{\text{even}}(T_H^2 X)$ into itself which to $\alpha \in \wedge^{2p}(T_H^2 X)$ associates $(2\pi i)^{-p}\alpha$.

**Theorem 4.1.** For any $u_0 > 0$, there exists $C > 0$ such that for $u \geq u_0$, $T \geq 1$,

$$\left| \mathrm{Tr}_3\left(N_H^X e^{-u(D^X + TV)}\right) - \mathrm{Tr}_3\left((1/2 \dim N + N_H^Y) e^{-uD^2}\right) \right| \leq \frac{C}{\sqrt{T}},$$

$$\left| \mathrm{Tr}_3\left((N^X - N_H^X) e^{-u(D^X + TV)^2}\right) - \mathrm{Tr}_3\left((N^Y - N_H^Y) e^{-uD^2}\right) \right| \leq \frac{C}{\sqrt{T}}.$$

**Theorem 4.2.** Let $\tilde{P}_T$ be the orthogonal projection operator from $\Omega(X, \wedge(\nabla^*) \otimes \xi)$ to $\text{Ker}(D^X + TV)$. There exist $c > 0$ and $C > 0$ such that, for any $u \geq 1$ and $T \geq 1$,

$$\left| \mathrm{Tr}_3((N^X - N_H^X) e^{-u(D^X + TV)^2}) - \mathrm{Tr}_3((N^X - N_H^X) \tilde{P}_T) \right| \leq c e^{-Cu},$$

**Theorem 4.3.** There exist $C > 0$ and $\gamma \in [0, 1]$ such that, for any $u \in [0, 1]$ and $0 \leq T \leq 1/u$,

$$\left| \mathrm{Tr}_3\left(N_H^X e^{-u(D^X + TV)^2}\right) - \int_X \mathrm{Td}(TX, g^{TX}) \Phi \mathrm{Tr}_3\left(N_H^X e^{-C_H^2}\right) \right| \leq C(u(1 + T))^{\gamma/2}.$$ 

There exists a constant $C' > 0$ such that for $u \in [0, 1]$ and $0 \leq T \leq 1$,

$$\left| \mathrm{Tr}_3\left(N_H^X e^{-u(D^X + TV)^2}\right) - \mathrm{Tr}_3\left(N_H^X e^{-u(D^X)^2}\right) \right| \leq C'T.$$

**Theorem 4.4.** For any $T > 0$,

$$\lim_{u \to 0} \mathrm{Tr}_3\left(N_H^X e^{-u(D^X + (T/u)V)^2}\right) = \int_Y \Phi \mathrm{Tr}_3\left(N_H^Y e^{-R^2_{T^2}}\right) \mathrm{ch}(\wedge(\nabla^*_Y), h^{\wedge(\nabla^*_Y)}) \mathrm{ch}(\xi, h^\xi).$$

**Theorem 4.5.** There exist $C > 0$ and $\delta \in [0, 1]$ such that, for any $u \in [0, 1]$ and $T \geq 1$,

$$\left| \mathrm{Tr}_3\left(N_H^X e^{-u(D^X + (T/u)V)^2}\right) - \mathrm{Tr}_3\left((1/2 \dim N + N_H^Y) e^{-uD^2}\right) \right| \leq \frac{C}{T^}\delta.$$
Theorem 4.6. As $T \to +\infty$,
\[
\log \left( \frac{1 |_{\lambda(V^*)}}{1 |_{\lambda(V^*)}} \right) = -\log |\rho|_{V^*} \cdot \text{Tr}((\dim N + 2N_H Q) \log T + O\left(\frac{1}{T}\right)).
\]

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