Pacific Journal of Mathematics

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Volume 219 No. 2 April 2005

PACIFIC JOURNAL OF MATHEMATICS
Vol. 219, No. 2, 2005

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We prove a Bott-type residue formula twisted by $\bigwedge(\mathbb{V}^*)$ with a holomorphic vector bundle $\mathbb{V}$, and relate certain analytic torsions on the total manifold to the analytic torsions on the zero set of a holomorphic section of $\mathbb{V}$.

Introduction

Beasley and Witten [2003], studying half-linear models, have described a compactification on any Calabi–Yau threefold Y that is a complete intersection in a compact toric variety X. In particular, a remarkable cancellation involving the instanton effect [Beasley and Witten 2003, (1.3)], involving certain determinants of the $\bar{\partial}$-operator, was derived directly from a residue theorem. One would like to understand its implications in mathematics, for example in Gromov–Witten theory. Bershadsky, Cecotti, Ooguri and Vafa [Bershadsky et al. 1993; 1994] predicted that the analytic torsion of Ray–Singer will play a role regarding the genus-1 Gromov–Witten invariant. Thus we naturally try to understand the results about analytic torsion first.

As an application of [Bismut and Lebeau 1991] and the localization formula (1–3) in this paper, we were able to relate certain analytic torsions on the total manifold with the zero set of a holomorphic transversal section of $\mathbb{V}$, generalizing [Bismut 2004, Theorem 6.6] and [Zhang n.d.] with $\mathbb{V} = TX$ therein. We expect our formula will be useful for understanding [Beasley and Witten 2003, (1.3)] from a mathematical point of view.

This paper is organized as follows. In Section 1 we prove a Bott-type residue formula. In Section 2 we get a localization formula for Quillen metrics. In Section 3 we get a localization formula for analytic torsions under extra conditions. In Section 4, for the reader's convenience, we write down six intermediate results, corresponding to [Bismut and Lebeau 1991, Theorems 6.4-6.9].

MSC2000: 58J52, 32L10, 58J20, 32C35, 57R20.
Keywords: analytic torsion, characteristic classes, characteristic numbers, residue formula.
Feng was partially supported by the NNSF of China (10271059).

1. A Bott-type residue formula

In this section, along the lines of [Bismut 1986, §1], we give a Bott-type residue formula (1–3) by assuming that the holomorphic section is transversal; compare to [Beasley and Witten 2003, (2.32), (2.34)].

Let X be a compact complex manifold with $\dim X = n$ and let $\mathbb{V}$ be a holomorphic vector bundle on X with $\dim \mathbb{V} = l$. We assume that the line bundles $\det TX$ and $\det \mathbb{V}$ are holomorphically isomorphic. We fix a holomorphic isomorphism $\phi : \det \mathbb{V}^* \simeq \det T^*X$, which is clearly unique up to a constant. Thus ϕ defines a map from the $\mathbb{Z}_2$-graded tensor product $\Lambda(\overline{T^*X}) \widehat{\otimes} \Lambda(\mathbb{V}^*)$ to $\Lambda(\overline{T^*X}) \widehat{\otimes} \Lambda^{\max}(T^*X) \subset \Lambda(T^*_{\mathbb{R}}X) \otimes_{\mathbb{R}} \mathbb{C}$. We can define the integral of an element α of $\Omega(X, \Lambda(\mathbb{V}^*))$, the set of smooth sections of $\Lambda(\overline{T^*X}) \widehat{\otimes} \Lambda(\mathbb{V}^*)$ on X, by

$$\int_X \alpha = \int_X \phi(\alpha).$$

Let v be a holomorphic section of $\mathbb{V}$ on X. Assume that v vanishes on a complex manifold $Y \subset X$. Then $\nabla v|_Y : TX|_Y \to \mathbb{V}|_Y$ mapping U to $\nabla_U v$ does not depend on the choice of a connection ∇ on $\mathbb{V}$, and $\nabla_U v|_Y = 0$ for $U \in TY$. Let N be the normal bundle to Y in X. Assume also that $\nabla v|_Y : N \to \mathbb{V}|_Y$ is injective, and there is a holomorphic vector subbundle $\mathbb{V}_1$ on Y such that

$$\mathbb{V}|_Y = \mathbb{V}_1 \oplus \operatorname{Im} \nabla v|_Y. \tag{1–1}$$

Let $P^{\mathbb{V}_1}$ and $P^{\operatorname{Im}\nabla v}$ be the natural projections from $\mathbb{V}$ onto $\mathbb{V}_1$ and $\operatorname{Im} \nabla v|_Y$.

Let $i(v)$ be the standard contraction operator acting on $\Lambda(\mathbb{V}^*)$. A natural question, posed in [Beasley and Witten 2003, §2], is how to express $\int_X \alpha$ using the local data near the zero set Y of v for a $(\bar\partial^X + i(v))$-closed form α, that is, a form satisfying $(\bar\partial^X + i(v))\alpha = 0$.

First we recall an idea due to Bismut [Bismut 1986]; see also [Zhang 1990].

Proposition 1.1. *Let $\alpha \in \Omega(X, \Lambda(\mathbb{V}^*))$ be a $(\bar\partial^X + i(v))$-closed form. Then*

$$\int_X \alpha = \int_X e^{-(\bar\partial^X + i(v))\omega/t} \alpha \quad \textit{for any } \omega \in \Omega(X, \Lambda(\mathbb{V}^*)) \textit{ and } t > 0.$$

Proof. For any $\omega \in \Omega(X, \Lambda(\mathbb{V}^*))$

$$\int_X \bar\partial^X \omega = \int_X \phi(\bar\partial^X \omega) = \int_X \bar\partial^X \phi(\omega) = \int_X d\phi(\omega) = 0. \tag{1–2}$$

From $(\bar\partial^X + i(v))^2 = 0$ and $(\bar\partial^X + i(v))\alpha = 0$, we have

$$\frac{\partial}{\partial s} \int_X e^{-s(\bar\partial^X + i(v))\omega} \alpha = -\int_X (\bar\partial^X + i(v)) \big(\omega\, e^{-s(\bar\partial^X + i(v))\omega} \alpha\big) = 0,$$

and the desired equality follows. □

Recall that $\nabla v|_Y : N \to \operatorname{Im}\nabla v|_Y$ is an isomorphism that induces isomorphisms of holomorphic line bundles $\phi_N = (\det \nabla v|_Y)^* : \det(\operatorname{Im}\nabla v|_Y)^* \to \det N^*$ and $\phi_Y = \phi|_Y/\big((\det\nabla v|_Y)^*\big) : \det \mathbb{V}_1^* \to \det T^*Y$. These two isomorphisms make the integral $\int_N$ along the normal bundle N and $\int_Y$ well defined.

Let $h^{\mathbb{V}}$ be a Hermitian metric on $\mathbb{V}$ such that $\mathbb{V}_1$ and $\operatorname{Im}\nabla v|_Y$ are orthogonal on Y. Let g_1^N be a Hermitian metric on N such that $\nabla.v|_Y : N \to \operatorname{Im}\nabla v|_Y$ is an isometry. Let $R^{\mathbb{V}}$ be the curvature of the holomorphic Hermitian connection $\nabla^{\mathbb{V}}$ on $(\mathbb{V}, h^{\mathbb{V}})$. Let $j : Y \to X$ be the natural embedding, and $\{Y_j\}_j$ the connected components of Y. On Y, define

$$R_v^{\mathbb{V}} = -(\nabla.v)^{-1} P^{\operatorname{Im}\nabla v} R^{\mathbb{V}}(\,\cdot\,, j_*\cdot) P^{\mathbb{V}_1}\cdot \in \overline{T^*Y}\,\widehat{\otimes}\,\mathbb{V}_1^* \otimes \operatorname{End} N.$$

$R_v^{\mathbb{V}}$ is well defined since $P^{\operatorname{Im}\nabla v} R^{\mathbb{V}}(j_*\cdot, j_*\cdot) P^{\mathbb{V}_1} = 0$. Thus, for $U \in TY$, $W \in \mathbb{V}_1$, $u_1, u_2 \in N$,

$$\big\langle R_v^{\mathbb{V}}(\overline{U}, W)u_1, u_2\big\rangle_{g_1^N} = -\big\langle R^{\mathbb{V}}(u_1, \overline{U})W, \nabla_{u_2} v\big\rangle = \big\langle W, R^{\mathbb{V}}(\overline{u_1}, U)\nabla_{u_2} v\big\rangle.$$

Certainly $\det_N((1+R_v^{\mathbb{V}})/2\pi i)$ is $\bar\partial^Y$-closed.

The following result verifies a formula of Beasley and Whitney [2003, (2.32), (2.34)] and generalizes corresponding results in [Zhang 1990], [Liu 1995] and [Bott 1967].

Theorem 1.2. *For any $(\bar\partial^X + i(v))$-closed form $\alpha \in \Omega(X, \bigwedge(\mathbb{V}^*))$,*

$$\int_X \alpha = \sum_j \int_{Y_j} \frac{(-1)^{(l-n)(n-\dim Y_j)}\alpha}{\det_N\big((1+R_v^{\mathbb{V}})/(-2\pi i)\big)}. \tag{1–3}$$

Proof. Set

$$S = \langle\,\cdot\,, v\rangle_{h^{\mathbb{V}}} \in C^\infty(X, \mathbb{V}^*).$$

By Proposition 1.1, for any $t \in\,]0, +\infty[$,

$$\int_X \alpha = \int_X e^{-\frac{1}{2t}(\bar\partial^X + i(v))S}\alpha = \int_X e^{-\frac{1}{2t}(\bar\partial^X S + |v|^2)}\alpha. \tag{1–4}$$

Thus, as $t \to 0$, the integral $\int_X \alpha$ is asymptotically equal to $\int_{\mathcal{U}} e^{-\frac{1}{2t}(\bar\partial^X S + |v|^2)}\alpha$ for any neighborhood $\mathcal{U}$ of Y.

Take $y \in Y$. Since Y is a complex submanifold, we can find holomorphic coordinates $\{z_i\}_{i=1}^n$ of a neighborhood U of y such that y corresponds to 0 and $\{(\partial/\partial z_i)(0)\}_{i=m+1}^n$ is an orthonormal basis of (N, g_1^N), and, moreover,

$$U \cap Y = \{p \in U, z_{m+1}(p) = \cdots = z_n(p) = 0\}.$$

Let $\{\mu_k\}_{k=1}^{l'}$ and $\{\mu_k\}_{k=l'+1}^{l}$ be holomorphic frames for $\mathbb{V}_1$ and $\operatorname{Im}\nabla v|_Y$ on $U \cap Y$, with

$$\nabla^{\mathbb{V}}_{\partial/\partial z_k(0)} v = \mu_k(0) \quad \text{for } l'+1 \le k \le l,$$

and for $z' = (z_1, \dots, z_m)$, $z'' = (z_{m+1}, \dots, z_n)$, $z = (z', z'')$, define $\mu_k(z)$ by parallel transport of $\mu_k(z', 0)$ with respect to $\nabla^{\mathbb{V}}$ along the curve $u \mapsto (z', uz'')$. Identify $\mathbb{V}_z$ with $\mathbb{V}_{(z',0)}$ by identifying $\mu_k(z)$ with $\mu_k(z', 0)$. Denote by $W_y(\varepsilon)$ the ε-neighborhood of y in the normal space N. Then

$$\int_{Y\cap U}\int_{W_y(\varepsilon)} e^{-\frac{1}{2t}(\bar\partial^X S+|v|^2)}\alpha = \int_{Y\cap U}\int_{z\in W_y(\varepsilon/\sqrt{t})} e^{-\frac{1}{2t}(|v(\sqrt{t}z)|^2+(\bar\partial^X S)(\sqrt{t}z))} t^{n-m}\alpha(y,\sqrt{t}z). \tag{1–5}$$

Define $z = \sum_j z_j(\partial/\partial z_j)$ and $\bar z = \sum_j \bar z_j(\partial/\partial \bar z_j)$. The tautological vector field is $Z = z + \bar z$. Then, for $z \in N_y$,

$$\frac{1}{2t}|v(\sqrt{t}z)|^2 = \frac{1}{2}|\nabla^{\mathbb{V}}_z v|^2 + O(\sqrt{t}) = \frac{1}{2}|z|^2 + O(\sqrt{t})$$

and

$$\bar\partial^X S = \sum_{k=1}^{l} \langle \mu_k, \nabla^{\mathbb{V}}_{\cdot} v\rangle \mu^k.$$

From now on, set $z = (0, z'')$ and $Z = z + \bar z$. Since $\nabla^{\mathbb{V}}_Z \mu_k(0) = 0$, we know that

$$\begin{aligned}\frac{1}{2t}\bar\partial^X S(\sqrt{t}z) &= \frac{1}{2t}\sum_{k=1}^{l} \langle \mu_k, \nabla^{\mathbb{V}}_{\cdot} v\rangle (\sqrt{t}z)\mu^k(0)\\ &= \frac{1}{2t}\sum_{k=1}^{l}\Big(\langle \mu_k, \nabla^{\mathbb{V}}_{\cdot} v\rangle(0) + \sqrt{t}\,\langle \mu_k, \nabla^{\mathbb{V}}_Z\nabla^{\mathbb{V}}_{\cdot} v\rangle(0)\\ &\qquad + \frac{t}{2}\big(\langle \nabla^{\mathbb{V}}_Z\nabla^{\mathbb{V}}_Z\mu_k, \nabla^{\mathbb{V}}_{\cdot} v\rangle + \langle \mu_k, \nabla^{\mathbb{V}}_Z\nabla^{\mathbb{V}}_Z\nabla^{\mathbb{V}}_{\cdot} v\rangle\big)(0) + O(t^{3/2})\Big)\mu^k(0).\end{aligned} \tag{1–6}$$

Because of the factor t^{n-m} in (1–5), it should be clear that in the limit, only those monomials in the vertical form

$$d\bar z_{m+1}\wedge\cdots\wedge d\bar z_n \,\widehat{\otimes}\, \mu^{l'+1}\wedge\cdots\wedge\mu^l$$

whose weight is exactly t^{m-n} should be kept. Now,

$$\nabla^{\mathbb{V}}_Z\nabla^{\mathbb{V}}_{\partial/\partial z_j} v = R^{\mathbb{V}}\Big(Z, \frac{\partial}{\partial z_j}\Big)v + \nabla^{\mathbb{V}}_{\partial/\partial z_j}\nabla^{\mathbb{V}}_Z v - 1_{[m,n]}(j)\nabla^{\mathbb{V}}_{\partial/\partial z_j} v,$$

$$\nabla^{\mathbb{V}}_{\bar z}\nabla^{\mathbb{V}}_{\partial/\partial z_j} v(0) = R^{\mathbb{V}}\Big(\bar z, \frac{\partial}{\partial z_j}\Big)v + \nabla^{\mathbb{V}}_{\partial/\partial z_j}\nabla^{\mathbb{V}}_{\bar z} v = 0,$$

where $1_{[m,n]}$ is the characteristic function of the interval $[m,n]$. Note that $\nabla^{\mathbb{V}} = \nabla^{\mathbb{V}_1} \oplus \nabla^{\operatorname{Im}\nabla v}$ on Y and that

$$\big\langle \mu_k, \nabla^{\mathbb{V}}_z \nabla^{\mathbb{V}}_{\partial/\partial z_j} v\big\rangle(0) = 0 \quad \text{for } 1 \le j \le m, 1 \le k \le l'.$$

It follows that in the expression

$$\frac{1}{2\sqrt{t}} \big\langle \mu_k, \nabla^{\mathbb{V}}_Z \nabla^{\mathbb{V}}_{\cdot} v\big\rangle(0)\mu^k(0)$$

a nonzero contribution can only appear in the term

$$\text{(1–7)} \qquad \frac{1}{2\sqrt{t}}\bigg(\sum_{j=1}^{m}\sum_{k=l'+1}^{l} + \sum_{j=m+1}^{n}\sum_{k=1}^{l'}\bigg)\big\langle \mu_k, \nabla^{\mathbb{V}}_z \nabla^{\mathbb{V}}_{\partial/\partial z_j} v\big\rangle(0)\, d\bar z^j \otimes \mu^k(0).$$

Similarly, in the last term of (1–6), the only term with a nonzero contribution is

$$\frac{1}{4}\sum_{j=1}^{m}\sum_{k=1}^{l'}\Big(\big\langle \nabla^{\mathbb{V}}_Z\nabla^{\mathbb{V}}_Z \mu_k, \nabla^{\mathbb{V}}_{\partial/\partial z_j} v\big\rangle(0) + \big\langle \mu_k, \nabla^{\mathbb{V}}_Z\nabla^{\mathbb{V}}_Z\nabla^{\mathbb{V}}_{\partial/\partial z_j} v\big\rangle(0)\Big)\, d\bar z^j \otimes \mu^k(0).$$

But for $1 \le j \le m$, both $\nabla^{\mathbb{V}}_{\partial/\partial z_j} v(0)$ and $\nabla^{\mathbb{V}}_{\partial/\partial z_j}\nabla^{\mathbb{V}}_{\bar z}\nabla^{\mathbb{V}}_z v(0) = \nabla^{\mathbb{V}}_{\partial/\partial z_j}(R^{\mathbb{V}}(\bar z, z)v)(0)$ vanish, since $v = 0$ on Y. Thus, for $1 \le j \le m$,

$$\nabla^{\mathbb{V}}_Z\nabla^{\mathbb{V}}_Z\nabla^{\mathbb{V}}_{\partial/\partial z_j} v(0) = 2R^{\mathbb{V}}\Big(\bar z, \frac{\partial}{\partial z_j}\Big)\nabla^{\mathbb{V}}_z v(0) + \nabla^{\mathbb{V}}_{\partial/\partial z_j}\nabla^{\mathbb{V}}_z\nabla^{\mathbb{V}}_z v(0).$$

By the preceding discussion, as $t \to 0$, in (1–5), we should replace $\frac{1}{2t}\bar\partial^X S(y, \sqrt{t}z)$ by the 2-form

$$\frac{1}{2}\sum_{k=1}^{l}\big\langle \mu_k, \nabla^{\mathbb{V}}_{\cdot} v\big\rangle(0)\mu^k(0) \;+\; \sqrt{t} \times \text{expression (1–7)}$$
$$+ \frac{1}{2}\sum_{j=1}^{m}\sum_{k=1}^{l'}\Big\langle \mu_k, R^{\mathbb{V}}\Big(\bar z, \frac{\partial}{\partial z_j}\Big)\nabla^{\mathbb{V}}_z v + \nabla^{\mathbb{V}}_{\partial/\partial z_j}\nabla^{\mathbb{V}}_z\nabla^{\mathbb{V}}_z v\Big\rangle(0)\, d\bar z^j \otimes \mu^k(0).$$

Set $\beta_Y = d\bar z_1 \cdots d\bar z_m \wedge \mu^1(0)\cdots\mu^{l'}(0)$, $\beta_N = d\bar z_{m+1}\cdots d\bar z_n \wedge \mu^{l'+1}(0)\cdots\mu^l(0)$, $\phi(\mu^1(0)\cdots\mu^l(0)) = f\,dz_1\cdots dz_n$. Then

$$\phi_Y(\mu^1(0)\cdots\mu^{l'}(0))\phi_N(\mu^{l'+1}(0)\cdots\mu^l(0)) = f\,dz_1\cdots dz_n.$$

Thus

$$\begin{aligned}\phi(\beta_Y \wedge \beta_N) &= (-1)^{l'(n-m)} f\,d\bar z_1\cdots d\bar z_n \wedge dz_1\cdots dz_n\\ &= (-1)^{(l'-m)(n-m)}\phi_Y(\beta_Y)\phi_N(\beta_N).\end{aligned}$$

Now, observing that $\int_{\mathbb{C}} \bar z^i e^{-|z|^2}\,dz\,d\bar z = 0$ for $i > 0$ and that $\nabla^{\mathbb{V}}_{\cdot} v : (N, g_1^N) \to \big(\operatorname{Im}\nabla v, h^{\operatorname{Im}\nabla v}\big)$ is an isometry and $l - l' = n - m$, we find that the limit of (1–4) as $t \to 0$ is the sum over j of

$$\int_{Y_j} (-1)^{(l-n)(n-m)} j^*\alpha \int_N \exp\Big(-\frac{1}{2}\sum_{k=1}^{l} \big\langle \mu_k, \nabla_{\cdot}^{\mathbb{V}} v\big\rangle(0)\,\mu^k(0) \\ -\tfrac{1}{2}\big\langle \cdot\,, P^{\mathbb{V}_1} R^{\mathbb{V}}(\bar z, j_*\cdot)\nabla_z^{\mathbb{V}} v\big\rangle(0) - \tfrac{1}{2}|\nabla_z^{\mathbb{V}} v|^2\Big). \tag{1–8}$$

The second integrand in this expression can be rewritten as

$$\exp\Big(-\frac{1}{2}\sum_{i=1}^{n-m} d\bar z_{m+i}\wedge \mu^{l'+i}(0) + \tfrac{1}{2}\big\langle R^{\mathbb{V}}(z, j_*\cdot)P^{\mathbb{V}_1}\cdot, \nabla_z^{\mathbb{V}} v\big\rangle(0) - \tfrac{1}{2}|z|^2\Big) \\ = \exp\Big(\tfrac{1}{2}\big\langle(\nabla^{\mathbb{V}} v)^{-1} R^{\mathbb{V}}(z, j_*\cdot)P^{\mathbb{V}_1}\cdot, z\big\rangle - \tfrac{1}{2}|z|^2\Big)\big(\tfrac{1}{2}\big)^{l-l'} dz_{m+1}\, d\bar z_{m+1}\cdots dz_n\, d\bar z_n.$$

Thus the expression in (1–8) is equal to

$$\int_{Y_j} \frac{(-1)^{(l-n)(n-m)}\alpha}{\det_N\big((1+R_v^{\mathbb{V}})/(-2\pi i)\big)},$$

which leads to (1–3). □

2. Localization of Quillen metrics via a transversal section

Let X be a compact complex manifold of dimension n. Let $\mathbb{V}$ and ξ be holomorphic vector bundles on X with $\dim \mathbb{V} = m$, and let v be a holomorphic section of $\mathbb{V}$. Assume that v vanishes on a complex manifold $Y \subset X$ and satisfies (1–1). Then we have a complex of holomorphic vector bundles on X,

$$0 \to \Lambda^m(\mathbb{V}^*) \xrightarrow{i(v)} \Lambda^{m-1}(\mathbb{V}^*) \xrightarrow{i(v)} \cdots \xrightarrow{i(v)} \Lambda^1(\mathbb{V}^*) \xrightarrow{i(v)} \Lambda^0(\mathbb{V}^*) \to 0. \tag{2–1}$$

Let $\big(\Omega(X, \Lambda(\mathbb{V}^*)\otimes\xi),\ \bar\partial^X\big)$ be the Dolbeault complex associated to the holomorphic vector bundle $\Lambda(\mathbb{V}^*)\otimes\xi$. Let $\mathcal{H}_v(X, \Lambda(\mathbb{V}^*)\otimes\xi)$ be the hypercohomologies of the bicomplex $\big(\Omega(X, \Lambda(\mathbb{V}^*)\otimes\xi), \bar\partial^X,\ i(v)\big)$. Let $j : Y \to X$ be the obvious embedding. Now the pullback map j^* induces naturally a map of complexes

$$j^* : \big(\Omega(X, \Lambda(\mathbb{V}^*)\otimes\xi),\ \bar\partial^X + i(v)\big) \to \big(\Omega(Y, \Lambda(\mathbb{V}_1^*)\otimes\xi),\ \bar\partial^Y\big). \tag{2–2}$$

Theorem 2.1. *The map j^* is a quasi-isomorphism of complexes. In particular, j^* induces an isomorphism*

$$\mathcal{H}_v(X, \Lambda(\mathbb{V}^*)\otimes\xi) \simeq H(Y, \Lambda(\mathbb{V}_1^*)\otimes\xi). \tag{2–3}$$

Proof. In [Feng 2003] there is an analytic proof of this theorem when $\mathbb{V} = TX$. There we used the twisted vector bundle $\Lambda(T^*X)$ and here $\Lambda(\mathbb{V}^*)$ takes its place; the proof works just the same. For an algebraic proof, we can modify the proof of [Bismut 2004, Theorem 5.1]. □

Let N^X, N_H^X be the number operators on $\bigwedge(T^*X)$, $\bigwedge(\mathbb{V}^*)$ corresponding to multiplication by p on $\bigwedge^p(T^*X)$, $\bigwedge^p(\mathbb{V}^*)$; do the same replacing X by Y and $\mathbb{V}^*$ by $\mathbb{V}_1^*$. Then $N^X - N_H^X$ and $N^Y - N_H^Y$ define $\mathbb{Z}$-gradings on $\Omega(X, \bigwedge(\mathbb{V}^*)\otimes\xi)$ and $\Omega(Y, \bigwedge(\mathbb{V}_1^*)\otimes\xi)$, which in turn induce $\mathbb{Z}$-gradings on $\mathscr{H}_v(X, \bigwedge(\mathbb{V}^*)\otimes\xi)$ and $H(Y, \bigwedge(\mathbb{V}_1^*)\otimes\xi)$, respectively. The isomorphism j^* preserves these $\mathbb{Z}$-gradings.

From [Bismut and Lebeau 1991, (1.24)], we define the complex lines $\lambda_v(\mathbb{V}^*)$ and $\lambda(\mathbb{V}_1^*)$ by

$$\lambda_v(\mathbb{V}^*) = \bigotimes_{p=-m}^{n} \big(\det \mathscr{H}_v^p(X, \textstyle\bigwedge(\mathbb{V}^*)\otimes\xi)\big)^{(-1)^{p+1}},$$
$$\lambda(\mathbb{V}_1^*) = \bigotimes_{p=0}^{n}\bigotimes_{q=0}^{m} \big(\det H^p(Y, \textstyle\bigwedge^q(\mathbb{V}_1^*)\otimes\xi)\big)^{(-1)^{p+q+1}}.$$

By (2–3), we have a canonical isomorphism of complex lines

$$\lambda_v(\mathbb{V}^*) \simeq \lambda(\mathbb{V}_1^*).$$

Let ρ be the nonzero section of $\lambda(\mathbb{V}_1^*)^{-1}\otimes\lambda_v(\mathbb{V}^*)$ associated with this canonical isomorphism.

Let g^{TX} be a Kähler metric on TX. We identify N with the bundle orthogonal to TY in $TX|_Y$. Let g^{TY} and g^N be the metrics on TY and N induced by g^{TX}. Let h^ξ be a Hermitian metric on ξ. Let $h^{\mathbb{V}}$ be a metric on $\mathbb{V}$ such that $\mathbb{V}_1$ and $\operatorname{Im}\nabla v|_Y$ are orthogonal on Y and $\nabla v|_Y : N \to \operatorname{Im}\nabla v|_Y$ is an isometry.

Let dv_X be the Riemannian volume form on (X, g^{TX}). Let $\langle\cdot,\cdot\rangle_0$ be the metric on $\bigwedge(\overline{T^*X})\,\widehat{\otimes}\,\bigwedge(\mathbb{V}^*)\otimes\xi$ induced by $g^{TX}, h^{\mathbb{V}}, h^\xi$. The Hermitian product on $\Omega(X, \bigwedge(\mathbb{V}^*)\otimes\xi)$ is defined by

$$\langle\alpha,\alpha'\rangle = \frac{1}{(2\pi)^n}\int_X \langle\alpha,\alpha'\rangle_0\, dv_X \quad \text{for } \alpha,\alpha' \in \Omega(X, \textstyle\bigwedge(\mathbb{V}^*)\otimes\xi). \tag{2–4}$$

Let $\bar\partial^{X*}$ and $v^*\wedge = i(v)^*$ be the adjoint of $\bar\partial^X$ and $i(v)$ with respect to $\langle\cdot,\cdot\rangle$. Set

$$V = i(v) + i(v)^*, \qquad D^X = \bar\partial^X + \bar\partial^{X*}.$$

By Hodge theory,

$$\mathscr{H}_v(X, \textstyle\bigwedge(\mathbb{V}^*)\otimes\xi) \simeq \operatorname{Ker}(D^X + V). \tag{2–5}$$

Denote by P be the operator of orthogonal projection from $\Omega(X, \bigwedge(\mathbb{V}^*)\otimes\xi)$ onto $\ker(D^X+V)$ and set $P^\perp = 1-P$. Let $h^{\mathscr{H}_v}$ be the L^2-metric on $\mathscr{H}_v(X, \bigwedge(\mathbb{V}^*)\otimes\xi)$ induced by the L^2-product (2–4) via the isomorphism (2–5). Define in the same way a Hermitian product on $\Omega(Y, \bigwedge(\mathbb{V}_1^*)\otimes\xi)$ associated to $g^{TY}, h^{\mathbb{V}_1}, h^\xi$. Let $\bar\partial^{Y*}$ be the adjoint of $\bar\partial^Y$, and $h^{H(Y,\bigwedge(\mathbb{V}_1^*)\otimes\xi)}$ the corresponding L^2-metric on

$H(Y, \bigwedge(\mathbb{V}_1^*) \otimes \xi)$. Set

$$D^Y = \bar{\partial}^Y + \bar{\partial}^{Y*}.$$

Let Q be the orthogonal projection operator from $\Omega(Y, \bigwedge(\mathbb{V}_1^*) \otimes \xi)$ on $\operatorname{Ker} D^Y$, and $Q^\perp = 1 - Q$. Let $|\cdot|_{\lambda_v(\mathbb{V}^*)}$ and $|\cdot|_{\lambda(\mathbb{V}^*)}$ be the L^2-metrics on $\lambda_v(\mathbb{V}^*)$ and $\lambda(\mathbb{V}^*)$ induced by $h^{\mathscr{H}_v}$ and $h^{H(Y, \wedge(\mathbb{V}_1^*) \otimes \xi)}$. Following [Bismut and Lebeau 1991, (1.49)], let

$$\theta_v^X(s) = -\operatorname{Tr_s}\big((N^X - N_H^X)((D^X + V)^2)^{-s} P^\perp\big).$$

Then $\theta_v^X(s)$ extends to a meromorphic function of $s \in \mathbb{C}$, which is holomorphic at $s = 0$.

The Quillen metric $\|\cdot\|_{\lambda_v(\mathbb{V}^*)}$ on the line $\lambda_v(\mathbb{V}^*)$ is defined by

$$\|\cdot\|_{\lambda_v(\mathbb{V}^*)} = |\cdot|_{\lambda_v(\mathbb{V}^*)} \exp\left(-\frac{1}{2} \frac{\partial \theta_v^X}{\partial s}(0)\right).$$

In the same way, the function

$$\theta^Y(s) = -\operatorname{Tr_s}\big((N^Y - N_H^Y)(D^{Y,2})^{-s} Q^\perp\big)$$

extends to a meromorphic function of $s \in \mathbb{C}$, holomorphic at $s = 0$. The Quillen metric $\|\cdot\|_{\lambda(\mathbb{V}_1^*)}$ on the line $\lambda(\mathbb{V}_1^*)$ is defined by

$$\|\cdot\|_{\lambda(\mathbb{V}_1^*)} = |\cdot|_{\lambda(\mathbb{V}_1^*)} \exp\left(-\frac{1}{2} \frac{\partial \theta^Y}{\partial s}(0)\right).$$

Let $\|\cdot\|_{\lambda(\mathbb{V}_1^*)^{-1} \otimes \lambda_v(\mathbb{V}^*)}$ be the Quillen metric on $\lambda(\mathbb{V}_1^*)^{-1} \otimes \lambda_v(\mathbb{V}^*)$ induced by $\|\cdot\|_{\lambda_v(\mathbb{V}^*)}$ and $\|\cdot\|_{\lambda(\mathbb{V}_1^*)}$ as in [Bismut and Lebeau 1991, §1e].

The purpose of this section is to give a formula for $\|\rho\|^2_{\lambda(\mathbb{V}_1^*)^{-1} \otimes \lambda_v(\mathbb{V}^*)}$. Now we introduce some notations.

For a holomorphic Hermitian vector bundle (E, h^E) on X, we denote by $\operatorname{Td}(E)$, $\operatorname{ch}(E)$, $c_{\max}(E)$ the Todd class, Chern character, and top Chern class of E, and by $\operatorname{Td}(E, h^E)$, $\operatorname{ch}(E, h^E)$, $c_{\max}(E, h^E)$ the Chern–Weil representatives of $\operatorname{Td}(E)$, $\operatorname{ch}(E)$, $c_{\max}(E)$ with respect to the holomorphic Hermitian connection ∇^E on (E, h^E).

Let δ_Y be the current of integration on Y. By [Bismut 1992, Theorem 3.6], a current $\tilde{c}_{\max}(\mathbb{V}, h^{\mathbb{V}})$ on X is well defined by the holomorphic section v (which induces an embedding $v : X \to \mathbb{V}$), and this current satisfies

$$\frac{\bar{\partial}\partial}{2\pi i} \tilde{c}_{\max}(\mathbb{V}, h^{\mathbb{V}}) = c_{\max}(\mathbb{V}_1, h^{\mathbb{V}_1}) \delta_Y - c_{\max}(\mathbb{V}, h^{\mathbb{V}}). \tag{2–6}$$

Let $\widetilde{\operatorname{Td}}(TY, TX, g^{TX|_Y})$ be the Bott–Chern current on Y associated to the exact sequence

$$0 \to TY \to TX|_Y \to N \to 0 \tag{2–7}$$

constructed in [Bismut et al. 1988a, §1f], which satisfies

$$\frac{\bar\partial\partial}{2\pi i}\,\widetilde{\mathrm{Td}}(TY, TX, g^{TX|_Y}) = \mathrm{Td}(TX|_Y, g^{TX|_Y}) - \mathrm{Td}(TY, g^{TY})\,\mathrm{Td}(N, g^N).$$

Finally, let $R(x)$ be the power series introduced in [Gillet and Soulé 1991], which is such that if $\zeta(s)$ is the Riemann zeta function, then

$$R(x) = \sum_{\substack{n\geq 1\\ n \text{ odd}}} \left(\sum_{j=1}^{n} \frac{1}{j}\zeta(-n) + 2\frac{\partial\zeta}{\partial s}(-n)\right)\frac{x^n}{n!}.$$

We identify R with the corresponding additive genus. We also set

$$\mathrm{ch}(\Lambda^*(\mathbb{V}_1^*)) = \sum_i (-1)^i\,\mathrm{ch}(\Lambda^i(\mathbb{V}_1^*)),$$

and denote by $\mathrm{ch}(\Lambda^*(\mathbb{V}_1^*), h^{\Lambda^*(\mathbb{V}_1^*)})$ its Chern–Weil representative.

Theorem 2.2. *The Quillen metric* $\|\rho\|^2_{\lambda(\mathbb{V}_1^*)^{-1}\otimes\lambda_v(\mathbb{V}^*)}$ *is given by the exponential of*

$$\begin{aligned}(2\text{–}8)\quad & -\int_X \mathrm{Td}(TX, g^{TX})\,\mathrm{Td}^{-1}(\mathbb{V}, h^{\mathbb{V}})\tilde{c}_{\max}(\mathbb{V}, h^{\mathbb{V}})\,\mathrm{ch}(\xi, h^{\xi})\\ &+\int_Y \mathrm{Td}^{-1}(N, g^N)\,\widetilde{\mathrm{Td}}(TY, TX|_Y, g^{TX|_Y})\,\mathrm{ch}(\Lambda^*(\mathbb{V}_1^*), h^{\Lambda^*(\mathbb{V}_1^*)})\,\mathrm{ch}(\xi, h^{\xi})\\ &-\int_Y \mathrm{Td}(TY)\,R(N)\,\mathrm{ch}(\Lambda^*(\mathbb{V}_1^*))\,\mathrm{ch}(\xi).\end{aligned}$$

Proof. Set

$$(2\text{–}9)\qquad T(\Lambda(\mathbb{V}^*), h^{\Lambda(\mathbb{V}^*)}) = \mathrm{Td}^{-1}(\mathbb{V}, h^{\mathbb{V}})\tilde{c}_{\max}(\mathbb{V}, h^{\mathbb{V}}).$$

By the same argument as in [Bismut et al. 1990, Theorem 3.17], the current

$$T(\Lambda(\mathbb{V}^*), h^{\Lambda(\mathbb{V}^*)})$$

is exactly the current on X associated to (2–1) (evaluated modulo irrelevant ∂ or $\bar\partial$ coboundaries).

Now, from the choice of our metric $h^{\mathbb{V}}$, the analogue of [Bismut and Lebeau 1991, Definition 1.21, assumption (A)] is satisfied for the complex (2–1). Then we verify that as far as local index theoretic computations are concerned, the situation is exactly the same as in [Bismut and Lebeau 1991]. Because of the quasi-isomorphism of Theorem 2.1, there are no “small” eigenvalues of the operator $D + TV$ when $T \to +\infty$. In Section 3, we write down the intermediate results corresponding to [Bismut and Lebeau 1991, §6c]. Comparing to [Bismut and Lebeau 1991, §§6c–6e], the proof of Theorem 2.2 is complete. □

Remark 2.3. Assume that Y consists only discrete points; then $l \geq n$ and the last two terms of (2–8) are zero. In this case, if $n = l$, then (2–1) is a resolution of $j_*(\mathscr{O}_Y)$ and Theorem 2.2 is a direct consequence of [Bismut and Lebeau 1991, Theorem 0.1]. By [Bismut 1992, Theorem 3.2, Definition 3.5], $\tilde{c}_{\max}(\mathbb{V}, h^{\mathbb{V}})$ is zero if $l > n+1$.

3. L^2 metrics on $H_v(X, \bigwedge(\mathbb{V}^*))$ and localization

We keep the assumptions and notations of Section 2.

Let g^{TX} be a Kähler metric on TX, and let g^{TY}, g^N be the metrics on TY, N induced by g^{TX}. Let $h^{\mathbb{V}}$ be a metric on $\mathbb{V}$ such that $\mathbb{V}_1$ and $\operatorname{Im} \nabla v|_Y$ are orthogonal on Y and $\nabla v|_Y : (N, g^N) \to \operatorname{Im} \nabla v|_Y$ is an isometry.

Let $\phi_1 : \det \mathbb{V}_1^* \to \det T^*Y$ be a nonzero holomorphic section. Let $h_1^{\mathbb{V}}$ be a metric on $\mathbb{V}$ such that on Y, $\mathbb{V}_1$ and $\operatorname{Im} \nabla v|_Y$ are orthogonal and

$$|\phi|_{\det \mathbb{V} \otimes \det T^*X, 1} = |\phi_1|_{\det \mathbb{V}_1 \otimes \det T^*Y, 1} = 1,$$

where $|\cdot|_{\det \mathbb{V} \otimes \det T^*X, 1}$ and $|\cdot|_{\det \mathbb{V}_1 \otimes \det T^*Y, 1}$ are the norms on the holomorphic line bundles $\det \mathbb{V} \otimes \det T^*X$ and $\det \mathbb{V}_1 \otimes \det T^*Y$ induced by $h_1^{\mathbb{V}}$ and g^{TX}.

We will add a subscript 1 to denote the objects induced by $h_1^{\mathbb{V}}$. For

$$\beta \in \bigwedge\nolimits^p(\overline{T^*X}) \mathbin{\widehat{\otimes}} \bigwedge\nolimits^q(\mathbb{V}^*),$$

we define $*_{\mathbb{V},1}\beta \in \bigwedge^{n-p}(\overline{T^*X}) \mathbin{\widehat{\otimes}} \bigwedge^{l-q}(\mathbb{V}^*)$ by

$$\langle \alpha, \beta \rangle_1 \, \phi^{-1}(dv_X) = \alpha \wedge *_{\mathbb{V},1}\beta.$$

It's useful to write down a local expression for $*_{\mathbb{V},1}$. if $\{w^i\}_{i=1}^n$ and $\{\mu^i\}_{i=1}^l$, are orthonormal bases of T^*X and $(\mathbb{V}^*, h_1^{\mathbb{V}})$, then

$$dv_X = (-1)^{n(n+1)/2}(\sqrt{-1})^n \overline{w}^1 \wedge \cdots \wedge \overline{w}^n \mathbin{\widehat{\otimes}} w^1 \wedge \cdots \wedge w^n$$

and $\phi^{-1}(w^1 \wedge \cdots \wedge w^n) = f\, \mu^1 \wedge \cdots \wedge \mu^l$ with $|f| = 1$. If

$$\beta = \overline{w}^1 \wedge \cdots \wedge \overline{w}^p \mathbin{\widehat{\otimes}} \mu^1 \wedge \cdots \wedge \mu^q,$$

then

$$*_{\mathbb{V},1}\beta = (-1)^{(n-p)q + n(n+1)/2}(\sqrt{-1})^n f\, \overline{w}^{p+1} \wedge \cdots \wedge \overline{w}^n \mathbin{\widehat{\otimes}} \mu^{q+1} \wedge \cdots \wedge \mu^l.$$

Thus $*_{\mathbb{V},1} *_{\mathbb{V},1}\beta = (-1)^{(p+q)(n+l+1)}\beta$, for any $\beta \in \bigwedge^p(\overline{T^*X}) \widehat{\otimes} \bigwedge^q(\mathbb{V}^*)$. Combining this with (1–2), we find that

$$\bar{\partial}^{X*}\beta = (-1)^{p+q+1} *_{\mathbb{V},1}^{-1} \bar{\partial}^X *_{\mathbb{V},1} \beta, \quad (i(v))^*\beta = (-1)^{p+q+1} *_{\mathbb{V},1}^{-1} i(v) *_{\mathbb{V},1} \beta.$$

Thus the antilinear map $*_{\mathbb{V},1}$ is an isometry from $\big(\mathscr{H}_v(X, \bigwedge(\mathbb{V}^*)), h_1^{\mathscr{H}_v}\big)$ to itself.

The bilinear form

$$\alpha,\beta\in\mathcal{H}_v(X,\textstyle\bigwedge(\mathbb{V}^*))\mapsto\frac{1}{(2\pi)^n}\int_X\alpha\wedge\beta \tag{3–1}$$

is nondegenerate; indeed, $\alpha\in\mathcal{H}_v(X,\bigwedge(\mathbb{V}^*))$ implies $*_{\mathbb{V},1}\alpha\in\mathcal{H}_v(X,\bigwedge(\mathbb{V}^*))$, so $\alpha\neq 0$ implies

$$\int_X\alpha\wedge *_{\mathbb{V},1}\alpha>0.$$

Thus the metric $|\cdot|_{\lambda_v(\mathbb{V}^*),1}$ on $\lambda_v(\mathbb{V}^*)$ only depends on the nondegenerate bilinear form (3–1) on $\mathcal{H}_v(X,\bigwedge(\mathbb{V}^*))$, which is metric-independent.

Recall the definition of $\det\nabla v|_Y$ from Section 1. Now,

$$\frac{\phi|_Y/((\det\nabla v|_Y)^*)}{\phi_1}$$

is a holomorphic function on Y. Since Y is compact, this function is locally constant. Then we have the following extension of [Bismut 2004, Theorem 5.7].

Theorem 3.1.

$$\log\big(|\rho|_{\lambda(\mathbb{V}_1^*)^{-1}\otimes\lambda_v(\mathbb{V}^*),1}\big)^2=\int_Y\operatorname{Td}(TY)\operatorname{ch}(\textstyle\bigwedge(\mathbb{V}_1^*))\log\left|\dfrac{\phi|_Y/((\det\nabla v|_Y)^*)}{\phi_1}\right|. \tag{3–2}$$

Proof. We use ϕ_1 to define the integral $\int_Y\gamma$ for $\gamma\in H(Y,\bigwedge(\mathbb{V}_1^*))$. Since

$$|\phi_1|_{\det\mathbb{V}_1\otimes\det T^*Y,1}=1,$$

following the same considerations as above, we find that the antilinear operator $*_{\mathbb{V}_1,1}$ maps $H(Y,\bigwedge(\mathbb{V}_1^*))$ into itself isometrically. Therefore, to evaluate the left-hand side of (3–2), we only need to compare the bilinear forms (3–1) with

$$a,b\in H(Y,\textstyle\bigwedge(\mathbb{V}_1^*))\mapsto\dfrac{1}{(2\pi)^m}\displaystyle\int_Y a\wedge b.$$

Let $A_v\in\operatorname{End}^{\mathrm{even}}H(Y,\bigwedge(\mathbb{V}_1^*))$ be given by

$$a\to\frac{(-1)^{(l-n)(n-m)}a}{(2\pi)^{n-m}\det_N\big((1+R_v^{\mathbb{V}})/(-2\pi i)\big)}\,\frac{\phi|_Y/((\det\nabla v|_Y)^*)}{\phi_1}. \tag{3–3}$$

Set

$$\det A_v=\frac{\det A_v|_{H^{\mathrm{even}}(Y,\bigwedge(\mathbb{V}_1^*))}}{\det A_v|_{H^{\mathrm{odd}}(Y,\bigwedge(\mathbb{V}_1^*))}};$$

then

$$\big(|\rho|_{\lambda(\mathbb{V}_1^*)^{-1}\otimes\lambda_v(\mathbb{V}^*),1}\big)^2=|\det A_v|.$$

Now, A_v is a degree-increasing operator in $H(Y,\bigwedge(\mathbb{V}_1^*))$. Therefore it acts like a triangular matrix whose diagonal part is just multiplication by the locally constant function $\dfrac{\phi|_Y/((\det\nabla v|_Y)^*)}{\phi_1}$. Using (3–3), we get

$$\det A_v = \Big(\frac{\phi|_Y/((\det\nabla v|_Y)^*)}{\phi_1}\Big)^{\chi(Y,\wedge(\mathbb{V}_1^*))}.$$

But $\chi(Y,\wedge(\mathbb{V}_1^*)) = \int_Y \mathrm{Td}(TY)\,\mathrm{ch}(\wedge(\mathbb{V}_1^*))$; thus we get (3–2). □

Let g_1^N be the metric on N such that $\nabla v|_Y : (N, g_1^N) \to (\mathrm{Im}(\nabla v), h_1^{\mathrm{Im}(\nabla v)})$ is an isometry. Let $\widetilde{\mathrm{Td}}^{-1}(N, g^N, g_1^N)$ be the Bott–Chern class constructed in [Bismut et al. 1988a, §1f] such that

$$\frac{\bar\partial\partial}{2\pi i}\widetilde{\mathrm{Td}}^{-1}(N, g^N, g_1^N) = \mathrm{Td}^{-1}(N, g_1^N) - \mathrm{Td}^{-1}(N, g^N).$$

Finally, we can compute the analytic torsion on the total manifold via the zero set of a transversal section v.

Theorem 3.2. *If $h_1^{\mathbb{V}_1} = h^{\mathbb{V}_1}$ on Y, then*

$$\begin{aligned}(3\text{–}4)\quad -\frac{\partial\theta^X_{v,1}}{\partial s}(0) + \frac{\partial\theta^Y}{\partial s}(0) &= -\int_X \mathrm{Td}(TX, g^{TX})\,\mathrm{Td}^{-1}(\mathbb{V}, h_1^{\mathbb{V}})\tilde{c}_{\max}(\mathbb{V}, h_1^{\mathbb{V}})\\ &\quad + \int_Y \Big(\mathrm{Td}^{-1}(N, g^N)\,\widetilde{\mathrm{Td}}\big(TY, TX|_Y, g^{TX|_Y}\big)\\ &\qquad + \mathrm{Td}(TX, g^{TX})\,\widetilde{\mathrm{Td}}^{-1}(N, g^N, g_1^N)\Big)\,\mathrm{ch}\big(\wedge^*(\mathbb{V}_1^*), h^{\wedge^*(\mathbb{V}_1^*)}\big)\\ &\quad - \int_Y \mathrm{Td}(TY)\,\mathrm{ch}(\wedge^*(\mathbb{V}_1^*))\Big(R(N) + \log\Big|\frac{\phi|_Y/((\det\nabla v|_Y)^*)}{\phi_1}\Big|\Big).\end{aligned}$$

Proof. Since $h_1^{\mathbb{V}_1} = h^{\mathbb{V}_1}$, we have $|\cdot|_{\lambda(\mathbb{V}_1^*)} = |\cdot|_{\lambda(\mathbb{V}_1^*),1}$ and $\|\cdot\|_{\lambda(\mathbb{V}_1^*)} = \|\cdot\|_{\lambda(\mathbb{V}_1^*),1}$. Let $\widetilde{\mathrm{ch}}\big(\wedge(\mathbb{V}^*), h_1^{\wedge(\mathbb{V}^*)}, h^{\wedge(\mathbb{V}^*)}\big)$ be the Bott–Chern class constructed in [Bismut et al. 1988a, §1f], so that

$$\frac{\bar\partial\partial}{2\pi i}\widetilde{\mathrm{ch}}\big(\wedge(\mathbb{V}^*), h_1^{\wedge(\mathbb{V}^*)}, h^{\wedge(\mathbb{V}^*)}\big) = \mathrm{ch}\big(\wedge(\mathbb{V}^*), h^{\wedge(\mathbb{V}^*)}\big) - \mathrm{ch}\big(\wedge(\mathbb{V}^*), h_1^{\wedge(\mathbb{V}^*)}\big).$$

Then by the anomaly formula [Bismut et al. 1988b, Theorem 1.23],

$$\log\Big(\frac{\|\cdot\|^2_{\lambda_v(\mathbb{V}^*)}}{\|\cdot\|^2_{\lambda_v(\mathbb{V}^*),1}}\Big) = \int_X \mathrm{Td}(TX, g^{TX})\,\widetilde{\mathrm{ch}}\big(\wedge(\mathbb{V}^*), h_1^{\wedge(\mathbb{V}^*)}, h^{\wedge(\mathbb{V}^*)}\big).$$

By [Bismut et al. 1990, Theorem 2.5],

$$\begin{aligned}(3\text{–}5)\quad & T\big(\wedge(\mathbb{V}^*), h^{\wedge(\mathbb{V}^*)}\big) - T\big(\wedge(\mathbb{V}^*), h_1^{\wedge(\mathbb{V}^*)}\big)\\ &= \mathrm{ch}\big(\wedge^*(\mathbb{V}_1^*), h^{\wedge^*(\mathbb{V}_1^*)}\big)\widetilde{\mathrm{Td}}^{-1}(N, g_1^N, g^N)\delta_Y - \widetilde{\mathrm{ch}}\big(\wedge(\mathbb{V}^*), h_1^{\wedge(\mathbb{V}^*)}, h^{\wedge(\mathbb{V}^*)}\big).\end{aligned}$$

By (2–9), Theorems 2.2 and 3.1, and the preceding equations, the proof of Theorem 3.2 is complete. □

Remark 3.3. If Y consists only of discrete points and $n=l$, then $\phi_1=\mathrm{Id}$. In this case let $g^{\det N}$ and $g_1^{\det N}$ be the metrics on $\det N=\det TX$ induced by g^N and g_1^N. By Remark 2.3 and Theorem 3.2,

$$-\frac{\partial\theta^X_{v,1}}{\partial s}(0)=-\int_X \mathrm{Td}(TX,g^{TX})\,\mathrm{Td}^{-1}(\mathbb{V},h_1^{\mathbb{V}})\,\tilde{c}_{\max}(\mathbb{V},h_1^{\mathbb{V}}) + \sum_{p\in Y}\Bigl(\tfrac{1}{2}\log(g^{\det N}/g_1^{\det N})-\log|\phi/(\det\nabla v|_Y)^*|\Bigr).$$

Remark 3.4. If $\mathbb{V}=TX$ and v is a holomorphic Killing vector field, (3–4) is a special case of [Bismut 1992, Theorems 6.2 and 7.7]. In this case, $h_1^{\mathbb{V}}=g^{TX}$, and on Y, we have a holomorphic and orthogonal splitting $TX|_Y=TY\oplus N$. Thus $\widetilde{\mathrm{Td}}(TY,TX|_Y,g^{TX|_Y})=0$. To compute $\widetilde{\mathrm{Td}}^{-1}(N,g^N,g_1^N)$, note that $g_1^N=g^N((\nabla v)\cdot,(\nabla v)\cdot)$, as $A=(\nabla v)^*(\nabla v)$ is positive and self-adjoint; thus $(A)^s$ is well defined for $s\in[0,1]$. Taking $g_s^N=g^N((A)^s\cdot,\cdot)$, we obtain by [Bismut et al. 1988a, Theorem 1.30]

$$\widetilde{\mathrm{Td}}^{-1}(N,g^N,g_1^N)=\int_0^1\bigl\langle(\mathrm{Td}^{-1})'(N,g_s^N),\log A\bigr\rangle\,ds.$$

But ∇v is holomorphic, so the curvature R_s^N associated to the holomorphic connection on (N,g_s^N) is $R_s^N=R^N$ for $s\in[0,1]$. Thus

$$\widetilde{\mathrm{Td}}^{-1}(N,g^N,g_1^N)=\bigl\langle(\mathrm{Td}^{-1})'(N,g^N),\log A\bigr\rangle. \tag{3–6}$$

Now

$$\mathrm{Td}(TX,g^{TX})\,T\bigl(\Lambda(T^*X),h^{\Lambda(T^*X)}\bigr)=\tilde{c}_{\max}(TX,g^{TX}) \tag{3–7}$$

is an $(n-1,n-1)$-form on X.

In this case, we get easily the special case of [Bismut 2004, Theorem 4.15] directly from [Ray and Singer 1973] by using Poincaré duality:

$$\frac{\partial\theta^Y}{\partial s}(0)=0. \tag{3–8}$$

From (3–4), (3–6), (3–7), and the vanishing of the constant terms of $R(N)$ and $\frac{\mathrm{Td}'}{\mathrm{Td}}(N,g^N)-\frac{1}{2}$, we get

$$-\frac{\partial\theta^X_{v,1}}{\partial s}(0)=\int_Y c_{\max}(TY)\Bigl(R(N)-\Bigl\langle\frac{\mathrm{Td}'}{\mathrm{Td}}(N,g^N)-\tfrac{1}{2},\log A\Bigr\rangle\Bigr)=0. \tag{3–9}$$

4. Appendix: six intermediate results

In this section, to help readers understand how to obtain Theorem 2.2, we write down the corresponding intermediate results from [Bismut and Lebeau 1991, Theorems 6.4-6.9].

Let $\nabla^{\Lambda(\mathbb{V}^*)}$ be the connection on $\Lambda(\mathbb{V}^*)$ induced by $\nabla^{\mathbb{V}^*}$. Set $C_u = \nabla^{\Lambda(\mathbb{V}^*)} + \sqrt{u}V$. Let $\mathscr{B}^2_{T^2}$ and $\operatorname{Tr}_s\big(N^Y_H \exp(-\mathscr{B}^2_{T^2})\big)$ be the operator and the generalized trace associated to the complex (2–7) as in [Bismut and Lebeau 1991, §5]. Let Φ be the homomorphism from $\Lambda^{\text{even}}(T^*_{\mathbb{R}}X)$ into itself which to $\alpha \in \Lambda^{2p}(T^*_{\mathbb{R}}X)$ associates $(2\pi i)^{-p}\alpha$.

Theorem 4.1. *For any $u_0 > 0$, there exists $C > 0$ such that for $u \ge u_0$, $T \ge 1$,*

$$\Big|\operatorname{Tr}_s\big(N^X_H e^{-u(D^X+TV)^2}\big) - \operatorname{Tr}_s\big((\tfrac{1}{2}\dim N + N^Y_H)e^{-uD^{Y,2}}\big)\Big| \le \frac{C}{\sqrt{T}},$$

$$\Big|\operatorname{Tr}_s\big((N^X - N^X_H)e^{-u(D^X+TV)^2}\big) - \operatorname{Tr}_s\big((N^Y - N^Y_H)e^{-uD^{Y,2}}\big)\Big| \le \frac{C}{\sqrt{T}}.$$

Theorem 4.2. *Let $\tilde{P}_T$ be the orthogonal projection operator from $\Omega(X, \Lambda(\mathbb{V}^*)\otimes\xi)$ to $\operatorname{Ker}(D^X + TV)$. There exist $c > 0$ and $C > 0$ such that, for any $u \ge 1$ and $T \ge 1$,*

$$\Big|\operatorname{Tr}_s\big((N^X - N^X_H)e^{-u(D^X+TV)^2}\big) - \operatorname{Tr}_s\big((N^X - N^X_H)\tilde{P}_T\big)\Big| \le ce^{-Cu},$$

Theorem 4.3. *There exist $C > 0$ and $\gamma \in]0, 1]$ such that, for any $u \in]0, 1]$ and $0 \le T \le 1/u$,*

$$\Big|\operatorname{Tr}_s\big(N^X_H e^{-(uD^X+TV)^2}\big) - \int_X \operatorname{Td}(TX, g^{TX})\Phi\operatorname{Tr}_s\big(N^X_H e^{-C^2_{T^2}}\big)\Big| \le C(u(1+T))^\gamma.$$

There exists a constant $C' > 0$ such that for $u \in]0, 1]$ and $0 \le T \le 1$,

$$\Big|\operatorname{Tr}_s\Big(N^X_H e^{-(uD^X+TV)^2}\Big) - \operatorname{Tr}_s\Big(N^X_H e^{-(uD^X)^2}\Big)\Big| \le C'T.$$

Theorem 4.4. *For any $T > 0$,*

$$\lim_{u\to 0}\operatorname{Tr}_s\big(N^X_H e^{-(uD^X+(T/u)V)^2}\big) = \int_Y \Phi\operatorname{Tr}_s\big(N^Y_H e^{-\mathscr{B}^2_{T^2}}\big)\operatorname{ch}\big(\Lambda(\mathbb{V}^*_1), h^{\Lambda(\mathbb{V}^*_1)}\big)\operatorname{ch}(\xi, h^\xi).$$

Theorem 4.5. *There exist $C > 0$ and $\delta \in]0, 1]$ such that, for any $u \in]0, 1]$ and $T \ge 1$,*

$$\Big|\operatorname{Tr}_s\big(N^X_H e^{-(uD^X+(T/u)V)^2}\big) - \operatorname{Tr}_s\big((\tfrac{1}{2}\dim N + N^Y_H)e^{-uD^{Y,2}}\big)\Big| \le \frac{C}{T^\delta}.$$

Let $|\cdot|^2_{\lambda_v(\mathbb{V}^*),T}$ be the L^2-metric on $\lambda_v(\mathbb{V}^*)$ induced by g^{TX}, $T^2h^{\mathbb{V}}$ as in (2–5).

Theorem 4.6. *As* $T \to +\infty$,

$$\log\left(\frac{|\cdot|^2_{\lambda_v(\mathbb{V}^*),T}}{|\cdot|^2_{\lambda_v(\mathbb{V}^*)}}\right) = -\log|\rho|^2_{\lambda(\mathbb{V}_1^*)^{-1}\otimes\lambda_v(\mathbb{V}^*)} + \mathrm{Tr}_s\big((\dim N + 2N_H^Y)Q\big)\log T + O\Big(\frac{1}{T}\Big).$$

Acknowledgements

Feng thanks Jean-Pierre Bourguignon and the IHES, where part of this research was performed, for their hospitality. Thanks also to K. Liu for drawing our attention to [Beasley and Witten 2003].

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Received July 25, 2003. Revised February 13, 2004.

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