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BOSONIC REALIZATIONS OF HIGHER-LEVEL TOROIDAL LIE ALGEBRAS

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We construct realizations for the 2-toroidal Lie algebra associated with the Lie algebra A_1 using vertex operators based on bosonic fields. In particular our construction realizes higher-level representations of the 2-toroidal algebra for any given pair of levels (k_0, k_1) with $k_0 \neq 0$. We also construct a smaller module of level $(k_0, 0)$ for the toroidal algebra from the Fock space using certain screening vertex operator, and this later representation generalizes the higher-level construction of the affine Lie algebra $\widehat{\mathfrak{sl}}_2$.

1. Introduction

Toroidal Lie algebras are a natural generalization of the affine Kac–Moody algebras introduced by Moody, Rao and Yokonuma [Moody et al. 1990]. Let $A = \mathbb{C}[s, s^{-1}, t, t^{-1}]$ be the ring of Laurent polynomials in commuting variables. By definition a 2-toroidal Lie algebra is a perfect central extension of the iterated loop algebra $\mathfrak{g} \otimes A$, where $\mathfrak{g}$ is a finite-dimensional simple Lie algebra over $\mathbb{C}$.

Let Ω_A/dA be the Kähler differentials of A modulo the exact forms. The universal central extension of the iterated loop algebra is given by

$$T(\mathfrak{g}) = (\mathfrak{g} \otimes A) \oplus \Omega_A/dA.$$

Any 2-toroidal Lie algebra is a homomorphic image of this toroidal Lie algebra. The center of $T(\mathfrak{g})$ is Ω_A/dA, which is a infinite-dimensional vector space. The Laurent polynomial ring A induces a natural $\mathbb{Z}^2$-gradation on $T(\mathfrak{g})$. For the center we have $\Omega_A/dA = \bigoplus_{\sigma\in\mathbb{Z}^2} \mathscr{Z}(\mathfrak{g})_\sigma$, with $\dim \mathscr{Z}_\sigma = 1$ if $\sigma \neq (0, 0)$ and 2 if $\sigma = (0, 0)$. We denote by c_0 and c_1 the two standard degree-zero central elements in the toroidal Lie algebra $T(\mathfrak{g})$. A module of $T(\mathfrak{g})$ is called a level-(k_0, k_1) module if the standard center (c_0, c_1) acts as (k_0, k_1) for some complex numbers k_0 and k_1. Here we study the level-(k_0, k_1) modules for $k_0 \neq 0$.

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In various constructions of the affine Lie algebra $\widehat{\mathfrak{sl}}_2$ the free field representation is of particular use in its applications. Wakimoto [1986] and Feigen and Frenkel [1988] first gave a general construction for the general case, and later Nemenschansky [1989] gave an invariant form in the special case. Though the two forms can be interchanged by a nontrivial map, we realized that the later form is better for our purpose in the toroidal cases. The operators in question have the form $e^A(B+C)$, where A, B, C are generating functions of the scaled Heisenberg operators. One of the nice things is that all root generators in the toroidal algebra associated with the Lie algebra $\mathfrak{sl}_2$ can be represented by this type of vertex operators. In our construction we have fully used this simplicity and make all calculations in a uniform manner.

As we mentioned earlier, toroidal algebras are generalizations of finite-dimensional Lie algebras, like affine Lie algebras. This similarity is constantly kept in mind as we study their structure and representation theory. Some other basic references related to our work include [Berman and Billig 1999; Eswara Rao and Moody 1994; Fabbri and Moody 1994; Larsson 1999; Moody et al. 1990; Tan 1999]. Our aim in this paper is to give a higher-level representation for the simplest nontrivial example: the 2-toroidal Lie algebra. Our construction generalizes previous work on higher-level representations of the affine Lie algebra $\widehat{\mathfrak{sl}}_2$.

In Section 2 we define the toroidal Lie algebra and state the MRY-presentation [Moody et al. 1990] of the toroidal algebra in terms of generators and relations. The algebra structure is expressed in terms of formal power series identities. We also state some results in this section to be used later. In Section 3 we start with a finite-rank lattice with a symmetric bilinear form and define a Fock space and some vertex operators, which in turn give representations of the toroidal Lie algebra of type A_1, and also a level-$(k_0, 0)$ module with $k_0 \neq 0$ for the double affine algebra of type A_1. In Section 4 we study the structure of the Fock space for the toroidal Lie algebra by using certain screening vertex operators, thus generalizing the higher-level representation of the affine algebra $\widehat{\mathfrak{sl}}_2$ to the toroidal Lie algebra.

2. Toroidal Lie algebras

Let $\mathfrak{sl}_2$ be the 3-dimensional simple Lie algebra over the complex numbers and

$$A = \mathbb{C}[s, s^{-1}, t, t^{-1}]$$

the ring of Laurent polynomials in commuting variables. We consider the iterated loop algebra

$$\mathfrak{g} = \mathfrak{sl}_2 \otimes A.$$

A toroidal Lie algebra of type A_1 is a perfect central extension of the iterated loop algebra $\mathfrak{g}$, which is often an infinite-dimensional central extension. Let Ω_A be the

A-module of differentials with differential mapping $d : A \to \Omega_A$, such that

$$d(f_1 f_2) = (df_1) f_2 + f_1(df_2) \quad \text{for all } f_1, f_2 \text{ in } A.$$

Let $\overline{} : \Omega_A \to \Omega_A/dA$ be the canonical linear map for which $\overline{df} = 0$ for all $f \in A$. Endow the vector space

$$T(A_1) := (\mathfrak{sl}_2 \otimes A) \oplus \Omega_A/dA$$

with the bracket operation defined by

$$[x \otimes f_1, y \otimes f_2] = [x, y] \otimes f_1 f_2 + (x, y)\, \overline{f_2 df_1},$$

for $x, y \in \mathfrak{sl}_2$, $f_1, f_2 \in A$, where $(\cdot\,,\cdot)$ is the trace form and Ω_A/dA is central. From [Moody et al. 1990] we know that $T(A_1)$ is a perfect Lie algebra and is the universal central extension of the iterated loop algebra $\mathfrak{sl}_2 \otimes A$. Therefore any toroidal Lie algebra of type A_1 is a homomorphic image of $T(A_1)$. The gradation of the polynomial ring A gives a natural $\mathbb{Z}^2$-gradation to the toroidal Lie algebra

$$T(A_1) := \bigoplus_{\sigma \in \mathbb{Z}^2} T(A_1)_\sigma,$$

where $T(A_1)_\sigma$ is spanned by $x \otimes s^{m_0} t^{m_1}$, $\overline{s^{m_0} t^{m_1} s^{-1} ds}$ and $\overline{s^{m_0} t^{m_1} t^{-1} dt}$ for $\sigma = (m_0, m_1) \in \mathbb{Z}^2$ and $x \in \mathfrak{sl}_2$. The condition $\overline{df} = 0$ for all $f \in A$ implies that $m_0 \overline{s^{m_0} t^{m_1} s^{-1} ds} + m_1 \overline{s^{m_0} t^{m_1} t^{-1} dt} = 0$ for all $m_0, m_1 \in \mathbb{Z}$. Therefore the dimension of $T(A_1)_\sigma$ is 4 if $\sigma \neq (0,0)$ and 5 if $\sigma = (0,0)$. In particular, $T(A_1)_{(0,0)}$ is spanned by $x \otimes 1$ for $x \in \mathfrak{sl}_2$, and central elements $\overline{s^{-1} ds}$, $\overline{t^{-1} dt}$. We denote these two degree-zero central elements by c_0 and c_1.

The most interesting quotient algebra of the toroidal Lie algebra $T(A_1)$ is the double affine algebra denoted by $T_0(A_1)$, that is, the toroidal Lie algebra of type A_1 with a two-dimensional center. The double affine algebra is the quotient of $T(A_1)$ modulo all the central elements with degree other than zero. In fact, $T_0(A_1)$ has the realization

$$T_0(A_1) = (\mathfrak{sl}_2 \otimes A) \oplus \mathbb{C}c_0 \oplus \mathbb{C}c_1$$

with the Lie product

$$[x \otimes f_1, y \otimes f_2] = [x, y] \otimes f_1 f_2 + \Phi(f_2 \partial_s f_1) c_0 + \Phi(f_2 \partial_t f_1) c_1$$

for all $x, y \in \mathfrak{sl}_2$ and $f_1, f_2 \in A$, where Φ is the linear functional on A defined by

$$\Phi(s^k t^m) = \begin{cases} 0, & \text{if } (k, m) \neq (0,0) \\ 1, & \text{if } (k, m) = (0,0) \end{cases}$$

for all $k, m \in \mathbb{Z}$.

Definition 2.1. If M is a module for a toroidal Lie algebra of type A_1, we call M a level-(k_0, k_1) module for some complex numbers k_0, k_1 if the degree-zero central elements c_0, c_1 act on M as constants k_0, k_1.

In this paper we give a concrete construction for a level-(k_0, k_1) module with $k_0 \neq 0$ for the toroidal Lie algebra $T(A_1)$ and for the double affine algebra $T_0(A_1)$.

Let $\{x_\pm, h\}$ be the standard basis of $\mathfrak{sl}_2$. Also let $(a_{ij})_{2\times 2}$ be the generalized Cartan matrix of the affine algebra $A_1^{(1)}$ and

$$Q := \mathbb{Z}\alpha_0 + \mathbb{Z}\alpha_1$$

its root lattice. The toroidal Lie algebra $T(A_1)$ has a presentation [Moody et al. 1990] with generators $¢$, $\alpha_i(k)$ and $x_k(\pm\alpha_i)$, for $k \in \mathbb{Z}$ and $i = 0, 1$, and the following relations, for $k, m \in \mathbb{Z}$ and $i, j = 0, 1$:

(R0) $\big[¢, \alpha_i(k)\big] = 0 = \big[¢, x_k(\pm\alpha_i)\big]$;

(R1) $\big[\alpha_i(k), \alpha_j(m)\big] = k a_{ij} \delta_{k+m,0} ¢$;

(R2) $\big[\alpha_i(k), x_m(\pm\alpha_j)\big] = \pm a_{ij} x_{k+m}(\pm\alpha_j)$;

(R3) $\big[x_k(\alpha_i), x_m(-\alpha_j)\big] = -\delta_{ij}\,\big\{\alpha_i(k+m) + k\delta_{k+m,0} ¢\big\}$;

(R4) $\big[x_k(\alpha_i), x_m(\alpha_i)\big] = 0 = \big[x_k(-\alpha_i), x_m(-\alpha_i)\big]$;

$$(\mathrm{ad}\, x_0(\alpha_i))^3\, x_m(\alpha_j) = 0 \text{ if } i \neq j; \quad (\mathrm{ad}\, x_0(-\alpha_i))^3\, x_m(-\alpha_j) = 0 \text{ if } i \neq j.$$

The Lie algebra isomorphism ψ between the two presentations of $T(A_1)$ is given by

$$\begin{aligned} ¢ &\mapsto \overline{s^{-1}ds}, \\ x_m(\pm\alpha_1) &\mapsto \pm x_\pm \otimes s^m, \\ x_m(\pm\alpha_0) &\mapsto \pm x_\mp \otimes s^m t^{\pm 1}, \\ \alpha_1(k) &\mapsto h \otimes s^k, \\ \alpha_0(k) &\mapsto -h \otimes s^k + \overline{s^k t^{-1} dt}. \end{aligned}$$

Therefore, the degree-zero central elements are $c_0 = ¢$ and $c_1 = \delta(0)$, where $\delta = \alpha_0 + \alpha_1$ is the null root in Q. We will identify the two presentations of the toroidal Lie algebra $T(A_1)$ via this isomorphism ψ.

Following [Moody et al. 1990], we introduce a $\mathbb{Z} \times Q$-gradation on $T(A_1)$ by assigning $\deg ¢ = (0, 0)$, $\deg \alpha_i(k) = (k, 0)$, $\deg x_k(\pm\alpha_i) = (k, \pm\alpha_i)$, with $i = 0, 1$ and $k \in \mathbb{Z}$. We denote by T_k^α the subspace of $T(A_1)$ spanned by the elements with degree (k, α) for $k \in \mathbb{Z}$, $\alpha \in Q$. Then, under the isomorphism ψ, we have $\psi^{-1}(\overline{s^k t^{-1} dt}) = \delta(k) \in T_k^0$ and $\psi^{-1}(\overline{s^k t^r s^{-1} ds}) \in T_k^{r\delta}$.

Let $z, w, z_1, z_2, \ldots$ be formal variables. We define formal power series with coefficients from the toroidal Lie algebra $T(A_1)$:

$$\alpha_i(z) = \sum_{n\in\mathbb{Z}} \alpha_i(n) z^{-n-1},$$
$$x(\pm\alpha_i, z) = \sum_{n\in\mathbb{Z}} x_n(\pm\alpha_i) z^{-n-1},$$

for $i = 0, 1$. Then the Lie algebra structure of $T(A_1)$ can be expressed in terms of the following power series identities:

(R0′) $\big[\not{c}, \alpha_i(z)\big] = 0 = \big[\not{c}, x(\pm\alpha_i, z)\big]$;

(R1′) $\big[\alpha_i(z), \alpha_j(w)\big] = a_{ij} z^{-1}\partial_w \delta(\frac{w}{z})\not{c}$;

(R2′) $\big[\alpha_i(z), x(\pm\alpha_j, w)\big] = \pm a_{ij} x(\pm\alpha_j, w) z^{-1}\delta(\frac{w}{z})$;

(R3′) $\big[x(\alpha_i, z), x(-\alpha_j, w)\big] = -\delta_{ij}\left\{\alpha_i(w) z^{-1}\delta(\frac{w}{z}) + z^{-1}\partial_w\delta(\frac{w}{z})\not{c}\right\}$;

(R4′) $\big[x(\alpha_i, z), x(\alpha_i, w)\big] = 0 = \big[x(-\alpha_i, z), x(-\alpha_i, w)\big]$;

$$(\operatorname{ad} x(\alpha_i, z_1))(\operatorname{ad} x(\alpha_i, z_2))(\operatorname{ad} x(\alpha_i, z_3))\, x(\alpha_j, z_4) = 0 \quad \text{if } i \neq j;$$
$$(\operatorname{ad} x(-\alpha_i, z_1))(\operatorname{ad} x(-\alpha_i, z_2))(\operatorname{ad} x(-\alpha_i, z_3))\, x(-\alpha_j, z_4) = 0 \quad \text{if } i \neq j.$$

Finally, we recall a result from [Moody et al. 1990] that will be used in the next section.

Proposition 2.2. *Suppose $\mathcal{L}$ is a Lie algebra over $\mathbb{C}$ graded by $\mathbb{Z}\otimes Q$, and $\phi : T(A_1) \to \mathcal{L}$ is a surjective graded homomorphism of Lie algebras such that*

(i) *ϕ is injective on T_n^α for all $n \in \mathbb{Z}$ and real root α,*

(ii) *$\phi(\delta(k)) \neq 0$ for all k and $\phi|_{\mathbb{C}\delta(0)+\mathbb{C}\not{c}}$ is injective, and*

(iii) *for all nonzero integers k, m,*

$$\phi\big([x_m(\alpha_1 + k\delta), x_0(-\alpha_1)] - [x_0(\alpha_1 + k\delta), x_m(-\alpha_1)]\big) \neq 0,$$
$$\phi\big([x_1(\alpha_1 + k\delta), x_{-1}(-\alpha_1)] - [x_{-1}(\alpha_1 + k\delta), x_1(-\alpha_1)]\big) \neq 0.$$

Then ϕ is an isomorphism, where $x_m(\pm\alpha_1 + k\delta) := \psi^{-1}(\pm x_\pm \otimes s^m t^k)$.

Proposition 2.3. *Suppose $\mathcal{L}$ is a Lie algebra over $\mathbb{C}$ graded by $\mathbb{Z}\otimes Q$, and $\phi : T(A_1) \to \mathcal{L}$ is a surjective graded homomorphism of Lie algebras such that*

(i) *ϕ is injective on T_n^α for all $n \in \mathbb{Z}$ and real root α,*

(ii) *$\phi(\delta(k)) = 0$ for all $k \neq 0$ and $\phi|_{\mathbb{C}\delta(0)+\mathbb{C}\not{c}}$ is injective, and*

(iii) *for all nonzero integers k, m,*

$$\phi\big([x_m(\alpha_1 + k\delta), x_0(-\alpha_1)] - [x_0(\alpha_1 + k\delta), x_m(-\alpha_1)]\big) = 0,$$
$$\phi\big([x_1(\alpha_1 + k\delta), x_{-1}(-\alpha_1)] - [x_{-1}(\alpha_1 + k\delta), x_1(-\alpha_1)]\big) = 0,$$

Then $\mathcal{L}$ is isomorphic to the double affine algebra $T_0(A_1)$.

Proof. We only need to show that the set of nonzero-degree central elements of the toroidal Lie algebra $T(A_1)$ is in the kernel of ϕ. Indeed, under the isomorphism ψ of the toroidal Lie algebras, we see that $\delta(k) = \psi^{-1}(\overline{s^k t^{-1} dt})$ and

$$\big[x_m(\alpha_1+k\delta), x_0(-\alpha_1)\big] - \big[x_0(\alpha_1+k\delta), x_m(-\alpha_1)\big] = -m\psi^{-1}(\overline{s^m t^k s^{-1} ds}),$$
$$\big[x_1(\alpha_1+k\delta), x_{-1}(-\alpha_1)\big] - \big[x_{-1}(\alpha_1+k\delta), x_1(-\alpha_1)\big] = -2\psi^{-1}(\overline{t^k s^{-1} ds}),$$

but, from [Moody et al. 1990], the elements $\overline{s^p t^q s^{-1} ds}$, $\overline{s^p t^{-1} dt}$ and $\overline{s^{-1} ds}$ for $(p, q) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ form a basis of the center for the toroidal Lie algebra $T(A_1)$. The assumption implies that the nonzero-degree central elements $\psi^{-1}(\overline{s^p t^q s^{-1} ds})$ and $\psi^{-1}(\overline{s^q t^{-1} dt})$ are in the kernel of the homomorphism ϕ for

$$(p, q) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}). \qquad \square$$

3. Representations of the toroidal algebra

In this section we give two bosonic realizations for the toroidal Lie algebra $T(A_1)$. Let k_0 be a fixed complex number with $k_0 \neq 0$, and Γ a finite rank lattice with a symmetric $\mathbb{C}$-valued $\mathbb{Z}$-bilinear form $(\cdot\,,\cdot)$. We extend the form to a $\mathbb{C}$-bilinear form on the vector space $H = \mathbb{C} \otimes_{\mathbb{Z}} \Gamma$. Let Γ_0 be a fixed integral sublattice of Γ. We define

$$\Gamma_0^\star = \{\alpha \in H;\ (\alpha, \Gamma_0) \subset \mathbb{Z}\}.$$

Then $\Gamma_0 \subset \Gamma_0^\star$. Let

$$\mathcal{H} = \langle h(n), c\!\!/ \,|\, h \in H, n \in \mathbb{Z}\rangle,$$

with $H = \mathbb{C} \otimes_{\mathbb{Z}} \Gamma$, be the affinization of the vector space H, defined with the Lie product

$$[\alpha(m), \beta(n)] = m(\alpha, \beta)\delta_{m+n,0} c\!\!/$$

for $m, n \in \mathbb{Z}$, $\alpha, \beta \in \Gamma$, and $c\!\!/$ central. We define the Fock space

$$V := \mathbb{C}[\Gamma_0^\star] \otimes S(\mathcal{H}^-),$$

where $S(\mathcal{H}^-)$ is the symmetric algebra on $\mathcal{H}^- := \langle h(n) \mid n < 0\rangle$, and

$$\mathbb{C}[\Gamma_0^\star] = \bigoplus_{\alpha \in \Gamma_0^\star} \mathbb{C}e^\alpha$$

is the group algebra on the additive subgroup $\Gamma_0^\star$ of the vector space H. Then V has a natural module structure for the Lie algebra $\mathcal{H}$ and the group algebra $\mathbb{C}[\Gamma_0^\star]$ with the actions defined by making $c\!\!/$ act as k_0, $h(-n)$ act as multiplication, and $h(n)$ act as a partial differential operator, for $n > 0$, $h \in H$, so that

$$[\alpha(m), \beta(n)] = mk_0(\alpha, \beta)\delta_{m+n,0}$$

for all $\alpha, \beta \in H$ and $m, n \in \mathbb{Z}$. Moreover $\alpha(0)$ acts as a partial differential operator on $\mathbb{C}[\Gamma_0^\star]$ for which $[\alpha(0), e^\beta] = (\alpha, \beta)e^\beta$. Therefore $\alpha(0).\beta = (\alpha, \beta)$ for $\alpha, \beta \in H$.

With a formal variable z, and $\alpha, \beta \in H$, we define fields

$$\alpha(z) = \sum_{n\in\mathbb{Z}} \alpha(n)z^{-n-1},$$
$$\alpha(z)_+ = \sum_{n<0} \alpha(n)z^{-n-1},$$
$$\overline{\beta(z)} = \beta + \beta(0)\log z - \sum_{n\neq 0} \frac{\beta(n)}{n} z^{-n},$$
$$\overline{\beta(z)}_+ = \beta - \sum_{n<0} \frac{\beta(n)}{n} z^{-n}.$$

It is easy to see that $\partial_z \overline{\beta(z)} = \beta(z)$ and $\partial_z \overline{\beta(z)}_+ = \beta(z)_+$. For

$$A, B \in \{\alpha(z), \overline{\beta(z)} \mid \alpha, \beta \in H\},$$

we define $\langle A, B\rangle = [A, B_+]$. Then it is easy to show (see [Frenkel et al. 1988]) that $\langle\overline{\alpha(z)}, \overline{\beta(w)}\rangle = (\alpha, \beta)\log(z - w)$ for $\alpha, \beta \in H$, which then implies

$$\langle\alpha(z), \overline{\beta(w)}\rangle = (\alpha, \beta)(z - w)^{-1},$$
$$\langle\overline{\alpha(z)}, \beta(w)\rangle = -(\alpha, \beta)(z - w)^{-1},$$
$$\langle\alpha(z), \beta(w)\rangle = (\alpha, \beta)(z - w)^{-2},$$

where the formal power series in z and w are understood to be expanded in the second variable w.

Define the usual normal ordering : : as in [Frenkel et al. 1988]. Then we have for $\alpha \in H$

$$:\alpha(z)\beta(w): = \alpha(z)\beta(w) - \langle\alpha(z), \beta(w)\rangle,$$

and, for $\alpha \in \Gamma_0$,

$$:e^{\overline{\alpha(z)}}: = e^\alpha z^{\alpha(0)} \exp\Bigl(-\sum_{n<0} \frac{\alpha(n)}{n} z^{-n}\Bigr) \exp\Bigl(-\sum_{n>0} \frac{\alpha(n)}{n} z^{-n}\Bigr).$$

It is clear that the vertex operators $:e^{\overline{\alpha(z)}}:$, for $\alpha \in \Gamma_0$, can be formally expanded as a power series in z for which the coefficients are well defined operators acting on the Fock space V.

We will need the following result in the study of the bosonic realizations for the toroidal Lie algebra $T(A_1)$; see [Jing and Lyerly 1999].

Lemma 3.1. *Let $P_i(z)$, $Q_i(w)$, for $i = 1, 2$, be fields such that the contractions $\langle P_i, Q_j\rangle$ commute with all fields $P_i(z)$, $Q_i(w)$. Then*

$$:e^{P_1}P_2::e^{Q_1}Q_2: = :e^{P_1}P_2e^{Q_1}Q_2:e^{\langle P_1,Q_1\rangle} + :e^{P_1}P_2e^{Q_1}:e^{\langle P_1,Q_1\rangle}\langle P_1, Q_2\rangle$$
$$+:e^{P_1}e^{Q_1}Q_2:e^{\langle P_1,Q_1\rangle}\langle P_2, Q_1\rangle + :e^{P_1}e^{Q_1}:e^{\langle P_1,Q_1\rangle}(\langle P_2, Q_2\rangle + \langle P_1, Q_2\rangle\langle P_2, Q_1\rangle).$$

For $\alpha, \beta \in \Gamma_1$, we have, from [Frenkel et al. 1988], the identity

$$:e^{\overline{\alpha(z)}}::e^{\overline{\beta(w)}}: = :e^{\overline{\alpha(z)}}e^{\overline{\beta(w)}}:(z-w)^{(\alpha,\beta)}.$$

Inductively one can show, for $\beta_1, \dots, \beta_k \in \Gamma_0$, the following Wick theorem

$$:e^{\overline{\beta_1(z_1)}}:\cdots:e^{\overline{\beta_k(z_k)}}: = :e^{\overline{\beta_1(z_1)}}\cdots e^{\overline{\beta_k(z_k)}}:\prod_{i<j}(z_i-z_j)^{(\beta_i,\beta_j)}.$$

Corollary 3.2. *For $\alpha, \beta \in \Gamma_0$ and $\gamma, \tau \in H$, suppose $(\alpha, \beta) = 0$. Then*

$$\big[:e^{\overline{\alpha(z)}}\gamma(z):, :e^{\overline{\beta(w)}}\tau(w):\big] = :e^{\overline{(\alpha+\beta)(z)}}A(z):z^{-1}\delta\Big(\frac{w}{z}\Big) + B:e^{\overline{(\alpha+\beta)(z)}}:z^{-1}\partial_w\delta\Big(\frac{w}{z}\Big),$$

where $A = (\gamma, \beta)\tau - (\alpha, \tau)\gamma - B\beta \in H$ and $B = (\gamma, \tau) - (\alpha, \tau)(\gamma, \beta) \in \mathbb{C}$.

To give our first representation of the toroidal Lie algebra $T(A_1)$ we consider the lattice

$$\Gamma := \frac{1}{k_0}(\mathbb{Z}a_0 \oplus \mathbb{Z}a_1 \oplus \mathbb{Z}b \oplus \mathbb{Z}r),$$

with a symmetric bilinear form determined by

$$(b, b) = -2k_0, \quad (r, r) = 2(k_0 + 2), \quad (a_i, a_j) = k_0 a_{ij} \quad \text{for } i, j = 0, 1,$$

the others being zero. Let $\Gamma_0 = \frac{1}{k_0}(\mathbb{Z}(a_0 - b) + \mathbb{Z}(a_1 + b))$, which is clearly an integral sublattice of Γ. On the corresponding Fock space $V := \mathbb{C}[\Gamma_0^\star] \otimes S(\mathcal{H}^-)$, we define vertex operators

$$X_0(\pm\alpha_1, z) = \tfrac{1}{2}:e^{\pm\frac{1}{k_0}\overline{(a_1+b)(z)}}(b(z) \mp r(z)):$$
$$X_0(\pm\alpha_0, z) = \tfrac{1}{2}:e^{\pm\frac{1}{k_0}\overline{(a_0-b)(z)}}(b(z) \pm r(z)):,$$

where α_0, α_1 are the simple roots of the affine Lie algebra $A_1^{(1)}$.

Theorem 3.3. *Let k_0 be any nonzero complex number. Then on the Fock space V we have a representation for the toroidal Lie algebra $T(A_1)$. The homomorphism is given by $\not{c} \mapsto k_0$, $\alpha_i(z) \mapsto a_i(z)$, $x(\pm\alpha_i, z) \mapsto X_0(\pm\alpha_i, z)$, for $i = 0, 1$.*

Proof. We first write the vertex operators in the form

$$X_0(\pm\alpha_i, z) = \tfrac{1}{2}:e^{\pm\frac{1}{k_0}\overline{(a_i-\epsilon_i b)(z)}}(b(z) \pm \epsilon_i r(z)):,$$

where $\epsilon_i = (-1)^i$ for $i = 0, 1$. We will now show that the operators $a_i(z)$ and $X_0(\pm\alpha_i, z)$ satisfy the relations (R0′)–(R4′) of the toroidal Lie algebra $T(A_1)$. In fact, (R0′) and (R1′) are obvious. For (R2′) we have

$$\begin{aligned}[][a_i(z), X_0(\pm\alpha_j, w)] &= \tfrac{1}{2}[:a_i(z):, :e^{\pm\frac{1}{k_0}\overline{(a_j-\epsilon_j b)(z)}}(b(z)\pm\epsilon_j r(z)):] \\ &= \tfrac{1}{2}:e^{\pm\frac{1}{k_0}\overline{(a_j-\epsilon_j b)(z)}}A(z):z^{-1}\delta\Big(\frac{w}{z}\Big),\end{aligned}$$

where $A = (a_i, \pm\frac{1}{k_0}(a_j-\epsilon_j b))(b\pm\epsilon_j r) = \pm a_{ij}(b\pm\epsilon_j r)$. Therefore

$$\begin{aligned}[][a_i(z), X_0(\pm\alpha_j, w)] &= \pm\tfrac{1}{2}a_{ij}:e^{\pm\frac{1}{k_0}\overline{(a_j-\epsilon_j b)(z)}}(b(z)\pm\epsilon_j r(z)):z^{-1}\delta\Big(\frac{w}{z}\Big) \\ &= \pm a_{ij}X_0(\pm\alpha_j, z)z^{-1}\delta\Big(\frac{w}{z}\Big),\end{aligned}$$

which is the required relation. To prove relation (R3′) we have

$$\begin{aligned}&[X_0(\alpha_i, z), X_0(\alpha_j, w)] \\ &\quad= \tfrac{1}{4}\Big[:e^{\frac{1}{k_0}\overline{(a_i-\epsilon_i b)(z)}}(b(z)+\epsilon_i r(z)):, :e^{-\frac{1}{k_0}\overline{(a_j-\epsilon_j b)(w)}}(b(w)-\epsilon_j r(w)):\Big] \\ &\quad= \tfrac{1}{4}\Big(:e^{\frac{1}{k_0}\overline{(a_i-a_j-\epsilon_i b+\epsilon_j b)(z)}}A(z):z^{-1}\delta\big(\tfrac{w}{z}\big) + B:e^{\frac{1}{k_0}\overline{(a_i-a_j-\epsilon_i b+\epsilon_j b)(z)}}:z^{-1}\partial_w\delta\big(\tfrac{w}{z}\big)\Big),\end{aligned}$$

where, by applying Corollary 3.2,

$$\begin{aligned}B &= (b+\epsilon_i r, b-\epsilon_j r) - \Big(\frac{a_i-\epsilon_i b}{k_0}, b-\epsilon_j r\Big)\Big(b+\epsilon_i r, -\frac{a_j-\epsilon_j b}{k_0}\Big) \\ &= -2k_0 - 2\epsilon_i\epsilon_j k_0, \\ A &= \Big(b+\epsilon_i r, -\frac{a_j-\epsilon_j b}{k_0}\Big)(b-\epsilon_j r) \\ &\qquad - \Big(\frac{a_i-\epsilon_i b}{k_0}, b-\epsilon_j r\Big)(b+\epsilon_i r) - \Big(-2k_0-2\epsilon_i\epsilon_j k_0\Big)\Big(-\frac{a_j-\epsilon_j b}{k_0}\Big) \\ &= -2(1+\epsilon_i\epsilon_j)a_j.\end{aligned}$$

Therefore, we get

$$\begin{aligned}[][X_0(\alpha_i, z), X_0(-\alpha_j, w)] &= -\tfrac{1}{2}(1+\epsilon_i\epsilon_j)\Big(:e^{\frac{1}{k_0}\overline{(a_i-a_j-\epsilon_i b+\epsilon_j b)(z)}}a_j(z):z^{-1}\delta\big(\tfrac{w}{z}\big) \\ &\qquad + k_0:e^{\frac{1}{k_0}\overline{(a_i-a_j-\epsilon_i b+\epsilon_j b)(z)}}:z^{-1}\partial_w\delta\big(\tfrac{w}{z}\big)\Big) \\ &= -\delta_{ij}\left(a_j(z)z^{-1}\delta\big(\tfrac{w}{z}\big) + k_0 z^{-1}\partial_w\delta\big(\tfrac{w}{z}\big)\right),\end{aligned}$$

as required.

(R4′) contains two types of relations. We give only the proof for the "positive" case. The "negative" case can be proved similarly.

$$\begin{aligned}
&[X_0(\alpha_i, z), X_0(\alpha_j, w)]\\
&= \tfrac{1}{4}\big[:e^{\frac{1}{k_0}\overline{(a_i-\epsilon_i b)(z)}}(b(z)+\epsilon_i r(z)):,\ :e^{\frac{1}{k_0}\overline{(a_j-\epsilon_j b)(w)}}(b(w)+\epsilon_j r(w)):\big]\\
&= \frac{1}{4}\Big(:e^{\frac{1}{k_0}\overline{(a_i+a_j-\epsilon_i b-\epsilon_j b)(z)}}A(z):z^{-1}\delta\big(\tfrac{w}{z}\big)+B:e^{\frac{1}{k_0}\overline{(a_i+a_j-\epsilon_i b-\epsilon_j b)(z)}}:z^{-1}\partial_w\delta\big(\tfrac{w}{z}\big)\Big),
\end{aligned}$$

where, by applying Corollary 3.2,

$$B=(b+\epsilon_i r, b+\epsilon_j r)-\Big(\frac{a_i-\epsilon_i b}{k_0}, b+\epsilon_j r\Big)\Big(b+\epsilon_i r, \frac{a_j-\epsilon_j b}{k_0}\Big)=2k_0(\epsilon_i\epsilon_j-1),$$

$$\begin{aligned}
A&=\Big(b+\epsilon_i r, \frac{a_j-\epsilon_j b}{k_0}\Big)(b+\epsilon_j r)\\
&\qquad -\Big(\frac{a_i-\epsilon_i b}{k_0}, b+\epsilon_j r\Big)\Big(b+\epsilon_i r\Big)-2k_0(\epsilon_i\epsilon_j-1)\Big(\frac{a_j-\epsilon_j b}{k_0}\Big)\\
&=2(1-\epsilon_i\epsilon_j)a_j.
\end{aligned}$$

Therefore $[X_0(\alpha_i, z), X_0(\alpha_i, w)]=0$ and, for $i\neq j$,

$$\begin{aligned}
&[X_0(\alpha_i, z), X_0(\alpha_j, w)]\\
&\qquad = :e^{\frac{1}{k_0}\overline{(a_i+a_j)(z)}}a_j(z):z^{-1}\delta\Big(\frac{w}{z}\Big)-k_0:e^{\frac{1}{k_0}\overline{(a_i+a_j)(z)}}:z^{-1}\partial_w\delta\Big(\frac{w}{z}\Big).
\end{aligned}$$

Clearly, for $i\neq j$, the vertex operator $X_0(\alpha_i, z)$ commutes with

$$:e^{\frac{1}{k_0}\overline{(a_i+a_j)(z)}}:.$$

Therefore to complete the proof of relation (R4′) we only need to show the identity

$$\big[X_0(\alpha_i, z_1), [X_0(\alpha_i, z_2),\ :e^{\frac{1}{k_0}\overline{(a_i+a_j)(z_3)}}a_j(z_3):]\big]=0 \tag{1}$$

for $i\neq j$. Indeed,

$$\begin{aligned}
&\big[X_0(\alpha_i, z),\ :e^{\frac{1}{k_0}\overline{(a_i+a_j)(w)}}a_j(w):\big]\\
&\qquad = \tfrac{1}{2}\big[:e^{\frac{1}{k_0}\overline{(a_i-\epsilon_i b)}(z)}(b+\epsilon_i r)(z):,\ :e^{\frac{1}{k_0}\overline{(a_i+a_j)}(w)}a_j(w):\big]\\
&\qquad = \tfrac{1}{2}\Big(:e^{\frac{1}{k_0}\overline{(2a_i+a_j-\epsilon_i b)(z)}}A(z):z^{-1}\delta\big(\tfrac{w}{z}\big)+B:e^{\frac{1}{k_0}\overline{(2a_i+a_j-\epsilon_i b)(z)}}:z^{-1}\partial_w\delta\big(\tfrac{w}{z}\big)\Big),
\end{aligned}$$

where, by applying Corollary 3.2,

$$B=(b+\epsilon_i r, a_j)-\Big(\frac{a_i-\epsilon_i b}{k_0}, a_j\Big)\Big(b+\epsilon_i r, \frac{a_i+a_j}{k_0}\Big)=0$$

and

$$A=\Big(b+\epsilon_i r, \frac{a_i+a_j}{k_0}\Big)a_j-\Big(\frac{a_i-\epsilon_i b}{k_0}, a_j\Big)(b+\epsilon_i r)=2(b+\epsilon_i r);$$

that is

$$\left[X_0(\alpha_i, z),\ :e^{\frac{1}{k_0}\overline{(a_i+a_j)}(w)}a_j(w):\right] = :e^{\frac{1}{k_0}\overline{(2a_i+a_j-\epsilon_i b)}(z)}(b+\epsilon_i r):z^{-1}\delta\Big(\frac{w}{z}\Big).$$

Therefore (1) is reduced to the identity

$$\left[X_0(\alpha_i, z),\ :e^{\frac{1}{k_0}\overline{(2a_i+a_j-\epsilon_i b)}(w)}(b+\epsilon_i r)(w):\right] = 0$$

for $i \neq j$. The left side is equal to

$$\begin{aligned}&\tfrac{1}{2}\left[:e^{\frac{1}{k_0}\overline{(a_i-\epsilon_i b)}(z)}(b+\epsilon_i r)(z):,\ :e^{\frac{1}{k_0}\overline{(2a_i+a_j-\epsilon_i b)}(w)}(b+\epsilon_i r)(w):\right]\\ &\quad = \tfrac{1}{2}\left(:e^{\frac{1}{k_0}\overline{(3a_i+a_j-2\epsilon_i b)(z)}}A(z):z^{-1}\delta\big(\tfrac{w}{z}\big) + B:e^{\frac{1}{k_0}\overline{(3a_i+a_j-2\epsilon_i b)(z)}}:z^{-1}\partial_w\delta\big(\tfrac{w}{z}\big)\right),\end{aligned}$$

where, by applying Corollary 3.2,

$$B = (b+\epsilon_i r, b+\epsilon_i r) - \Big(\frac{a_i-\epsilon_i b}{k_0}, b+\epsilon_i r\Big)\Big(b+\epsilon_i r, \frac{2a_i+a_j-\epsilon_i b}{k_0}\Big) = 0$$

and

$$A = \Big(b+\epsilon_i r, \frac{2a_i+a_j-\epsilon_i b}{k_0}\Big)(b+\epsilon_i r) - \Big(\frac{a_i-\epsilon_i b}{k_0}, b+\epsilon_i r\Big)(b+\epsilon_i r) = 0,$$

giving the desired identity. □

From the construction of the representation for the toroidal Lie algebra given in the previous theorem, it is easy to see that the operators $\alpha_1(k)+\alpha_0(k)$ act on the Fock space V trivially for all positive integers k, which in turn implies that the central elements $\psi(\delta(k))$ act as the zero operator for $k > 0$. Therefore the representation is not faithful. Indeed, the quotient space $V(0)$ of the Fock space

$$\mathbb{C}[\Gamma_0^*] \otimes S(\mathcal{H}^-)$$

defines a representation for the double affine Lie algebra $T_0(A_1)$, which is isomorphic to the Lie algebra $T(A_1)$ modulo all central elements of degree other then zero (see Section 2).

Corollary 3.4. *The vector space $V(0)$ is endowed with a representation of the double affine Lie algebra $T_0(A_1)$ with level-$(k_0, 0)$, under the formula given before Theorem 3.3.*

We will study this module structure again in the next section.

To give a faithful representation of the toroidal Lie algebra, we consider the rank-six lattice

$$\Gamma := \frac{1}{k_0}(\mathbb{Z}a_0 \oplus \mathbb{Z}a_1 \oplus \mathbb{Z}b \oplus \mathbb{Z}c \oplus \mathbb{Z}d) \oplus \frac{1}{k_0+2}\mathbb{Z}r,$$

with the symmetric bilinear form determined by

$$(b,b)=-2k_0,\quad (r,r)=2(k_0+2),\quad (c,d)=k_0,\quad (a_i,a_j)=k_0a_{ij}\quad \text{for } i,\, j=0,1,$$

all others being zero. Then

$$\Gamma_0 := \frac{1}{k_0}\mathbb{Z}(a_0-b)+\frac{1}{k_0}\mathbb{Z}(a_1+b)+\frac{1}{k_0}\mathbb{Z}c$$

is clearly an integral sublattice of Γ. Let $\Gamma_0^\star$ be the corresponding additive subgroup of $H=\mathbb{C}\otimes_{\mathbb{Z}}\Gamma$, and V the corresponding Fock space.

We also modify the vertex operators from the previous theorem to the form

$$X(\pm\alpha_1,z)=\tfrac{1}{2}:e^{\pm\frac{1}{k_0}\overline{(a_1+b)(z)}}(b(z)\mp r(z)):,$$
$$X(\pm\alpha_0,z)=\tfrac{1}{2}:e^{\pm\frac{1}{k_0}\overline{(a_0-b+c)(z)}}(b(z)\pm r(z)):.$$

Theorem 3.5. *The coefficient operators of the vertex operators $a_i(z)$, $X(\pm\alpha_i,z)$, for $i=0,1$, acting on the Fock space V, generate a Lie algebra $\mathscr{L}(A_1)$ isomorphic to the toroidal Lie algebra $T(A_1)$, the isomorphism begin given by the linear map ϕ defined by*

$$\begin{aligned} \not{c} &\mapsto k_0,\\ \alpha_1(z) &\mapsto a_1(z),\\ \alpha_0(z) &\mapsto a_0(z)+c(z),\\ x(\pm\alpha_i,z) &\mapsto X(\pm\alpha_i,z)\quad \text{for } i=0,1. \end{aligned}$$

Therefore, on the Fock space V, we have a faithful representation of the toroidal Lie algebra $T(A_1)$.

Proof. We first need to show that the surjective mapping ϕ defines a Lie algebra homomorphism from $T(A_1)$ to $\mathscr{L}(A_1)$. It suffices to show that the vertex operators $a_i(z)$, $X(\pm\alpha_i,z)$ satisfy the corresponding power series identities (R0′)–(R4′). The argument is just as in the proof of Theorem 3.3, and we omit it for brevity's sake.

We next use Proposition 2.2 to show that the mapping ϕ is indeed an injective homomorphism. For $\alpha=\mu_1a_0+\mu_2a_1+\mu_3b+\mu_4c\in\Gamma_0^\star$ with $\mu_i\in\frac{1}{k_0}\mathbb{Z}$, let

$$e^\alpha\otimes\lambda_1(-n_1)\cdots\lambda_k(-n_k)\in V.$$

We define a $\mathbb{Z}\times Q$-gradation on the Fock space V by setting

$$\deg\bigl(e^\alpha\otimes\lambda_1(-n_1)\cdots\lambda_k(-n_k)\bigr)=(n_1+\cdots+n_k,\, k_0\mu_1\alpha_0+k_0\mu_2\alpha_1).$$

With this gradation, the operator $a(n)$, for $a\in H$, is a homogeneous operator of degree $(-n,0)$. Moreover, if the vertex operator $X(\pm\alpha_i,z)$ is formally expanded into power series as

$$X(\pm\alpha_i, z) = \sum_{m\in\mathbb{Z}} X_m(\pm\alpha_i)z^{-m-1},$$

the coefficient operator $X_m(\pm\alpha_i)$ is a homogeneous operator of degree $(-m, \pm\alpha_i)$. Thus the map ϕ is a $(\mathbb{Z}\times Q)$-graded Lie algebra homomorphism. To finish the proof of this theorem, we need only show that ϕ satisfies the three conditions of Proposition 2.2.

Recall the notation $x_m(\pm\alpha_1 + k\delta) = \psi^{-1}(\pm s^m t^k \otimes x_\pm)$, where $\delta = \alpha_0 + \alpha_1$ is the null root in Q. Let

$$x(\alpha, z) = \sum_{m\in\mathbb{Z}} x_m(\alpha)z^{-m-1} \qquad \text{for } \alpha = \pm\alpha_1 + k\delta.$$

Then it is easy to show that $\phi : x(\alpha, z) \mapsto X(\alpha, z)$, where $\alpha = \pm\alpha_1 + k\delta$, and

$$X(\pm\alpha_1 + k\delta, z) = \tfrac{1}{2} :e^{\pm\frac{1}{k_0}\overline{(a_1+b+k(a_0+a_1)+kc)(z)}}(b(z) \mp r(z)): .$$

Applying Corollary 3.2 again we have

$$\begin{aligned}&\big[X(\alpha_1 + k\delta, z), X(-\alpha_1 - k\delta, w)\big]\\ &\qquad = -k_0 z^{-1}\partial_w \delta\Big(\frac{w}{z}\Big) - (a_1 + k(a_0 + a_1) + kc)(z)z^{-1}\delta\Big(\frac{w}{z}\Big).\end{aligned}$$

This gives

$$\big[X_m(\alpha_1 + k\delta), X_{-m}(-\alpha_1 - k\delta)\big] = -a_1(0) - k(a_0 + a_1)(0) - kc(0) - mk_0,$$

which is clearly a nonzero operator for any $m, k \in \mathbb{Z}$. Thus ϕ is injective on the one-dimensional subspace $T_m^\alpha = \mathbb{C}x_m(\alpha)$ for any real root $\alpha = \pm\alpha_1 + k\delta$ and $k, m \in \mathbb{Z}$. Moreover,

$$\phi(\delta(k)) = a_0(k) + a_1(k) + c(k)$$

is also a nonzero operator, and ϕ is clearly injective on $\mathbb{C}\delta(0) + \mathbb{C}\not{c}$.

Finally, we need to show that, for $m, k \neq 0$,

$$\begin{aligned}\big[X_m(\alpha_1 + k\delta), X_0(-\alpha_1)\big] - \big[X_0(\alpha_1 + k\delta), X_m(-\alpha_1)\big] \neq 0,\\ \big[X_1(\alpha_1 + k\delta), X_{-1}(-\alpha_1)\big] - \big[X_{-1}(\alpha_1 + k\delta), X_1(-\alpha_1)\big] \neq 0.\end{aligned} \tag{2}$$

By Corollary 3.2,

$$\begin{aligned}&\big[X(\alpha_1 + k\delta, z), X(-\alpha_1, w)\big] + \big[X(-\alpha_1, z), X(\alpha_1 + k\delta, w)\big]\\ &\qquad = -2k_0 :e^{\frac{1}{k_0}\overline{(ka_0+ka_1+kc)(z)}}: z^{-1}\partial_w\delta\Big(\frac{w}{z}\Big)\\ &\qquad\qquad + k :e^{\frac{1}{k_0}\overline{(ka_0+ka_1+kc)(z)}}(a_0(z) + a_1(z) + c(z)): z^{-1}\delta\Big(\frac{w}{z}\Big),\end{aligned}$$

which gives

$$\left[X(\alpha_1+k\delta,z),\,X_0(-\alpha_1)\right]-\left[X_0(\alpha_1+k\delta),\,X(-\alpha_1,z)\right]$$
$$=k:e^{\frac{1}{k}_0\overline{(ka_0+ka_1+kc)(z)}}(a_0(z)+a_1(z)+c(z)):.$$

To see that the coefficient of z^{-m-1} in the expression on the right is nonzero for $m\neq 0$, we notice that

$$\left[:e^{\frac{1}{k_0}\overline{(ka_0+ka_1+kc)(z)}}(a_0(z)+a_1(z)+c(z)):,\,:e^{-\frac{1}{k_0}\overline{(ka_0+ka_1+kc)(z)}}d(z):\right]$$
$$=k_0z^{-1}\partial_w\delta\left(\frac{w}{z}\right).$$

The coefficient of z^{-m-1} on the right-hand side of the previous identity is k_0mw^{m-1}, which is nonzero whenever $m\neq 0$. This proves the first line in (2), while the second can be proved by a similar argument which is omitted here. Therefore ϕ is an isomorphism of Lie algebras. □

Corollary 3.6. *For any fixed $k_1\in\mathbb{Z}$, define*

$$V(k_1)=e^{k_1d+\Gamma_0}\otimes S(\mathcal{H}^-).$$

Then the vector space $V(k_1)$ is endowed with a representation of the toroidal Lie algebra $T(A_1)$ with level-(k_0,k_1).

4. Module structure

We now define a smaller module from our Fock space representation via the so-called screening operator. We will only consider the case when $c=0$.

For given $j_0,\,j_1,\,l_1,\,l_2\in\mathbb{C}$ with $j_0+j_1\in\mathbb{Z}\frac{k_0}{2}$, set

$$v_{j_0,j_1,l_1,l_2}:=e^{j_0\frac{a_0}{k_0}}e^{j_1\frac{a_1}{k_0}}e^{l_1\frac{b}{k_0}}e^{-l_2\frac{r}{k_0+2}}.$$

We define the Fock space F_{j_0,j_1,l_1,l_2} to be the space $S(\mathcal{H}^-)v_{j_0,j_1,l_1,l_2}$. Then the vertex operators $X(\pm\alpha_i,z)$ are well defined on F_{j_0,j_1,l_1,l_2}, provided that $2(j_1-l_1)$ and $2(j_0+l_1)$ are integers. It is clear that the vertex operators satisfy

$$X(\pm\alpha_0,z):F_{j_0,j_1,l_1,l_2}\longrightarrow F_{j_0\pm1,j_1,l_1\mp1,l_2},$$
$$X(\pm\alpha_1,z):F_{j_0,j_1,l_1,l_2}\longrightarrow F_{j_0,j_1\pm1,l_1\pm1,l_2}.$$

Introduce a screening operator $S_0\colon F_{j_0,j_1,l_1,l_2}\to F_{j_0,j_1,l_1+\frac{k_0}{2},l_2+\frac{k_0+2}{2}}$ by setting

$$S(z)=:e^{\frac{1}{2}\overline{(b(z)-r(z))}}:=\sum_n S_nz^{-n-1}.$$

This is well defined provided that $l_1-l_2\in\mathbb{Z}$.

Proposition 4.1.

$$\{X(\alpha_1,z),S(w)\}=\frac{\partial}{\partial w}\left(:e^{\overline{\frac{1}{k_0},(a_1+b)(w)+\frac{1}{2}(b-r)(w)}}:\frac{1}{z-w}\right),$$

$$\{X(-\alpha_1,z),S(w)\}=0,$$

$$\{X(-\alpha_0,z),S(w)\}=0,$$

$$\{X(\alpha_0,z),S(w)\}=\frac{\partial}{\partial w}\left(:e^{\overline{\frac{1}{k_0}(a_0+b)(w)+\frac{1}{2}(b-r)(w)}}:\frac{1}{z-w}\right).$$

Proof. Let

$$\phi(\alpha_1,z)=\phi(-\alpha_0,z):=\tfrac{1}{2}:e^{\overline{\frac{1}{k_0}b(z)}}(b(z)-r(z)):,$$

$$\phi(\alpha_0,z)=\phi(-\alpha_1,z):=\tfrac{1}{2}:e^{\overline{-\frac{1}{k_0}b(z)}}(b(z)+r(z)):$$

be the parafermions. It follows from Lemma 3.1 that

$$\begin{aligned}\phi(\alpha_1,z)S(w)&\\ &\sim\tfrac{1}{2}:e^{\overline{\frac{1}{k_0}b(z)}}(b-r)(z)e^{\overline{\frac{1}{2}(b-r)(w)}}:\frac{1}{z-w}+\tfrac{1}{2}:e^{\overline{\frac{1}{k_0}b(z)+\frac{1}{2}(b-r)(w)}}:\frac{2}{(z-w)^2}\\ &\sim\frac{\partial}{\partial w}\left(:e^{\overline{\frac{1}{k_0}b(w)+\frac{1}{2}(b-r)(w)}}:\frac{1}{z-w}\right).\end{aligned}$$

□

Let d be the zero mode of $S(z)$: $d=\int S(z)dz$. It is easy to check that the anticommutator $\{S(z),S(z)\}=0$, thus d gives rise to a complex of vector spaces:

$$\cdots\longrightarrow F_{j_0,j_1,l_1-\frac{k_0}{2},l_2-\frac{k_0+2}{2}}\longrightarrow F_{j_0,j_1,l_1,l_2}\longrightarrow F_{j_0,j_1,l_1+\frac{k_0}{2},l_2+\frac{k_0+2}{2}}\longrightarrow F_{j_0,j_1,l_1+k_0,l_2+k_0+2}\longrightarrow\cdots$$

We can define the restricted $T(A)$-submodule using Proposition 4.1. Given l we define a $T(A)$-submodule

$$F_l=\bigoplus_{j_1\in l+\mathbb{Z},\,j_0\in -l+\mathbb{Z}}\ker\Big(d:F_{j_0,j_1,j_1,l}\to F_{j_0,j_1,j_1+\frac{k_0}{2},l+\frac{k_0+2}{2}}\Big).$$

Theorem 4.2. *The operator d commutes or anticommutes with elements of the toroidal algebra $T(A_1)$ and $d^2=0$. Moreover we have the long exact sequence*

$$0\longrightarrow F_l\longrightarrow\bigoplus_{j_0,j_1}F_{j_0,j_1,j_1,l}\longrightarrow\bigoplus_{j_0,j_1}F_{j_0,j_1,j_1+\frac{k_0}{2},l+\frac{k_0+2}{2}}\longrightarrow\bigoplus_{j_0,j_1}F_{j_0,j_1,j_1+k_0,l+k_0+2}\longrightarrow\cdots,$$

where the maps from $\bigoplus_{j_0,j_1}F_{j_0,j_1,j_1,l}$ onward are $\bigoplus d$ and the summations run through $j_0\in -l+\mathbb{Z}$ and $j_1\in l+\mathbb{Z}$.

Proof. We introduce the operator $S^\star(z) = e^{\overline{-\frac{1}{2}(b(z)-r(z))}} = \sum_n S^*_n z^{-n}$, and set $d^\star = S^\star_0$. It is easy to see that $\{S(z), S^\star(w)\} = 1$. Hence $dd^\star + d^\star d = 1$, and we already knew that $d^2 = 0$. Thus the following long sequence of vector spaces is exact:

$$0 \to \ker_{F_{j_0,j_1,j_1,l}} d \to F_{j_0,j_1,j_1,l} \to F_{j_0,j_1,j_1+\frac{k_0}{2},l+\frac{k_0+2}{2}} \to F_{j_0,j_1,j_1+k_0,l+k_0+2} \to \cdots .$$

Taking the direct sum we obtain Theorem 4.2. □

Since $a_0(n)+a_1(n)$ acts trivially we can modulo the relation and define

$$\tilde{F}_l = F_l/(a_0(n)+a_1(n);\, -n \in \mathbb{N});$$

then it is also a $T(A_1)$-module and the results in Proposition 4.1 obviously hold for the module $\tilde{F}_l$. If we further modulo $a_1(0)+a_0(0)$ we will obtain the Verma module for the affine Lie algebra generically.

Using the exact sequence we can compute the character for the module $\tilde{F}_l$ as follows.

Theorem 4.3. *The character of the $T(A_1)$-module $\tilde{F}_l$ is given by*

$$\mathrm{ch}(\tilde{F}_l) = \sum_{s=0}^{\infty} (-1)^s \frac{\sum_{\alpha\in\bar{Q}} e^{-l\frac{r}{k_0+2}+s(\frac{k_0+2}{2}r+\frac{k_0}{2}b)} e^{\alpha}}{\prod(e^{-\delta_1})\prod(e^{-\delta_b})\prod(e^{-\delta_r})}$$

where

$$\prod(x) = \prod_{m>0}(1-x^m) \quad \textit{and} \quad \bar{Q} = \frac{1}{k_0}(\mathbb{Z}\alpha_1 + \mathbb{Z}b).$$ □

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