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The kernel of Burau(4) $\otimes \mathbb{Z}_p$, the reduced Burau representation with coefficients in $\mathbb{Z}_p$ of the 4-braid group B_4, consists only of pseudo-Anosov braids.

1. Introduction

Given two pseudo-Anosov homeomorphisms with distinct invariant measured foliations, some powers of their isotopy classes generate a rank two free subgroup of the mapping class group of the surface [Long 1986]. This construction gives an example of all pseudo-Anosov subgroup of the mapping class group. A positive answer is given in [Whittlesey 2000] to the natural question of the existence of all pseudo-Anosov *normal* subgroups by showing that the Brunnian mapping classes on a sphere with at least five punctures are neither periodic nor reducible. Not every Brunnian n-braid maps to a Brunnian mapping class on an $(n+1)$-punctured sphere. One can however show that a nontrivial Brunnian n-braid should be pseudo-Anosov for $n \geq 3$, by adapting the arguments in [Whittlesey 2000].

In this note we show that the kernel of Burau(4) $\otimes \mathbb{Z}_p$, the reduced Burau representation with coefficients in $\mathbb{Z}_p$ of the 4-braid group B_4, consists only of pseudo-Anosov braids. Our result also implies that the kernel of Burau(4), if nontrivial, is all pseudo-Anosov. By [Cooper and Long 1997; 1998], Burau(4) $\otimes \mathbb{Z}_p$ for $p = 2, 3$ is not faithful. It is straightforward to check that there exist non-Brunnian braids in the kernels, hence giving new examples of all pseudo-Anosov normal subgroups of B_4 that are not contained in the example of Whittlesey.

For the proof, assume that we are given a nontrivial 4-braid that is not pseudo-Anosov. If it is periodic, it is conjugate to a rigid rotation [Brouwer 1919], whose Burau action is clearly nontrivial. If it is reducible, then in many ways it is similar to a 3-braid so that its Burau action is fairly predictable, for which case an automaton that records the polynomial degrees suffices to prove faithfulness. Our argument is similar to that of the ping-pong lemma. We construct an automaton whose states are disjoint subsets of $\mathbb{Z}_p[t, t^{-1}]^3$ and whose arrows are braid actions that map the subsets into the subsets.

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For braids with more than four strands, this approach immediately faces obstacles. Since Burau(4) $\otimes \mathbb{Z}_2$ is not faithful, the kernel of Burau(5) $\otimes \mathbb{Z}_2$ contains reducible braids. Taking other representations or taking intersection with other subgroups to get rid of such reducible braids then makes the proof more difficult.

We remark that the present result is a byproduct of working on the faithfulness question of Burau(4) [Moody 1991; 1993; Long and Paton 1993; Bigelow 1999].

2. No periodic or reducible braids

The n-braid group B_n consists of the mapping classes on the n-punctured disk. The center of B_n is the infinite cyclic group generated by the Dehn twist along the boundary. A braid is called *periodic* if some of its powers are contained in the center. A braid is called *reducible* if it is represented by a disk homeomorphism that fixes a collection of disjoint essential curves. If a braid is neither periodic nor reducible, the Nielsen–Thurston classification of surface homeomorphisms [Thurston 1988; Fathi et al. 1979] implies that it is represented by a pseudo-Anosov homeomorphism. Such a braid is called *pseudo-Anosov*. A subgroup of B_n is called *all pseudo-Anosov* if its nontrivial elements are all pseudo-Anosov.

The n-braid group B_n has the presentation

$$B_n = \left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i, & |i-j| \geq 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j, & |i-j| = 1 \end{array} \right\rangle$$

The reduced Burau representation

$$\rho_n = \mathrm{Burau}(n) : B_n \to \mathrm{GL}_{n-1}(\mathbb{Z}[t, t^{-1}])$$

is defined by the action on the first homology of the cyclic cover of the punctured disk. For the purpose of this note, it suffices to define ρ_4 by the three matrices

$$\rho_4(\sigma_1) = \begin{pmatrix} -t & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho_4(\sigma_2) = \begin{pmatrix} 1 & t & 0 \\ 0 & -t & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad \rho_4(\sigma_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & -t \end{pmatrix}.$$

We use the convention that B_4 acts on $\mathbb{Z}[t, t^{-1}]^3$ from the right. We denote by $\boldsymbol{v} *_\rho \beta$, or more simply by $\boldsymbol{v} * \beta$, the matrix multiplication $\boldsymbol{v}\rho(\beta)$ for a row vector $\boldsymbol{v}$, a representation ρ and a braid β. For example, $(f, g, h) *_{\rho_4} \sigma_1 = (-tf + g, g, h)$ for $f, g, h \in \mathbb{Z}[t, t^{-1}]$.

Theorem 1. *The kernel of $(\rho_4 \otimes \mathbb{Z}_p) : B_4 \to \mathrm{GL}_3(\mathbb{Z}_p[t, t^{-1}])$ for $p \geq 2$ does not contain a nontrivial periodic or reducible braid. In particular if $\rho_4 \otimes \mathbb{Z}_p$ is not faithful, its kernel is an all pseudo-Anosov normal subgroup of B_4.*

The proof will involve several lemmas.

Lemma 2. *$\rho_n \otimes \mathbb{Z}_p$ is faithful for periodic braids.*

Proof. If $\beta\in B_n$ is a periodic n-braid, then it is represented by a rigid rotation on the punctured disk [Brouwer 1919] so that it is conjugate to $(\sigma_{n-1}\cdots\sigma_2\sigma_1)^k$ or to $(\sigma_{n-1}\cdots\sigma_2\sigma_1\sigma_1)^k$ for some $k\in\mathbb{Z}$. Since $\det\big((\rho_n\otimes\mathbb{Z}_p)(\beta)\big)=(-t)^{e(\beta)}$, where the exponent sum $e(\beta)$ is $k(n-1)$ or kn, we see that if β is in the kernel of $\rho_n\otimes\mathbb{Z}_p$, then $k=0$ and β is trivial. $\square$

Let $\Delta_3=\sigma_1\sigma_2\sigma_1\in B_3$ and $\Delta_4=\sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1\in B_4$ be the square roots of the generator of the center of B_3 and B_4, respectively. For a Laurent polynomial $f(t)=\sum_m a_m t^m$, define $\deg f=\max\{m : a_m\neq 0\}$. By convention we define $\deg f=-\infty$ if $f=0$.

Lemma 3. *$\rho_3\otimes\mathbb{Z}_p$ is faithful.*

Proof. Let $\rho=\rho_3\otimes\mathbb{Z}_p$ be the reduced Burau representation of B_3 with coefficients in $\mathbb{Z}_p$. It is given by the matrices

$$\rho(\sigma_1)=\begin{pmatrix}-t & 0\\ 1 & 1\end{pmatrix},\qquad \rho(\sigma_2)=\begin{pmatrix}1 & t\\ 0 & -t\end{pmatrix}.$$

Suppose that $\rho(\beta)$ is trivial for some nontrivial 3-braid β. By Lemma 2, it is either reducible or pseudo-Anosov. If β is reducible, it is conjugate to $\Delta_3^{2m}\sigma_1^k$ for some integers k and m, which is an arbitrary 3-braid with an invariant curve standardly embedded in the disk enclosing the first two punctures as in Figure 1, right. Since $\rho(\beta)$ is trivial,

$$\rho(\Delta_3^{2m}\sigma_1^k)=t^{3m}\begin{pmatrix}(-t)^k & 0\\ * & 1\end{pmatrix}$$

must be the identity matrix. So $m=0$ and $k=0$ hence β is trivial, which contradicts the assumption.

If β is pseudo-Anosov, it is conjugate to $P(\sigma_1^{-1},\sigma_2)\Delta_3^{2k}$ where P is a positive word on two letters [Murasugi 1974; Song et al. 2002]. By taking inverse or conjugation by Δ_3 if necessary, we can assume that $P(\sigma_1^{-1},\sigma_2)$ starts with σ_2. In other words, β or β^{-1} is conjugate to $\alpha=\sigma_2 Q(\sigma_1^{-1},\sigma_2)\Delta_3^{2k}$ for some positive word Q. The ρ-actions of σ_1^{-1}, σ_2 and Δ_3^2 on $\mathbb{Z}_p[t,t^{-1}]^2$ are given as follows: for $\boldsymbol{v}=(f,g)\in\mathbb{Z}_p[t,t^{-1}]^2$,

$$\boldsymbol{v}*\sigma_1^{-1}=(-t^{-1}(f-g),g),\quad \boldsymbol{v}*\sigma_2=(f,t(f-g))\quad\text{and}\quad \boldsymbol{v}*\Delta_3^2=(t^3f,t^3g).$$

Consider the subset $V_0=\big\{(f,g)\in\mathbb{Z}_p[t,t^{-1}]^2 \mid \deg f<\deg g\big\}$. It is easy to check that V_0 is invariant under the action of σ_1^{-1}, σ_2 and Δ_3^2. Let $\boldsymbol{v}_0=(1,0)$. Then $\boldsymbol{v}_0*\sigma_2=(1,t)\in V_0$, so that $\boldsymbol{v}_0*\alpha=(1,t)*Q(\sigma_1^{-1},\sigma_2)\Delta_3^{2k}\in V_0$. Since $\boldsymbol{v}_0\notin V_0$, we have $\boldsymbol{v}_0*\alpha\neq\boldsymbol{v}_0$, which contradicts the assumption that β is in the kernel of ρ. $\square$

Figure 1. Up to homeomorphisms on a 4-punctured disk, there are only two essential curves.

Proof of Theorem 1. Let $\rho = \rho_4 \otimes \mathbb{Z}_p$ be the reduced Burau representation of B_4 with coefficients in $\mathbb{Z}_p$. Assume $\rho(\beta)$ is trivial for some nontrivial 4-braid $\beta \in B_4$. The braid β is either reducible or pseudo-Anosov by Lemma 2. We need to show that β is not reducible.

Suppose that β is reducible. By taking some power of β if necessary, we may assume that β is represented by a homeomorphism that fixes an essential simple closed curve C. By applying a conjugation by a braid that sends C to one of the curves in Figure 1, we assume that C is one of the two standardly embedded curves and the homeomorphism representing β fixes C.

Let C be the curve enclosing the first three punctures as Figure 1, left. Then β can be written as $\beta = \Delta_4^{2m} W(\sigma_1, \sigma_2)$ for an integer m and a word W on two letters. Observing that the ρ-action by a 3-braid leaves the third coordinate invariant, i.e., $(f, g, h) * W(\sigma_1, \sigma_2) = (f_1, g_1, h)$, we have $(0, 0, 1) * \beta = (f, g, t^{4m})$ for some $f, g \in \mathbb{Z}_p[t, t^{-1}]$. Since $\rho(\beta)$ is trivial, we obtain $m = 0$, which in turn implies that β is in $\langle \sigma_1, \sigma_2 \rangle = B_3 \subset B_4$. The faithfulness of $\rho_3 \otimes \mathbb{Z}_p$ by Lemma 3 leads to a contradiction.

Now assume that C contains the first two punctures as Figure 1, right. The 4-braids represented by homeomorphisms that fix C form a subgroup of B_4 generated by σ_1, $x = \sigma_2 \sigma_1^2 \sigma_2$ and $y = \sigma_3$. Since σ_1 commutes with both x and y, we write

$$\beta = \sigma_1^k W(x, y)$$

for an integer k and a word W on two letters.

By using the relations $xyxy = yxyx$, $(xyxy)\sigma_1^2 = \Delta_4^2$ and that $xyxy$ commutes with x, y and σ_1, we rewrite β into another form by which we will track $(0, 0, 1) * \beta$.

By replacing x^{-1} with $(yxy)(xyxy)^{-1}$ and y^{-1} with $(xyx)(xyxy)^{-1}$ and then collecting $(xyxy)^{\pm 1}$ to the left, we have $W(x, y) = (xyxy)^m P(x, y)$ for some integer m and a positive word P on two letters. We can assume that we have moved $(xyxy)$ to the left as many as possible so that neither $xyxy$ nor $yxyx$ occurs in P as a subword. We have

$$\beta = \sigma_1^k (xyxy)^m P(x, y) = \Delta_4^{2m} \sigma_1^{k-2m} P(x, y).$$

We claim that P contains both x and y as a subword. If P does not contain y, i.e., $P = x^l$ for some $l \geq 0$, then $\beta = \Delta_4^{2m} \sigma_1^{k-2m} x^l = \Delta_4^{2m} \sigma_1^{k-2m} (\sigma_2 \sigma_1^2 \sigma_2)^l$ fixes the curve in Figure 1, left. By the previous argument β is trivial. If P does not contain x, i.e., $P = y^l$ for some $l \geq 0$, then $\beta = \Delta_4^{2m}\sigma_1^{k-2m}y^l$. From the equalities

$$(0, 0, 1) * \beta = (0, 0, (-t)^{4m+l}), \qquad (1, 0, 0) * \beta = ((-t)^{4m+(k-2m)}, 0, 0),$$

we deduce $l = -4m$ and $k = -2m$. The exponent sum $e(\beta) = 12m + (k-2m) + l = 4m$ should equal zero since $\rho(\beta)$ is trivial. Therefore we have $m = l = k = 0$, which implies that β is trivial.

Next, since x and y both commute with σ_1 and Δ_4^2, by applying a conjugation we may assume that P starts with y and ends with x. In Figure 2, left, we construct an automaton that accepts a positive word in x, y without any occurrence of $xyxy$ and $yxyx$. Arbitrary paths following the arrows give words accepted by the automaton. Now we have

$$\beta = \Delta_4^{2m}\sigma_1^{k-2m}Q(x, y, xy, yx, yxy, xyx)$$

for some positive word Q accepted by the automaton in Figure 2, left. Note that Q starts with one of y, yxy, yx and ends with one of x, xyx, yx. In other words, Q is represented by a path starting at the state Y and ending at the state X.

We replace xyx by $y^{-1}(xyxy)$, yxy by $x^{-1}(xyxy)$ and then collect all $(xyxy)$'s to the left to obtain

$$\beta = \Delta_4^{2m_1}\sigma_1^{k_1}Q(x, y, xy, yx, x^{-1}, y^{-1})$$

for some k_1 and m_1.

Consider the subsets of $\mathbb{Z}_p[t, t^{-1}]^3$ given by

$$V_X = \{(f, g, h) \in \mathbb{Z}_p[t, t^{-1}]^3 \mid \deg g > \deg f,\ \deg g \geq \deg h\},$$
$$V_Y = \{(f, g, h) \in \mathbb{Z}_p[t, t^{-1}]^3 \mid \deg h > \deg f,\ \deg h > \deg g\}.$$

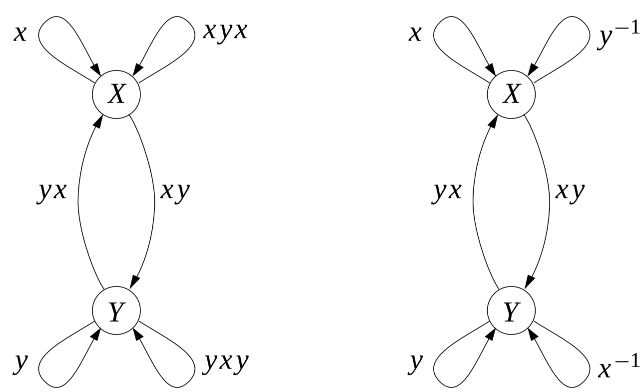

Figure 2. Left: an automaton that accepts exactly those words not containing $xyxy$ or $yxyx$. Right: see next page.

The ρ-action of each arrow of the automaton in Figure 2, right, is given as follows. Let $\boldsymbol{v} = (f, g, h) \in \mathbb{Z}_p[t, t^{-1}]^3$ be an arbitrary vector.

$$\begin{aligned}
\boldsymbol{v} * x &= \big(tf + (t^2 - t)g + (1-t)h,\ t^3 g + (1-t^2)h,\ h\big),\\
\boldsymbol{v} * y &= \big(f,\ g,\ tg - th\big),\\
\boldsymbol{v} * (xy) &= \big(tf + (t^2 - t)g + (1-t)h,\ t^3 g + (1-t^2)h,\ t^4 g - t^3 h\big),\\
\boldsymbol{v} * (yx) &= \big(tf + (t^2 - t)h,\ tg + (t^3 - t)h,\ tg - th\big),\\
\boldsymbol{v} * x^{-1} &= \big(t^{-1} f + (t^{-3} - t^{-2})g + (t^{-2} - t^{-3})h,\ t^{-3} g + (t^{-2} - t^{-3})h,\ h\big),\\
\boldsymbol{v} * y^{-1} &= \big(f,\ g,\ g - t^{-1} h\big).
\end{aligned}$$

Then it is routine to check from these formulae that

$$\begin{aligned}
&V_X * x \subset V_X, \quad V_X * y^{-1} \subset V_X, \quad V_X * (xy) \subset V_Y,\\
&V_Y * y \subset V_Y, \quad V_Y * x^{-1} \subset V_Y, \quad V_Y * (yx) \subset V_X.
\end{aligned}$$

These relations are compatible with the automaton in Figure 2, right. If a path starts at Y and ends at X then the ρ-action of its braid word maps V_Y into V_X. So we have $V_Y * Q \subset V_X$ for $Q = Q(x, y, xy, yx, x^{-1}, y^{-1})$.

Since $(0, 0, t^{4m_1}) \in V_Y$, we have

$$\begin{aligned}
(0, 0, 1) * \beta &= (0, 0, 1) * \Delta_4^{2m_1} \sigma_1^{k_1} Q\\
&= (0, 0, t^{4m_1}) * \sigma_1^{k_1} Q\\
&= (0, 0, t^{4m_1}) * Q,
\end{aligned}$$

which lies in V_X. Since $(0, 0, 1) \in V_Y$ and $V_X \cap V_Y = \varnothing$, the condition $(0, 0, 1) * \beta \in V_X$ implies that $\rho(\beta)$ is nontrivial. □

We remark that the group generated by x and y is the Artin group of Coxeter type B_2 and that $xyxy = yxyx$ is the defining relation of the subgroup generated by x and y. So the subgroup generated by x, y and σ_1 is the direct product of the infinite cyclic subgroup generated by σ_1 and the subgroup generated by x and y.

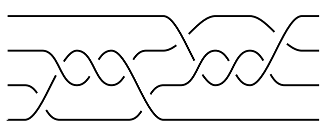

Figure 3. The braid $\sigma_1^{-1}\sigma_2^3\sigma_1\sigma_3\sigma_2^{-3}\sigma_3^{-1}$, whose fourth power is in the kernel of $\mathrm{Burau}(4) \otimes \mathbb{Z}_2$.

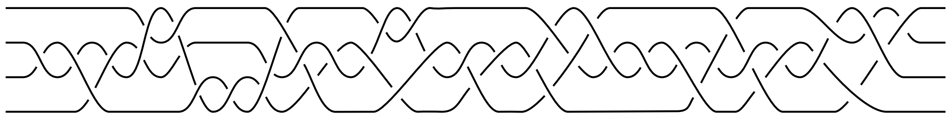

Figure 4. A braid in the kernel of Burau(4) $\otimes \mathbb{Z}_3$.

3. Non-Brunnian elements

Cooper and Long [1997] obtained a presentation of the image of $\rho_4 \otimes \mathbb{Z}_2$. As a corollary, $\rho_4 \otimes \mathbb{Z}_2$ is not faithful. The same authors computed in [Cooper and Long 1998] a presentation of a group containing the image of $\rho_4 \otimes \mathbb{Z}_3$ as a finite index subgroup and gave a nontrivial braid in the kernel explicitly. In this section we show that the examples of Cooper and Long are not Brunnian.

Let $\alpha_k = (\sigma_1^{-1}\sigma_2^k\sigma_1\sigma_3\sigma_2^{-k}\sigma_3^{-1})^4$ for $k \neq 0$. (See Figure 3 for the expression in parentheses, with $k = 3$.) The braid α_k comes from the fourth relation of [Cooper and Long 1997, Theorem 1.4] and is in the kernel of $\beta_4 \otimes \mathbb{Z}_2$. α_k is not Brunnian because we obtain σ_1^{4k} by forgetting the second and the fourth strands.

Now let α be the braid

$$\sigma_2^2\sigma_1\sigma_2^{-2}\sigma_3^{-2}\sigma_2\sigma_1^{-3}\sigma_2^{-1}\sigma_3\sigma_2^{-1}\sigma_1\sigma_2^2\sigma_3^{-2}\sigma_1^{-1}\sigma_2^{-2} \cdot \sigma_1\sigma_2^{-2}\sigma_1\sigma_3\sigma_2^{-1}\sigma_3\sigma_2^3\sigma_1\sigma_2^{-1}\sigma_3\sigma_2^{-1}\sigma_1\sigma_2^{-2}\sigma_1\sigma_3^2\sigma_2\sigma_3^{-1},$$

as in Figure 4. It is conjugate to the braid given by [Cooper and Long 1998] as a nontrivial element of $\ker \mathrm{Burau}(4) \otimes \mathbb{Z}_3$. It is easy to see that α is not Brunnian. If we forget the fourth strand from α as Figure 5, we get a nontrivial 3-braid

$$\begin{aligned}\alpha' &= \sigma_2^2\sigma_1\sigma_2^{-1}\sigma_1^{-3}\sigma_2^{-1}\sigma_1^2\sigma_2^{-2}\sigma_1^{-2}\sigma_2^3\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2^2 \\ &= (\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2^2)^3\Delta_3^{-2}.\end{aligned}$$

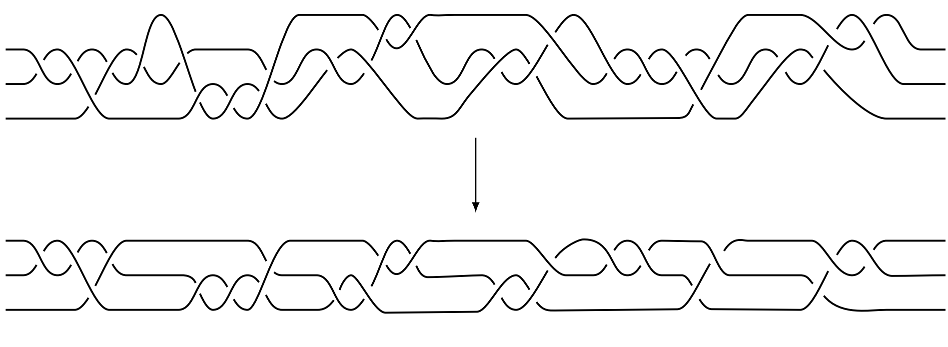

Figure 5. Forgetting the fourth strand.

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