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INDECOMPOSABILITY OF FREE GROUP FACTORS OVER NONPRIME SUBFACTORS AND ABELIAN SUBALGEBRAS

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We use the free entropy defined by D. Voiculescu to prove that the free group factors cannot be decomposed as closed linear spans of noncommutative monomials in elements of nonprime subfactors or abelian $*$-subalgebras, if the degrees of monomials have an upper bound depending on the number of generators. The resulting estimates for the hyperfinite and abelian dimensions of free group factors settle in the affirmative a conjecture of L. Ge and S. Popa (for infinitely many generators).

1. Introduction

L. Ge and S. Popa [1998] defined for a given type II_1-factor $\mathcal{M}$ the two quantities

$$\ell_h(\mathcal{M}) = \min\{ f \in \mathbb{N} \mid \exists \text{ hyperfinite } \mathcal{R}_1, \ldots, \mathcal{R}_f \subset \mathcal{M} \text{ s.t. } \overline{\mathrm{sp}}^w \mathcal{R}_1\mathcal{R}_2 \cdots \mathcal{R}_f = \mathcal{M}\},$$

$$\ell_a(\mathcal{M}) = \min\{ f \in \mathbb{N} \mid \exists \text{ abelian } \mathcal{A}_1, \ldots, \mathcal{A}_f \subset \mathcal{M} \text{ s.t. } \overline{\mathrm{sp}}^w \mathcal{A}_1\mathcal{A}_2 \cdots \mathcal{A}_f = \mathcal{M}\}$$

(the min being ∞ if $\mathcal{M}$ cannot be generated as stated) and conjectured that

$$\ell_h(\mathcal{L}(\mathbb{F}_n)) = \ell_a(\mathcal{L}(\mathbb{F}_n)) = \infty \quad \text{for } n \geq 2,$$

where $\mathcal{L}(\mathbb{F}_n)$ is the type II_1-factor associated to the free group with n generators.

We use the concept of free entropy introduced by D. Voiculescu in his breakthrough paper [1994] to prove that the conjecture mentioned above is true at least partially (for $n = \infty$) that is, $\ell_h(\mathcal{L}(\mathbb{F}_n)), \ell_a(\mathcal{L}(\mathbb{F}_n)) \geq \left[\frac{n-2}{2}\right] + 1$ for $4 \leq n \leq \infty$. Actually, our result is more general and it states that the free group factor with n generators cannot be asymptotically generated (Definitions 3.2 and 4.2) as

$$\lim_{\omega \to 0}{}^{\|\cdot\|_2} \sum_{\substack{1 \leq j_1, \ldots, j_{t+1} \leq f \\ 1 \leq t \leq d}} \mathcal{N}^\omega_{j_1} \mathfrak{X}^\omega \mathcal{N}^\omega_{j_2} \mathfrak{X}^\omega \cdots \mathcal{N}^\omega_{j_t} \mathfrak{X}^\omega \mathcal{N}^\omega_{j_{t+1}}$$

or

$$\lim_{\omega \to 0}{}^{\|\cdot\|_2} \sum_{\substack{1 \leq j_1, \ldots, j_{t+1} \leq f \\ 1 \leq t \leq d}} \mathcal{A}^\omega_{j_1} \mathfrak{X}^\omega \mathcal{A}^\omega_{j_2} \mathfrak{X}^\omega \cdots \mathcal{A}^\omega_{j_t} \mathfrak{X}^\omega \mathcal{A}^\omega_{j_{t+1}}$$

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if the $\mathcal{N}_1^\omega, \ldots, \mathcal{N}_f^\omega$ (for all ω) are nonprime subfactors, the $\mathcal{A}_1^\omega, \ldots, \mathcal{A}_f^\omega$ are abelian $*$-subalgebras, the $\mathcal{X}^\omega \subset \mathcal{L}(\mathbb{F}_n)$ are subsets containing p self-adjoint elements, and $f, d \geq 1$ are integers such that $n \geq p + 2f + 1$. Note that $\mathcal{L}(\mathbb{F}_n)$ admits decompositions of this sort if we allow $d = \infty$, for example if $\mathcal{X}^\omega = \mathcal{X} = \{1\}$, $f = n$, $\mathcal{N}_1^\omega = \mathcal{N}_1, \ldots, \mathcal{N}_n^\omega = \mathcal{N}_n$ are n distinct copies of the hyperfinite type II_1-factor $\mathcal{R}$ and $\mathcal{A}_1^\omega = \mathcal{A}_1, \ldots, \mathcal{A}_n^\omega = \mathcal{A}_n$ are n distinct copies of $L^\infty([0,1])$ (since $\mathcal{L}(\mathbb{F}_n)$ is both the free product of n copies of $\mathcal{R}$ and the free product of n copies of $L^\infty([0,1])$; see [Voiculescu et al. 1992]). The indecomposability of $\mathcal{L}(\mathbb{F}_n)$ as $\overline{\mathrm{sp}}^w \mathcal{N}\mathcal{X}\mathcal{N}$ implies the primeness of its finite-index subfactors; more generally, all subfactors of finite index in the interpolated free group factors of Dykema [1994] and Rădulescu [1994] are prime [Ştefan 1998]. Indeed, according to V. Jones [1983], if $\mathcal{N}$ is a subfactor of finite index in $\mathcal{M}$ then $\mathcal{M}$ decomposes as $\mathcal{N}e\mathcal{N}$, where e is the Jones projection. In particular, the indecomposability properties of $\mathcal{L}(\mathbb{F}_n)$ over nonprime subfactors and abelian subalgebras are preserved to its subfactors of finite index. Recall that the Haagerup approximation property [Haagerup 1978/79] is another property preserved to the free group subfactors. A first example of a prime II_1-factor (with a nonseparable predual, though) was given by Popa [1983] and then Ge [1998] proved (with a free entropy estimate) that the free group factor $\mathcal{L}(\mathbb{F}_n)$ is prime for all n with $2 \leq n < \infty$, thus answering a question from [Popa 1995].

Our results are based on estimates of free entropy, that is, estimates of volumes of various sets of matrix approximants (matricial microstates). Voiculescu [1996] pioneered this technique in his proof of the absence of Cartan subalgebras in the free group factors. Subsequently, Ge [1997] and Dykema [1997] were able to prove that the free group factors do not have abelian subalgebras of finite multiplicity.

The paper has four parts. In Section 2 we prove the first estimate of free entropy and recover a result of Voiculescu [1994]: if a free family of m self-adjoint noncommutative random variables can be generated by noncommutative power series by another family of n self-adjoint noncommutative random variables, then $n \geq m$ (Theorem 2.3). However, we show that the assumption of freeness from [Voiculescu 1994] is not essential and can be dropped. As a consequence, the number of self-adjoint generators with finite entropy that generate a $*$-algebra $\mathcal{A}$ *algebraically*, is constant. In Section 3 we prove the indecomposability of $\mathcal{L}(\mathbb{F}_n)$ (and of its subfactors of finite index) over nonprime subfactors (Theorem 3.5), and in Section 4 the indecomposability over abelian subalgebras (Theorem 4.4).

We give next a short account of Voiculescu's free probability theory [Voiculescu 1990; Voiculescu et al. 1992] and of his original concept of free entropy [Voiculescu 1994; 1996]. A type II_1-factor $\mathcal{M}$ endowed with its unique normalized, faithful, normal trace τ is sometimes called a W^*-probability space. The trace τ determines the 2-norm on $\mathcal{M}$ by the formula $\|x\|_2 = \tau(x^*x)^{1/2}$, for all $x \in M$, and the completion of $\mathcal{M}$ with respect to $\|\cdot\|_2$ is denoted $L^2(\mathcal{M}, \tau)$. An element $x \in \mathcal{M}$ is a semicircular element if it is self-adjoint and if its distribution is given by the semicircle law:

$$\tau(x^k) = \frac{2}{\pi}\int_{-1}^{1} t^k\sqrt{1-t^2}\,dt \quad \text{for all } k \in \mathbb{N}.$$

A family $(\mathcal{A}_i)_{i\in I}$ of unital $*$-subalgebras of $\mathcal{M}$ is a free family if the conditions $n \in \mathbb{N}$, $i_1, \dots, i_n \in I$, $i_1 \neq i_2 \neq \cdots \neq i_n$, $x_k \in \mathcal{A}_{i_k}$ and $\tau(x_k) = 0$ for $1 \le k \le n$ imply $\tau(x_1x_2\cdots x_n) = 0$. A set $\{x_i\}_{i\in I} \subset \mathcal{M}$ is free if the family $(*\text{-alg}\{1, x_i\})_{i\in I}$ is free. A free set $\{x_i\}_{i\in I} \subset \mathcal{M}$ consisting of semicircular elements is called a semicircular system. If $\mathbb{F}_n$ is the free group with n generators $(2 \le n \le \infty)$ then $\mathcal{L}(\mathbb{F}_n)$ denotes the von Neumann algebra generated by the left regular representation $\lambda : \mathbb{F}_n \to \mathcal{B}(l^2(\mathbb{F}_n))$; see [Murray and von Neumann 1943]. $\mathcal{L}(\mathbb{F}_n)$ is a factor of type II_1 — the free group factor on n generators. It has a canonical trace $\tau(\,\cdot\,) = (\,\cdot\,\delta_e, \delta_e)$, where $\{\delta_g\}_{g\in\mathbb{F}_n}$ is the standard orthonormal basis in $l^2(\mathbb{F}_n)$. Every $\mathcal{L}(\mathbb{F}_n)$ is generated as a von Neumann algebra by a semicircular system with n elements [Voiculescu et al. 1992]. We denote by $\mathcal{M}_k^{\mathrm{sa}} = \mathcal{M}_k^{\mathrm{sa}}(\mathbb{C})$ the set of $k \times k$ self-adjoint complex matrices and by τ_k its unique normalized trace. τ_k induces the 2-norm $\|\cdot\|_2 : \mathcal{M}_k^{\mathrm{sa}} \to \mathbb{R}_+$ and the euclidean norm $\|\cdot\|_e := \sqrt{k}\,\|\cdot\|_2$. If B is a measurable subset of an m-dimensional (real) manifold, $\mathrm{vol}_m(B)$ denotes the Lebesgue measure of B. The free entropy $\chi(x_1, \dots, x_n)$ of a finite family of self-adjoint elements was introduced in [Voiculescu 1994], but we will recall the definition of the modified free entropy [Voiculescu 1996], which is better suited for applications. For self-adjoint elements $x_1, \dots, x_{n+m} \in \mathcal{M}$ one defines first the set of matricial microstates: Fixing $R, \epsilon > 0$ and $p, k \in \mathbb{N}$ we define $\Gamma_R(x_1, \dots, x_n : x_{n+1}, \dots, x_{n+m}; p, k, \epsilon)$ to be the set

$$\begin{aligned}\Big\{(A_1, \dots, A_n) \in (\mathcal{M}_k^{\mathrm{sa}})^n \;\Big|\; &\text{there exist } A_{n+1}, \dots, A_{n+m} \in \mathcal{M}_k^{\mathrm{sa}}\\ &\text{such that } \|A_j\| \le R \text{ and } \big|\tau(x_{i_1}\cdots x_{i_q}) - \tau_k(A_{i_1}\cdots A_{i_q})\big| < \epsilon\\ &\text{for all } q = 1, \dots, p \text{ and all } j, i_1, \dots, i_q \in \{1, \dots, n+m\}\Big\}.\end{aligned}$$

Next we define

$$\chi_R(x_1, \dots, x_n : x_{n+1}, \dots, x_{n+m}; p, k, \epsilon) := \log\big(\mathrm{vol}_{nk^2}(\Gamma_R(x_1, \dots, x_n : x_{n+1}, \dots, x_{n+m}; p, k, \epsilon))\big),$$

$$\chi_R(x_1, \dots, x_n : x_{n+1}, \dots, x_{n+m}; p, \epsilon) := \limsup_{k\to\infty}\Big(\frac{1}{k^2}\chi_R(x_1, \dots, x_n : x_{n+1}, \dots, x_{n+m}; p, k, \epsilon) + \frac{n}{2}\log k\Big),$$

$$\chi_R(x_1, \dots, x_n : x_{n+1}, \dots, x_{n+m}) := \inf\big\{\chi_R(x_1, \dots, x_n : x_{n+1}, \dots, x_{n+m}; p, \epsilon) \;\big|\; p \in \mathbb{N}, \epsilon > 0\big\},$$

$$\chi(x_1, \dots, x_n : x_{n+1}, \dots, x_{n+m}) := \sup\big\{\chi_R(x_1, \dots, x_n : x_{n+1}, \dots, x_{n+m}) \;\big|\; R > 0\big\}.$$

When taking the last sup it suffices to assume $0 < R \leq \max\{\|x_1\|, \ldots, \|x_{n+m}\|\}$ rather than $0 < R < \infty$ [Voiculescu 1994; 1996]. The quantity

$$\chi(x_1, \ldots, x_n : x_{n+1}, \ldots, x_{n+m})$$

is the free entropy of $x_1, \ldots, x_n$ in the presence of $x_{n+1}, \ldots, x_{n+m}$. If $m = 0$, it is simply called the free entropy of $x_1, \ldots, x_n$ and written $\chi(x_1, \ldots, x_n)$. If $\{x_{n+1}, \ldots, x_{n+m}\} \subset \{x_1, \ldots, x_n\}''$ we have

$$\chi(x_1, \ldots, x_n : x_{n+1}, \ldots, x_{n+m}) = \chi(x_1, \ldots, x_n);$$

see [Voiculescu 1996]. For a single self-adjoint element $x = x^* \in \mathcal{M}$ one has:

$$\chi(x) = \tfrac{3}{4} + \tfrac{1}{2}\log 2\pi + \iint \log|s - t|\, d\mu(s)\, d\mu(t),$$

where μ is the distribution of x; see [Voiculescu 1994]. If $x_1, \ldots, x_n$ are n self-adjoint free elements of $\mathcal{M}$ then $\chi(x_1, \ldots, x_n) = \chi(x_1) + \cdots + \chi(x_n)$ [Voiculescu 1994]. The converse is also true [Voiculescu 1997], provided that $\chi(x_i) > -\infty$ for $1 \leq i \leq n$. In particular, the free entropy of a finite semicircular system is finite; hence the free group factor $\mathcal{L}(\mathbb{F}_n)$ has a system of generators with finite free entropy for $2 \leq n < \infty$.

2. Noncommutative power series and free entropy

The main result of this section is that if a (not necessarily free) family of m self-adjoint noncommutative random variables with finite free entropy can be generated as noncommutative power series by another family of n self-adjoint noncommutative random variables, then $n \geq m$. In other words, a finite system with finite free entropy has minimal cardinality among all finite systems of self-adjoint elements that are equivalent under the noncommutative analytic functional calculus. Thus, we recover Voiculescu's result [1994], with the observation that our approach does not require the assumption of freeness.

We review first a few facts concerning the theory of systems of algebraic equations [van der Waerden 1949], necessary in the proof of Lemma 2.1. If $g_1, \ldots, g_n$ are forms in n variables, there exists a polynomial (the resolvent) in their coefficients, $R(g_1, \ldots, g_n)$, with the property that $R(g_1, \ldots, g_n) = 0$ if and only if the system $g_1(\xi_1, \ldots, \xi_n) = \cdots = g_n(\xi_1, \ldots, \xi_n) = 0$ has a nontrivial solution. If $h_1, \ldots, h_{n-1}$ are $n-1$ forms in n variables and

$$h_n(u)(\xi_1, \ldots, \xi_n) := u_1\xi_1 + \cdots + u_n\xi_n,$$

then $R_u(h_1, \ldots, h_{n-1}) := R(h_1, \ldots, h_{n-1}, h_n(u))$ (the u-resolvent) is either identically 0, or a form of degree $\deg h_1 \times \cdots \times \deg h_{n-1}$ in $u = (u_1, \ldots, u_n)$. In the first case, the system $h_1 = \cdots = h_{n-1} = 0$ has infinitely many solutions

$[(\xi_1, \dots, \xi_n)] \in \mathbb{PC}^{n-1}$; in the second, all the solutions $[(\xi_1, \dots, \xi_n)] \in \mathbb{PC}^{n-1}$ are given by the factorization of $R_u(h_1, \dots, h_{n-1})$ (and thus, the system admits at most $\deg h_1 \times \dots \times \deg h_{n-1}$ solutions, as predicted by Bézout's Theorem).

Let $f_1, \dots, f_n \in \mathbb{R}[\Xi_1, \dots, \Xi_n]$ be n polynomials in n indeterminates, of degrees $d_1, \dots, d_n$, respectively. For $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ define

$$F_{i,a_i}(\xi_1, \dots, \xi_{n+1}) = \xi_{n+1}^{d_i} \left(f_i\Big(\frac{\xi_1}{\xi_{n+1}}, \dots, \frac{\xi_n}{\xi_{n+1}}\Big) - a_i \right) \qquad \text{for } i = 1, \dots, n.$$

Bézout's Theorem implies that the system of equations

$$f_1(\xi_1, \dots, \xi_n) = a_1, \dots, f_n(\xi_1, \dots, \xi_n) = a_n$$

admits at most $d_1 \dots d_n$ solutions $(\xi_1, \dots, \xi_n) \in \mathbb{C}^n$ if $R_u(F_{1,a_1}, \dots, F_{n,a_n}) \not\equiv 0$. Note also that the set

$$S_u(f_1, \dots, f_n) := \left\{(a_1, \dots, a_n) \in \mathbb{R}^n \mid R_u(F_{1,a_1}, \dots, F_{n,a_n}) \not\equiv 0\right\}$$

is either open and dense in $\mathbb{R}^n$, or empty.

We proceed now with Lemma 2.1, which gives an upper bound for the Lebesgue measure of the intersection of an algebraically parameterized manifold embedded in $\mathbb{R}^m$ with the unit ball of $\mathbb{R}^m$. This lemma will be of further use in estimating the volumes of various sets of matricial microstates that will appear as sets of points within a given distance from such manifolds.

Lemma 2.1. *For integers $n \leq m$ and polynomials $f_1, \dots, f_m \in \mathbb{R}[\Xi_1, \dots, \Xi_n]$ define $f = (f_1, \dots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$. If the polynomials*

$$\det\Big(\frac{\partial f_J}{\partial \xi}\Big)$$

are not identically 0 *for all multiindices $J \in \{(i_1, \dots, i_n) \mid 1 \leq i_1 < \dots < i_n \leq m\}$ and if $S_u = S_u(f_1, \dots, f_n) \neq \varnothing$, then*

$$\int_{f^{-1}\left(\overline{B(0,1)}\right)} \Big(\sum_{|J|=n} \det^2\Big(\frac{\partial f_J}{\partial \xi}\Big)\Big)^{1/2} d\xi \leq \binom{m}{n} \cdot C \cdot \mathrm{vol}_n(B(0,1)), \tag{1}$$

where $C = C(\deg f) = \max\{\deg f_{i_1} \times \dots \times \deg f_{i_n} \mid 1 \leq i_1 < \dots < i_n \leq m\}$ and $B(0,1) = B_n(0,1)$ is the unit ball in $\mathbb{R}^n$.

Proof. We consider first the case $m = n$. Let S denote the set of all irregular values of f, that is,

$$S = f\left(\{\xi \in \mathbb{R}^n \mid \mathrm{rank}(df_\xi) < n\}\right).$$

It suffices to show that (1) holds with $f^{-1}\big(\overline{B(0,1)} \setminus S_\epsilon\big)$ replacing $f^{-1}\big(\overline{B(0,1)}\big)$, where S_ϵ is an arbitrary open set that contains $S \cup (\mathbb{R}^n \setminus S_u)$. For any

$$a = (a_1, \dots, a_n) \in \mathrm{Range}\, f \cap \overline{B(0,1)} \setminus S_\epsilon$$

the set $f^{-1}(\{a\})$ has at most $C = \deg f_1 \times \cdots \times \deg f_n$ elements, say $f^{-1}(\{a\}) = \{b_1, \ldots, b_{p(a)}\}$ for some $1 \le p(a) \le C$. There exist an open ball $B_a \ni a$ and open neighborhoods $V_1^a \ni b_1, \ldots, V_{p(a)}^a \ni b_{p(a)}$ such that B_a and V_i^a are diffeomorphic via f for $1 \le i \le p(a)$ and $f^{-1}(B_a) = \bigcup_{i=1}^{p(a)} V_i^a$. Since it is compact, we can cover Range $f \cap \overline{B(0,1)} \setminus S_\epsilon$ with a finite set of such open balls $B_{a_1}, \ldots, B_{a_k}$. This covering determines a finite partition of Range $f \cap \overline{B(0,1)} \setminus S_\epsilon$, say $W_1, \ldots, W_t$. For each $1 \le j \le t$ choose a unique $1 \le l = l(j) \le k$ such that $W_j \subset B_{a_l}$ and $f^{-1}(W_j) = T_{j1} \cup \cdots \cup T_{jp(a_l)}$, where $T_{ji} \subset V_i^{a_l}$ and W_j and T_{ji} are diffeomorphic via f for all $1 \le i \le p(a_l)$. We have

$$\begin{aligned}
\int_{f^{-1}\left(\overline{B(0,1)}\setminus S_\epsilon\right)} \left|\det\left(\frac{\partial f}{\partial \xi}\right)\right| d\xi &= \sum_{j=1}^{t} \int_{f^{-1}(W_j)} \left|\det\left(\frac{\partial f}{\partial \xi}\right)\right| d\xi \\
&= \sum_{j=1}^{t} \sum_{i=1}^{p(a_{l(j)})} \int_{T_{ji}} \left|\det\left(\frac{\partial f}{\partial \xi}\right)\right| d\xi \\
&= \sum_{j=1}^{t} \sum_{i=1}^{p(a_{l(j)})} \mathrm{vol}_n(W_j) \\
&\le C \sum_{j=1}^{t} \mathrm{vol}_n(W_j) = C \cdot \mathrm{vol}_n\left(\overline{B(0,1)} \setminus S_\epsilon\right).
\end{aligned}$$

In the case $m > n$ one has the estimates

$$\begin{aligned}
\int_{f^{-1}(\overline{B(0,1)})} \left(\sum_{|J|=n} \det{}^2\left(\frac{\partial f_J}{\partial \xi}\right)\right)^{1/2} d\xi &\le \int_{f^{-1}(\overline{B(0,1)})} \sum_{|J|=n} \left|\det\left(\frac{\partial f_J}{\partial \xi}\right)\right| d\xi \\
&\le \sum_{|J|=n} \int_{f_J^{-1}(\overline{B(0,1)})} \left|\det\left(\frac{\partial f_J}{\partial \xi}\right)\right| d\xi \\
&\le \binom{m}{n} \cdot C \cdot \mathrm{vol}_n(B(0,1)). \qquad \square
\end{aligned}$$

Lemma 2.1 will be used in the proof of Proposition 2.2. The $k \times k$ matricial microstates of $x_1, \ldots, x_m$ are points within euclidean distance $2\omega\sqrt{mk}$ from the range of a polynomial function in the matricial microstates of $y_1, \ldots, y_n$ provided that each x_i is within $\|\cdot\|_2$-distance ω from noncommutative polynomials in $y_1, \ldots, y_n$.

Proposition 2.2. *Let $P_1, \ldots, P_m \in \mathbb{C}\langle Y_1, \ldots, Y_n\rangle$ be complex polynomials in n noncommutative self-adjoint variables. Assume that $(\mathcal{M}, \tau)$ is a II_1-factor and that $\{x_1, \ldots, x_m\} \subset \mathcal{M}$ is a finite set of self-adjoint generators of $\mathcal{M}$. Set*

$$a = \max\left\{\|x_1\|_2 + 1, \ldots, \|x_m\|_2 + 1\right\}$$

and $d = \max\{\deg P_1, \dots, \deg P_m\}$. *If* $\{y_1, \dots, y_n\} \subset \mathcal{M}$ *is another finite set of self-adjoint generators of* $\mathcal{M}$ *with* $n < m$ *and such that*

$$\|x_i - P_i(y_1, \dots, y_n)\|_2 < \omega \quad \textit{for all } i = 1, \dots, m \textit{ and some } \omega \in (0, a],$$

then

$$\chi(x_1, \dots, x_m) \le C(m, n, a) + (m - n)\log \omega + n \log d,$$

where $C(m, n, a)$ *is a constant that depends only on* m, n, a.

Proof. Replacing each P_i by $\frac{1}{2}(P_i + P_i^*)$ if necessary, we can assume that $P_i = P_i^*$ for $i = 1, \dots, m$. Given $R > 0$, $\epsilon > 0$ and an integer $p \ge 1$, consider

$$(A_1, \dots, A_m) \in \Gamma_R(x_1, \dots, x_m : y_1, \dots, y_n;\, p, k, \epsilon).$$

If p is large enough and $\epsilon > 0$ is sufficiently small, one can find matrices $B_1, \dots, B_n$ in $\mathcal{M}_k^{\mathrm{sa}}$ such that $\|B_1\|, \dots, \|B_n\| \le R$ and

$$\|A_i - P_i(B_1, \dots, B_n)\|_2 < \omega \quad \text{for } i = 1, \dots, m,$$

or, equivalently,

$$\|A_i - P_i(B_1, \dots, B_n)\|_e < \omega\sqrt{k} \quad \text{for } i = 1, \dots, m.$$

With the identifications $g = (g_1, \dots, g_{mk^2}) : (\mathcal{M}_k^{\mathrm{sa}})^n \cong \mathbb{R}^{nk^2} \to (\mathcal{M}_k^{\mathrm{sa}})^m \cong \mathbb{R}^{mk^2}$, $(B_1, \dots, B_n) = (\xi_1, \dots, \xi_{nk^2}) \in \mathbb{R}^{nk^2}$, and

$$g(B_1, \dots, B_n) = \big(P_1(B_1, \dots, B_n), \dots, P_m(B_1, \dots, B_n)\big),$$

the previous inequalities imply

$$\big\|(A_i)_{1 \le i \le m} - g(\xi_1, \dots, \xi_{nk^2})\big\|_e < \omega\sqrt{mk}.$$

At the cost of introducing an additional variable $\xi_{nk^2+1} \in \mathbb{R}$, we can assume that the components of g are mk^2 homogeneous polynomial functions in the variables $\xi_1, \dots, \xi_{nk^2+1}$, all having degrees at most d.

Now let $f_1, \dots, f_{mk^2}$ be arbitrary homogeneous polynomial functions in $\xi_1, \dots, \xi_{nk^2+1}$, such that $\deg f_j = \deg g_j$ for $j = 1, \dots, mk^2$. For every multiindex $J = (j_1, \dots, j_{nk^2+1})$ with $1 \le j_1 < \cdots < j_{nk^2+1} \le mk^2$, saying that $S_u(f_{j_1}, \dots, f_{j_{nk^2+1}})$ is empty is equivalent to saying that the coefficients of $f_{j_1}, \dots, f_{j_{nk^2+1}}$ satisfy a certain system of algebraic equations. Hence the set

$$\Omega_1 = \Big\{ f = (f_1, \dots, f_{mk^2}) \;\Big|\; \deg f_j = \deg g_j \text{ for } j = 1, \dots, mk^2, \\ S_u(f_{j_1}, \dots, f_{j_{nk^2+1}}) \neq \varnothing \text{ for all } J = (j_1, \dots, j_{nk^2+1}) \Big\}$$

is open and dense in its natural ambient linear space. Similarly, the set

$$\Omega_2 = \Big\{ f = (f_1, \ldots, f_{mk^2}) \;\Big|\; \deg f_j = \deg g_j \text{ for } j = 1, \ldots, mk^2, \\ \det\Big(\frac{\partial f_J}{\partial \xi}\Big) \not\equiv 0 \text{ for all } J = (j_1, \ldots, j_{nk^2+1}) \Big\}$$

is also open and dense in the same linear space.

The matrix $d f_\xi$ has $\binom{mk^2}{nk^2+1}$ minors of dimension $(nk^2+1)\times(nk^2+1)$ and all these minors have a nontrivial common zero only if a certain system of algebraic equations in the coefficients of $f_1, \ldots, f_{mk^2}$ has a solution [van der Waerden 1949]. Not all the polynomials appearing in this system are identically equal to 0. It follows that the set

$$\Omega_3 = \big\{ f = (f_1, \ldots, f_{mk^2}) \;\big|\; \deg f_j = \deg g_j \text{ for } j = 1, \ldots, mk^2, \\ \operatorname{rank}(d f_\xi) = nk^2 + 1 \ \forall \xi \in \mathbb{R}^{nk^2+1} \setminus \{0\} \big\}$$

contains a subset that is open and dense in the linear space previously considered. Therefore there exists an element $f \in \Omega_1 \cap \Omega_2 \cap \Omega_3$ such that

$$\big\| f(\xi_1, \ldots, \xi_{nk^2+1}) - g(\xi_1, \ldots, \xi_{nk^2+1}) \big\|_e < \omega\sqrt{mk} \quad \text{if } |\xi_i| \le R \text{ for } 1 \le i \le nk^2+1;$$

hence $\big\|(A_i)_{1\le i\le m} - f(\xi_1, \ldots, \xi_{nk^2+1})\big\|_e < 2\omega\sqrt{mk}$. The function f satisfies the hypotheses of Lemma 2.1 and its components are homogeneous polynomials. It has the property that $\operatorname{dist}_e\big((A_i)_{1\le i\le m}, \text{ Range } f\big) < 2\omega\sqrt{mk}$ and it does not depend on the system $(A_i)_{1\le i\le m}$.

We have $\|(A_1, \ldots, A_m)\|_e \le a\sqrt{mk}$ (if $\epsilon > 0$ is small enough); hence the set of matricial microstates $(A_1, \ldots, A_m)$ of $(x_1, \ldots, x_m)$ such that

$$\operatorname{dist}_e\big((A_1, \ldots, A_m), \text{ Range } f\big) < 2\omega\sqrt{mk}$$

is contained in the (mk^2, nk^2+1)-tube of radius $2\omega\sqrt{mk}$ around

$$\text{Range } f \cap B_{mk^2}\big(0, (a+2\omega)\sqrt{mk}\,\big).$$

If B is a small ball in $\mathbb{R}^{nk^2+1} \setminus \{0\}$ and if $V_B(2\omega\sqrt{mk})$ denotes the (mk^2, nk^2+1)-tube of radius $2\omega\sqrt{mk}$ around $f(B)$, the formula for volumes of tubes [Weyl 1939] implies

$$\operatorname{vol}_{mk^2}\big(V_B(2\omega\sqrt{mk})\big) = \operatorname{vol}_{mk^2-nk^2-1}\big(B_{mk^2-nk^2-1}(0,1)\big) \\ \cdot \sum_{\substack{e \text{ even} \\ 0\le e\le nk^2+1}} \frac{\big(2\omega\sqrt{mk}\,\big)^{e+mk^2-nk^2-1} k_{B,e}}{(mk^2-nk^2+1)(mk^2-nk^2+3)\cdots(mk^2-nk^2-1+e)}.$$

With the notations from [Weyl 1939] one has $k_{B,e} = \int_{f(B)} H_e ds$ and

$$H_e = \frac{1}{2^e (e/2)!} \sum_{\sigma \in \Sigma_e} \operatorname{sgn} \sigma \sum_{\alpha_1, \dots, \alpha_e = 1}^{nk^2+1} H_{\alpha_1 \alpha_2}^{\alpha_{\sigma(1)} \alpha_{\sigma(2)}} H_{\alpha_3 \alpha_4}^{\alpha_{\sigma(3)} \alpha_{\sigma(4)}} \dots,$$

where $H_{\alpha\beta}^{\lambda\mu}$ denotes the Riemann tensor of $f(B)$. Assuming without loss of generality that $\deg f_j = d$ for $j = 1, \dots, mk^2$, one can verify that each $H_{\alpha\beta}^{\lambda\mu}(f(\xi))$ is a sum of quotients of homogeneous polynomials where all numerators have degree $6(d-1)(nk^2+1)-2d$ and all denominators have degree $6(d-1)(nk^2+1)$. Hence H_e is a rational function in ξ and in the coefficients of $f(\xi)$. Due to its intrinsic nature, H_e is independent of the embedding of Range f in $\mathbb{R}^{mk^2+1}$; in particular it is invariant under orthogonal transformations in $\mathbb{R}^{mk^2+1}$. Since there exist sufficiently many polynomials $f(\xi)$ such that Range f is flat, this entails $H_e = 0$ for even e such that $2 \le e \le nk^2 + 1$. Therefore the volume of the (mk^2, nk^2+1)-tube of radius $2\omega\sqrt{mk}$ around $f(B)$ is

$$\operatorname{vol}_{mk^2} V_B(2\omega\sqrt{mk}) = (\operatorname{vol}_{mk^2-nk^2-1} B_{mk^2-nk^2-1}(0,1))\,(2\omega\sqrt{mk})^{mk^2-nk^2-1} \int_{f(B)} ds,$$

and with Lemma 2.1 and the inequality

$$\frac{1}{\Gamma\Bigl(1+\dfrac{nk^2+1}{2}\Bigr)} \cdot \frac{1}{\Gamma\Bigl(1+\dfrac{mk^2-nk^2-1}{2}\Bigr)} \le \frac{2^{mk^2/2}}{\Gamma\Bigl(1+\dfrac{mk^2}{2}\Bigr)} \tag{2}$$

we obtain the estimate

$$\begin{aligned}
&\operatorname{vol}_{mk^2} \Gamma_R(x_1, \dots, x_m : y_1, \dots, y_n;\, p, k, \epsilon) \\
&\quad \le \binom{mk^2}{nk^2+1} \cdot C(d) \cdot \operatorname{vol}_{nk^2+1} B(0,\, (a+2\omega)\sqrt{mk}) \\
&\qquad\qquad \cdot \operatorname{vol}_{mk^2-nk^2-1} B(0,1) \cdot (2\omega\sqrt{mk})^{mk^2-nk^2-1} \\
&\quad = \binom{mk^2}{nk^2+1} \cdot C(d) \cdot \pi^{(nk^2+1)/2} \\
&\qquad \cdot \frac{(a+2\omega)^{nk^2+1} (mk)^{(nk^2+1)/2} \pi^{(mk^2-nk^2-1)/2} (2\omega)^{mk^2-nk^2-1} (mk)^{(mk^2-nk^2-1)/2}}{\Gamma\Bigl(1+\dfrac{nk^2+1}{2}\Bigr)\Gamma\Bigl(1+\dfrac{mk^2-nk^2-1}{2}\Bigr)} \\
&\quad \le \binom{mk^2}{nk^2+1} \cdot C(d) \cdot \frac{\pi^{mk^2/2} (mk)^{mk^2/2} 2^{mk^2/2} (3a)^{nk^2+1} (2\omega)^{mk^2-nk^2-1}}{\Gamma\Bigl(1+\dfrac{mk^2}{2}\Bigr)}.
\end{aligned}$$

The last inequality implies further

$$\begin{aligned}\chi_R(x_1,\dots,x_m : y_1,\dots,y_n;\ p,k,\epsilon)\\ \le \frac{1}{k^2}\log\binom{mk^2}{nk^2+1}+\frac{1}{k^2}\log C(d)+\frac{m}{2}\log\pi+\Big(\frac{3m}{2}-n\Big)\log 2+n\log(3a)\\ +\frac{m}{2}\log(mk)+(m-n)\log\omega-\frac{1}{k^2}\log\Gamma\Big(1+\frac{mk^2}{2}\Big)+\frac{m}{2}\log k+o(1).\end{aligned}$$

Note that

$$\frac{1}{k^2}\log\Gamma\Big(1+\frac{mk^2}{2}\Big)=\frac{m}{2}\log\frac{mk^2}{2e}+o(1),$$

$C(d)\le d^{nk^2+1}$ and $\frac{1}{k^2}\log\binom{mk^2}{nk^2+1}=m\log m-n\log n-(m-n)\log(m-n)+o(1)$; therefore

$$\begin{aligned}\chi_R(x_1,\dots,x_m : y_1,\dots,y_n;\ p,k,\epsilon)\\ \le m\log m-n\log n+n\log d-(m-n)\log(m-n)+\frac{m}{2}\log\pi\\ +\Big(\frac{3m}{2}-n\Big)\log 2+n\log(3a)+\frac{m}{2}\log m+\frac{m}{2}\log k+(m-n)\log\omega\\ -\frac{m}{2}\log\frac{m}{2e}-m\log k+\frac{m}{2}\log k+o(1)\\ =C(m,n,a)+(m-n)\log\omega+n\log d+o(1).\end{aligned}$$

By taking the appropriate limits after k, p, ϵ, we finally obtain

$$\chi_R(x_1,\dots,x_m : y_1,\dots,y_n)\le C(m,n,a)+(m-n)\log\omega+n\log d,$$

and since $R>0$ is arbitrary,

$$\chi(x_1,\dots,x_m : y_1,\dots,y_n)\le C(m,n,a)+(m-n)\log\omega+n\log d.$$

Now recall that $\{x_1,\dots,x_m\}$ is a system of generators of $\mathcal{M}$; hence $\chi(x_1,\dots,x_m)=\chi(x_1,\dots,x_m : y_1,\dots,y_n)$. □

Let $Y_1,\dots,Y_n$ be noncommutative indeterminates and let

$$P(Y_1,\dots,Y_n)=\sum_{k=0}^{\infty}\sum_{1\le i_1,\dots,i_k\le n} a_{i_1\cdots i_k}Y_{i_1}\cdots Y_{i_k}$$

be a noncommutative power series in $Y_1,\dots,Y_n$, with complex coefficients. Following [Voiculescu 1994], we say that $R>0$ is a radius of convergence of P if

$$\sum_{k=0}^{\infty}\sum_{1\le i_1,\dots,i_k\le n}|a_{i_1\cdots i_k}|R^k<\infty.$$

It is well known from the theory of power series that if $0 < R_0 < R$, then

$$\sum_{k=q+1}^{\infty} \sum_{1\le i_1,\dots,i_k\le n} |a_{i_1\cdots i_k}| R_0^k = O\left(\left(\frac{R_0}{R}\right)^{q+1}\right).$$

The next result is basically [Voiculescu 1994, Corollary 6.12], with the observation that the freeness of $\{x_1, \dots, x_m\}$ assumed there has been dropped.

Theorem 2.3. *Let $x_1, \dots, x_m$ and $y_1, \dots, y_n$ be self-adjoint noncommutative random variables in a II_1-factor $(\mathcal{M}, \tau)$ such that $y_1, \dots, y_n \in \{x_1, \dots, x_m\}''$ and $\chi(x_1, \dots, x_m) > -\infty$. If $x_i = P_i(y_1, \dots, y_n)$ for $i = 1, \dots, m$, where the P_i are noncommutative power series having a common radius of convergence $R > b = \max\{\|y_1\|, \dots, \|y_n\|\}$, then $n \ge m$.*

Proof. Suppose that $m > n$. For $1 \le i \le m$, x_i is a noncommutative power series of $y_1, \dots, y_n$, i.e.,

$$x_i = \sum_{k=0}^{\infty} \sum_{1\le i_1,\dots,i_k\le n} a^{(i)}_{i_1\cdots i_k} y_{i_1} \cdots y_{i_k}.$$

For every integer $q \ge 0$, $P_{i,q}(y_1, \dots, y_n) := \sum_{k=0}^{q} \sum_{1\le i_1,\dots,i_k\le n} a^{(i)}_{i_1\cdots i_k} y_{i_1} \cdots y_{i_k}$ is a noncommutative polynomial of degree at most q, and

$$\begin{aligned}\left\|x_i - P_{i,q}(y_1, \dots, y_n)\right\|_2 &= \left\| \sum_{k=q+1}^{\infty} \sum_{1\le i_1,\dots,i_k\le n} a^{(i)}_{i_1\cdots i_k} y_{i_1} \cdots y_{i_k} \right\|_2 \\ &\le \sum_{k=q+1}^{\infty} \sum_{1\le i_1,\dots,i_k\le n} |a^{(i)}_{i_1\cdots i_k}| b^k = O\left(\left(\frac{b}{R}\right)^{q+1}\right).\end{aligned}$$

The estimate of free entropy from Proposition 2.2 implies

$$\chi(x_1, \dots, x_m) \le C(m, n, a) + (m-n)\log\left(\frac{b}{R}\right)^{q+1} + n\log q + O(1)$$

and letting q tend to ∞, one obtains $\chi(x_1, \dots, x_m) = -\infty$, a contradiction. □

Let $\mathcal{N}$ be a $*$-algebra in a W^*-probability space $(\mathcal{M}, \tau)$. Suppose that $\mathcal{N}$ is finitely generated and let $\{x_1, \dots, x_m\}$ be a system of self-adjoint generators. Let also $\{y_1, \dots, y_n\}$ be another set of self-adjoint elements that generate $\mathcal{N}$ *algebraically* as a $*$-algebra. In particular, there exist noncommutative polynomials $(P_i)_{1\le i\le m}$ such that $x_i = P_i(y_1, \dots, y_n)$ for $i = 1, \dots, \le m$. In this context, Corollary 2.4 is an immediate consequence of Theorem 2.3.

Corollary 2.4. *If $\chi(x_1, \dots, x_m) > -\infty$ and $*$-alg$\{y_1, \dots, y_n\} = *$-alg$\{x_1, \dots, x_m\}$ then $n \ge m$, so any 2 systems of self-adjoint elements with finite free entropy that generate $\mathcal{N}$ algebraically as a $*$-algebra have the same cardinality.*

Voiculescu [1998] proved that the modified free entropy dimension [Voiculescu 1996] of a finite set of self-adjoint elements that generate algebraically a $*$-algebra $\mathcal{N}$ is independent of the set of generators. It is still an open question whether the free entropy dimension is a von Neumann algebra invariant. Voiculescu [1999] also showed that sets of generators satisfying sequential commutation in certain property T factors have modified free entropy dimension ≤ 1. L. Ge and J. Shen ([2000]) proved then that the estimate $\delta_0 \leq 1$ is true for any set of generators, as long as the factor has one set of generators satisfying sequential commutation. Recall from [Voiculescu 1996] the definition of the modified free entropy dimension:

$$\delta_0(x_1, \ldots, x_m) = m + \limsup_{\omega \to 0} \frac{\chi(x_1 + \omega s_1, \ldots, x_1 + \omega s_m : s_1, \ldots, s_m)}{|\log \omega|},$$

where $\{s_1, \ldots, s_m\}$ is a semicircular system free from $\{x_1, \ldots, x_m\}$. In general one has $\delta_0(x_1, \ldots, x_m) \leq m$, and also $0 \leq \delta_0(x_1, \ldots, x_m)$ if $\{x_1, \ldots, x_m\} \subset \mathcal{L}(\mathbb{F}_p)$ for some p. Considering two sets $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_n\}$ of self-adjoint elements that generate algebraically the $*$-algebra $\mathcal{N}$ and noticing that $\{y_1, \ldots, y_n\} \subset \{x_1 + \omega s_1, \ldots, x_1 + \omega s_m, s_1, \ldots, s_m\}''$, one has

$$\begin{aligned}\delta_0(x_1, \ldots, x_m) &= m + \limsup_{\omega \to 0} \frac{\chi(x_1 + \omega s_1, \ldots, x_1 + \omega s_m : s_1, \ldots, s_m, y_1, \ldots, y_n)}{|\log \omega|} \\ &\leq m + \limsup_{\omega \to 0} \frac{\chi(x_1 + \omega s_1, \ldots, x_1 + \omega s_m : y_1, \ldots, y_n)}{|\log \omega|}.\end{aligned}$$

Also, $\left\| x_i + \omega s_i - P_i(y_1, \ldots, y_n) \right\| = \|\omega s_i\| \leq \omega$ for $i = 1, \ldots, m$, and with Proposition 2.2 we obtain

$$\begin{aligned}\delta_0(x_1, \ldots, x_m) &\leq m + \limsup_{\omega \to 0} \frac{C(m, n, a) + (m - n)\log \omega + n \log d}{|\log \omega|} \\ &\leq m + n - m = n,\end{aligned}$$

where $a = \max\{\|x_1\|_2 + 1, \ldots, \|x_m\|_2 + 1, \|y_1\|_2 + 1, \ldots, \|y_n\|_2 + 1\}$ and $d = \max\{\deg P_i \mid 1 \leq i \leq m\}$. In particular, if there exists a set $\{y_1, \ldots, y_n\}$ with $\delta_0(y_1, \ldots, y_n) = n$ which generates $\mathcal{N}$ algebraically, then

$$\sup\left\{\delta_0(x_1, \ldots, x_m) \;\middle|\; *\text{-alg}\{x_1, \ldots, x_m\} = \mathcal{N}\right\} = n.$$

3. Indecomposability over nonprime subfactors

In this section we prove that the free group factor $\mathcal{L}(\mathbb{F}_n)$ does not admit an asymptotic decomposition of the form

$$\lim_{\omega \to 0}{}^{\|\cdot\|_2} \sum_{\substack{1 \leq j_1, \ldots, j_{t+1} \leq f \\ 1 \leq t \leq d}} \mathcal{N}_{j_1}^{\omega} \mathfrak{X}^{\omega} \mathcal{N}_{j_2}^{\omega} \mathfrak{X}^{\omega} \cdots \mathcal{N}_{j_t}^{\omega} \mathfrak{X}^{\omega} \mathcal{N}_{j_{t+1}}^{\omega},$$

where (for each ω) $\mathscr{X}^\omega \subset \mathscr{L}(\mathbb{F}_n)$ is a subset with p self-adjoint elements, $\mathcal{N}_1^\omega, \dots, \mathcal{N}_f^\omega$ are nonprime subfactors of $\mathscr{L}(\mathbb{F}_n)$, the integer d is at least 1, and $n \geq p+2f+1$. A nonprime II_1-factor is just a factor isomorphic to the tensor product of two factors of type II_1. For free group subfactors one has the following: if $n \geq p+2f+2$ and $\mathscr{P} \subset \mathscr{L}(\mathbb{F}_n)$ is a subfactor of finite index, then $\mathscr{P}$ does not admit such an asymptotic decomposition either. In particular, the hyperfinite dimension of $\mathscr{L}(\mathbb{F}_n)$ is at least $\left[\frac{n-2}{2}\right]+1$ and that of $\mathscr{P}$ is at least $\left[\frac{n-3}{2}\right]+1$. For $n=\infty$ this settles a conjecture of Ge and Popa [1998]: the hyperfinite dimension of free group factors is infinite. The definitions of hyperfinite dimension and of asymptotic decomposition over nonprime subfactors are given next.

Definition 3.1 [Ge and Popa 1998]. If $\mathcal{M}$ is a type II_1-factor, the hyperfinite dimension of $\mathcal{M}$, denoted $\ell_h(\mathcal{M})$, is by definition the smallest positive integer $f \in \mathbb{N}$ with the property that there exist hyperfinite subalgebras $\mathscr{R}_1, \dots, \mathscr{R}_f \subset \mathcal{M}$ such that $\overline{\mathrm{sp}}^w \mathscr{R}_1\mathscr{R}_2 \cdots \mathscr{R}_f = \mathcal{M}$. If there is no such positive integer, $\ell_h(\mathcal{M}) = +\infty$.

Definition 3.2. A type II_1-factor $\mathcal{M}$ admits an asymptotic decomposition over nonprime subfactors if, for any $n \geq 1$, any $x_1, \dots, x_n \in \mathcal{M}$, and any $\omega > 0$, there exist nonprime subfactors $\mathcal{N}_1^\omega = \mathcal{N}_1(x_1, \dots, x_n; \omega), \dots, \mathcal{N}_f^\omega = \mathcal{N}_f(x_1, \dots, x_n; \omega)$ of $\mathcal{M}$ and also a set $\mathscr{X}^\omega = \mathscr{X}(x_1, \dots, x_n; \omega) \subset \mathcal{M}$ containing p self-adjoint elements, such that

$$\mathrm{dist}_{\|\cdot\|_2}\left(x_j, \sum_{\substack{1\leq j_1,\dots,j_{t+1}\leq f\\ 1\leq t\leq d}} \mathcal{N}_{j_1}^\omega \mathscr{X}^\omega \mathcal{N}_{j_2}^\omega \mathscr{X}^\omega \cdots \mathcal{N}_{j_t}^\omega \mathscr{X}^\omega \mathcal{N}_{j_{t+1}}^\omega\right) < \omega \quad \text{for } j = 1, \dots, n.$$

In this situation we write

$$\mathcal{M} = \lim_{\omega\to 0}{}^{\|\cdot\|_2} \sum_{\substack{1\leq j_1,\dots,j_{t+1}\leq f\\ 1\leq t\leq d}} \mathcal{N}_{j_1}^\omega \mathscr{X}^\omega \mathcal{N}_{j_2}^\omega \mathscr{X}^\omega \cdots \mathcal{N}_{j_t}^\omega \mathscr{X}^\omega \mathcal{N}_{j_{t+1}}^\omega.$$

If $\mathscr{L}(\mathbb{F}_n)$ admitted an asymptotic decomposition over nonprime subfactors as in this definition, the situation described in Proposition 3.4 (with $\mathcal{M} = \mathscr{L}(\mathbb{F}_n)$) would take place for arbitrary $\omega > 0$, since any II_1-factor is generated by its projections of given trace ($\frac{1}{2}$, for example). The following is a result from [Ge 1998, p. 155] (see also [Kadison and Ringrose 1986, Exercise 12.4.11]); we include a proof for completeness.

Lemma 3.3. *Any type* II_1*-factor* $\mathcal{M}$ *with separable predual is generated by a countable family of projections of given trace.*

Proof. Every II_1-factor with separable predual is generated by a countable family of abelian subalgebras, so there exist abelian subalgebras $\mathscr{A}_1, \mathscr{A}_2, \dots$ of $\mathcal{M}$ generating $\mathcal{M}$ as a von Neumann algebra. If necessary, one can replace each $\mathscr{A}_n$ by a maximal abelian subalgebra of $\mathcal{M}$ containing it, hence $\mathscr{A}_n$ can assume to be a maximal abelian subalgebra of $\mathcal{M}$ for $1 \le n < \infty$. Being a maximal abelian subalgebra of a type II_1-factor, each $\mathcal{A}_n$ has no atoms and thus it is generated by a countable subset of projections of given trace. $\square$

Proposition 3.4. *Let $z_1, \dots, z_p$ be self-adjoint elements of a II_1-factor $\mathcal{M}$ and let $(\mathcal{N}_v)_{1\le v\le f}$ be a family of subfactors of $\mathcal{M}$. Assume that $\mathcal{N}_v = \mathcal{R}_1^{(v)} \vee \mathcal{R}_2^{(v)} \simeq \mathcal{R}_1^{(v)} \otimes \mathcal{R}_2^{(v)}$, where $\mathcal{R}_1^{(v)}, \mathcal{R}_2^{(v)}$ are II_1-factors, and assume that $x_1, \dots, x_n$ are self-adjoint generators of $\mathcal{M}$. Assume moreover that there exist projections $p_1^{(v)}, \dots, p_{r_v}^{(v)} \in \mathcal{R}_2^{(v)}$ and $q_1^{(v)}, \dots, q_{s_v}^{(v)} \in \mathcal{R}_1^{(v)}$ of trace $\frac{1}{2}$ and complex noncommutative polynomials $(\phi_j)_{1\le j\le n}$ of degree at most d (where $d \ge 1$ is fixed) in the variables $(z_u)_{1\le u\le p}$ such that*

$$\left\| x_j - \phi_j \left((p_i^{(v)})_{\substack{1\le i\le r_v \\ 1\le v\le f}}, (q_l^{(v)})_{\substack{1\le l\le s_v \\ 1\le v\le f}}, (z_u)_{1\le u\le p} \right) \right\|_2 < \omega \quad \text{for } j = 1, \dots, n, \tag{3}$$

where $\omega \in (0, a]$ is a given positive number, and such that in all the monomials of each ϕ_j the projections $p_i^{(v)}, q_l^{(v)}$ and $p_k^{(w)}, q_s^{(w)}$ are separated by some z_u if $v \ne w$. Then

$$\chi(x_1, \dots, x_n) \le C(n, p, a, d, f) + (n - p - 2f) \log \omega, \tag{4}$$

where $a = \max\left\{ \|x_j\|_2 + 1 \;\middle|\; 1 \le j \le n \right\}$ and $C(n, p, a, d, f)$ is a constant that depends only on n, p, a, d, f.

Proof. All variables involved are self-adjoint, so we can assume that $\phi_j = \phi_j^*$ for $j = 1, \dots, n$. Fix an integer $k_0 \ge 1$ and let $R > 0$. Suppose $\mathcal{M}_{k_0}(\mathbb{C}) \cong \mathcal{M}_1^{(v)} \subset \mathcal{R}_1^{(v)}$ and $\mathcal{M}_{k_0}(\mathbb{C}) \cong \mathcal{M}_2^{(v)} \subset \mathcal{R}_2^{(v)}$, and let $\{e_{jl}^{(v)}\}_{j,l}$, $\{f_{jl}^{(v)}\}_{j,l}$ be matrix units for $\mathcal{M}_1^{(v)}$ and $\mathcal{M}_2^{(v)}$ respectively. If

$$\left((A_j)_{1\le j\le n}, (G_i^{(v)})_{\substack{1\le i\le r_v \\ 1\le v\le f}}, (H_l^{(v)})_{\substack{1\le l\le s_v \\ 1\le v\le f}}, \{E_{jl}^{(v)}\}_{j,l,v}, \{F_{jl}^{(v)}\}_{j,l,v}, (Z_u)_{1\le u\le p} \right)$$

is an arbitrary microstate in the set of matricial microstates

$$\Gamma_R\left((x_j)_{1\le j\le n}, (p_i^{(v)})_{\substack{1\le i\le r_v \\ 1\le v\le f}}, (q_l^{(v)})_{\substack{1\le l\le s_v \\ 1\le v\le f}}, \{e_{jl}^{(v)}\}_{j,l,v}, \{f_{jl}^{(v)}\}_{j,l,v}, (z_u)_{1\le u\le p}; m, k, \epsilon \right)$$

and if m is large and ϵ is small enough, then

$$\left\| A_j - \phi_j \left((G_i^{(v)})_{\substack{1\le i\le r_v \\ 1\le v\le f}}, (H_l^{(v)})_{\substack{1\le l\le s_v \\ 1\le v\le f}}, (Z_u)_{1\le u\le p} \right) \right\|_2 < \omega \quad \text{for } j = 1, \dots, n.$$

Let $\delta > 0$ and write $k = k_0^2 t + w$ for some integers w, t with $0 \le w \le k_0^2 - 1$. If m, ϵ are suitably chosen, there exist $\mathcal{M}_1^{(v)} \cong \tilde{\mathcal{M}}_1^{(v)} \subset \mathcal{M}_k(\mathbb{C})$, $\mathcal{M}_2^{(v)} \cong \tilde{\mathcal{M}}_2^{(v)} \subset \mathcal{M}_k(\mathbb{C})$ (not necessarily unital inclusions) and matrix units $\{\tilde{E}_{jl}^{(v)}\}_{j,l,v} \subset \tilde{\mathcal{M}}_1^{(v)}$, $\{\tilde{F}_{jl}^{(v)}\}_{j,l,v} \subset \tilde{\mathcal{M}}_2^{(v)}$ such that

$$\|\tilde{E}_{jl}^{(v)} - E_{jl}^{(v)}\|_2 < \delta \quad \text{and} \quad \|\tilde{F}_{jl}^{(v)} - F_{jl}^{(v)}\|_2 < \delta \quad \text{for } j, l = 1, \dots, k_0,$$

and $\tilde{\mathcal{M}}_1^{(v)} \subset \big(\tilde{\mathcal{M}}_2^{(v)}\big)' \cap \mathcal{M}_k(\mathbb{C})$. The relative commutants of $\tilde{\mathcal{M}}_1^{(v)}$ and $\tilde{\mathcal{M}}_2^{(v)}$ in $\mathcal{M}_k(\mathbb{C})$ satisfy

$$\big(\tilde{\mathcal{M}}_1^{(v)}\big)' \cap \mathcal{M}_k(\mathbb{C}) \cong \big(\mathcal{M}_{k_0}(\mathbb{C}) \otimes 1 \otimes \mathcal{M}_t(\mathbb{C})\big) \oplus \mathcal{M}_w(\mathbb{C}),$$
$$\big(\tilde{\mathcal{M}}_2^{(v)}\big)' \cap \mathcal{M}_k(\mathbb{C}) \cong \big(1 \otimes \mathcal{M}_{k_0}(\mathbb{C}) \otimes \mathcal{M}_t(\mathbb{C})\big) \oplus \mathcal{M}_w(\mathbb{C}).$$

Let

$$\eta^{(v)}\big(x, \{e_{jl}^{(v)}\}_{j,l}\big) := \frac{1}{k_0} \sum_{j,l=1}^{k_0} e_{jl}^{(v)} x e_{lj}^{(v)} \in \mathbb{C}\langle X_1, \ldots, X_{k_0^2+1}\rangle$$

be the polynomial in k_0^2+1 indeterminates that gives the conditional expectation $E_{(\mathcal{M}_1^{(v)})' \cap \mathcal{M}} : \mathcal{M} \to (\mathcal{M}_1^{(v)})' \cap \mathcal{M}$, that is,

$$E_{(\mathcal{M}_1^{(v)})' \cap \mathcal{M}}(x) = \eta^{(v)}\big(x, \{e_{jl}^{(v)}\}_{j,l}\big).$$

Then $G_1^{(v,1)} := \eta^{(v)}\big(G_1^{(v)}, \{\tilde{E}_{jl}^{(v)}\}_{j,l}\big) \in \big(\tilde{\mathcal{M}}_1^{(v)}\big)' \cap \mathcal{M}_k(\mathbb{C})$, and since

$$p_1^{(v)} = E_{(\mathcal{M}_1^{(v)})' \cap \mathcal{M}}(p_1^{(v)}) = \eta^{(v)}\big(p_1^{(v)}, \{e_{jl}^{(v)}\}_{j,l}\big),$$

it follows that

$$\Big|\tau_k\big((G_1^{(v,1)})^l\big) - \tau\big((p_1^{(v)})^l\big)\Big| < \delta_1 \quad \text{for all } l = 1, \ldots, m_1$$

for any given δ_1, m_1, provided that ϵ, δ are small and m is large enough. For suitable m_1, δ_1 there exists a projection $P_1^{(v,1)} \in \big(\tilde{\mathcal{M}}_1^{(v)}\big)' \cap \mathcal{M}_k(\mathbb{C})$ of rank $\big[\frac{k_0 t + w}{2}\big]$ such that $\big\|P_1^{(v,1)} - G_1^{(v,1)}\big\|_2 < \delta_2$. Then $\big\|G_1^{(v)} - P_1^{(v,1)}\big\|_2 \le \big\|G_1^{(v)} - G_1^{(v,1)}\big\|_2 + \big\|G_1^{(v,1)} - P_1^{(v,1)}\big\|_2 < 2\delta_2$, since $\big\|G_1^{(v)} - G_1^{(v,1)}\big\|_2 < \delta_2$ for convenient m, ϵ, δ. With this procedure we can find projections

$$P_1^{(v,1)}, \ldots, P_{r_v}^{(v,1)} \in \big(\tilde{\mathcal{M}}_1^{(v)}\big)' \cap \mathcal{M}_k(\mathbb{C}) \quad \text{and} \quad Q_1^{(v,1)}, \ldots, Q_{s_v}^{(v,1)} \in \big(\tilde{\mathcal{M}}_2^{(v)}\big)' \cap \mathcal{M}_k(\mathbb{C}),$$

all of rank $\big[\frac{k_0 t + w}{2}\big]$, such that $\big\|G_i^{(v)} - P_i^{(v,1)}\big\|_2 < 2\delta_2$ and $\big\|H_j^{(v)} - Q_j^{(v,1)}\big\|_2 < 2\delta_2$ for all indices i, j, v. Moreover,

$$\Big\|A_j - \phi_j\Big((P_i^{(v,1)})_{\substack{1\le i\le r_v\\ 1\le v\le f}}, (Q_l^{(v,1)})_{\substack{1\le l\le s_v\\ 1\le v\le f}}, (Z_u)_{1\le u\le p}\Big)\Big\|_2 < \omega \quad \text{for all } j = 1, \ldots, n$$

if we choose a sufficiently small $\delta_2 > 0$. Let $\mathcal{G}_1^{(v)}(k) \subset \big(\tilde{\mathcal{M}}_1^{(v)}\big)' \cap \mathcal{M}_k(\mathbb{C})$ and $\mathcal{G}_2^{(v)}(k) \subset \big(\tilde{\mathcal{M}}_2^{(v)}\big)' \cap \mathcal{M}_k(\mathbb{C})$ be two fixed copies of the Grassmann manifold $\mathcal{G}\big(k_0 t + w, \big[\frac{k_0 t + w}{2}\big]\big)$ of projections in $\mathcal{M}_{k_0 t + w}(\mathbb{C})$ of rank $\big[\frac{k_0 t + w}{2}\big]$. There exists a unitary $U^{(v)} \in \mathcal{U}(k)$ such that

$$U^{(v)} P_1^{(v,1)} U^{(v)*}, \ldots, U^{(v)} P_{r_v}^{(v,1)} U^{(v)*} \in \mathcal{G}_1^{(v)}(k),$$
$$U^{(v)} Q_1^{(v,1)} U^{(v)*}, \ldots, U^{(v)} Q_{s_v}^{(v,1)} U^{(v)*} \in \mathcal{G}_2^{(v)}(k).$$

The previous inequality becomes

$$\Big\| A_j - \phi_j\Big(\big(U^{(v)}P_i^{(v,1)}U^{(v)*}\big)_{\substack{1\le i\le r_v\\ 1\le v\le f}}, \big(U^{(v)}Q_l^{(v,1)}U^{(v)*}\big)_{\substack{1\le l\le s_v\\ 1\le v\le f}}, (Z_u)_{1\le u\le p}, \big(\operatorname{Re} U^{(v)}, \operatorname{Im} U^{(v)}\big)_{1\le v\le f}\Big)\Big\|_2 < \omega$$

for all $j = 1, \dots, n$. The euclidean norm on $\mathcal{M}_k^{\mathrm{sa}}$ induces a $\mathcal{U}(k_0t+w)$-invariant metric on the manifold $\mathcal{G}\Big(k_0t+w, \Big[\frac{k_0t+w}{2}\Big]\Big)$, and if $\{P_a\}_{a\in A(k)}$ is a minimal θ-net in the manifold with respect to this metric, it follows from [Szarek 1982] that $|A(k)| \le (Ch_k/\theta)^{g_k}$, where C is a universal constant, $g_k = 2\big[\frac{k_0t+w}{2}\big]\cdot\big(k_0t+w-\big[\frac{k_0t+w}{2}\big]\big)$ is the dimension of $\mathcal{G}\big(k_0t+w, \big[\frac{k_0t+w}{2}\big]\big)$ and $h_k \le \sqrt{2k}$ is the diameter of the Grassmann manifold $\mathcal{G}\big(k_0t+w, \big[\frac{k_0t+w}{2}\big]\big)$ in $\mathcal{M}_k^{\mathrm{sa}}$. There exist $\alpha := (a_1^{(v)}, \dots, a_{r_v}^{(v)})_{1\le v\le f}$ and $\beta := (b_1^{(v)}, \dots, b_{s_v}^{(v)})_{1\le v\le f}$ with entries from $A(k)$ such that

$$\big\| P_{a_i^{(v)}}^{(v)} - U^{(v)}P_i^{(v,1)}U^{(v)*}\big\|_e \le \theta \quad \text{and} \quad \big\| P_{b_l^{(v)}}^{(v)} - U^{(v)}Q_l^{(v,1)}U^{(v)*}\big\|_e \le \theta$$

for $1 \le i \le r_v,\ 1 \le l \le s_v,\ 1 \le v \le f$. In particular, the polynomials $(\phi_j)_{1\le j\le n}$ are Lipschitz functions; hence there exists a constant $D = D\big((\phi_j)_{1\le j\le n}, R\big) > 0$ (note that $|\alpha| = r_1 + \cdots + r_f$ and $|\beta| = s_1 + \cdots + s_f$) such that

$$\begin{aligned}\big\|\phi_j(V_1, \dots, V_{|\alpha|+|\beta|+p+2f}) - \phi_j(W_1, \dots, W_{|\alpha|+|\beta|+p+2f})\big\|_e \\ \le D\big\|(V_1, \dots, V_{|\alpha|+|\beta|+p+2f}) - (W_1, \dots, W_{|\alpha|+|\beta|+p+2f})\big\|_e\end{aligned}$$

for all $1 \le j \le n$ and all

$$V_1, \dots, V_{|\alpha|+|\beta|+p+2f}, W_1, \dots, W_{|\alpha|+|\beta|+p+2f} \in \{V \in M_k \mid \|V\| \le R\}.$$

We then have

$$\begin{aligned}
&\Big\| A_j - \phi_j\Big((P_a)_{a\in\alpha}, (P_b)_{b\in\beta}, (Z_u)_{1\le u\le p}, \big(\operatorname{Re} U^{(v)}, \operatorname{Im} U^{(v)}\big)_{1\le v\le f}\Big)\Big\|_e \\
&\quad < \omega\sqrt{k} + D\Big\|\Big(\big(U^{(v)}P_i^{(v,1)}U^{(v)*}\big)_{\substack{1\le i\le r_v\\ 1\le v\le f}}, \big(U^{(v)}Q_l^{(v,1)}U^{(v)*}\big)_{\substack{1\le l\le s_v\\ 1\le v\le f}}, \\
&\qquad\qquad (Z_u)_{1\le u\le p}, \big(\operatorname{Re} U^{(v)}, \operatorname{Im} U^{(v)}\big)_{1\le v\le f}\Big) \\
&\qquad\qquad - \Big((P_a)_{a\in\alpha}, (P_b)_{b\in\beta}, (Z_u)_{1\le u\le p}, \big(\operatorname{Re} U^{(v)}, \operatorname{Im} U^{(v)}\big)_{1\le v\le f}\Big)\Big\|_e \\
&\quad < \omega\sqrt{k} + D\theta\sqrt{|\alpha|+|\beta|} \\
&\quad = 2\omega\sqrt{k},
\end{aligned}$$

if we choose

$$\theta := \frac{\omega}{D}\sqrt{\frac{k}{|\alpha|+|\beta|}}.$$

Define $F_{\alpha,\beta} : (\mathcal{M}_k^{\mathrm{sa}})^{p+2f} \to (\mathcal{M}_k^{\mathrm{sa}})^n$ by

$$F_{\alpha,\beta}\Big((W_u)_{1\le u\le p},\big(W_1^{(v)},W_2^{(v)}\big)_{1\le v\le f}\Big) = \Big(\phi_j\big((P_a)_{a\in\alpha},(P_b)_{b\in\beta},(W_u)_{1\le u\le p},(W_1^{(v)},W_2^{(v)})_{1\le v\le f}\big)\Big)_{1\le j\le n},$$

and note that $\operatorname{dist}_e\big((A_j)_{1\le j\le n},\operatorname{Range}F_{\alpha,\beta}\big) < 2\omega\sqrt{nk}$. Note also that all the components of $F_{\alpha,\beta}$ are polynomial functions of degrees at most $3d+2$. Now use Lemma 2.1 as in the proof of Proposition 2.2 to obtain the estimates

$$\begin{aligned}
&\operatorname{vol}_{nk^2}\Gamma_R\Big((x_j)_{1\le j\le n}:(p_i^{(v)})_{\substack{1\le i\le r_v\\1\le v\le f}},(q_l^{(v)})_{\substack{1\le l\le s_v\\1\le v\le f}},\{e_{jl}^{(v)}\}_{j,l,v},\{f_{jl}^{(v)}\}_{j,l,v},(z_u)_{1\le u\le p};\, m,k,\epsilon\Big)\\
&\le\Big(\Big(\frac{Ch_k}{\theta}\Big)^{g_k}\Big)^{|\alpha|+|\beta|}\cdot\binom{nk^2}{(p+2f)k^2}\cdot C(d)\\
&\qquad\cdot\operatorname{vol}_{(p+2f)k^2}B\big(0,(a+2\omega)\sqrt{nk}\,\big)\cdot\operatorname{vol}_{nk^2-(p+2f)k^2}B\big(0,2\omega\sqrt{nk}\,\big)\\
&=\Big(\frac{CDh_k}{\omega}\sqrt{\frac{|\alpha|+|\beta|}{k}}\Big)^{(|\alpha|+|\beta|)g_k}\cdot\binom{nk^2}{(p+2f)k^2}\cdot C(d)\\
&\qquad\cdot\frac{(\pi nk)^{(p+2f)k^2/2}(2\omega+a)^{(p+2f)k^2}}{\Gamma\Big(1+\frac{(p+2f)k^2}{2}\Big)}\cdot\frac{(\pi nk)^{(nk^2-(p+2f)k^2)/2}(2\omega)^{nk^2-(p+2f)k^2}}{\Gamma\Big(1+\frac{nk^2-(p+2f)k^2}{2}\Big)}.
\end{aligned}$$

This estimate, inequality (2) on page 373, and the inequalities $h_k\le\sqrt{2k},\ 0<\omega\le a$,

$$\begin{aligned}
g_k&=2\Big[\frac{k_0t+w}{2}\Big]\Big(k_0t+w-\Big[\frac{k_0t+w}{2}\Big]\Big)\\
&\le 2\,\frac{k_0t+w}{2}\Big(k_0t+w-\frac{k_0t+w}{2}\Big)=\frac{(k_0t+w)^2}{2}=\frac{(k+k_0w-w)^2}{2k_0^2},
\end{aligned}$$

together with $C(d)\le(3d+2)^{(p+2f)k^2}$, imply

$$\begin{aligned}
&\operatorname{vol}_{nk^2}\Gamma_R\Big((x_j)_{1\le j\le n}:(p_i^{(v)})_{\substack{1\le i\le r_v\\1\le v\le f}},(q_l^{(v)})_{\substack{1\le l\le s_v\\1\le v\le f}},\{e_{jl}^{(v)}\}_{j,l,v},\{f_{jl}^{(v)}\}_{j,l,v},(z_u)_{1\le u\le p};\, m,k,\epsilon\Big)\\
&\le\Big(\frac{CD\sqrt{2(|\alpha|+|\beta|)}}{\omega}\Big)^{\frac{(k+k_0w-w)^2}{2k_0^2}(|\alpha|+|\beta|)}\\
&\qquad\cdot\frac{2^{nk^2/2}(\pi nk)^{nk^2/2}(3a)^{(p+2f)k^2}(2\omega)^{(n-p-2f)k^2}}{\Gamma\Big(1+\frac{nk^2}{2}\Big)}\binom{nk^2}{(p+2f)k^2}(3d+2)^{(p+2f)k^2}.
\end{aligned}$$

Therefore

$$\frac{1}{k^2}\chi_R\Big((x_j)_{1\le j\le n} : (p_i^{(v)})_{\substack{1\le i\le r_v\\ 1\le v\le f}}, (q_l^{(v)})_{\substack{1\le l\le s_v\\ 1\le v\le f}}, \{e_{jl}^{(v)}\}_{j,l,v}, \{f_{jl}^{(v)}\}_{j,l,v}, (z_u)_{1\le u\le p}; m, k, \epsilon\Big)+\frac{n}{2}\log k$$
$$\le C(n,p,a,d,f)+n\log k+\frac{|\alpha|+|\beta|}{2k_0^2}\Big(1+\frac{k_0w-w}{k}\Big)^2\log\frac{CD\sqrt{2(|\alpha|+|\beta|)}}{\omega}$$
$$+(n-p-2f)\log\omega-\frac{1}{k^2}\log\Gamma\Big(1+\frac{nk^2}{2}\Big)+\frac{1}{k^2}\log\binom{nk^2}{(p+2f)k^2}.$$

Use the asymptotics

$$\frac{1}{k^2}\log\binom{nk^2}{(p+2f)k^2} = n\log n-(p+2f)\log(p+2f)-(n-p-2f)\log(n-p-2f)+o(1)$$

and Stirling's formula

$$\frac{1}{k^2}\log\Gamma\Big(1+\frac{nk^2}{2}\Big)=\frac{n}{2}\log\frac{nk^2}{2e}+o(1)$$

to conclude that

$$\chi_R\Big((x_j)_{1\le j\le n} : (p_i^{(v)})_{\substack{1\le i\le r_v\\ 1\le v\le f}}, (q_l^{(v)})_{\substack{1\le l\le s_v\\ 1\le v\le f}}, \{e_{jl}^{(v)}\}_{j,l,v}, \{f_{jl}^{(v)}\}_{j,l,v}, (z_u)_{1\le u\le p}; m, \epsilon\Big)$$
$$\le \frac{|\alpha|+|\beta|}{2k_0^2}\log(CD\sqrt{2(|\alpha|+|\beta|)})+C(n,p,a,d,f)+\Big(n-p-2f-\frac{|\alpha|+|\beta|}{2k_0^2}\Big)\log\omega. \tag{5}$$

The last inequality shows that the free entropy of $\{x_1,\dots,x_n\}$ does not exceed $C(n,p,a,d,f)+(n-p-2f)\log\omega$, since k_0 is an arbitrary integer, R is an arbitrary positive number and $x_1,\dots,x_n$ generate M. □

3.1. *Hyperfinite dimension of free group factors.*

Theorem 3.5. *If $n\ge p+2f+1$, the free group factor $\mathcal{L}(\mathbb{F}_n)$ cannot be asymptotically decomposed as*

$$\lim_{\omega\to 0}{}^{\|\cdot\|_2}\sum_{\substack{1\le j_1,\dots,j_{t+1}\le f\\ 1\le t\le d}}\mathcal{N}_{j_1}^{\omega}\mathfrak{X}^{\omega}\mathcal{N}_{j_2}^{\omega}\mathfrak{X}^{\omega}\cdots\mathcal{N}_{j_t}^{\omega}\mathfrak{X}^{\omega}\mathcal{N}_{j_{t+1}}^{\omega},$$

where (for each ω) $\mathfrak{X}^{\omega}\subset\mathcal{L}(\mathbb{F}_n)$ contains p self-adjoint elements, $\mathcal{N}_1^{\omega},\dots,\mathcal{N}_f^{\omega}$ are nonprime subfactors of $\mathcal{L}(\mathbb{F}_n)$, and $d\ge 1$ is an integer.

Proof. Suppose first that $\infty > n \geq p + 2f + 1$ and consider a semicircular system $\{x_1, \dots, x_n\}$ that generates $\mathscr{L}(\mathbb{F}_n)$ as a von Neumann algebra. If there were a decomposition as in the theorem, one could find for every $\omega > 0$ noncommutative polynomials and projections as in Proposition 3.4 satisfying the inequalities (3). But then the estimate of the free entropy (4) would imply that $\chi(x_1, \dots, x_n) = -\infty$ as ω tends to 0, a contradiction.

If $n = \infty$ then $\mathscr{L}(\mathbb{F}_\infty)$ is generated by an infinite semicircular system $\{x_t\}_{t\geq 1}$. If we fix an integer $k \geq p + 2f + 1$, we can approximate $x_1, \dots, x_k$ by polynomials $(\phi_j)_{1\leq j\leq k}$ as in (3), getting the estimate of the modified free entropy (5) with k instead of n. Taking m, $1/\epsilon$, R, $k_0 \to \infty$ and $\omega \to 0$ in this estimate, one obtains

$$\chi\Big((x_j)_{1\leq j\leq k} : (p_i^{(v)})_{\substack{1\leq i\leq r_v \\ 1\leq v\leq f}}, (q_l^{(v)})_{\substack{1\leq l\leq s_v \\ 1\leq v\leq f}}, \{e_{jl}^{(v)}\}_{j,l,v}, \{f_{jl}^{(v)}\}_{j,l,v}, (z_u)_{1\leq u\leq p}\Big) < \chi(x_1, \dots, x_k),$$

where $(p_i^{(v)})_{1\leq i\leq r_v, 1\leq v\leq f}$, $(q_l^{(v)})_{1\leq l\leq s_v, 1\leq v\leq f}$, $\{e_{jl}^{(v)}\}_{j,l,v}$, $\{f_{jl}^{(v)}\}_{j,l,v}$, and $(z_u)_{1\leq u\leq p}$ are as in Proposition 3.4. If $\mathscr{A}_t$ denotes the von Neumann algebra $\{x_1, \dots, x_t\}''$ and E_t the conditional expectation onto it, then

$$\Big((x_j)_{1\leq j\leq k}, (E_t(p_i^{(v)}))_{\substack{1\leq i\leq r_v \\ 1\leq v\leq f}}, (E_t(q_l^{(v)}))_{\substack{1\leq l\leq s_v \\ 1\leq v\leq f}}, \{E_t(e_{jl}^{(v)})\}_{j,l,v}, \{E_t(f_{jl}^{(v)})\}_{j,l,v}, (E_t(z_u))_{1\leq u\leq p}\Big)_{t\geq 1}$$

converges in distribution as $t \to \infty$ to

$$\Big((x_j)_{1\leq j\leq k}, (p_i^{(v)})_{\substack{1\leq i\leq r_v \\ 1\leq v\leq f}}, (q_l^{(v)})_{\substack{1\leq l\leq s_v \\ 1\leq v\leq f}}, \{e_{jl}^{(v)}\}_{j,l,v}, \{f_{jl}^{(v)}\}_{j,l,v}, (z_u)_{1\leq u\leq p}\Big).$$

Therefore

$$\chi\Big((x_j)_{1\leq j\leq k} : (E_t(p_i^{(v)}))_{\substack{1\leq i\leq r_v \\ 1\leq v\leq f}}, (E_t(q_l^{(v)}))_{\substack{1\leq l\leq s_v \\ 1\leq v\leq f}}, \{E_t(e_{jl}^{(v)})\}_{j,l,v}, \{E_t(f_{jl}^{(v)})\}_{j,l,v}, (E_t(z_u))_{1\leq u\leq p}\Big) < \chi(x_1, \dots, x_k)$$

for some large integer $t > k$. But this leads to a contradiction:

$$\begin{aligned}\chi(x_1, \dots, x_t) &= \chi\Big((x_j)_{1\leq j\leq t} : (E_t(p_i^{(v)}))_{\substack{1\leq i\leq r_v \\ 1\leq v\leq f}}, (E_t(q_l^{(v)}))_{\substack{1\leq l\leq s_v \\ 1\leq v\leq f}}, \{E_t(e_{jl}^{(v)})\}_{j,l,v}, \{E_t(f_{jl}^{(v)})\}_{j,l,v}, (E_t(z_u))_{1\leq u\leq p}\Big) \\ &\leq \chi\Big((x_j)_{1\leq j\leq k} : (E_t(p_i^{(v)}))_{\substack{1\leq i\leq r_v \\ 1\leq v\leq f}}, (E_t(q_l^{(v)}))_{\substack{1\leq l\leq s_v \\ 1\leq v\leq f}}, \{E_t(e_{jl}^{(v)})\}_{j,l,v}, \{E_t(f_{jl}^{(v)})\}_{j,l,v}, (E_t(z_u))_{1\leq u\leq p}\Big) \\ &\quad + \chi(x_{k+1}, \dots, x_t) \\ &< \chi(x_1, \dots, x_k) + \chi(x_{k+1}, \dots, x_t) = \chi(x_1, \dots, x_t). \qquad \square\end{aligned}$$

Corollary 3.6. *If $\mathcal{P} \subset \mathcal{L}(\mathbb{F}_n)$ is a subfactor of finite index and if $n \geq p + 2f + 2$, then $\mathcal{P}$ cannot be asymptotically decomposed as*

$$\lim_{\omega \to 0}{}^{\|\cdot\|_2} \sum_{\substack{1 \leq j_1, \ldots, j_{t+1} \leq f \\ 1 \leq t \leq d}} \mathcal{N}_{j_1}^{\omega} \mathfrak{X}^{\omega} \mathcal{N}_{j_2}^{\omega} \mathfrak{X}^{\omega} \cdots \mathcal{N}_{j_t}^{\omega} \mathfrak{X}^{\omega} \mathcal{N}_{j_{t+1}}^{\omega},$$

where (for each ω) $\mathfrak{X}^{\omega}$ contains p self-adjoint elements of $\mathcal{P}$, the $\mathcal{N}_1^{\omega}, \ldots, \mathcal{N}_f^{\omega}$ are nonprime subfactors of $\mathcal{P}$, and $d \geq 1$ is an integer.

Proof. Since $\mathcal{P} \subset \mathcal{L}(\mathbb{F}_n)$ is a subfactor of finite index, $\mathcal{L}(\mathbb{F}_n)$ can be obtained from $\mathcal{P}$ with the basic construction [Jones 1983; Jones and Sunder 1997]: there exists a subfactor $\mathcal{Q} \subset \mathcal{P}$ such that $\mathcal{L}(\mathbb{F}_n) = \langle \mathcal{P}, e_{\mathcal{Q}} \rangle$, where $e_{\mathcal{Q}}$ is the Jones projection associated to the inclusion $\mathcal{Q} \subset \mathcal{P}$. But $\langle \mathcal{P}, e_{\mathcal{Q}} \rangle = \mathcal{P} e_{\mathcal{Q}} \mathcal{P}$ [Jones and Sunder 1997]; hence $\mathcal{L}(\mathbb{F}_n)$ can be decomposed as $\mathcal{P} e_{\mathcal{Q}} \mathcal{P}$. Now apply Theorem 3.5. □

Corollary 3.7. *If $n \geq p+2f+1$, the free group factor $\mathcal{L}(\mathbb{F}_n)$ cannot be decomposed as*

$$\overline{\mathrm{sp}}^{w} \sum_{\substack{1 \leq j_1, \ldots, j_{t+1} \leq f \\ 1 \leq t \leq d}} \mathcal{N}_{j_1} \mathfrak{X} \mathcal{N}_{j_2} \mathfrak{X} \cdots \mathcal{N}_{j_t} \mathfrak{X} \mathcal{N}_{j_{t+1}},$$

where $\mathfrak{X} \subset \mathcal{L}(\mathbb{F}_n)$ contains p self-adjoint elements, $\mathcal{N}_1, \ldots, \mathcal{N}_f$ are nonprime subfactors of $\mathcal{L}(\mathbb{F}_n)$, and $d \geq 1$ is an integer. Moreover, if $\mathcal{P} \subset \mathcal{L}(\mathbb{F}_n)$ is a subfactor of finite index and if $n \geq p + 2f + 2$, then $\mathcal{P}$ also cannot be decomposed as

$$\overline{\mathrm{sp}}^{w} \sum_{\substack{1 \leq j_1, \ldots, j_{t+1} \leq f \\ 1 \leq t \leq d}} \mathcal{N}_{j_1} \mathfrak{X} \mathcal{N}_{j_2} \mathfrak{X} \cdots \mathcal{N}_{j_t} \mathfrak{X} \mathcal{N}_{j_{t+1}},$$

for any subset $\mathfrak{X}$ containing p self-adjoint elements of $\mathcal{P}$, any $\mathcal{N}_1, \ldots, \mathcal{N}_f$ nonprime subfactors of $\mathcal{P}$, and any integer $d \geq 1$.

Proof. This follows from Theorem 3.5 and Corollary 3.6, with $\mathfrak{X}^{\omega} = \mathfrak{X}$, $\mathcal{N}_1^{\omega} = \mathcal{N}_1$, $\ldots$, $\mathcal{N}_f^{\omega} = \mathcal{N}_f$. □

Corollary 3.8 settles a conjecture from [Ge and Popa 1998] in the case $n = \infty$. Recall that for a type II_1-factor $\mathcal{M}$ one defines

$$\ell_h(\mathcal{M}) = \min\{ f \in \mathbb{N} \mid \exists \text{ hyperfinite } \mathcal{R}_1, \ldots, \mathcal{R}_f \subset M \text{ s.t. } \overline{\mathrm{sp}}^{w} \mathcal{R}_1 \mathcal{R}_2 \cdots \mathcal{R}_f = \mathcal{M} \}.$$

Note that the definition of hyperfinite dimension is given in terms of hyperfinite subalgebras. If one defined the hyperfinite dimension in terms of hyperfinite subfactors instead of hyperfinite subalgebras, the proof of Corollary 3.8 would have followed immediately from Corollary 3.7. But with Definition 3.1, we need the asymptotic indecomposability result from Theorem 3.5.

Corollary 3.8. $\ell_h(\mathcal{L}(\mathbb{F}_n)) \geq \left[\frac{n-2}{2}\right] + 1$ *for* $4 \leq n \leq \infty$.

Proof. If $\ell_h(\mathscr{L}(\mathbb{F}_n)) \le \left[\frac{n-2}{2}\right]$, then $\mathscr{L}(\mathbb{F}_n) = \overline{\text{sp}}^w \mathscr{R}_1\mathscr{R}_2 \cdots \mathscr{R}_f$ for some hyperfinite subalgebras $\mathscr{R}_1, \dots, \mathscr{R}_f$ and some integer f with $n \ge 2f+2$. Let $m \ge 1$, $y_1, \dots, y_m \in \mathscr{L}(\mathbb{F}_n)$ and $\omega > 0$ be fixed. There exist finite dimensional subalgebras $\mathscr{B}_v^\omega = \mathscr{B}_v(y_1, \dots, y_m; \omega) \subset \mathscr{R}_v$, for $1 \le v \le f$, such that

$$\text{dist}_{\|\cdot\|_2}\big(y_j, \mathscr{B}_1^\omega\mathscr{B}_2^\omega \cdots \mathscr{B}_f^\omega\big) < \omega \quad \text{for } 1 \le j \le m.$$

Each finite dimensional subalgebra $\mathscr{B}_v^\omega$ is contained in a copy of the hyperfinite II_1-factor, say $\mathscr{B}_v^\omega \subset \mathscr{R}_v^\omega = \mathscr{R}_v^\omega(y_1, \dots, y_m; \omega) \subset \mathscr{L}(\mathbb{F}_n)$. Consequently,

$$\text{dist}_{\|\cdot\|_2}\big(y_j, \mathscr{R}_1^\omega\mathscr{R}_2^\omega \cdots \mathscr{R}_f^\omega\big) < \omega \quad \text{for } 1 \le j \le m;$$

hence $\mathscr{L}(\mathbb{F}_n)$ admits an asymptotic decomposition of the form

$$\lim_{\omega\to 0}{}^{\|\cdot\|_2}\, \mathscr{R}_1^\omega\mathscr{R}_2^\omega \cdots \mathscr{R}_f^\omega,$$

contradicting Theorem 3.5 since $\mathscr{R}_1^\omega, \dots, \mathscr{R}_f^\omega$ are nonprime and $n \ge 2f+2$. □

Corollary 3.9. *If $\mathscr{P} \subset \mathscr{L}(\mathbb{F}_n)$ is a subfactor of finite index and $5 \le n \le \infty$, then $\ell_h(\mathscr{P}) \ge \left[\frac{n-3}{2}\right] + 1$.*

Proof. Follows from Corollary 3.6. □

4. Indecomposability over abelian subalgebras

Another estimate of free entropy is used to prove that the free group factor $\mathscr{L}(\mathbb{F}_n)$ does not admit an asymptotic decomposition of the form

$$\lim_{\omega\to 0}{}^{\|\cdot\|_2} \sum_{\substack{1\le j_1,\dots,j_{t+1}\le f\\ 1\le t\le d}} \mathscr{A}_{j_1}^\omega \mathfrak{X}^\omega \mathscr{A}_{j_2}^\omega \mathfrak{X}^\omega \cdots \mathscr{A}_{j_t}^\omega \mathfrak{X}^\omega \mathscr{A}_{j_{t+1}}^\omega,$$

where (for each ω) the $\mathscr{A}_1^\omega, \dots, \mathscr{A}_f^\omega$ are abelian subalgebras of $\mathscr{L}(\mathbb{F}_n)$, $\mathfrak{X}^\omega \subset \mathscr{L}(\mathbb{F}_n)$ is a subset with p self-adjoint elements, $d \ge 1$ is an arbitrary integer, and $n \ge p+2f+1$. Similarly, for free group subfactors one has the following: if $n \ge p+2f+2$ and $\mathscr{P} \subset \mathscr{L}(\mathbb{F}_n)$ is a subfactor of finite index, then $\mathscr{P}$ does not admit such an asymptotic decomposition either. In particular, the abelian dimension of $\mathscr{L}(\mathbb{F}_n)$ is $\ge \left[\frac{n-2}{2}\right]+1$ and the abelian dimension of $\mathscr{P}$ is $\ge \left[\frac{n-3}{2}\right]+1$. For $n = \infty$ this proves the second part of Ge and Popa's conjecture [Ge and Popa 1998]: the abelian dimension of free group factors is infinite. The definitions of abelian dimension and asymptotic decomposition over abelian subalgebras are given next.

Definition 4.1 [Ge and Popa 1998]. If $\mathscr{M}$ is a II_1-factor, the abelian dimension of $\mathscr{M}$, denoted $\ell_a(\mathscr{M})$, is defined as the smallest positive integer $f \in \mathbb{N}$ with the property that there exist abelian subalgebras $\mathscr{A}_1, \dots, \mathscr{A}_f \subset \mathscr{M}$ such that $\overline{\text{sp}}^w \mathscr{A}_1\mathscr{A}_2 \cdots \mathscr{A}_f = \mathscr{M}$. If there is no such positive integer, $\ell_a(\mathscr{M}) = +\infty$.

Definition 4.2. A type II_1-factor $\mathcal{M}$ admits an asymptotic decomposition over abelian subalgebras if, for any $n \geq 1$, any $x_1, \ldots, x_n \in \mathcal{M}$, and any $\omega > 0$, there exist abelian $*$-subalgebras $\mathcal{A}_1^\omega = \mathcal{A}_1(x_1, \ldots, x_n; \omega), \ldots, \mathcal{A}_f^\omega = \mathcal{A}_f(x_1, \ldots, x_n; \omega)$ of $\mathcal{M}$ and also a set $\mathfrak{X}^\omega = \mathfrak{X}(x_1, \ldots, x_n; \omega) \subset \mathcal{M}$ containing p self-adjoint elements, such that

$$\operatorname{dist}_{\|\cdot\|_2}\Biggl(x_j, \sum_{\substack{1 \leq j_1, \ldots, j_{t+1} \leq f \\ 1 \leq t \leq d}} \mathcal{A}_{j_1}^\omega \mathfrak{X}^\omega \mathcal{A}_{j_2}^\omega \mathfrak{X}^\omega \cdots \mathcal{A}_{j_t}^\omega \mathfrak{X}^\omega \mathcal{A}_{j_{t+1}}^\omega\Biggr) < \omega \quad \text{for } 1 \leq j \leq n.$$

In this situation we write

$$\mathcal{M} = \lim_{\omega \to 0}{}^{\|\cdot\|_2} \sum_{\substack{1 \leq j_1, \ldots, j_{t+1} \leq f \\ 1 \leq t \leq d}} \mathcal{A}_{j_1}^\omega \mathfrak{X}^\omega \mathcal{A}_{j_2}^\omega \mathfrak{X}^\omega \cdots \mathcal{A}_{j_t}^\omega \mathfrak{X}^\omega \mathcal{A}_{j_{t+1}}^\omega.$$

Proposition 4.3 gives an estimate of the free entropy of a (finite) system of generators of a II_1-factor $\mathcal{M}$ that can be asymptotically decomposed as

$$\lim_{\omega \to 0}{}^{\|\cdot\|_2} \sum_{\substack{1 \leq j_1, \ldots, j_{t+1} \leq f \\ 1 \leq t \leq d}} \mathcal{A}_{j_1}^\omega \mathfrak{X}^\omega \mathcal{A}_{j_2}^\omega \mathfrak{X}^\omega \cdots \mathcal{A}_{j_t}^\omega \mathfrak{X}^\omega \mathcal{A}_{j_{t+1}}^\omega.$$

As in the statement of Proposition 3.4, the approximations in the $\|\cdot\|_2$-norm (6) hold for every $\omega > 0$ if the II_1-factor can be decomposed as above.

Proposition 4.3. *Let $z_1, \ldots, z_p$ be self-adjoint elements of a II_1-factor $\mathcal{M}$ and let $(\mathcal{A}_v)_{1 \leq v \leq f}$ be a family of abelian subalgebras of $\mathcal{M}$. Let $x_1, \ldots, x_n$ be self-adjoint generators of $\mathcal{M}$ and assume that there exist projections $p_1^{(v)}, \ldots, p_{r_v}^{(v)} \in \mathcal{A}_v$ and complex noncommutative polynomials $(\phi_j)_{1 \leq j \leq n}$ of degree at most d (where $d \geq 1$ is fixed) in the variables $(z_u)_{1 \leq u \leq p}$ such that*

$$\Bigl\| x_j - \phi_j\Bigl((p_i^{(v)})_{\substack{1 \leq i \leq r_v \\ 1 \leq v \leq f}}, (z_u)_{1 \leq u \leq p}\Bigr)\Bigr\|_2 < \omega \quad \textit{for } j = 1, \ldots, n, \tag{6}$$

where $\omega \in (0, a]$ is a given positive number, and such that in all monomials of every ϕ_j the projections $p_i^{(v)}$ and $p_k^{(w)}$ are separated by some z_u if $v \neq w$. Then

$$\chi(x_1, \ldots, x_n) \leq C(n, p, a, d, f) + (n - p - 2f)\log \omega, \tag{7}$$

where $a = \max\bigl\{\|x_j\|_2 + 1 \mid 1 \leq j \leq n\bigr\}$ and $C(n, p, a, d, f)$ is a constant that depends only on n, p, a, d, f.

Proof. As in the proof of Proposition 3.4 we can assume that $\phi_j = \phi_j^*$ for $1 \leq j \leq n$, and fix $R > 0$. Consider an arbitrary element

$$\Bigl((B_j)_{1 \leq j \leq n}, (P_i^{(v)})_{\substack{1 \leq i \leq r_v \\ 1 \leq v \leq f}}, (Z_u)_{1 \leq u \leq p}\Bigr)$$

of

$$\Gamma_R\Big((x_j)_{1\le j\le n}, (p_i^{(v)})_{\substack{1\le i\le r_v\\ 1\le v\le f}}, (z_u)_{1\le u\le p}; m, k, \epsilon\Big)$$

for some large integers m, k and small $\epsilon > 0$. Possibly after further restricting m and ϵ, we can find mutually orthogonal projections $Q_1^{(v)}, \dots, Q_{r_v}^{(v)} \in \mathcal{M}_k^{\mathrm{sa}}$ with rank $Q_i^{(v)} = \big[\tau(p_i^{(v)})k\big]$ for $i = 1, \dots, r_v$, such that

$$\Big\| B_j - \phi_j\Big((Q_i^{(v)})_{\substack{1\le i\le r_v\\ 1\le v\le f}}, (Z_u)_{1\le u\le p}\Big)\Big\|_2 < \omega \quad \text{for all } 1 \le j \le n.$$

If $S_1^{(v)}, \dots, S_r^{(v)} \in \mathcal{M}_k^{\mathrm{sa}}$ are fixed, mutually orthogonal projections with rank $S_i^{(v)} = \big[\tau(p_i^{(v)})k\big]$ for every $1 \le i \le r_v$, then there exists a unitary $U^{(v)} \in \mathcal{U}(k)$ such that $Q_i^{(v)} = U^{(v)*} S_i U^{(v)}$ for every $1 \le i \le r_v$. The previous inequality becomes

$$\Big\| B_j - \phi_j\Big((S_i^{(v)})_{\substack{1\le i\le r_v\\ 1\le v\le f}}, (Z_u)_{1\le u\le p}, \big(\operatorname{Re} U^{(v)}, \operatorname{Im} U^{(v)}\big)_{1\le v\le f}\Big)\Big\|_2 < \omega,$$

and all the components of ϕ_j are polynomials of degrees $\le 3d+2$ in the last $p+2f$ variables. Reasoning as in the last part of the proof of Proposition 3.4 we can easily obtain now the estimate $\chi(x_1, \dots, x_n) \le C(n, p, a, d, f) + (n-p-2f)\log\omega$. □

Abelian dimension of free group factors.

Theorem 4.4. *If $n \ge p + 2f + 1$, the free group factor $\mathcal{L}(\mathbb{F}_n)$ does not admit an asymptotic decomposition of the form*

$$\lim_{\omega\to 0}{}^{\|\cdot\|_2} \sum_{\substack{1\le j_1,\dots,j_{t+1}\le f\\ 1\le t\le d}} \mathcal{A}_{j_1}^{\omega}\mathfrak{X}^{\omega}\mathcal{A}_{j_2}^{\omega}\mathfrak{X}^{\omega}\cdots\mathcal{A}_{j_t}^{\omega}\mathfrak{X}^{\omega}\mathcal{A}_{j_{t+1}}^{\omega},$$

where each subset $\mathfrak{X}^{\omega}$ contains p self-adjoint elements, $\mathcal{A}_1^{\omega}, \dots, \mathcal{A}_f^{\omega} \subset \mathcal{L}(\mathbb{F}_n)$ are abelian $*$-subalgebras and $d \ge 1$ is an integer.

Proof. Apply Proposition 4.3 in the same manner that Proposition 3.4 was used in the proof of Theorem 3.5. □

Corollary 4.5. *If $\mathcal{P} \subset \mathcal{L}(\mathbb{F}_n)$ is a subfactor of finite index and if $n \ge p + 2f + 2$, then $\mathcal{P}$ cannot be asymptotically decomposed as*

$$\lim_{\omega\to 0}{}^{\|\cdot\|_2} \sum_{\substack{1\le j_1,\dots,j_{t+1}\le f\\ 1\le t\le d}} \mathcal{A}_{j_1}^{\omega}\mathfrak{X}^{\omega}\mathcal{A}_{j_2}^{\omega}\mathfrak{X}^{\omega}\cdots\mathcal{A}_{j_t}^{\omega}\mathfrak{X}^{\omega}\mathcal{A}_{j_{t+1}}^{\omega},$$

where each subset $\mathfrak{X}^{\omega}$ contains p self-adjoint elements of $\mathcal{P}$, the $\mathcal{A}_1^{\omega}, \dots, \mathcal{A}_f^{\omega} \subset \mathcal{P}$ are abelian $*$-subalgebras, and $d \ge 1$ is an integer.

Proof. This is a direct consequence of Theorem 4.4 and of decomposition $\mathcal{L}(\mathbb{F}_n) = \mathcal{P}e_{\mathcal{Q}}\mathcal{P}$ (see the proof of Corollary 3.6). □

Corollary 4.6. *If $n \geq p+2f+1$, the free group factor $\mathcal{L}(\mathbb{F}_n)$ cannot be decomposed as*

$$\overline{\mathrm{sp}}^{w} \sum_{\substack{1\leq j_1,\ldots,j_{t+1}\leq f\\ 1\leq t\leq d}} \mathcal{A}_{j_1}\mathfrak{X}\mathcal{A}_{j_2}\mathfrak{X}\cdots\mathcal{A}_{j_t}\mathfrak{X}\mathcal{A}_{j_{t+1}},$$

where $\mathfrak{X} \subset \mathcal{L}(\mathbb{F}_n)$ contains p self-adjoint elements, $\mathcal{A}_1,\ldots,\mathcal{A}_f$ are abelian $*$-subalgebras of $\mathcal{L}(\mathbb{F}_n)$, and $d \geq 1$ is an integer. Moreover, if $\mathcal{P} \subset \mathcal{L}(\mathbb{F}_n)$ is a subfactor of finite index and if $n \geq p+2f+2$, then $\mathcal{P}$ also cannot be decomposed as

$$\overline{\mathrm{sp}}^{w} \sum_{\substack{1\leq j_1,\ldots,j_{t+1}\leq f\\ 1\leq t\leq d}} \mathcal{A}_{j_1}\mathfrak{X}\mathcal{A}_{j_2}\mathfrak{X}\cdots\mathcal{A}_{j_t}\mathfrak{X}\mathcal{A}_{j_{t+1}},$$

for any subset $\mathfrak{X}$ containing p self-adjoint elements of $\mathcal{P}$, any $\mathcal{A}_1,\ldots,\mathcal{A}_f$ abelian $*$-subalgebras of $\mathcal{P}$, and any integer $d \geq 1$.

Proof. Apply Theorem 4.4 and Corollary 4.5 for $\mathfrak{X}^{\omega} = \mathfrak{X}$, $\mathcal{A}_1^{\omega} = \mathcal{A}_1,\ldots,\mathcal{A}_f^{\omega} = \mathcal{A}_f$. □

Corollary 4.7 settles the second part of the conjecture of Ge and Popa [1998], in the case $n = \infty$. As a reminder, $\ell_a(\mathcal{M})$ is defined as

$$\min\left\{ f \in \mathbb{N} \mid \exists \text{ abelian } *\text{-algebras } \mathcal{A}_1,\ldots,\mathcal{A}_f \subset \mathcal{M} \text{ s.t. } \overline{\mathrm{sp}}^{w}\mathcal{A}_1\mathcal{A}_2\cdots\mathcal{A}_f = \mathcal{M}\right\}$$

for every type II_1-factor $\mathcal{M}$.

Corollary 4.7. $\ell_a(\mathcal{L}(\mathbb{F}_n)) \geq \left[\frac{n-2}{2}\right]+1$ *for* $4 \leq n \leq \infty$.

Proof. This follows from the first part of Corollary 4.6 with $\mathfrak{X} = \{1\}$. □

Corollary 4.8. *If $\mathcal{P} \subset \mathcal{L}(\mathbb{F}_n)$ is a subfactor of finite index and $5 \leq n \leq \infty$, then $\ell_a(\mathcal{P}) \geq \left[\frac{n-3}{2}\right]+1$.*

Proof. Apply the second part of Corollary 4.6. □

Remark 4.9. One can combine both indecomposability properties of $\mathcal{L}(\mathbb{F}_n)$ into a single statement: if $n \geq p+2f+1$, the free group factor $\mathcal{L}(\mathbb{F}_n)$ does not admit an asymptotic decomposition of the form

$$\lim_{\omega\to 0}{}^{\|\cdot\|_2} \sum_{\substack{1\leq j_1,\ldots,j_{t+1}\leq f\\ 1\leq t\leq d}} \mathcal{M}_{j_1}^{\omega}\mathfrak{X}^{\omega}\mathcal{M}_{j_2}^{\omega}\mathfrak{X}^{\omega}\cdots\mathcal{M}_{j_t}^{\omega}\mathfrak{X}^{\omega}\mathcal{M}_{j_{t+1}}^{\omega},$$

where each subset $\mathfrak{X}^{\omega}$ contains p self-adjoint elements, each $\mathcal{M}_1^{\omega},\ldots,\mathcal{M}_f^{\omega} \subset \mathcal{L}(\mathbb{F}_n)$ is either a nonprime subfactor or an abelian $*$-subalgebra and $d \geq 1$ is an integer.

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MARIUS B. ŞTEFAN
MATHEMATICS DEPARTMENT
UCLA
LOS ANGELES, CA 90095-1555
stefan@math.ucla.edu