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Motivated by Hartshorne's work on curves in $\mathbb{P}^3$, we study the properties of reflexive rank-2 sheaves on smooth projective threefolds.

1. Introduction

We work over an algebraically closed field of characteristic 0.

There has been a tremendous amount of interest in recent years in the study of curves on Calabi–Yau threefolds, and especially on the general quintic in $\mathbb{P}^4$. In this note, motivated by Hartshorne's work [1978; 1980] on curves in $\mathbb{P}^3$, we study the properties of reflexive rank-2 sheaves on smooth projective threefolds.

Some similar results are obtained in [Ballico and Miró-Roig 1997] for Fano threefolds (and somewhat more generally). The greatest advantage of our results is the determination of explicit effective bounds for the third Chern class, c_3, of a reflexive sheaf (Theorem 14) and of explicit bounds for vanishing of higher cohomology and the existence of global sections (Corollary 13). In Section 3 we write out these bounds for the case of a smooth threefold hypersurface of degree d.

We refer the reader to [Hartshorne 1980] for basic properties of reflexive sheaves. Recall the following *Serre correspondence* for reflexive sheaves (the referenced result is only for $\mathbb{P}^3$, but as noted in [Hartshorne 1978, 1.1.1] the general case follows immediately from the proof):

Theorem 1 [Hartshorne 1980, 4.1]. *Let X be a smooth projective threefold, M an invertible sheaf with $H^1(X, M^*) = H^2(X, M^*) = 0$. There is a one-to-one correspondence between*

(1) *pairs $(\mathscr{F}, s)$, where $\mathscr{F}$ is a rank-2 reflexive sheaf on X with $\bigwedge^2 \mathscr{F} = M$ and $s \in \Gamma(\mathscr{F})$ is a section whose zero set has codimension 2, and*

(2) *pairs (Y, ξ), where Y is a closed Cohen–Macaulay curve in X, generically a local complete intersection, and $\xi \in \Gamma(Y, \omega_Y \otimes \omega_X^* \otimes M^*)$ is a section that generates the sheaf $\omega_Y \otimes \omega_X^* \otimes M^*$ except at finitely many points.*

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Furthermore, $c_3(\mathscr{F}) = 2p_a(Y) - 2 - c_2(\mathscr{F})c_1(\omega_X) - c_2(\mathscr{F})c_1(\mathscr{F})$. □

The case where $\mathscr{F}$ is locally free corresponds the curve Y being a local complete intersection. Furthermore $\omega_Y \otimes \omega_X^* \otimes M^* \cong \mathscr{O}_Y$, ξ is a nonzero section and $c_3(\mathscr{F}) = 0$. In this case we say Y is *subcanonical*.

Example 2. Suppose $X \subset \mathbb{P}^4$ is a smooth hypersurface of degree d, $Y \subset X$ a smooth rational curve. Then Y is the zero locus of a section of some rank two vector bundle V if and only if Y is a line or a plane conic in the embedding given by $\mathscr{O}_X(1)$. If Y is a line, then $\bigwedge^2 V = \mathscr{O}_X(3-d)$; if Y is a plane conic, then $\bigwedge^2 V = \mathscr{O}_X(4-d)$.

Example 3. Suppose $X \subset \mathbb{P}^4$ is a smooth hypersurface of degree d, $Y \subset X$ a smooth elliptic curve. Then Y is the zero locus of a section of some rank two vector bundle V with $\bigwedge^2 V = \mathscr{O}_X(5-d)$.

Finally, we recall some basic formulae:

Proposition 4. *Let $\mathscr{F}$ be a coherent sheaf of rank r on a smooth threefold X. Then*

$$\chi(X,\mathscr{F}) = \tfrac{1}{6}c_1(\mathscr{F})^3 - \tfrac{1}{2}c_1(\mathscr{F})c_2(\mathscr{F}) - \tfrac{1}{2}c_1(X)c_2(\mathscr{F}) + \tfrac{1}{4}c_1(X)c_1(\mathscr{F})^2$$
$$+ \tfrac{1}{12}c_1(X)^2c_1(\mathscr{F}) + \tfrac{1}{12}c_2(X)c_1(\mathscr{F}) + \tfrac{r}{24}c_1(X)c_2(X) + \tfrac{1}{2}c_3(\mathscr{F}).$$

Note also that if $\mathscr{F}$ has rank two and L is an invertible sheaf, then

(1) $c_1(\mathscr{F} \otimes L) = c_1(\mathscr{F}) + 2c_1(L)$,

(2) $c_2(\mathscr{F} \otimes L) = c_2(\mathscr{F}) + c_1(L)c_1(\mathscr{F}) + c_1(L)^2$,

(3) $c_3(\mathscr{F} \otimes L) = c_3(\mathscr{F})$.

2. Stability and Boundedness

Definition 5. Let L be a very ample line bundle on a smooth projective variety X. A reflexive coherent sheaf $\mathscr{F}$ on X is *L-semistable* if for every coherent subsheaf $\mathscr{F}'$ of $\mathscr{F}$ with $0 < \operatorname{rank}\mathscr{F}' < \operatorname{rank}\mathscr{F}$, we have $\mu(\mathscr{F}', L) \le \mu(\mathscr{F}, L)$, where

$$\mu(\mathscr{F}, L) = \frac{c_1(\mathscr{F}).[L]^{\dim X-1}}{(\operatorname{rank}\mathscr{F})\,[L]^{\dim X}}.$$

If the inequality is strict, $\mathscr{F}$ is *L-stable*. Note that if $\operatorname{rank}\mathscr{F} = 2$, it suffices to take $\mathscr{F}'$ invertible.

Definition 6. We say that a reflexive sheaf $\mathscr{F}$ is *normalized with respect to L* if $-1 < \mu(\mathscr{F}, L) \le 0$. As L is typically fixed, we usually say simply that $\mathscr{F}$ is *normalized*. Note that since $\mu(\mathscr{F} \otimes L, L) = \mu(\mathscr{F}, L) + 1$, there exists, for any fixed $\mathscr{F}$, a unique $k \in \mathbb{Z}$ such that $\mathscr{F} \otimes L^k$ is normalized with respect to L.

For a fixed X, our goal is to give a bound on $c_3(\mathscr{F})$ in terms of $c_1(\mathscr{F})$ and $c_2(\mathscr{F})$. Note that the formula for c_3 in Theorem 1 gives:

Lemma 7. *Let X be a smooth threefold, L a very ample line bundle, $\mathscr{F}$ a rank two reflexive sheaf, $\bigwedge^2 \mathscr{F} = M$ a line bundle with $H^1(M^*) = H^2(M^*) = 0$. If $s \in \Gamma(\mathscr{F})$ is a section whose zero locus is a curve, then*

$$c_3(\mathscr{F}) \leq d^2 - 3d - c_2(\mathscr{F})c_1(\omega_X) - c_2(\mathscr{F})c_1(\mathscr{F})$$

where $d = c_2(\mathscr{F})c_1(L)$.

Proof. In light of Theorem 1, we need only note that the degree of the curve section in the embedding given by L is $d = c_2(\mathscr{F})c_1(L)$. The fact that $2p_a(Y) - 2 \leq d^2 - 3d$ is just the bound coming from the degree of a plane curve. □

The idea now is: given a very ample line bundle L, bound the twist of $\mathscr{F}$ by L^r needed to produce a section, and then use the bound in Lemma 7. First note the following elementary result:

Lemma 8. *Let $\mathscr{F}$ be a reflexive sheaf on a smooth projective variety X with a very ample line bundle L. If either*

(1) *$\mathscr{F}$ is L-stable and $\mu(\mathscr{F}, L) \leq 0$ or*

(2) *$\mathscr{F}$ is L-semistable and $\mu(\mathscr{F}, L) < 0$*

then $H^0(X, \mathscr{F}) = 0$.

Proof. Suppose otherwise that $\mathscr{F}$ has a section $\mathbb{O}_X \to \mathscr{F}$. Dualizing, we get a surjection $\mathscr{F}^* \to \mathscr{I}_Y \subset \mathbb{O}_X$; dualizing again we have $0 \to \mathscr{I}_Y^* \to \mathscr{F}$, but $\mathscr{I}_Y^*$ is invertible and $H^0(X, \mathscr{I}_Y^*) = \mathrm{Hom}_{\mathbb{O}_X}(\mathscr{I}_Y, \mathbb{O}_X) \neq 0$. Hence $\mu(\mathscr{I}_Y^*, L) \geq 0$ and the result follows. □

The main technical result is:

Proposition 9. *Let X be a smooth projective threefold with very ample line bundle L and with $\mathrm{Pic}\, X = \mathbb{Z}L$. Let $\mathscr{F}$ be a normalized L-semistable rank-2 reflexive sheaf, and D be a general member of the linear system $|L|$. Assume that the general member of the linear system $|L \otimes \mathbb{O}_D|$ is not rational, and that $m < 0$ is an integer satisfying*

$$2m < 3\mu(\Theta_X, L) - 2\mu(\mathscr{F}, L) - 2.$$

Then $H^0(\mathscr{F}_D(mD)) = 0$.

Remark 10. The assumption that the general member of the linear system $|L \otimes \mathbb{O}_D|$ is not rational can be dropped if we require that

$$2m < 3\mu(\Theta_X, L) - 2\mu(\mathscr{F}, L) - 4.$$

As this would impact all further estimates, we have chosen to add the extra hypothesis rather than explicitly keeping track of the two separate cases. The interested reader will have little trouble altering the bounds in subsequent arguments in cases where this is of interest (say a threefold quadric hypersurface).

Proof of Proposition 9. We proceed by contradiction. Let m be the smallest integer, if one exists, satisfying the inequality and such that $H^0(\mathscr{F}_D(mD))$ is nonzero for the general, hence every, member of $|L|$. We will show $m \geq 0$.

Fix a smooth member D such that $\mathscr{F}_D$ is locally free. The proposed section yields a sequence

$$0 \to \mathbb{O}_D \to \mathscr{F}_D(mD) \to \mathscr{I}_Z(2mD) \otimes \textstyle\bigwedge^2 \mathscr{F} \to 0,$$

where $Z \subset D$ is zero-dimensional of length $c_2(\mathscr{F}(mD))$. Choose a smooth curve C in the system $|L_D|$ (i.e., in the class of $D.[L]$) with $Z \cap C$ empty. Tensoring the sequence above by $\mathbb{O}_C$ yields an extension of line bundles.

The class of the extension lies in

$$\mathrm{Ext}^1_{\mathbb{O}_C}\big(\mathbb{O}_C, \mathbb{O}_C(-2mD) \otimes \textstyle\bigwedge^2 \mathscr{F}^*\big) = H^1\big(C, \mathbb{O}_C(-2mD) \otimes \textstyle\bigwedge^2 \mathscr{F}^*\big).$$

Note that $K_C = K_X \otimes \mathbb{O}_C(2D)$. Now, as the inequality in the hypotheses is easily seen to be equivalent to

$$-2m[L]^3 - c_1(\mathscr{F}).[L]^2 > K_X.[L]^2 + 2[L]^3 = 2g(C) - 2$$

the extension group vanishes, hence

$$\mathscr{F}_C(mD) = \mathbb{O}_C \oplus \big[\mathbb{O}_C(2mD) \otimes \textstyle\bigwedge^2 \mathscr{F}\big]$$

and $h^0(C, \mathscr{F}_C(mD)) = 1$. By minimality of m, we see also that

$$h^0(D, \mathscr{F}_D(mD)) = 1.$$

Now blow up X along C, and consider $\pi : \mathrm{Bl}_C(X) \to X$. We have a morphism $f : \mathrm{Bl}_C(X) \to \mathbb{P}^1$ given by the pencil of divisors in $|L|$ containing C. It is easy to see that for *every* one of these divisors, $h^0(\mathscr{F}_D(mD)) = 1$. Then, because $\pi^*\mathscr{F}$ is reflexive, $f_*\pi^*\mathscr{F}(mD)$ is invertible [Hartshorne 1980, 1.4,1.7]. However, we know that $H^0(\mathscr{F}_D(mD)) \to H^0(\mathscr{F}_C(mD))$ is an isomorphism and therefore $f_*\pi^*\mathscr{F}(mD) \cong f_*\pi^*\mathscr{F}_C(mD) \cong \mathbb{O}_{\mathbb{P}^1}$, where the last isomorphism follows directly from the splitting of $\mathscr{F}_C$.

Consequently, $H^0(X, \mathscr{F}(mD)) \neq 0$ and so $m \geq 0$ by Lemma 8, contradicting the assumption that m is negative. □

Corollary 11. *With notation and hypotheses as is Proposition 9, if*

$$2k > max\big\{0, 2 + 2\mu(\mathscr{F}, L) - 3\mu(\Theta_X, L)\big\},$$

then $H^2\big(D, K_D \otimes \mathscr{F}^*_D(kD)\big) = 0$ *for the general member* D. *If, furthermore,* k *is such that*

$$(6k^2+6k+2)-(6k+3)\big(2\mu(\mathscr{F},L)+3\mu(\Theta_X,L)\big) \geq \frac{\big(6c_2(\mathscr{F})-c_1(X)^2-3c_1(\mathscr{F})^2-3c_1(\mathscr{F})c_1(X)-c_2(X)\big)[L]}{[L]^3},$$

then $H^0\big(D, K_D \otimes \mathscr{F}^*_D(kD)\big) \neq 0$.

Proof. We can choose D smooth and so that $\mathscr{F}_D$ is locally free. Then

$$h^2\big(D, K_D \otimes \mathscr{F}^*_D(kD)\big) = h^0\big(D, \mathscr{F}_D(-kD)\big),$$

which is zero by Proposition 9.

Because of the vanishing of H^2 above, the second part follows directly from a computation of the Euler characteristic. □

Corollary 12. *With notation and hypotheses as is Proposition 9 there exists a constant* ρ *depending on* $c_1(\mathscr{F})$, $c_2(\mathscr{F})$, $c_1(L)$ *and* $c_i(\Theta_X)$ *such that if* $r \geq \rho$ *then* $H^1\big(D, K_D \otimes \mathscr{F}^*_D(rD)\big) = 0$.

Proof. By the previous corollary, there is a constant depending on the above parameters such that if k is larger than that constant, then $K_D \otimes \mathscr{F}^*_D(kD)$ has a section. Choosing the *smallest* such integer k we have a sequence

$$0 \to \mathbb{O}_D \to K_D \otimes \mathscr{F}^*_D(kD) \to \mathscr{I}_Z(2kD) \otimes K_D^2 \otimes \textstyle\bigwedge^2 \mathscr{F}^* \to 0,$$

where, as above, $Z \subset D$ is zero-dimensional of length

$$\ell = c_2(\mathscr{F}_D) - \big(c_1(K_D)+kc_1(\mathbb{O}(D))\big)c_1(\mathscr{F}_D) + \big(c_1(K_D)+kc_1(\mathbb{O}(D))\big)^2.$$

Let $\alpha \in \mathbb{Z}$ be such that $K_X^2 \otimes \bigwedge^2 \mathscr{F}^* = L^\alpha$. Because D is a smooth surface, $H^1(D, \mathbb{O}(pD)) = 0$ for $p \geq 3c_1(L)^3 - 5$ (by [Bertram et al. 1991, 1.10], for instance). Further, by the standard uniform regularity result [Mumford 1966, p.103], $H^1\big(D, \mathscr{I}_Z((2k+t)D) \otimes K_D^2 \otimes \bigwedge^2 \mathscr{F}^*\big)$ vanishes for $t \geq \ell - 2k - \alpha - 2$ and $t \geq 3c_1(L)^3 - 7 - 2k - \alpha$.

Consequently, $H^1\big(D, K_D \otimes \mathscr{F}^*_D(rD)\big) = 0$ for

$$r \geq max\,\{\ell - k - \alpha - 2,\ 3c_1(L)^3 - 7 - k - \alpha\}.$$ □

Corollary 13. *With notation and hypotheses as in Proposition 9, there exists an integer* ρ_2 *depending on* $c_1(\mathscr{F})$, $c_2(\mathscr{F})$, $c_1(L)$ *and* $c_i(\Theta_X)$ *such that if* $r \geq \rho_2$ *then* $H^0(X, K_X \otimes \mathscr{F}^* \otimes L^r) \neq 0$.

Proof. The vanishing of H^1 and H^2 on D described in the corollaries above gives $H^2(X, K_X \otimes \mathscr{F}^* \otimes L^r) = 0$. The result now follows by another Euler characteristic argument (see Proposition 4). □

Theorem 14. *Let X be a smooth projective threefold with very ample line bundle L and with* Pic $X = \mathbb{Z}L$. *Let $\mathscr{F}$ be an L-semistable rank-2 reflexive sheaf. Then there exists an integer C depending on $c_1(\mathscr{F})$, $c_2(\mathscr{F})$, $c_1(L)$ and $c_i(\Theta_X)$ such that $C \geq c_3(\mathscr{F})$.*

Proof. As $c_3(\mathscr{F})$ is unaffected by twisting by a line bundle, we may assume that $\mathscr{F}$ is normalized. The preceding results apply and we can take a section of $K_X \otimes \mathscr{F}^* \otimes L^k$ for some k, bounded as in Corollary 13. We then have an exact sequence

$$0 \to \mathscr{O}_X \to K_X \otimes \mathscr{F}^* \otimes L^k \to \mathscr{I}_Y \otimes K_X^2 \otimes L^{2k} \otimes \textstyle\bigwedge^2 \mathscr{F}^* \to 0,$$

where $Y \subset X$ is a curve. Computing Euler characteristics gives

$$2p_a(Y) - 2 = d_1 d_2 + c_3(\mathscr{F}) + c_1(\omega_X) d_2,$$

where

$$d_1 = c_1(K_X \otimes \mathscr{F}^* \otimes L^k) = -c_1(\mathscr{F}) - 2c_1(X) + 2kc_1(L)$$

and

$$\begin{aligned} d_2 &= c_2(K_X \otimes \mathscr{F}^* \otimes L^k) \\ &= c_2(\mathscr{F}) + c_1(\mathscr{F})c_1(X) - kc_1(\mathscr{F})c_1(L) + c_1(X)^2 - 2kc_1(X)c_1(L) + k^2 c_1(L)^2. \end{aligned}$$

In the embedding determined by L, the degree of the curve Y is precisely $d_2 c_1(L)$. This implies $d_2 c_1(L)(d_2 c_1(L) - 3) \geq 2p_a(Y) - 2$ and so

$$d_2 c_1(L)(d_2 c_1(L) - 3) - d_1 d_2 - d_2 c_1(\omega_X) \geq c_3(\mathscr{F}). \qquad \square$$

3. Explicit bounds

Let X be a smooth hypersurface in $\mathbb{P}^4$ of degree $d > 2$, and $\mathscr{F}$ a rank two L-semistable reflexive sheaf. In this case, we have $K_X = \mathscr{O}_X(d-5)$; since L-semistability is independent of the choice of L, we take $L = \mathscr{O}(1)$. Note that $[L]^3 = d$, that $c_2(\Theta_X) = (10 - 5d + d^2)c_1(L)^2$, and that $\mu(\Theta_X, L) = \frac{1}{2}(5-d)$. Further, if $\mathscr{F}$ is normalized then $\mu(\mathscr{F}, L) = 0$ or $\mu(\mathscr{F}, L) = -\frac{1}{2}$. We explicitly compute the bound in the case $\mu(\mathscr{F}, L) = 0$, the other case being exactly analogous, though a bit more notationally cluttered. For notational convenience we let $S = c_2(\mathscr{F})c_1(L)$.

The first bound in Corollary 11 becomes

$$k > \max\left\{0,\ \tfrac{1}{4}(3d-11)\right\},$$

so here it suffices to take $k > 0$ if $d < 5$ and $k > \frac{1}{4}(3d-11)$ if $d \geq 5$.

The second bound in Corollary 11 becomes

$$(6k^2 + 6k + 2) + \frac{(6k+3)(3d-15)}{2} - \frac{(6S - 35d + 15d^2 - 2d^3)}{d} > 0;$$

hence

$$k > \frac{-3d^2 + 13d + \sqrt{11d^4 - 150d^3 + 391d^2 + 48dS}}{4d}$$

when $S \geq \frac{1}{48}(-11d^3 + 150d^2 - 391d)$, otherwise the second bound in Corollary 11 is unnecessary.

In Corollary 12, note that $K_D = \mathbb{O}_D(d-4)$ and that the bound for p is irrelevant since the vanishing holds already for $p = 0$. The length of Z is at most

$$S + d(d - 4 + k)^2,$$

so for the vanishing $H^1\big(D, K_D \otimes \mathscr{F}^*_D(rD)\big) = 0$ we need

$$2r \geq S + d(d - 4 + k)^2 - 2(d - 4).$$

In Corollary 13, we compute the Euler characteristic of $\mathscr{F}^*(m)$ and take $m \geq r$ such that $\chi(K_X \otimes \mathscr{F}^*(m)) - \frac{1}{2}(c_3(\mathscr{F})) > 0$. We have

$$\begin{aligned}\chi(K_X \otimes \mathscr{F}^*(m)) &- \tfrac{1}{2}(c_3(\mathscr{F})) \\ &= \tfrac{1}{12}(2m + d - 5)(d^3 + 2md^2 - 5d^2 - 10md + 10d + 2m^2 d - 6S);\end{aligned}$$

hence we need

$$m > \frac{-d^2 + 5d + \sqrt{5d^2 - d^4 + 12dS}}{2d}.$$

As before, this bound is irrelevant unless $S \geq \frac{1}{12}(d^3 - 5d)$.

For example, in the case of the quintic we obtain

(1) for $S \geq 13$:

$$\begin{aligned}256c_3(\mathscr{F}) < \big(&320S^2 + 80S\sqrt{60S-525} - 4004S - 540\sqrt{60S-525} + 11955\big) \\ &\times \big(320S^2 + 80S\sqrt{60S-525} - 4068S - 548\sqrt{60S-525} + 12339\big);\end{aligned}$$

(2) for $S < 13$:

$$16c_3(\mathscr{F}) < (5S^2 + 184S + 1620)(5S^2 + 180S + 1536).$$

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