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We give a systematic description of many monomial bases for a specified quantized enveloping algebra and of many integral monomial bases for the associated Lusztig $\mathbb{Z}[v, v^{-1}]$-form. The relations among monomial bases, PBW bases and canonical bases are also discussed.

1. Introduction

Let $\mathfrak{g}$ be a (complex) semisimple Lie algebra and let $\boldsymbol{U}^+$ be the positive part of its associated quantized enveloping algebra $\boldsymbol{U} = \boldsymbol{U}_v(\mathfrak{g})$ over $\mathbb{Q}(v)$ with a Drinfeld–Jimbo presentation in the generators $E_i, F_i, K_i^{\pm 1}$ ($i \in I = [1, n]$). We denote by U^+ the Lusztig form of $\boldsymbol{U}^+$, that is, U^+ is generated by all the divided powers $E_i^{(m)}$ over $\mathscr{Z} := \mathbb{Z}[v, v^{-1}]$. Let Ω be the set of words on the alphabet I and, for $w = i_1^{e_1} i_2^{e_2} \cdots i_m^{e_m} \in \Omega$ with $i_{j-1} \neq i_j$ for all j, put $E_w = E_{i_1}^{e_1} \cdots E_{i_m}^{e_m}$ and $\mathfrak{m}^{(w)} = E_{i_1}^{(e_1)} \cdots E_{i_m}^{(e_m)}$. Further, let Λ denote the set of all functions from the set of positive roots of $\mathfrak{g}$ to nonnegative integers.

Certain monomial bases of the form $\mathfrak{m}^{(w)}$ have been introduced for U^+ in [Lusztig 1990, 7.8] and [Ringel 1995, Theorem 1′] for the simply laced case, and in [Chari and Xi 1999] in general, and are used in the elementary construction of canonical bases. In this paper, we present a systematic way to sort out bases from the monomials E_w for $\boldsymbol{U}^+$ and from the monomials $\mathfrak{m}^{(w)}$ for U^+, and relate them to PBW bases and canonical bases. The main result is:

Theorem 1.1. *Assume that $\mathfrak{g}$ is simply laced. There is a partition $\Omega = \bigcup_{\lambda \in \Lambda} \Omega_\lambda$ such that, by choosing an arbitrary word $w_\lambda \in \Omega_\lambda$ for every $\lambda \in \Lambda$, the set $\{E_{w_\lambda}\}_{\lambda \in \Lambda}$ of monomials forms a basis for $\boldsymbol{U}^+$. If all words w_λ are chosen to be distinguished* (see Section 5), *the set $\{\mathfrak{m}^{(w_\lambda)}\}_{\lambda \in \Lambda}$ forms a $\mathscr{Z}$-basis for U^+.*

We shall see from Remarks 6.5 that the monomial bases given in [Lusztig 1990], [Ringel 1995] and [Reineke 2001a, 4.2] can be obtained in this systematic description by a selection of the representatives w_λ. The assumption of simply laced types

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is made so that we may directly use the theory of quiver representations. See also [Deng and Du 2005] for a similar result in the affine $\mathfrak{sl}_n$ case. It is natural to expect that a similar result holds in the nonsimply laced case and to relate this theory to Kashiwara's crystal bases defined by using the monomials in Kashiwara operators [1991].

The main ingredients for the proof are Ringel's Hall algebra theory [Ringel 1995], the monoidal structure [Reineke 2001b] on the set $\mathcal{M}$ of isoclasses of finite-dimensional representations of a Dynkin quiver Q and the Bruhat–Chevalley type partial ordering on orbits in an affine space. These will be discussed separately in Sections 2, 3 and 4. Distinguished words are introduced and investigated in Section 5 and we prove the main result in Section 6. As an application of the theory, we mention an elementary construction [Reineke 2001b, §6] of the canonical bases for U^+ as the counterpart of a similar construction for the Hecke algebra in [Kazhdan and Lusztig 1979]. This construction uses the same order as the one used in the geometric construction, involving perverse sheaf and intersection cohomology theories. Finally, more explicit results on distinguished words are worked out for the case of type A in Section 7.

Throughout, k denotes a finite field unless otherwise specified. Let $q_k = |k|$. All modules are finite-dimensional over k. If M is a module, nM, $n \geqslant 0$, denotes the direct sum of n copies of M. Further, by $[M]$ we denote the class of modules isomorphic to M, i.e., the isoclass of M. For modules $M, N_1, \dots, N_t$, let $F^M_{N_1 \cdots N_t}$ denote the number of filtrations

$$M = M_0 \supset M_1 \supset \cdots \supset M_{t-1} \supset M_t = 0$$

such that $M_{i-1}/M_i \cong N_i$ for all $1 \leqslant i \leqslant t$.

2. Ringel–Hall algebras of Dynkin quivers

Let $Q = (I, Q_1)$ be a quiver, i.e., a finite directed graph, where $I = Q_0$ is the set of vertices $\{1, 2, \dots, n\}$ and Q_1 is the set of arrows. If $\rho \in Q_1$ is an arrow from tail i to head j, we write $h(\rho)$ for j and $t(\rho)$ for i. Thus we obtain functions $h, t : Q_1 \to I$. A vertex $i \in I$ is called a sink if there is no arrow ρ with $t(\rho) = i$, and a source if there is no ρ with $h(\rho) = i$.

Let kQ be the path algebra of Q. A (finite-dimensional) *representation* V of Q, consisting of a set of finite-dimensional vector spaces V_i for each $i \in I$ and a set of linear transformations $V_\rho : V_{t(\rho)} \to V_{h(\rho)}$ for each $\rho \in Q_1$, is identified with a (left) kQ-module. We call $\mathbf{dim}\, V := (\dim V_1, \dots, \dim V_n)$ the *dimension vector* of V and $\ell(V) := \sum_{i=1}^n \dim V_i$ the *length* of V (also called the dimension of V). If

Q contains no oriented cycles, there are exactly n pairwise nonisomorphic simple kQ-modules $S_1, \ldots, S_n$ corresponding bijectively to the vertices of Q.

From now on, we assume that Q is a Dynkin quiver, that is, a quiver whose underlying graph is a (simply laced) Dynkin graph. By Gabriel's theorem [1972], there is a bijection between the set of isoclasses of indecomposable kQ-modules and a positive system Φ^+ of the root system Φ associated with Q. For any $\beta \in \Phi^+$, let $M(\beta) = M_k(\beta)$ denote the corresponding indecomposable kQ-module. By the Krull–Remak–Schmidt theorem, every kQ-module M is isomorphic to

$$M(\lambda) = M_k(\lambda) := \bigoplus_{\beta \in \Phi^+} \lambda(\beta) M_k(\beta),$$

for some function $\lambda : \Phi^+ \to \mathbb{N}$. Thus the isoclasses of kQ-modules are indexed by the set

$$\Lambda = \{\lambda : \Phi^+ \to \mathbb{N}\} \cong \mathbb{N}^{|\Phi^+|}.$$

By a result of Ringel [1990], for $\lambda, \mu_1, \ldots, \mu_m$ in Λ, there is a polynomial $\varphi^{\lambda}_{\mu_1 \cdots \mu_m}(q) \in \mathbb{Z}[q]$, called a Hall polynomial, such that for any finite field k of q_k elements

$$\varphi^{\lambda}_{\mu_1 \cdots \mu_m}(q_k) = F^{M_k(\lambda)}_{M_k(\mu_1) \cdots M_k(\mu_m)}.$$

Let $\mathcal{A} = \mathbb{Z}[q]$ be the integral polynomial ring in the indeterminate q. The generic (untwisted) Ringel–Hall algebra $\mathcal{H} = \mathcal{H}_q(Q)$ of Q over $\mathcal{A}$ is by definition the free $\mathcal{A}$-module having basis $\{u_\lambda \mid \lambda \in \Lambda\}$, and satisfying the multiplicative relations

$$u_\mu u_\nu = \sum_{\lambda \in \Lambda} \varphi^{\lambda}_{\mu\nu}(q) u_\lambda.$$

We sometimes write $u_\lambda = u_{[M(\lambda)]}$ in order to make certain calculations in term of modules. For $i \in I$, we set $u_i = u_{[S_i]}$. Clearly, $\mathcal{H}$ admits a natural $\mathbb{N}^n$-grading by dimension vectors.

Following [Ringel 1993b], we can twist the multiplication of the Ringel–Hall algebra to obtain the positive part $\boldsymbol{U}^+$ of a quantized enveloping algebra.

Let $\mathcal{Z} = \mathbb{Z}[v, v^{-1}]$, where v is an indeterminate with $v^2 = q$. The *twisted* Ringel–Hall algebra $\mathcal{H}^\star = \mathcal{H}^\star_v(Q)$ of Q is by definition the free $\mathcal{Z}$-module having basis $\{u_\lambda = u_{[M(\lambda)]} \mid \lambda \in \Lambda\}$ and satisfying the multiplication rules

$$u_\mu \star u_\nu = v^{\langle \mu, \nu \rangle} u_\mu u_\nu = v^{\langle \mu, \nu \rangle} \sum_{\lambda \in \Lambda} \varphi^{\lambda}_{\mu\nu}(v^2) u_\lambda,$$

where $\langle \mu, \nu \rangle = \dim_k \mathrm{Hom}_{kQ}(M(\mu), N(\nu)) - \dim_k \mathrm{Ext}^1_{kQ}(M(\mu), N(\nu))$ is the Euler form associated with the quiver Q. Note that, if we define the bilinear form $\langle -, - \rangle : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ by

$$\langle \boldsymbol{a}, \boldsymbol{b} \rangle = \sum_{i \in I} a_i b_i - \sum_{\rho \in Q_1} a_{t(\rho)} b_{h(\rho)},$$

where $\boldsymbol{a} = (a_1, \cdots, a_n)$, $\boldsymbol{b} = (b_1, \ldots, b_n)$, then

$$\langle \mu, \nu \rangle = \big\langle \mathbf{dim}\, M(\mu), \mathbf{dim}\, M(\nu) \big\rangle.$$

For each $m \geqslant 1$, set $[m] = (v^m - v^{-m})/(v - v^{-1})$ and $[m]^! = [1][2] \cdots [m]$. We define, for each $i \in I$, the divided powers

$$u_i^{(\star m)} = \frac{u_i^{\star m}}{[m]^!} \quad \text{and} \quad E_i^{(m)} = \frac{E_i^m}{[m]^!},$$

in $\mathcal{H}^\star$ and $\boldsymbol{U}^+$, respectively. Here $u_i^{\star m} = \underbrace{u_i \star \cdots \star u_i}_{m} = v^{m(m-1)/2} u_i^m$.

Proposition 2.1 [Ringel 1995, §7]. *The algebra $\mathcal{H}^\star$ is generated by all $u_i^{(\star m)}$, for $i \in I$, $m \geqslant 1$. There is a natural isomorphism*

$$\Psi : U^+ \xrightarrow{\sim} \mathcal{H}^\star, \ E_i^{(m)} \mapsto u_i^{(\star m)} \quad (i \in I, m \geqslant 1).$$

We shall identify U^+ with $\mathcal{H}^\star$ under this isomorphism.

3. Generic extensions and the monoid $\mathcal{M}$

In this section, we collect some recent results on generic extensions for quiver representations over an *algebraically closed* field k.

Fix $\boldsymbol{d} = (d_i)_i \in \mathbb{N}^n$ and define the affine space

$$R(\boldsymbol{d}) = R(Q, \boldsymbol{d}) := \prod_{\alpha \in Q_1} \mathrm{Hom}_k(k^{d_{t(\alpha)}}, k^{d_{h(\alpha)}}) \cong \prod_{\alpha \in Q_1} k^{d_{h(\alpha)} \times d_{t(\alpha)}}.$$

Thus a point $x = (x_\alpha)_\alpha$ of $R(\boldsymbol{d})$ determines a representation $V(x)$ of Q. The algebraic group $\mathrm{GL}(\boldsymbol{d}) = \prod_{i=1}^n \mathrm{GL}_{d_i}(k)$ acts on $R(\boldsymbol{d})$ by conjugation:

$$(g_i)_i \cdot (x_\alpha)_\alpha = (g_{h(\alpha)} x_\alpha g_{t(\alpha)}^{-1})_\alpha.$$

The $\mathrm{GL}(\boldsymbol{d})$-orbits $\mathcal{O}_x$ in $R(\boldsymbol{d})$ correspond bijectively to the isoclasses $[V(x)]$ of representations of Q with dimension vector $\boldsymbol{d}$.

The stabilizer $\mathrm{GL}(\boldsymbol{d})_x = \{g \in \mathrm{GL}(\boldsymbol{d}) \mid gx = x\}$ of x is the group of automorphisms of $M := V(x)$ which is Zariski-open in $\mathrm{End}_{kQ}(M)$ and has dimension equal to $\dim \mathrm{End}_{kQ}(M)$. It follows that the orbit $\mathcal{O}_M := \mathcal{O}_x$ of M has dimension

$$\dim \mathcal{O}_M = \dim \mathrm{GL}(\boldsymbol{d}) - \dim \mathrm{End}_{kQ}(M).$$

Lemma 3.1 [Reineke 2001b]. *Let Q be a Dynkin quiver. For $x \in R(\boldsymbol{d}_1)$ and $y \in R(\boldsymbol{d}_2)$, let $\mathcal{E}(\mathcal{O}_x, \mathcal{O}_y)$ be the set of all $z \in R(\boldsymbol{d})$ where $\boldsymbol{d} = \boldsymbol{d}_1 + \boldsymbol{d}_2$ such that $V(z)$ is an extension of some $M \in \mathcal{O}_x$ by some $N \in \mathcal{O}_y$. Then $\mathcal{E}(\mathcal{O}_x, \mathcal{O}_y)$ is irreducible.*

Given representations M, N of Q, consider the extensions

$$0 \to N \to E \to M \to 0$$

of M by N. By the lemma, there is a unique (up to isomorphism) such extension G with $\dim \mathcal{O}_G$ maximal (i.e., with $\dim \mathrm{End}_{kQ}(G)$ minimal). We call G the *generic extension* of M by N, denoted by $M * N$.

For two representations M, N, we say that M degenerates to N, or that N is a degeneration of M, and write $[N] \leqslant [M]$ (or simply $N \leqslant M$), if $\mathcal{O}_N \subseteq \overline{\mathcal{O}}_M$, the closure of $\mathcal{O}_M$. Note that $N < M \iff \mathcal{O}_N \subseteq \overline{\mathcal{O}}_M \backslash \mathcal{O}_M$.

Remark 3.2. The relation $\leqslant$ on the isoclasses is independent of the field k. This is seen from the following equivalence proved in [Bongartz 1996, Proposition 3.2]:

$$N \leqslant M \iff \dim \mathrm{Hom}(X, N) \geqslant \dim \mathrm{Hom}(X, M) \text{ for all } X \tag{3–1}$$

and the fact that the dimension $\dim \mathrm{Hom}(X, Y)$ is the same over any field. Thus we may simply define a (characteristic-free) partial order on Λ by

$$\lambda \leqslant \mu \iff M_k(\lambda) \leqslant M_k(\mu).$$

for any given (algebraically closed) field k.

The first part of the following result is well-known (see, for example, [Bongartz 1996, 1.1]) and the other parts are proved in [Reineke 2001b].

Theorem 3.3. (1) *If $0 \to N \to E \to M \to 0$ is exact and nonsplit, then $M \oplus N < E$.*

(2) *Let M, N, X be representations of Q. Then $X \leqslant M * N$ if and only if there exist $M' \leqslant M$, $N' \leqslant N$ such that X is an extension of M' by N'. In particular, $M' \leqslant M,\ N' \leqslant N \implies M' * N' \leqslant M * N$.*

(3) *Let $\mathcal{M}$ be the set of isoclasses of kQ-modules and define a multiplication $*$ on $\mathcal{M}$ by $[M] * [N] = [M * N]$ for any $[M], [N] \in \mathcal{M}$. Then $\mathcal{M}$ is a monoid with identity $1 = [0]$ and the multiplication $*$ preserves the induced partial ordering on $\mathcal{M}$.*

(4) *$\mathcal{M}$ is generated by the simple modules $[S_i]$, $i \in I$.*

Let Ω be the set of words in the alphabet $I = \{1, \dots, n\}$. For $w = i_1 i_2 \cdots i_m \in \Omega$, let $\wp(w) \in \Lambda$ be the element defined by

$$[S_{i_1}] * \cdots * [S_{i_m}] = [M(\wp(w))]. \tag{3–2}$$

Thus we obtain a map $\wp : \Omega \to \Lambda$. The theorem shows that $\wp$ is surjective and induces a partition $\Omega = \bigcup_{\lambda \in \Lambda} \Omega_\lambda$ with $\Omega_\lambda = \wp^{-1}(\lambda)$. Each Ω_λ is called a *fibre* of $\wp$.

By Remark 3.2, if we set $\lambda * \mu := M(\lambda * \mu) \cong M(\lambda) * M(\mu)$ for $\lambda, \mu \in \Lambda$, the element $\lambda * \mu$ is well-defined, independent of the field k. Note that the multiplication $*$ on Λ depends on the orientation of Q.

4. The poset Λ

In this section we look at some properties of the poset $(\Lambda, \leqslant)$, where $\leqslant$ is defined in Remark 3.2.

For $w = i_1 i_2 \cdots i_m \in \Omega$ and $\lambda \in \Lambda$, let φ_w^λ denote the Hall polynomial $\varphi_{\mu_1 \cdots \mu_m}^\lambda$, where $M(\mu_r) \cong S_{i_r}$. Thus, for a finite field k,

$$\varphi_w^\lambda(q_k) = F_{S_{i_1 k} \cdots S_{i_m k}}^{M_k(\lambda)}$$

is the number of composition series of $M_k(\lambda)$:

$$M_k(\lambda) = M_0 \supset M_1 \supset \cdots \supset M_{m-1} \supset M_m = 0$$

with $M_{j-1}/M_j \cong S_{i_j k}$. Such a composition series is called a composition series of *type* w.

The following lemma is a bit stronger than [Deng and Du 2005, 6.2].

Lemma 4.1. *Let $w \in \Omega$ and $\mu \geqslant \lambda$ in Λ. Then $\varphi_w^\mu \neq 0$ implies $\varphi_w^\lambda \neq 0$.*

Proof. Let $w = i_1 i_2 \cdots i_m$ and $w' = i_2 \cdots i_m$. We apply induction on m. If $m = 1$ then $\mu \geqslant \lambda$ forces $M(\mu) = M(\lambda)$ and the result is clear. Now assume $m > 1$. If $\varphi_w^\mu \neq 0$, then $\varphi_w^\mu(q_k) \neq 0$ for some finite field k. Thus $M_k(\mu)$ has a submodule $M'_k \cong M_k(\mu')$ having a composition series of type w'. Hence $\varphi_{w'}^{\mu'} \neq 0$, since $\varphi_{w'}^{\mu'}(q_k) \neq 0$. Base change to the algebraic closure $\bar{k}$ of k gives an exact sequence over $\bar{k}$

$$0 \longrightarrow M' \longrightarrow M(\mu) \longrightarrow S_{i_1} \longrightarrow 0,$$

where we have dropped the subscripts $\bar{k}$. Thus

$$M(\lambda) \leqslant M(\mu) \leqslant S_{i_1} * M'.$$

By Theorem 3.3(2), there exist modules N', N'' such that $M(\lambda)$ is an extension of N' by N'' and $N' \leqslant M'$, $N'' \leqslant S_{i_1}$. So we obtain an exact sequence (over $\bar{k}$)

$$0 \longrightarrow N' \xrightarrow{f} M(\lambda) \xrightarrow{g} N'' \longrightarrow 0.$$

Now the condition $N' \leqslant M'$ means $\lambda' \leqslant \mu'$ where $N' \cong M(\lambda')$. Since $\varphi_{w'}^{\mu'} \neq 0$, it follows from induction that $\varphi_{w'}^{\lambda'} \neq 0$, that is, N' has a composition series of type w'. On the other hand, since S_{i_1} is simple, $N'' \leqslant S_{i_1}$ implies $N'' \cong S_{i_1}$. Therefore, $M(\lambda)$ has a composition series of type w, and consequently, $\varphi_w^\lambda \neq 0$. □

We now relate the partial order $\leqslant$ to certain nonzero Hall polynomials.

Theorem 4.2. *Let $\lambda, \mu \in \Lambda$. Then $\lambda \leqslant \mu$ if and only if there exists a word $w \in \wp^{-1}(\mu)$ with $\varphi_w^\lambda \neq 0$.*

Proof. Suppose $\lambda \leqslant \mu$. Since $\wp$ is surjective, $\mu = \wp(w)$ for some $w \in \Omega$. By (3–2), we see that $\varphi_w^{\wp(w)} \neq 0$. Thus Lemma 4.1 implies $\varphi_w^\lambda \neq 0$, as required.

Conversely, let $w = i_1 i_2 \cdots i_m \in \Omega$, $\lambda \in \Lambda$, and suppose $\varphi_w^\lambda \neq 0$. We use induction on m to prove that $\lambda \leqslant \wp(w)$. If $m = 1$, there is nothing to prove. Let $m > 1$ and $w' = i_2 \cdots i_m$ and assume $\lambda' \leqslant \wp(w')$ whenever $\varphi_{w'}^{\lambda'} \neq 0$. Since $\varphi_w^\lambda \neq 0$, there is a finite field k (of any given characteristic) such that $\varphi_w^\lambda(q_k) \neq 0$. Thus there is a submodule M_k' of $M_k(\lambda)$ having a composition series of type w'. This implies $\varphi_{w'}^{\lambda'} \neq 0$ where $M_k(\lambda') \cong M_k'$. By induction, we have $\lambda' \leqslant \wp(w')$.

On the other hand, base change to the exact sequence

$$0 \longrightarrow M_k' \longrightarrow M_k(\lambda) \longrightarrow S_{i_1 k} \longrightarrow 0$$

yields an exact sequence over $\bar{k}$

$$0 \longrightarrow M' \longrightarrow M(\lambda) \longrightarrow S_{i_1} \longrightarrow 0.$$

(Here again we dropped the subscripts $\bar{k}$.) By Theorem 3.3(2) we obtain

$$M(\lambda) \leqslant S_{i_1} * M(\lambda') \leqslant S_{i_1} * M(\wp(w')) = M(\wp(w)).$$

Therefore, $\lambda \leqslant \wp(w)$. □

5. Distinguished words

Let $w = i_1 i_2 \cdots i_m$ be a word in Ω. Then w can be uniquely expressed in the *tight form* $w = j_1^{e_1} j_2^{e_2} \cdots j_t^{e_t}$, where $e_r \geqslant 1$, $1 \leqslant r \leqslant t$, and $j_r \neq j_{r+1}$ for $1 \leqslant r \leqslant t-1$. Following [Ringel 1993a, 2.3], a filtration

$$M = M_0 \supset M_1 \supset \cdots \supset M_{t-1} \supset M_t = 0$$

of a module is called a *reduced* filtration of type w if $M_{r-1}/M_r \cong e_r S_{j_r}$ for all $1 \leqslant r \leqslant t$. Any reduced filtration of M of type w can be refined to a composition series of M of type w. Conversely, given a composition series of M of type w, there is a unique reduced filtration of M of type w such that the given composition series is a refinement of this reduced filtration. By $\gamma_w^\lambda(q)$ we denote the Hall polynomial $\varphi_{\mu_1 \cdots \mu_t}^\lambda(q)$, where $M(\mu_r) = e_r S_{j_r}$. Thus, for a finite field k of q_k elements, $\gamma_w^\lambda(q_k)$ is the number of the reduced filtrations of $M_k(\lambda)$ of type w. A word w is called *distinguished* if $\gamma_w^{\wp(w)}(q) = 1$; this is the case if and only if, for some algebraically closed field k, $M_k(\wp(w))$ has a unique reduced filtration of type w. See [Deng and Du 2005, §5].

Example 5.1. Let $w = j_1{}^{e_1} j_2{}^{e_2} \cdots j_t{}^{e_t}$ be in the tight form. If $j_1, \ldots, j_t$ are pairwise distinct and satisfy

$$\mathrm{Ext}^1_{kQ}(S_{j_r}, S_{j_s}) \neq 0 \implies r < s,$$

then $F^M_{N_1\cdots N_t} = 0$ or 1 for every kQ-module M, where $N_r = e_r S_{j_r}$. Thus w is distinguished.

Distinguished words will be used in the construction of integral monomial bases for the Lusztig form. The following lemma shows that these words are somehow evenly distributed.

Lemma 5.2. *Each fibre of $\wp$ contains at least one distinguished word.*

Proof. This follows directly from [Reineke 2001a, Lemma 4.5]. For completeness, we present here the construction of such distinguished words.

By $\mathcal{I}$ we denote the set of the isoclasses of indecomposable representations of Q. Let $\mathcal{I}_*$ be a *directed* partition of $\mathcal{I}$ [Reineke 2001a, §4], that is, a partition of the set $\mathcal{I}$ into subsets $\mathcal{I}_1, \dots, \mathcal{I}_m$ such that

(a) $\mathrm{Ext}^1_{kQ}(M, N) = 0$ for all M, N in the same part $\mathcal{I}_r$,

(b) $\mathrm{Ext}^1_{kQ}(M, N) = 0 = \mathrm{Hom}_{kQ}(N, M)$ if $M \in \mathcal{I}_r$, $N \in \mathcal{I}_s$, where $1 \leqslant r < s \leqslant m$.

Then, for each $\lambda \in \Lambda$, we have a unique decomposition

$$M(\lambda) = \bigoplus_{r=1}^{m} M_r,$$

where all the summands of M_r belong to $\mathcal{I}_r$, $1 \leqslant r \leqslant m$. Thus

$$\mathrm{Hom}_{kQ}(M_r, M_s) \neq 0 \implies r \leqslant s. \tag{5–1}$$

Further, since Q is a Dynkin quiver, we can order the vertices of Q in a sequence $i_1, i_2, \dots, i_n$ such that, for each $1 < j \leqslant n$, i_j is a sink in the full subquiver of Q with vertices $\{i_1, \dots, i_{j-1}, i_j\}$. Equivalently, $i_1, i_2, \dots, i_n$ are ordered to satisfy

$$\mathrm{Ext}^1_{kQ}(S_{i_j}, S_{i_l}) \neq 0 \implies j < l. \tag{5–2}$$

Let $\boldsymbol{d}^{(r)} = (d_1^{(r)}, \dots, d_n^{(r)}) = \mathbf{dim}\, M_r$, for $1 \leqslant r \leqslant m$, and set

$$w_r = \underbrace{i_1 \cdots i_1}_{d^{(r)}_{i_1}} \cdots\cdots \underbrace{i_n \cdots i_n}_{d^{(r)}_{i_n}}$$

and $w_\lambda = w_1 \cdots w_m \in \Omega$. Then [Reineke 2001a, Lemma 4.5] implies that $\wp(w_\lambda) = \lambda$ and $\gamma^\lambda_{w_\lambda}(q) = 1$, that is, w_λ is distinguished. □

We call the distinguished words constructed above *directed* distinguished words (with respect to the given directed partition $\mathcal{I}_*$).

We mention a special case of directed partitions $\mathcal{I}_*$ where each part $\mathcal{I}_r$ contains only one isoclass. This case is equivalent to ordering the indecomposable modules $M(\beta_1), M(\beta_2), \dots$ such that

$$\mathrm{Hom}_{kQ}(M(\beta_r), M(\beta_s)) \neq 0 \implies r \leqslant s. \tag{5–3}$$

Note that monomial bases associated to these special directed distinguished words have been constructed in [Lusztig 1990] and [Ringel 1995]; see Remarks 6.5 below.

The following example shows that a fibre of $\wp$ could contain many words other than directed distinguished ones.

Example 5.3. Let Q denote the quiver

$$\begin{tikzcd} 1 \arrow[dr] & 2 \arrow[d] & 3 \arrow[dl] \\ & 4 & \end{tikzcd}$$

Let $\lambda \in \Lambda$ be such that $M(\lambda)$ is the indecomposable kQ-module with dimension vector $(1, 1, 1, 2)$. Then $\wp^{-1}(\lambda)$ contains 12 words

$$1234^2,\ 1324^2,\ 2134^2,\ 2314^2,\ 3124^2,\ 3214^2,$$
$$12434,\ 13424,\ 21434,\ 23414,\ 31424,\ 32414$$

all distinguished. From the structure of the Auslander–Reiten quiver of kQ, one sees easily that the first 6 words are directed distinguished, but the last 6 are not.

6. Monomial and integral monomial bases

For $m \geqslant 1$, let $[\![m]\!]^! = [\![1]\!][\![2]\!] \cdots [\![m]\!]$, where $[\![e]\!] = (q^e - 1)/(q-1)$. Then $[\![m]\!] = v^{m-1}[m]$ and $[\![m]\!]^! = v^{m(m-1)/2}[m]^!$.

Lemma 6.1. *Let $w \in \Omega$ be a word with the tight form $j_1^{e_1} j_2^{e_2} \cdots j_t^{e_t}$. Then, for each $\lambda \in \Lambda$,*

$$\varphi_w^\lambda(q) = \gamma_w^\lambda(q) \prod_{r=1}^t [\![e_r]\!]^!.$$

In particular, $\varphi_w^{\wp(w)}(q) = \prod_{r=1}^t [\![e_r]\!]^!$ if w is distinguished.

Proof. The result follows from the definition of a distinguished word and the fact that the number of composition series of eS_i is $[\![e]\!]^!$ (see [Ringel 1993b, 8.2]). $\square$

To each word $w = i_1 i_2 \cdots i_m \in \Omega$, we associate a monomial

$$u_w = u_{i_1} u_{i_2} \cdots u_{i_m} \in \mathcal{H}.$$

Theorem 4.2 and Lemma 6.1 give:

Proposition 6.2. *For each $w \in \Omega$ with the tight form ${j_1}^{e_1} {j_2}^{e_2} \cdots {j_t}^{e_t}$, we have*

$$u_w = \sum_{\lambda \leqslant \wp(w)} \varphi_w^\lambda(q) u_\lambda = \prod_{r=1}^t [\![e_r]\!]^! \sum_{\lambda \leqslant \wp(w)} \gamma_w^\lambda(q) u_\lambda. \tag{6–1}$$

Moreover, the coefficients appearing in the sum are all nonzero.

This improves [Ringel 1995, Theorem 1, p. 96] in two ways: it generalizes the formula from certain directed distinguished words to all words, and it replaces the lexicographical order by the Bruhat type partial order $\leqslant$.

For any commutative ring $\mathcal{A}'$ which is an $\mathcal{A}$-algebra and any $\mathcal{A}$-module M, let $M_{\mathcal{A}'} = \mathcal{A}' \otimes_{\mathcal{A}} M$ denote the $\mathcal{A}'$-module obtained from M by base change to $\mathcal{A}'$.

Theorem 6.3. *For every $\lambda \in \Lambda$, choose an arbitrary word $w_\lambda \in \wp^{-1}(\lambda)$. The set $\{u_{w_\lambda} \mid \lambda \in \Lambda\}$ is a $\mathbb{Q}(q)$-basis of $\mathcal{H}_{\mathbb{Q}(q)}$. If all the w_λ are chosen to be distinguished, then this set is an $\mathcal{A}_{(q-1)}$-basis of $\mathcal{H}_{\mathcal{A}_{(q-1)}}$ where $\mathcal{A}_{(q-1)}$ denotes the localization of $\mathcal{A}$ at the maximal ideal generated by $q-1$.*

Proof. This follows from Proposition 6.2 and the fact that $\varphi_{w_\lambda}^{\wp(w_\lambda)}$ is invertible in $\mathcal{A}_{(q-1)}$ if w_λ is distinguished. □

Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be the Lie algebra over $\mathbb{Q}$ of type Q with generators e_i, f_i, h_i. Let $\mathfrak{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. Define monomials e_w similarly for $w \in \Omega$ in $\mathfrak{U}(\mathfrak{n}_+)$.

Corollary 6.4. *For every $\lambda \in \Lambda$, choose an arbitrary distinguished word $w_\lambda \in \wp^{-1}(\lambda)$. The set $\{e_{w_\lambda} \mid \lambda \in \Lambda\}$ is a $\mathbb{Q}$-basis of $\mathfrak{U}(\mathfrak{n}_+)$.*

Proof. The result follows from the isomorphism $\mathcal{H}_{\mathcal{A}'}/(q-1)\mathcal{H}_{\mathcal{A}'} \cong \mathfrak{U}(\mathfrak{n}_+)$, where $\mathcal{A}' = \mathcal{A}_{(q-1)}$, and Theorem 6.3. □

Proof of Theorem 1.1. For each $w = i_1 i_2 \cdots i_m \in \Omega$ we have

$$u_{i_1} \star \cdots \star u_{i_m} = v^{\varepsilon(w)} u_w,$$

where

$$\varepsilon(w) = \sum_{1 \leqslant r < s \leqslant m} \langle \mathbf{dim}\, S_{i_r}, \mathbf{dim}\, S_{i_s} \rangle.$$

Let, for $w = j_1^{e_1} \cdots j_t^{e_t}$ in tight form,

$$\mathfrak{m}^{(w)} := E_{j_1}^{(e_1)} \cdots E_{j_t}^{(e_t)} = \left(\prod_{r=1}^{t} [e_r]^! \right)^{-1} u_{j_1}^{\star e_1} \star \cdots \star u_{j_t}^{\star e_t}.$$

Since $\prod_{r=1}^t [e_r]^! = v^{-\delta(w)} \prod_{r=1}^t [\![e_r]\!]^!$, where $\delta(w) = \sum_{r=1}^t e_r(e_r - 1)/2$, it follows from Proposition 6.2 that

$$\mathfrak{m}^{(w)} = \left(\prod_{r=1}^{t} [\![e_r]\!]^! \right)^{-1} v^{\delta(w)+\varepsilon(w)} u_w = v^{\delta(w)+\varepsilon(w)} \sum_{\lambda \leqslant \wp(w)} \gamma_w^\lambda(v^2) u_\lambda. \tag{6–2}$$

Together with Proposition 2.1 and Theorem 6.3, this implies Theorem 1.1 with $\Omega_\lambda = \wp^{-1}(\lambda)$ for all $\lambda \in \Lambda$. □

Remarks 6.5. (a) It is clear, from the definition, that the monomial basis $\{E^{(M)}\}$ constructed in [Reineke 2001a, Theorem 4.2] involves only directed distinguished words w_λ.

(b) As a special case of [Reineke 2001a, Theorem 4.2], the monomial bases constructed in [Lusztig 1990, 7.8; Ringel 1995, pp. 101–2] involve only those directed distinguished words defined with respect to the special directed partition $\mathcal{I}_*$ satisfying conditions (5–3) and (5–2); see [Ringel 1995, Theorem 1] and [Lusztig 1990, 4.12(c), 4.13].[1]

We now look briefly at the elementary and algebraic construction of the canonical basis for U^+ [Reineke 2001b, §6]. Note that the elementary constructions given in, e.g., [Lusztig 1990; Kashiwara 1991; Ringel 1995; Chari and Xi 1999] used a finer order than the one used in the geometric construction. We now use the same order which has an algebraic interpretation (3–1).

For each $\lambda \in \Lambda$, set

$$\tilde{\mathfrak{u}}_\lambda = v^{-\dim M(\lambda)+\dim \operatorname{End}(M(\lambda))} u_\lambda.$$

Then, by Proposition 2.1, U^+ is $\mathcal{Z}$-free with basis $\mathcal{E} = \{\tilde{\mathfrak{u}}_\lambda : \lambda \in \Lambda\}$. Note that $U^+ = \bigoplus_{\boldsymbol{d}} U^+_{\boldsymbol{d}}$ is $\mathbb{N}I$-graded according to the dimension vectors, and each $U^+_{\boldsymbol{d}}$ is $\mathcal{Z}$-free with basis $\mathcal{E} \cap U^+_{\boldsymbol{d}} = \{\tilde{\mathfrak{u}}_\lambda : \lambda \in \Lambda_{\boldsymbol{d}}\}$. Clearly, each $\Lambda_{\boldsymbol{d}}$ together with $\leqslant$ is a poset.

Define a ring homomorphism $\iota : U^+ \to U^+$ by setting $\iota(E_i^{(m)}) = E_i^{(m)}$ and $\iota(v) = v^{-1}$. Clearly, ι preserves the grading of U^+. Write, for any $\tilde{\mathfrak{u}}_\lambda \in U^+_{\boldsymbol{d}}$,

$$\iota(\tilde{\mathfrak{u}}_\mu) = \sum_\lambda r_{\lambda,\mu} \tilde{\mathfrak{u}}_\lambda. \tag{6–3}$$

By [Lusztig 1990, 9.10] (see [Du 1994] for more details), the existence of the canonical bases for $U^+_{\boldsymbol{d}}$ follows from the property

$$r_{\lambda,\lambda} = 1,\, r_{\lambda,\mu} = 0 \quad \text{unless } \lambda \leqslant \mu. \tag{6–4}$$

of the coefficients $r_{\lambda,\mu}$. We use (6–2) to derive (6–4). We first calculate $\delta(w)+\varepsilon(w)$ for directed distinguished words; compare [Ringel 1995, Lemma, p. 102].

Lemma 6.6. *We have for any directed distinguished word $w \in \Omega$*

$$\delta(w) + \varepsilon(w) = -\dim M(\wp(w)) + \dim \operatorname{End} M(\wp(w)).$$

Proof. Let $w \in \Omega$ be a directed distinguished word. Then, by definition, there is a directed partition $\mathcal{I}_*$ of $\mathcal{I}$ and a $\lambda \in \Lambda$ such that w has the form $w = w_\lambda = w_1 \cdots w_m$

[1] It seems to us that condition (5–2) was implicitly used in [Lusztig 1990, 7.2], though it was not explicitly stated in the paper.

with

$$w_r = \underbrace{i_1 \cdots i_1}_{d_{i_1}^{(r)}} \cdots\cdots \underbrace{i_n \cdots i_n}_{d_{i_n}^{(r)}},$$

where $M(\lambda) = M_1 \oplus M_2 \oplus \cdots \oplus M_m$, $\boldsymbol{d}^{(r)} = (d_1^{(r)}, \ldots, d_n^{(r)}) = \mathbf{dim}\, M_r$ for $1 \leqslant r \leqslant m$, and the sequence $i_1, i_2, \ldots, i_n$ of vertices are ordered to satisfy (5–2). Clearly,

$$\delta(w) = \sum_{r=1}^{m} \sum_{j=1}^{n} \frac{d_{i_j}^{(r)}\big(d_{i_j}^{(r)} - 1\big)}{2}.$$

Since $\langle \mathbf{dim}\, S_{i_j}, \mathbf{dim}\, S_{i_l} \rangle = 0$ for $j > l$ and $\mathrm{Ext}^1(M_r, M_s) = 0$ for all $1 \leqslant r \leqslant s \leqslant m$, we obtain, for each $1 \leqslant r \leqslant m$,

$$\begin{aligned}
\varepsilon(w_r) &= \sum_{j=1}^{n} \frac{d_{i_j}^{(r)}(d_{i_j}^{(r)} - 1)}{2} \langle \mathbf{dim}\, S_{i_j}, \mathbf{dim}\, S_{i_j} \rangle + \sum_{1 \leqslant j < l \leqslant n} \big\langle \mathbf{dim}\, d_{i_j}^{(r)} S_{i_j}, \mathbf{dim}\, d_{i_l}^{(r)} S_{i_l} \big\rangle \\
&= \langle \mathbf{dim}\, M_r, \mathbf{dim}\, M_r \rangle - \sum_{j=1}^{n} \frac{\big(d_{i_j}^{(r)}\big)^2}{2} - \sum_{j=1}^{n} \frac{d_{i_j}^{(r)}}{2} \\
&= \dim \mathrm{End}(M_r) - \sum_{j=1}^{n} \frac{d_{i_j}^{(r)}\big(d_{i_j}^{(r)} + 1\big)}{2}
\end{aligned}$$

and therefore,

$$\begin{aligned}
\varepsilon(w) &= \sum_{r=1}^{m} \varepsilon(w_r) + \sum_{1 \leqslant r < s \leqslant m} \langle \mathbf{dim}\, M_r, \mathbf{dim}\, M_s \rangle \\
&= \sum_{r=1}^{m} \varepsilon(w_r) + \sum_{1 \leqslant r < s \leqslant m} \dim \mathrm{Hom}(M_r, M_s).
\end{aligned}$$

Noting that $\mathrm{Hom}(M_r, M_s) = 0$ for $r > s$, we finally obtain

$$\begin{aligned}
\delta(w) + \varepsilon(w) &= \sum_{r=1}^{m} \dim \mathrm{End}(M_r) + \sum_{1 \leqslant r < s \leqslant m} \dim \mathrm{Hom}(M_r, M_s) - \sum_{r=1}^{m} \sum_{j=1}^{n} d_{i_j}^{(r)} \\
&= \dim \mathrm{End}(M(\lambda)) - \dim M(\lambda).
\end{aligned}$$

This completes the proof. □

By Lemma 6.6 and (6–2), any directed distinguished word w satisfies

$$\mathfrak{m}^{(w)} = \tilde{\mathfrak{u}}_{\wp(w)} + \sum_{\lambda < \wp(w)} f_{\lambda, \wp(w)} \tilde{\mathfrak{u}}_{\lambda}, \tag{6–5}$$

where $0 \neq f_{\lambda,\wp(w)} \in \mathscr{Z}$. If we fix a representative set $\Lambda' = \{w_\lambda : \lambda \in \Lambda\}$, where $w_\lambda \in \Omega_\lambda$, consisting of directed distinguished words, the relation above implies that, for any $\mu \in \Lambda$,

$$\tilde{\mathfrak{u}}_\mu \in \mathfrak{m}^{(w_\mu)} + \sum_{\lambda<\mu} \mathscr{Z}\mathfrak{m}^{(w_\lambda)}.$$

Restricting to $\Lambda_{\boldsymbol{d}}$, where $\boldsymbol{d}$ is a fixed dimension vector, we obtain the transition matrix $(f_{\lambda,\mu})_{\lambda,\mu\in\Lambda_{\boldsymbol{d}}}$. This matrix has an inverse $(g_{\lambda,\mu})_{\lambda,\mu\in\Lambda_{\boldsymbol{d}}}$ satisfying $g_{\lambda,\lambda} = 1$ and $g_{\lambda,\mu} = 0$ unless $\lambda \leqslant \mu$. Thus

$$\tilde{\mathfrak{u}}_\mu = \mathfrak{m}^{(w_\mu)} + \sum_{\lambda<\mu} g_{\lambda,\mu}\mathfrak{m}^{(w_\lambda)}.$$

Applying ι, we obtain by (6–5)

$$\iota(\tilde{\mathfrak{u}}_\mu) = \mathfrak{m}^{(w_\mu)} + \sum_{\lambda<\mu} \bar{g}_{\lambda,\mu}\mathfrak{m}^{(w_\lambda)} = \tilde{\mathfrak{u}}_\mu + \sum_{\lambda<\mu} r_{\lambda,\mu}\tilde{\mathfrak{u}}_\lambda. \tag{6–6}$$

This proves that the coefficients in (6–3) satisfy (6–4). Thus the corresponding canonical basis $\{\mathfrak{c}_\lambda\}_{\lambda\in\Lambda}$ is uniquely defined.

Remarks 6.7. (a) The canonical basis defined above is the same as Lusztig's canonical basis. This is because the basis $\mathscr{E}$ is a PBW type basis (see [Ringel 1996, Theorem 7]). We also note that, as in the Hecke algebra case [Kazhdan and Lusztig 1979; 1980], the partial order used in this construction is the same as the one used in the geometric construction (see [Lusztig 1990, §9]).

(b) The relation (6–6) is derived via directed distinguished words. However, it can be used to prove the following result,[2] which generalizes the formula given in Lemma 6.6 to all distinguished words. Thus we may also use nondirected distinguished words in the construction above to obtain canonical bases.

Proposition 6.8. *For any distinguished word $w \in \Omega$, we have*

$$\delta(w) + \varepsilon(w) = -\dim M(\wp(w)) + \dim \operatorname{End} M(\wp(w)).$$

Proof. Let $w \in \Omega$ be distinguished. By (6–2), we have

$$\mathfrak{m}^{(w)} = v^s\tilde{\mathfrak{u}}_{\wp(w)} + \sum_{\lambda<\wp(w)} h_{\lambda,\wp(w)}\tilde{\mathfrak{u}}_\lambda, \tag{6–7}$$

where $s = \delta(w)+\varepsilon(w)+\dim M(\wp(w))-\dim \operatorname{End} M(\wp(w))$ and $0 \neq h_{\lambda,\wp(w)} \in \mathscr{Z}$ for $\lambda < \wp(w)$. By applying ι to (6–7), we deduce from (6–6) that

$$\iota(\mathfrak{m}^{(w)}) = v^{-s}\tilde{\mathfrak{u}}_{\wp(w)} + \sum_{\lambda<\wp(w)} d_{\lambda,\wp(w)}\tilde{\mathfrak{u}}_\lambda$$

[2] We thank the referee for pointing out the proof.

for some $d_{\lambda,\wp(w)} \in \mathcal{Z}$. Since $\iota(\mathfrak{m}^{(w)}) = \mathfrak{m}^{(w)}$, equating coefficients yields $v^s = v^{-s}$. This implies $s = 0$, that is,

$$\delta(w) + \varepsilon(w) = -\dim M(\wp(w)) + \dim \operatorname{End} M(\wp(w)). \qquad \square$$

7. The type A case

We now give a combinatorial description of the map $\wp : \Omega \to \Lambda$ for the linear quiver

$$Q = A_n : 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n-1 \longrightarrow n.$$

We also give an explicit description of the distinguished words in this case. Since A_n is a subquiver of a cyclic quiver, the results obtained below and their proofs are similar to (or even simpler than) those given in [Deng and Du 2005], and the proofs will mostly be omitted.

It is known that, for $1 \leqslant i \leqslant j \leqslant n$, there is a unique (up to isomorphism) indecomposable kA_n-module M_{ij} with top S_i and of length $j - i + 1$, and all M_{ij}, $1 \leqslant i \leqslant j \leqslant n$, form a complete set of nonisomorphic indecomposable kA_n-modules. By Gabriel's theorem, each M_{ij} corresponds to a positive root β_{ij}. Thus $\Phi^+ = \{\beta_{ij} \mid 1 \leqslant i \leqslant j \leqslant n\}$. For each map $\lambda \in \Lambda$, we set $\lambda_{ij} = \lambda(\beta_{ij})$. First, we have the following positivity result, which can be proved by counting and induction on the length of w (compare [Deng and Du 2005, Proposition 9.1]).

Proposition 7.1. *For each $w \in \Omega$ and each $\lambda \in \Lambda$, the polynomial φ_w^λ lies in $\mathbb{N}[q]$.*

Now, for each $i \in I$, we define a map $\sigma_i : \Lambda \to \Lambda$ as follows. For $\lambda \in \Lambda$, if S_{i+1} is not a summand of $M(\lambda)/\operatorname{rad} M(\lambda)$ (i.e., $\lambda_{i+1,l} = 0$ for all l), then $\sigma_i\lambda$ is obtained by adding 1 to λ_{ii} so that $M(\sigma_i\lambda) = M(\lambda) \oplus S_i$; otherwise, $\sigma_i\lambda$ is defined by

$$(\sigma_i\lambda)_{rs} = \begin{cases} \lambda_{rs} & \text{if } (r, s) \neq (i, j), (i+1, j), \\ \lambda_{ij} + 1 & \text{if } (r, s) = (i, j), \\ \lambda_{i+1,j} - 1 & \text{if } (r, s) = (i+1, j), \end{cases}$$

where j is the maximal index with $\lambda_{i+1,j} \neq 0$. We have the following (compare [Deng and Du 2005, Proposition 3.7]).

Proposition 7.2. *Let $i \in I$ and $\lambda \in \Lambda$. Then $S_i * M(\lambda) \cong M(\sigma_i\lambda)$. Therefore $\wp(w) = \sigma_{i_1} \cdots \sigma_{i_m}(0)$ for any $w = i_1 \cdots i_m \in \Omega$.*

Let $w = j_1^{e_1} j_2^{e_2} \cdots j_t^{e_t} \in \Omega$ be in the tight form. For each $0 \leqslant r \leqslant t$, we put $w_r = j_{r+1}^{e_{r+1}} \cdots j_t^{e_t}$ and $\lambda^{(r)} = \wp(w_r)$. In particular, $w_0 = w$ and $w_t = 1$. Further, for $r \geqslant 1$, we have

$$\lambda^{(r-1)} = \wp(w_{r-1}) = \underbrace{\sigma_{j_r} \cdots \sigma_{j_r}}_{e_r}(\lambda^{(r)}).$$

The following result gives a combinatorial description of distinguished words (compare [Deng and Du 2005, 5.5]).

Proposition 7.3. *Let $w = j_1^{e_1} j_2^{e_2} \cdots j_t^{e_t} \in \Omega$ and $\lambda^{(r)}$, with $0 \leqslant r \leqslant t$, be given as above. Then w is distinguished if and only if, for each $1 \leqslant r \leqslant t$, either $\lambda^{(r)}_{j_r j} = 0$ for all $j_r \leqslant j \leqslant n$, or $e_r \leqslant \sum_{a=l_r+1}^{n} \lambda^{(r)}_{j_r+1,a}$ where l_r is the maximal index for which $\lambda^{(r)}_{j_r l_r} \neq 0$.*

Proof. Using a similar argument as in [Deng and Du 2005, Theorem 5.5], one can show that w is distinguished if and only if, for each $1 \leqslant r \leqslant t$, $M(\lambda^{(r-1)})$ admits a unique submodule isomorphic to $M(\lambda^{(r)})$. However, the latter condition is equivalent to the described combinatorial condition, as shown in [Deng and Du 2005, Lemma 5.4]. □

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