Pacific Journal of Mathematics

KEPLER'S SMALL STELLATED DODECAHEDRON AS A RIEMANN SURFACE

MATTHIAS WEBER

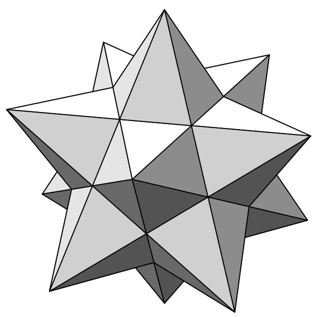
Volume 220 No. 1 May 2005

PACIFIC JOURNAL OF MATHEMATICS
Vol. 220, No. 1, 2005

KEPLER'S SMALL STELLATED DODECAHEDRON AS A RIEMANN SURFACE

MATTHIAS WEBER

We provide a new geometric computation for the Jacobian of the Riemann surface of genus 4 associated to the small stellated dodecahedron. Starting with Threlfall's description, we introduce other flat conformal geometries on this surface which are related to holomorphic 1-forms. They allow us to show that the Jacobian is isogenous to a fourfold product of a single elliptic curve whose lattice constant can be determined in two essentially different ways, yielding an unexpected relation between hypergeometric integrals. We also obtain a new platonic tessellation of the surface.



1. Introduction

In his *Harmonice Mundi*, Kepler [1619] considers regular shapes in 2 and 3 dimensions. Besides the classical convex regular polygons he describes regular star polygons, so it is natural to allow also polyhedra that have such star polygons as faces. He comes up with several examples, among them the small stellated dodecahedron. It is therefore plausible that he didn't consider the 60 triangles of the stellated dodecahedron as its natural faces but the 12 star pentagons. This given, the polyhedron has 12 vertices and 30 edges, so the Euler formula gives

$$V - E + F = 12 - 30 + 12 = -6$$

MSC2003: 30F30.
Keywords: Jacobians, flat structures, small stellated dodecahedron.

which is not the Euler characteristic of the sphere but of a Riemann surface of genus 4. This was first observed by Poinsot and started some confusion about the validity of Euler's formula; see [Lakatos 1976].

All this can be resolved by viewing each star pentagon as a Riemann surface with a branch point in the center: The same way a regular pentagon is composed of 5 isosceles triangles with angle $2\pi/5$, the regular pentagram is composed by 5 isosceles triangles with angle $4\pi/5$. In fact, one can try to imagine the stellated dodecahedron as an immersed surface where each star pentagon is realized as a branched pentagon whose center branch point is hidden by a stellating pyramid. In this way, the stellated dodecahedron inherits from its singular euclidean metric a conformal structure and becomes a compact Riemann surface Σ of genus 4 whose automorphism group contains at least the icosahedral group.

This possibility was probably first observed by Klein [1877], who showed that the Riemann surface defined in $\mathbb{P}^4$ as the complete intersection

$$\sum_{i=1}^{5} z_i = 0, \quad \sum_{i=1}^{5} z_i^2 = 0, \quad \sum_{i=1}^{5} z_i^3 = 0$$

is biholomorphic to Kepler's small stellated dodecahedron. We will briefly discuss this in Section 4.

Threlfall [1932] gives a detailed description of the pentagon tessellation of this genus 4 surface Σ in terms of hyperbolic geometry. In particular, he finds another tessellation of the same surface by quadrilaterals such that 10 meet in one vertex. Because he is working in hyperbolic geometry, it is clear a priori that these two tessellations live on the same Riemann surface. Though Threlfall mentions the term Riemann surface frequently, he is interested neither in the properties of this particular surface as an algebraic curve nor in its automorphism group.

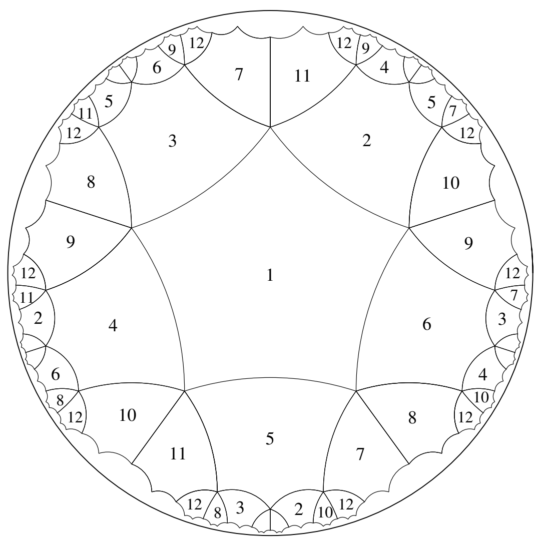
We will conformally replace the quadrilaterals in Threlfalls's description by other euclidean quadrilaterals to obtain new locally flat structures on the surface. These lead directly to a basis of holomorphic 1-forms by taking the exterior derivative of the developing maps of the flat structures. As the periods of the 1-forms are determined by the geometric data of the new metrics, we obtain easily a period matrix for the surface. In particular:

Theorem 1.1. *The Jacobian of Σ is isogenous to a 4-fold product of a rhombic torus. Its lattice constant can be computed either using the Schwarz–Christoffel formula for the new quadrilaterals or via the modular invariant of this torus.*

Remark. G. Riera and R. E. Rodríguez [1992] follow quite a different approach to compute the Jacobian of Σ: They first show that some 1-parameter family of polarized abelian varieties of dimension 4 is stabilized under the only 4-dimensional symplectic irreducible representation of S_5. Then they determine the parameter (implicitly) using an algebraic characterization of the quotient tori $\Sigma/\langle\phi\rangle$ and $\Sigma/(\mathbb{Z}/2\mathbb{Z})^2$ that differs from our description in Section 6.

2. A hyperbolic metric on the stellated dodecahedron

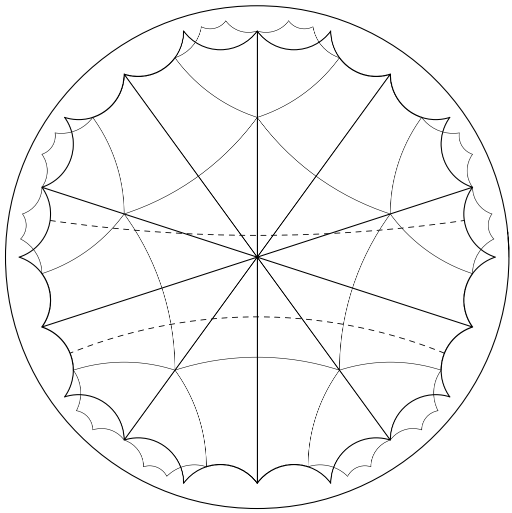
We now view the small stellated dodecahedron as a surface of genus 4, which comes with a natural tessellation by 12 star pentagons. Each star pentagon can be obtained by gluing together 5 isosceles euclidean triangles with obtuse angle $4\pi/5$. Map such a triangle conformally to a hyperbolic $(2\pi/5, 2\pi/10, 2\pi/10)$-triangle and continue this map by reflection first to the star pentagon. We obtain a conformal map from the star pentagon to a regular hyperbolic $2\pi/5$-pentagon. Continuing again by reflection to the whole surface yields a nonsingular conformal hyperbolic metric on the surface which is now tessellated by these hyperbolic pentagons. Here is the lift of this tessellation to the hyperbolic plane; the numbers designate the 12 faces:



Our next goal is to derive Threlfall's tessellation of the surface by hyperbolic quadrilaterals. The key for this is the rotation ρ of order-5 of the stellated dodecahedron around the axes through two opposite vertices. These vertices are two fixed points, but there are two more, namely the branch points of the dodecahedron faces which are intersected by the rotation axes. Hence the quotient $\Sigma/\langle\rho\rangle$ is a four-punctured sphere. More precisely:

Lemma 2.1. Σ *is a fivefold cyclic branched covering over the four-punctured sphere whose conformal structure is obtained by doubling a square. Using four branch slits* γ_i *from the center of one of the squares to the corners, the covering is* *given by gluing together five copies of the sphere thus slit, so that the left edge of slit γ_i of copy j is glued to the right edge of slit γ_i of copy $j+d_i$, where $d_i = 1, 2, 4, 3$.*

Sketch of proof. This statement can be proved by analyzing the next figure, where we have added to the 72° pentagon tessellation 10 fat hyperbolic $2\pi/10$-squares.



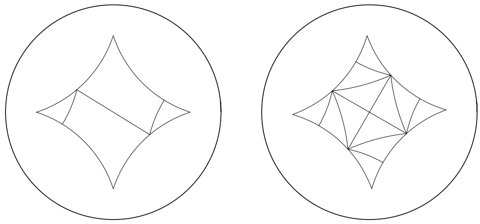
Using the figure on the previous page, one checks that these 10 squares constitute a fundamental domain for the surface. The edges are identified according to the two dashed geodesics and the order-5 rotational symmetry around the center of the figure. Now it is clear that two adjacent squares constitute a fundamental domain of the group $\langle\rho\rangle$ on Σ. The faces of these two squares have to be glued together by "flipping over", i.e., the quotient has the conformal structure claimed.

To see that the description of the covering in the lemma gives the same fundamental domain is straightforward; see [Threlfall 1932]. $\square$

We digress a bit to discuss also the other natural automorphisms of the surface:

The order-3 rotation around an axes through two opposite vertices of the *unstellated* dodecahedron defines a fixed point free automorphism τ of Σ which can be seen in the hyperbolic picture as a translation along the lower identification geodesic by 1/3 of its length. The quotient surface $\Sigma/\langle\tau\rangle$ is a nonsingular surface of genus 2 which comes with a tessellation by 4 hyperbolic 72°-pentagons; it is discussed in detail in [Threlfall 1932].

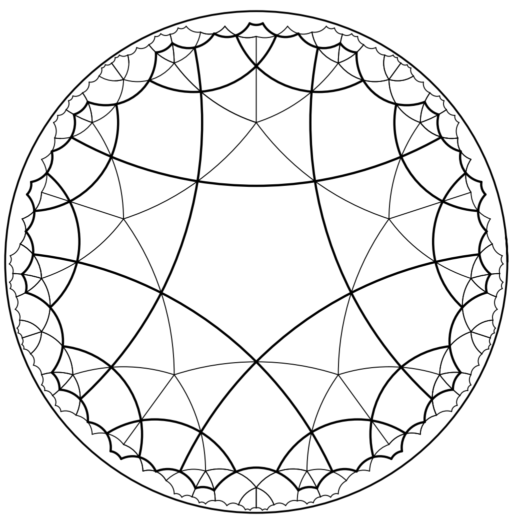
One can also obtain an order-2 rotation around the midpoints of the dodecahedron edges. But it turns out that this automorphism is actually the square of an order-4 rotation ϕ which is (of course) not an automorphism of the euclidean polyhedral structure on Σ but a conformal automorphism. That this rotation is really well defined on Σ becomes clear if we convince ourselves that the midpoints of some pentagon edges are also the centers of the quadrilaterals:



The left picture shows one of the quadrilaterals moved to a central position with the pentagon geodesics inside. Comparing the angles of the (congruent) triangles in the right picture with the two triangles in the left one shows easily the claimed symmetry.

To actually define this automorphism ϕ one can check that an order-4 rotation of one square is compatible with the identifications. One also finds a second fixed point, so that by the Riemann–Hurwitz formula, the quotient surface $\Sigma/\langle\phi\rangle$ is a torus. Because there are many different such automorphisms, this observation is the first indication that the Jacobian of Σ might be quite interesting. The investigation of this torus will be one of our primary goals.

Another way to see this automorphism is by looking at a new platonic tessellation of Σ by 24 right-angled regular pentagons:

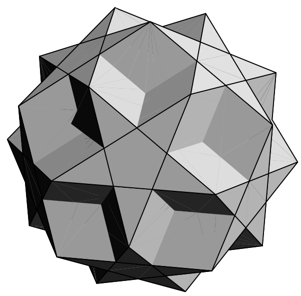


The figure shows the previous pentagon tessellation and the new one with thick lines. The order-4 rotation becomes a rotation around a vertex of this (preserved) tessellation. From this picture one can also deduce that ϕ has two fixed points.

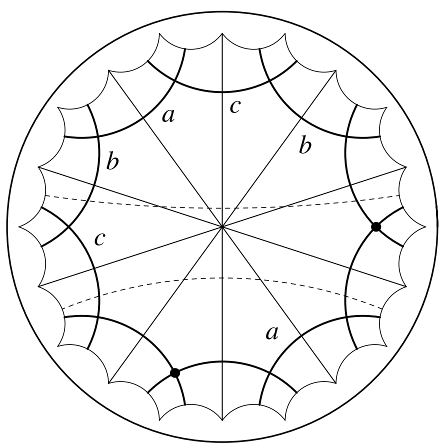
Furthermore, the thick lines are defined as geodesics connecting midpoints of adjacent pentagon edges: The sequence of edges hit by such a geodesic constitutes a *Petri polygon*; see [Coxeter and Moser 1972] for details.

The vertices of the 90° pentagons are either centers of the quadrilaterals or midpoints of the 72° pentagon edges.

This tessellation has also a euclidean realization as a euclidean uniform polyhedron, the so-called *dodecadodecahedron*, which is thus recognized as another (new) conformal version of Kepler's dodecahedron. This polyhedron has both regular pentagons and star-pentagons as faces:



The central right-angled regular decagon in the next figure shows a fundamental domain for the rotation ϕ on Σ. The fixed points are marked by a dot, and the nonadjacent edges are to be identified according to the labels.



This fundamental domain allows us to construct a degree-5 map from the quotient torus $T = \Sigma/\langle\phi\rangle$ to the sphere which is branched only over 3 points, as follows. Decompose the regular decagon into ten $(45°, 45°, 36°)$-triangles with vertices at the decagon vertices and its center. Map one of these triangles to the upper half-plane and continue by reflection. In principle, such a map pins down the conformal structure of the torus, but in general it is very hard to determine (say) the modular invariant of the torus from this map.

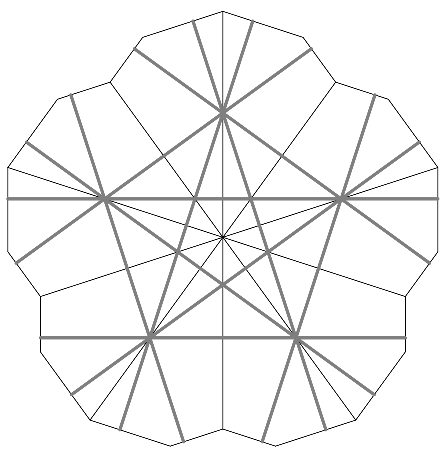
Proposition 2.2. *The automorphism group of* Σ *is* S_5, *the symmetric group of* 5 *elements*.

Proof. We know that Aut Σ contains the icosahedral group A_5 and has order at least 120. Assume that the automorphism group is strictly larger, that is, at least of order 240. Now the standard proof of Hurwitz's theorem about the order of the automorphism group of a compact Riemann surface forces Aut Σ to be a (2, 3, 7)-triangle group. But S_5 contains no element of order 7, so Aut Σ had to have at least $7 \cdot 120$ elements which contradicts the conclusion of Hurwitz's theorem. $\square$

3. Σ as an algebraic curve

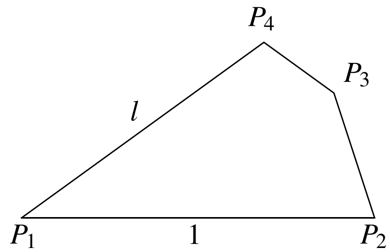
In this section, we construct a base of holomorphic 1-forms on Σ and derive an algebraic equation.

The first holomorphic 1-form ω_1 can be visualized by the following figure:



This is another fundamental domain of Σ, using euclidean quadrilaterals instead of hyperbolic $2\pi/10$-squares as in the figure on page 170. The identifications (which are indicated by the shaded lines) are realized by euclidean parallel translations. This is because we have chosen the quadrilateral with angles $\pi/5$, $2\pi/5$, $4\pi/5$, $3\pi/5$. Hence this description gives a singular flat metric on Σ with trivial linear holonomy. This means that the exterior derivative of the locally defined developing map of this flat metric is a globally well-defined holomorphic 1-form on Σ. Its zeros coincide with the singular points of this metric: Whenever the angles at a point add up to $k \cdot 2\pi$, the holomorphic 1-form will have a zero of order $k - 1$.

Hence the 1-form ω_1 defined by the preceding figure has divisor $P_2+3P_3+2P_4$, where the points are located as follows:



Unfortunately, up to now we haven't proved that the fundamental domain above defines the correct conformal structure on Σ. In fact, this is impossible, because we haven't really specified *which* quadrilateral we are going to use for this construction. To guarantee that the resulting surface is biholomorphic to Σ, it is sufficient to ensure that the chosen quadrilateral is biholomorphic to any square, or, by the Riemann mapping theorem, to the upper half-plane with vertices at -1, 0, 1, ∞.

We do not know how explicitly it is possible to find such a quadrilateral, but at least we know these data in terms of Schwarz–Christoffel integrals. Denote by e_i the edge P_iP_{i+1}. Then

$$\begin{aligned} e_1 &= \int_{-1}^{0} (t-1)^{-1/5} t^{-3/5} (t+1)^{-4/5} dt, \\ e_2 &= \int_{0}^{1} (t-1)^{-1/5} t^{-3/5} (t+1)^{-4/5} dt, \\ e_3 &= \int_{1}^{\infty} (t-1)^{-1/5} t^{-3/5} (t+1)^{-4/5} dt, \\ e_4 &= \int_{-\infty}^{-1} (t-1)^{-1/5} t^{-3/5} (t+1)^{-4/5} dt. \end{aligned}$$

Denote by $l_i = |e_i/e_1|$ the corresponding normalized edge lengths, with $l = l_4$. By trigonometry,

$$\begin{aligned} l_1 &= 1, & l_2 &= -1 + l\frac{\sqrt{5}+1}{2} \approx 0.373129, \\ l_3 &= \frac{\sqrt{5}+1}{2}(1-l) \approx 0.244905, & l_4 &= l \approx 0.848641. \end{aligned}$$

Now three more holomorphic 1-forms ω_i can be defined using *the same* quadrilateral: Because it is conformally a square, we can permute the vertices cyclically.

This results in cyclically permuted divisors:

	P_1	P_2	P_3	P_4
ω_1	0	1	3	2
ω_2	1	3	2	0
ω_3	3	2	0	1
ω_4	2	0	1	3

Using this, we can derive an algebraic equation for Σ:

Proposition 3.1. *Σ is biholomorphic to the algebraic curve defined by the affine equation*

$$y^5 = (x+1)x^2(x-1)^{-1}.$$

Proof. Denote by $x : \Sigma \to \mathbb{P}^1$ the branched quotient map $\Sigma \to \Sigma/\rho$, where we choose the images of the branch points to be $-1, 0, 1, \infty$, which is possible by symmetry. Hence

$$\big((x+1)x^2(x-1)^{-1}\big) = P_1^5 + P_2^{10} + P_3^{-5} - P_4^{-10}.$$

Now put $y = \omega_2/\omega_1$ and obtain the same divisor for y^5. After scaling y appropriately, the equation follows. □

The function y will be explained geometrically in the next section.

4. Excursion: Bring's curve

In this section we show why the small stellated dodecahedron is biholomorphic to Bring's curve B, which is the complete intersection in $\mathbb{P}^4$ of the three hypersurfaces

$$\sum_{i=1}^{5} z_i = 0, \quad \sum_{i=1}^{5} z_i^2 = 0, \quad \sum_{i=1}^{5} z_i^3 = 0,$$

This was first shown by Klein [1877; 1884]. Bring's curve B occurs naturally as the locus of solutions of the reduced quintic equation

$$z^5 + pz + q = 0$$

because the vanishing of the coefficients of z^2, z^3, z^4 is equivalent to the equations above.

For projective properties of B, see [Edge 1978].

Following Klein, we first construct a threefold branched covering

$$\pi_1 : \Sigma \to \mathbb{P}^1$$

which is branched twice at all 72°-pentagon vertices. This is done by mapping the hyperbolic $(2\pi/5, 2\pi/10, 2\pi/10)$-triangle that constitutes one fifth of the tesselating 72°-pentagon onto a spherical $(2\pi/5, 2\pi/5, 2\pi/5)$-triangle, and continuing this map by reflection. The image of all the triangles will form the icosahedral tessellation of the sphere. Each vertex has two preimages: one is a branched pentagon vertex, the other an unbranched pentagon midpoint.

There is also a second such map π_2, using the dual 72°-pentagon tessellation instead. Both of these maps can be given explicitly in terms of the 1-forms ω_i: By considering divisors we see easily that (up to normalization)

$$\omega_1 \omega_3 = \omega_2 \omega_4,$$

so that we have an explicit equation of the quadric Q on which the canonical curve of Σ lies. Now the projections on the respective factors of $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$ are given by the meromorphic functions

$$z \mapsto \omega_2/\omega_1 \quad \text{and} \quad z \mapsto \omega_4/\omega_1,$$

which have precisely the same branching behavior as the functions π_i above. This shows also that π_1 is proportional to the function y from the last section. We leave to the reader the transformation of the ω_i to the z_j and the proof that the latter then satisfy the cubic equation as well. See also [Edge 1978; Klein 1884, 1877, Slodowy 1986].

5. The Jacobian of Σ

In this section, we compute the Jacobian of Σ in terms of tenth roots of unity and the constant l of Section 3, which is the ratio of two hypergeometric functions. This also allows us to compute the lattice of the quotient tori.

To compute the Jacobian, we first choose an appropriate base for the homology of Σ. This base will not be canonical but adapted to our representation of Σ as a branched covering over a 4-punctured sphere. Denote by c_k the curve on Y that winds k times around P_1, then once around P_2 and finally as often around P_1 as is necessary to lift to a closed curve on Σ. Similarly, denote by $\tilde{c}_k$ the curve on Y that winds k times around P_2, then once around P_3 and finally as often around P_2 as is necessary to lift to a closed curve on Σ.

For the holomorphic 1-forms, we take the ω_j of Section 3. Here we are still free to choose a normalization. Because we intend to compute also the lattice of the quotient torus of Σ by the order-4 rotation subgroup $\langle\phi\rangle$, we will eventually need a nonzero holomorphic 1-form that is invariant under this rotation ϕ and whose periods we can compute. If we normalize the ω_i in such a way that $\phi^*\omega_i = \omega_{i+1}$, the 1-form $\omega = \omega_1 + \omega_2 + \omega_3 + \omega_4$ will do. This normalization can be achieved by

(1) taking the same sized quadrilateral for the different 1-forms and just relabeling the vertices, and

(2) fixing the developing map for all of them simultaneously.

Using these two normalizations, we obtain

Lemma 5.1. *Denote by* $\zeta = e^{2\pi i/10}$ *and by* $\Phi = \frac{1}{2}(\sqrt{5}+1)$. *For* $i = 1, 2, 3, 4$, *set* $\alpha_i = 2^i \pi/5$ *reduced modulo* 2π. *Indices are to be taken cyclically. Then*

$$\int_{c_k} \omega_j = e^{ki\alpha_j} e_j (1 - e^{i\alpha_{j+1}}),$$
$$\int_{\tilde{c}_k} \omega_j = e^{ki\alpha_{j+1}} e_{j+1} (1 - e^{i\alpha_{j+2}})$$

Hence the period matrix of the Jacobian with respect to the ω_j *and the cycles* $c_0, \ldots, c_3, \tilde{c}_0, \ldots, \tilde{c}_3$ *is given by*

$$\Omega = \left(\begin{array}{c} \zeta^{2k}(1-\zeta^4) \\ \zeta^{4k+7}(1-\zeta^8)(-1+l\Phi) \\ \zeta^{8k+6}(1-\zeta^6)\Phi(1-l) \\ \zeta^{6k+4}(1-\zeta^2)l \end{array} \Bigg|_{k=0}^{3} \quad \begin{array}{c} \zeta^{4k+7}(1-\zeta^8)(-1+l\Phi) \\ \zeta^{8k+6}(1-\zeta^6)\Phi(1-l) \\ \zeta^{6k+4}(1-\zeta^2)l \\ \zeta^{2k}(1-\zeta^4) \end{array} \Bigg|_{k=0}^{3} \right)$$

Proof. To compute the period of an ω_k, we use the definition of ω_k by a flat metric on the 4-punctured sphere which is given by doubling the quadrilateral of figure 9. Because the developing map of the flat metric is the integral of the corresponding 1-form, the period can be read off from the picture: Winding around a vertex P_j changes the direction into which we develop by the cone angle at P_j, and the loop from P_j to P_{j+1}, around this point and back to P_j contributes the factor $e_j(1 - e^{i\alpha_{j+1}})$. The rest is straightforward computation. □

For a similar computation, see [Karcher and Weber 1999].

This construction also shows that ρ acts on the 1-forms by multiplication with roots of unity:

$$\omega_1 \mapsto \zeta^2\omega_1, \quad \omega_2 \mapsto \zeta^4\omega_2, \quad \omega_3 \mapsto \zeta^8\omega_3, \quad \omega_4 \mapsto \zeta^6\omega_4.$$

This is because ρ changes the direction of the developing map by a rotation of order 5 if we choose the base point for the development in one of the fixed points, and the amount depends on the respective cone angle in this point.

Because we haven't normalized our homology base, the polarization of the Jacobian still has to be computed. We do this by giving the intersection matrix of the cycles:

Lemma 5.2. *The intersection matrix of the cycles* $c_0, \dots, c_3, \tilde{c}_0, \dots, \tilde{c}_3$ *is given by*

$$I = \begin{pmatrix} 0 & 1 & 1 & -1 & -1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 1 & 0 & 1 & 0 & -1 \\ -1 & -1 & 0 & 1 & 0 & -1 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & -1 & 1 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & -1 & -1 & 0 & 1 \\ -1 & 1 & 0 & 0 & 1 & -1 & -1 & 0 \end{pmatrix}.$$

The proof is straightforward but tedious and we omit it.

The claims may be checked by verifying the Riemann period conditions

$$\Omega I^{-1}\Omega^t = 0 \quad \text{and} \quad -i\Omega I^{-1}\overline{\Omega}^t > 0.$$

In fact,

$$\begin{aligned} -i\Omega I^{-1}\overline{\Omega}^t &= \left(-5\zeta^2 - 5\zeta^3 + 10\,l\,\Phi\,\left(\zeta^2+\zeta^3\right) - 5\,l^2\,(1+\Phi)\,\left(\zeta+\zeta^4\right)\right) Id \\ &\approx 5.52531 Id. \end{aligned}$$

Corollary 5.3. *The lattice of the quotient torus* $\Sigma/\langle\phi\rangle$ *is spanned by*

$$\begin{aligned} \tau_1 &= (1+\zeta)^2\left(-1+l+\zeta-\zeta^2\right) \approx 1.79303 - 0.321884i, \\ \tau_2 &= (1+\zeta)\left(-1+2l-l\zeta+\zeta^2+l\zeta^2-l\zeta^3\right) \approx 1.26139 + 1.31433i, \\ \tau_2/\tau_1 &= \frac{-1+\zeta^2+(1+\zeta^{-1})l}{-1+\zeta^{-2}+(1+\zeta)l} = \bar{\zeta}\cdot\frac{l-\zeta(1-\zeta)}{l-\bar{\zeta}(1-\bar{\zeta})} \approx 0.554051 + 0.832482i. \end{aligned}$$

Proof. We have to show that the periods $\pi_j, \tilde{\pi}_j$ of $\omega = \omega_1+\omega_2+\omega_3+\omega_4$ constitute this lattice. By Lemma 5.1 we have $\tilde{\pi}_j = \pi_j$ and $\pi_0 = \tau_1,\ \pi_1 = \tau_2,\ \pi_2 = -2\tau_1+\tau_2,$ $\pi_3 = 0,\ \pi_4 = \tau_1 - 2\tau_2.$ □

Remark. The specific value of l is only defined by the condition that our euclidean quadrilateral has to be a square. This also means that the formulas above do not make sense for any other surface.

We have computed the Jacobian of Σ and found at least three different quotient maps from Σ to tori. The relationship between all these tori will now be clarified.

Lemma 5.4. *Let* Γ *be a lattice in* $\mathbb{C}^n$ *and* $\alpha_1, \dots, \alpha_n$ *be* n *linearly independent linear functionals on* $\mathbb{C}^n$ *such that* $\Gamma_i = \alpha_i(\Gamma)$ *is a lattice in* $\mathbb{C}$. *Then* $\mathbb{C}^n/\Gamma$ *is isogenous to the product* $\mathbb{C}/\Gamma_1 \times \cdots \times \mathbb{C}/\Gamma_n$.

Proof. The regular linear map $\alpha_1 \times \cdots \times \alpha_n : \mathbb{C}^n \to \mathbb{C}^n$ induces a holomorphic Lie group homomorphism $\mathbb{C}^n/\Gamma \to \mathbb{C}/\Gamma_1 \times \cdots \times \mathbb{C}/\Gamma_n$. If this map had a nondiscrete kernel, there would be a $v \in \mathbb{C}^n - \{0\}$ such that $\alpha_i(v) = 0$ for all i, contradicting the linear independence of the α_i. $\square$

Corollary 5.5. Jac Σ *is isogenous to the product* $T \times T \times T \times T$.

Proof. The idea is to conjugate the map ϕ by ρ to obtain enough different quotient maps to the same torus. In our base of the lattice, the functional $z \mapsto z_1+z_2+z_3+z_4$ describes the map to the quotient torus induced by the quotient map $\Sigma \to \Sigma/\langle\phi\rangle$. Now we can as well consider the quotient maps associated to the conjugate maps $\rho^{-k}\phi\rho^k$ which are different quotient maps to the same torus. By the definition of the ω_i, ρ acts on them by multiplication as

$$\omega_i \mapsto \zeta^{2^i}\omega_i.$$

Thus $\rho^{-1}\phi\rho$ acts as

$$\omega_1 \mapsto \zeta^8\omega_2, \quad \omega_2 \mapsto \zeta^6\omega_3, \quad \omega_3 \mapsto \zeta^2\omega_4, \quad \omega_4 \mapsto \zeta^4\omega_1$$

and hence the induced map from Jac $\Sigma \to$ Jac T is described by the functional $z \mapsto \zeta^4 z_1+\zeta^8 z_2+\zeta^6 z_3+\zeta^2 z_4$. Similarly, the functionals $z \mapsto \zeta^8 z_1+\zeta^6 z_2+\zeta^2 z_3+\zeta^4 z_4$ and $z \mapsto \zeta^2 z_1 + \zeta^4 z_2 + \zeta^8 z_3 + \zeta^6 z_4$ describe the maps induced by $\rho^{-2}\phi\rho^2$ and $\rho^{-3}\phi\rho^3$. These 4 functionals are clearly independent, and the claim follows from the previous lemma. $\square$

Corollary 5.6. *All holomorphic image tori of* Σ *are isogenous.*

Proof. Any holomorphic surjective map $f : \Sigma \to E$ to an elliptic curve induces a group homomorphism f : Jac $\Sigma \to$ Jac $E = E$. This map cannot be trivial on all factors of Jac Σ; hence there is a nontrivial restriction $f_1 : T \to E$ that is necessarily a covering. $\square$

6. An algebraic equation for the quotient torus

In this section we derive an algebraic equation for the quotient torus $T = \Sigma/\langle\phi\rangle$ and compute its modular invariant. The arithmetic nature of this torus has been investigated by Serre [1980], and an equation is given (without proof) in [Slodowy 1986].

Our strategy for producing such an equation is as follows: Using the representation of Σ as a branched covering over the four-punctured sphere, we construct a degree-3 function y and a degree-4 function w on Σ having poles of order at most 2 and 3, respectively, and only at the branch points of the covering $\pi : \Sigma \to \Sigma/\langle\rho\rangle$. Averaging this function over the action of ϕ yields functions of degrees 2 and 3 on the quotient torus T. To determine an equation, we investigate these functions at their poles.

To start, we need to understand the action of ϕ in terms of the equation

$$y^5 = (x+1)x^2(x-1)^{-1}$$

(see Section 3). Recall that y represents a function on Σ with divisor $P_1 + 2P_2 - P_3 - 2P_4$ and x has branch points of order 5 with values $-1, 0, 1, \infty$ at the P_i. This implies that the new function

$$z = y^2/x$$

has divisor $2P_1 - P_2 - 2P_3 + P_4$ and is therefore proportional to the function π_2 from Section 5. From the two equations above one easily obtains

$$(*) \qquad yz^2 = \frac{y^2+z}{y^2-z}$$

and this equation reflects the order-4 automorphism ϕ as the map

$$y \mapsto z \quad z \mapsto -1/y.$$

Hence the average

$$Y = y + z - 1/y - 1/z$$

of Y will descend to T as a function with one double-order pole at the image of the P_i. Similarly, the function

$$w = y/z$$

on Σ has divisor $-P_1 + 3P_2 + P_3 - 3P_4$ and the average

$$W = \frac{y}{z} - \frac{1}{yz} + \frac{z}{y} - yz$$

descends to T as a function with one triple-order pole at the image of the P_i. We keep the names Y and W for the functions on T.

This means that there are constants $a, b, c, d, e, f \in \mathbb{C}$ such that

$$(**) \qquad (W - aY)^2 - bY^3 - cY^2 - dW - eY - f \equiv 0.$$

To determine them, we compute this expression on Σ in a neighborhood of P_1, using y as a local coordinate. Note that

$$z = -y^2 + O(y^7)$$

because $x = z/y^2$ has a branch point of order-5 with value -1 at P_1. This leads to

$$\frac{1-b}{y^6} + \frac{-2a+3b}{y^5} + \frac{-2+2a+a^2-3b-c}{y^4} + \frac{2a-2a^2-2b+2c-d}{y^3} + \frac{-1-4a+a^2+9b-c-e}{y^2} + O(y^{-1}) = 0,$$

which determines the first 5 constants as

$$a = \tfrac{3}{2}, \quad b = 1, \quad c = \tfrac{1}{4}, \quad d = -3, \quad e = 4.$$

Putting this back into $(**)$ gives

$$h = (y^3+y\,z+y^5\,z+y^2\,z^2+y^3\,z^2-y^4\,z^2+z^3+y^2\,z^3-4\,y^3\,z^3-y^4\,z^3-y^6\,z^3 \\ -y^2\,z^4-y^3\,z^4+y^4\,z^4+y\,z^5+y^5\,z^5-y^3\,z^6)/(y^3\,z^3),$$

which reduces to -4 using $(*)$.

Hence we obtain the desired equation in Y and W:

$$4-4\,Y-\frac{Y^2}{4}-Y^3+3\,W+\Big(\frac{-3\,Y}{2}+W\Big)^2=0.$$

In new variables this equation can be brought into the form

$$y^2 = 4x^3 - 75x - 1475.$$

These equations allow to compute the modular invariant λ of T as the cross ratio of ∞ and the three algebraic numbers

$$\tfrac{1}{8}\Big(\big(\tfrac{11}{5}\big)^{1/3}(59-24\sqrt{6})^{1/3}+5^{2/3}(59+24\sqrt{6})^{1/3}-13\Big),$$
$$(5^{2/3}/16)\,\Big((-1+i\sqrt{3})(59-24\sqrt{6})^{1/3}-(1+i\sqrt{3})(59+24\sqrt{6})^{1/3}-26\Big),$$
$$(5^{2/3}/16)\,\Big(-(1+i\sqrt{3})(59-24\sqrt{6})^{1/3}-(1-i\sqrt{3})(59+24\sqrt{6})^{1/3}-26\Big),$$

which gives roughly

$$\lambda \approx 0.660609 - 0.75073i.$$

This modular invariant can be used to compute the periods of the quotient torus in a different way. One obtains the period quotient τ_2/τ_1 of T as a quotient of two hypergeometric integrals, but this time as

$$\frac{\tau_2}{\tau_1} = \frac{\int_1^\infty u^{-1/2}(u-1)^{-1/2}(u-\lambda)^{-1/2}du}{\int_0^1 u^{-1/2}(u-1)^{-1/2}(u-\lambda)^{-1/2}du}.$$

Combining this expression with Corollary 5.3 gives an unexpected identity between hypergeometric integrals.

Acknowledgment

The author thanks Hermann Karcher for his interest.

References

[Coxeter and Moser 1972] H. S. M. Coxeter and W. O. J. Moser, *Generators and relations for discrete groups*, Third ed., Ergebnisse der Math. und ihrer Grenzgebiete **14**, Springer, New York, 1972. MR 50 #2313 Zbl 0239.20040

[Edge 1978] W. L. Edge, "Bring's curve", *J. London Math. Soc.* (2) **18**:3 (1978), 539–545. MR 80a: 14014 Zbl 0397.51013

[Karcher and Weber 1999] H. Karcher and M. Weber, "The geometry of Klein's Riemann surface", pp. 9–49 in *The eightfold way*, Math. Sci. Res. Inst. Publ. **35**, Cambridge Univ. Press, Cambridge, 1999. MR 2001a:14027 Zbl 0964.14029

[Kepler 1619] J. Kepler, *Harmonices mundi libri V*, Linz, 1619. Translated as *The harmony of the world* by E. J. Aiton et al., Philadelphia, Amer. Philos. Society, 1997.

[Klein 1877] F. Klein, "Weitere Untersuchungen über das Ikosaeder", *Math. Annalen* **12** (1877), 321–384.

[Klein 1884] F. Klein, *Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade*, Teubner, Leipzig, 1884. Reprinted Birkhäuser, Basel, 1993 (edited by P. Slodowy); translated as *Lectures on the icosahedron and the solution of equations of the fifth degree*, Kegan Paul, London, 1913 (2nd edition); reprinted by Dover, 1953. MR 96g:01046

[Lakatos 1976] I. Lakatos, *Proofs and refutations: the logic of mathematical discovery*, Cambridge University Press, Cambridge, 1976. MR 58 #122 Zbl 0334.00022

[Riera and Rodríguez 1992] G. Riera and R. E. Rodríguez, "The period matrix of Bring's curve", *Pacific J. Math.* **154**:1 (1992), 179–200. MR 93e:14033 Zbl 0734.30036

[Serre 1980] J.-P. Serre, "Extensions icosaedriques", in *Séminaire de Théorie des Nombres de Bordeaux, Année* 1979–1980, exposé no. 19, 1980. Reprinted as pp. 550–554 in *Œuvres*, vol. III, Springer, Berlin, 1985. Zbl 0474.12011

[Slodowy 1986] P. Slodowy, "Das Ikosaeder und die Gleichungen fünften Grades", pp. 71–113 in *Arithmetik und Geometrie*, Math. Miniaturen **3**, Birkhäuser, Basel, 1986. MR 88i:01056

[Threlfall 1932] W. Threlfall, "Gruppenbilder", *Abh. Sächs. Akad. Wiss. Leipzig Math.-Natur. Kl.* **41**:6 (1932), 1–59. Zbl 0004.02202 JFM 58.0132.01

Received June 3, 1999. Revised January 27, 2004.

MATTHIAS WEBER
DEPARTMENT OF MATHEMATICS
DEPARTMENT OF MATHEMATICS
RAWLES HALL
INDIANA UNIVERSITY
BLOOMINGTON, IN 47405
matweber@indiana.edu