EQUIVARIANT SPECTRAL FLOW AND A LEFSCHETZ THEOREM ON ODD-DIMENSIONAL SPIN MANIFOLDS

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We present a heat kernel proof of an equivariant index theorem on odd-dimensional spin manifolds, stated for Toeplitz operators. A notion of equivariant spectral flow is raised and plays an important role in our proof.

Introduction

Atiyah–Singer index theorems have profound applications and consequences, and can be proved in several ways. Of particular interest is the heat kernel proof, which allows one to obtain refinements such as local index theorems for Dirac operators. See [Berline et al. 1992] for a comprehensive treatment of the heat kernel method on even-dimensional manifolds. The heat kernel method also leads to direct analytic proofs of the equivariant index theorem for Dirac operators on even-dimensional spin manifolds; see [Bismut 1984; Berline and Vergne 1985; Lafferty et al. 1992].

This paper presents a heat kernel proof of an equivariant index theorem on odd-dimensional spin manifolds, stated for Toeplitz operators.

Baum and Douglas [1982] first stated and proved an odd-index theorem for Toeplitz operators using the general Atiyah–Singer index theorem for elliptic pseudodifferential operators. It is known that one can give a heat kernel proof of such a theorem; we describe briefly the basic ideas involved. The first step is to apply a result of Booß and Wojciechowski [1993] to identify the index of the Toeplitz operator to the spectral flow of a certain family of self-dual elliptic operator with positive order. The second step is then to use the relationship between spectral flows and variations of \( \eta \)-invariants to evaluate this spectral flow; see [Getzler 1993].

Our proof of the equivariant odd-index theorem follows the same strategy. We need to introduce a concept of equivariant spectral flow and establish an equivariant version of the Booß–Wojciechowski theorem mentioned above. We then extend the relationship between the spectral flow and variations of \( \eta \) invariants to the

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equivariant setting. Finally, we use the local index techniques to evaluate these variations.

For simplicity we follow the method of Lafferty, Yu and Zhang [1992], but there is no difficulty in applying other methods, such as those of [Bismut 1984] or [Berline and Vergne 1985].

Dai and Zhang [1998] introduced the concept of higher spectral flow and gave a heat kernel treatment to the family index problem for Toeplitz operators. They also proved recently, after a draft of this paper was finished, an index theorem for Toeplitz operators on odd-dimensional manifolds with boundary [Dai and Zhang 2001]. It would be interesting to prove an equivariant version of their theorem, which might result in new fixed-point theorems.

This paper is organized as follows. In Section 1 we review the basic definition of the Toeplitz operators associated to Dirac operators on odd-dimensional spin manifolds and prove the equivariant odd index theorem by using the Baum–Douglas trick [1982] and also the general Atiyah–Singer [1968] Lefschetz fixed-point theorem for elliptic pseudodifferential operators. In Section 2 we introduce the equivariant spectral flow and prove an equivariant extension of the Booß–Wojciechowski theorem [1993]. In Section 3 we establish a relation between the equivariant spectral flow and the variations of equivariant \( \eta \) invariants. This in turn gives a heat kernel formula for the equivariant index of the Toeplitz operators. In Section 4 we evaluate these variations by adopting the local index theorem techniques.

1. Toeplitz operators and a Lefschetz fix point theorem

We begin by fixing notations on odd-dimensional Clifford algebras that are used in this paper. From now on, we fix \( n = 2m + 1 \), where \( m \) is a positive integer.

Let \( V \) be a \( n \)-dimensional real vector space with a positive inner product and some orthonormal basis \( e_1, \ldots, e_n \). Set

\[
T(V) = \mathbb{R} \oplus V \oplus V \otimes V \oplus \cdots ,
\]

and let \( I \) to be the two-sided ideal of \( T(V) \) generated by \( \{ x \otimes x + (x, x)1 : x \in V \} \).

The Clifford algebra associated to \( V \) is defined as

\[
C(V) = T(V)/I ,
\]

and is also denoted \( C(n) \). Clearly \( \{ c_i = e_i f I \in C(V) \} \) is the set of generators of \( C(V) \) satisfying the relations

\[
c_i c_j + c_j c_i = -2\delta_{ij}.
\]
Define the chirality operator of $C(n) \otimes \mathbb{C}$ to be
$$\Gamma = (\sqrt{-1})^{m+1}c_1 \ldots c_n;$$
it lies in the center of $C(n) \otimes \mathbb{C}$. There is a unique irreducible complex $C(n)$-representation $S$ of dimension $2^m$ such that $\Gamma = \text{Id}_S$ on $S$.

For future use, we define the symbol map $\sigma : C(n) \to \text{End} \wedge^n \mathbb{C}^n$ by
\begin{equation}
\sigma(c_i) = e_i \wedge -\iota(e^*_i),
\end{equation}
with $\iota$ denoting contraction with elements of $V^*$. Thus $\sigma$ is a complex representation of $C(n)$. For $x \in C(n)$ nonscalar, we have
\begin{equation}
\text{Tr}_S(x) = -\sqrt{-1}(\sqrt{-1}^{2m}\sigma(x) 1)_{[n]},
\end{equation}
where $(\cdot)_{[d]}$, for any integer $d$, denotes the $d$-dimensional part of an exterior form.

We proceed to define the Toeplitz operator of interest. Throughout this paper we assume $M$ to be a closed oriented spin manifold of dimension $n = 2m + 1$, with a fixed spin structure. We also fix a Riemannian metric $g_{TM}$ on $M$.

Let $S(M)$ be the canonical complex spinor bundle of $M$; this is also a $C(T^*M)$-module. The canonical Levi-Civita connection induces a natural connection $\nabla^S$ on $S$. Choose a local orthonormal basis $e_1, \ldots, e_n$ is for $TM$, with dual basis $e^1, \ldots, e^n \in T^*M$. The canonical Dirac operator on $S$ can be defined as
\begin{equation}
D^S = \sum c(e^j)\nabla^S_{e_i};
\end{equation}
it is a self-adjoint first-order elliptic differential operator acting on $S(M)$, and so it induces a spectral decomposition of $\Gamma_{L^2}(S)$. Denote by $L^2_+(S)$ the direct sum of eigenspaces of $D$ associated to nonnegative eigenvalues, and by $P_+$ the orthogonal projection operator from $L^2(S)$ to $L^2_+(S)$. Set $P_- = \text{Id} - P_+$ and
\begin{equation}
P = P_+ - P_-.
\end{equation}

Given a trivial complex vector bundle $\mathbb{C}^N$ over $M$ carrying the trivial metric and connection, $D$ and $P$ extend trivially as operators acting on $\Gamma(S \otimes \mathbb{C}^N)$. Let $g : M \to U(N)$ be a smooth map. Then $g$ extends to an action on $S(M) \otimes \mathbb{C}^N$ as $\text{Id}_{S(M)} \otimes g$, still denoted by $g$.

**Definition 1.1.** The Toeplitz operator associated to $D$ and $g$ is
$$T_g = (P_+ \otimes \text{Id}_{\mathbb{C}^N})g(P_+ \otimes \text{Id}_{\mathbb{C}^N}) : L^2_+(S(M) \otimes \mathbb{C}^N) \to L^2_+(S(M) \otimes \mathbb{C}^N).$$

It is a classical fact that $T_g$ is a bounded Fredholm operator between the given Hilbert spaces. If we define $\Gamma_\lambda$ to be the eigenspace of $D$ with eigenvalue $\lambda$, $\Gamma_\lambda$ is of finite dimension for each $\lambda$.

We then describe the equivariant index problem for Toeplitz operators.
Consider a compact group $H$ of isometries of $M$ preserving the orientation and spin structure. $H$ also acts on $\Gamma(S(M) \otimes \mathbb{C}^N)$; since its action commutes with the Dirac operator $D$, it also commutes with $P_+$ and $P$. Furthermore each $\Gamma_\lambda$ is $H$-invariant. But to ensure the $H$-invariance of Toeplitz operator, we need an additional assumption on $H$:

\[(1–5) \quad g(hx) = g(x) \quad \text{for any } h \in H \text{ and any } x \in M.\]

As a consequence,

\[T_g h_{\Gamma(S(M) \otimes \mathbb{C}^N)} = h_{\Gamma(S(M) \otimes \mathbb{C}^N)} T_g.\]

**Definition 1.2.** Given $T_g$ and $H$ as above and satisfying (1–5), the equivariant index of $T_g$ associated with $H$ is the following virtual representation of $H$ in $R(H)$, the representation ring of $H$:

\[\text{Ind}_H(T_g) = \ker T_g - \text{coker } T_g.\]

We also write, for any $h \in H$,

\[(1–6) \quad \text{Ind}(h, T_g) = \text{Tr}(h, \text{Ind}_H(T_g)).\]

An application of the general Atiyah–Singer index theorem [1968] as in [Baum and Douglas 1982] gives:

**Theorem 1.3.** For $T_g$ as above, let $F_i$’s be the fixed, connected submanifolds of $M$ under the action of any $h \in H$, and $v_i$ the normal bundle of $F_i$ in $TM$. Then

\[
\text{Ind}(h, T_g) = \sum_i \left( \left( \frac{-\sqrt{-1}}{2\pi} \right)^{(1+\dim F_i)/2} \times \hat{A}(F) \, \text{ch}(g) \left[ \text{Pf} \left( 2 \sin \left( \frac{1}{2} \sqrt{-1} (R^v(F_i) + \Theta_i) \right) \right) \right]^{-1} [F_i] \right),
\]

where, in any local coordinate system, $\Theta_i$ is the logarithm of the Jacobian matrix of $h|_{v_i}$, $R^v(F_i)$ is the curvature matrix of the bundle $v_i$, and

\[(1–7) \quad \text{ch}(g) = \int_0^1 \text{Tr}[g^{-1}dg \exp(u(1-u)(g^{-1}dg)^2)] du \]

is the odd Chern character for the differentiable map $g : M \to U(N)$.

In Section 4 we will prove a local version of this theorem.

2. Equivariant spectral flow and equivariant index problem

Set $I = [0, 1]$. Let $\{D_u\}_{u \in I}$ be a continuous family of self-dual elliptic operators of positive order on the Hilbert space $\mathcal{H} = L^2(S \otimes \mathbb{C}^N)$. For any fixed $u \in I$, $\text{Spec } D_u$
is discrete, and we denote the corresponding eigenspaces by \(\Gamma_{u,\lambda}\) for \(\lambda \in \text{Spec } D_u\). For any open \(U \subset \mathbb{R}\), define \(\Gamma_{u,U} = \bigoplus_{\lambda \in U} \Gamma_{u,\lambda}\).

Recall first the usual (scalar) spectral flow [Atiyah et al. 1976].

**Definition 2.1.** For \(D_u(u \in I)\) a continuous family of self-dual elliptic operator of positive order, consider the graph of \(\text{Spec } D_u\):

\[
\mathcal{S} = \bigcup \text{Spec } D_u,
\]

which is a closed set of \(\mathbb{R} \times I\). The spectral flow \(\text{sf}(\{D_u\})\) of \(\{D_u\}\) is the intersection number of \(\mathcal{S}\) with the line \(\{-\delta\} \times I\) for sufficiently small positive \(\delta\). (If both \(D_0\) and \(D_1\) are invertible, we can replace \(\delta\) by 0 in this definition.)

We would like to extend this notion to the equivariant case. Let \(H\) be as in Section 1 and \(R(H)\) its representation ring. Assume that each \(D_u\) in the preceding discussion is compatible with the action of \(H\). Thus every \(\Gamma_{u,\lambda}\) can be viewed as an element of \(R(H)\).

The next result is an extension of the continuity of the spectra for a family of self-dual operators.

**Lemma 2.2.** Let \(\{D_u\}\) be as above. For a fixed \(u_0 \in I\) and any \(\lambda \in \text{Spec } D_{u_0}\) with \(\dim \Gamma_{u_0,\lambda} = k\), we can find a positive \(\epsilon\) such that for any \(u \in I\) satisfying \(|u - u_0| < \epsilon\) there is an open set \(U\), containing \(\lambda\) and depending only on \(u_0\), such that \(\dim \Gamma_{u,U} = k\).

Furthermore,

\[
(2\text{-}1) \quad \Gamma_{u,U} = \Gamma_{u_0,\lambda}
\]

as elements of \(R(H)\).

**Proof.** The equality \(\dim \Gamma_{u,U} = k\) is actually proved in [Booß-Bavnbek and Wojciechowski 1993, Lemma 17.1]. More precisely, the discussion there shows that there exist a \(\epsilon > 0\) and \(k\) continuous functions \(f_1, \ldots, f_k : (u_0 - \epsilon, u_0 + \epsilon) \to \mathbb{R}\) such that \(f_j(u_0) = \lambda\); furthermore, for any \(u \in (u_0 - \epsilon, u_0 + \epsilon)\), there exists an open set \(U\) that depends only on \(u_0\) and contains \(\lambda\) satisfying

\[
\{f_j(u)\}_{j=1}^k = \text{Spec } D_u \cap U.
\]

Let \(Q_u\), for \(u_0 - \epsilon < u < u_0 + \epsilon\), be the orthonormal projections of \(\Gamma_{u,U}\) onto \(\Gamma_{u_0,U}\). By the continuity of \(\{D_u\}\) and the \(f_j\)'s, \(\{Q_u\}\) is a continuous family of self-adjoint projections. Thus it is possible to readjust \(\epsilon\) if necessary so that

\[
\|Q_u - Q_{u_0}\| < 1,
\]
for \( u_0 - \epsilon < u < u_0 + \epsilon \). Now using a trick from [Reed and Simon 1978, p. 72], if we define
\[
W_u = \left( 1 - (Q_u - Q_{u_0})^2 \right)^{-1/2} \left( Q_u Q_{u_0} + (1 - Q_u)(1 - Q_{u_0}) \right),
\]
it is easy to verify that \( W_u \) is unitary and
\[
W_u^{-1} Q_u W_u = Q_{u_0}.
\]
The image of \( Q_u \) is \( \Gamma_{u, u} \) and the construction above is \( H \)-compatible, so (2–1) easily follows. \( \square \)

We proceed to define the equivariant spectral flow. Given \( \{D_u\} \) as above, set
\[
\text{Spec}_H D_u = \{(\lambda, \Gamma_{u, \lambda}) : \lambda \in \text{Spec} D_u\}.
\]
By Lemma 2.2 and the fact that \( R(H) \) has only countable many irreducible elements, there exist \( f_j(u) \in C(I) \) and \( R_j \in R(H) \), for \( j \in \mathbb{N} \), such that
\[
\bigcup_{u \in I} \text{Spec} D_u = \bigcup_j (f_j(u), R_j).
\]

**Definition 2.3.** Given \( D_u \) as above, the equivariant spectral flow of \( D_u \) is
\[
\text{sf}_H(\{D_u\}) = \sum_j \epsilon(f_j) R_j,
\]
where \( \epsilon(f_j) \) is the intersection number of the graph \( f_j \) with the line \( u = -\delta \) for sufficiently small positive \( \delta \). Also set
\[
\text{sf}(h, \{D_u\}) = \text{Tr}(h, \text{sf}_H(\{D_u\})).
\]

**Remark 2.4.** It is not hard to see that, as in the scalar case, only finite many \( \epsilon(f_k) \)'s in this definition are nonzero. If both \( D_0 \) and \( D_1 \) are invertible, \( \delta \) can be replaced by 0.

**Remark 2.5.** As in the scalar case, the equivariant spectral flow is a homotopy invariant. In particular, let \( E_\sigma \) be the affine space of all the elliptic, positive-ordered operators in \( \mathcal{H} \) with the same symbol \( \sigma \). Then \( E_\sigma \) is convex, and hence contractible. Therefore, given any two fixed \( H \)-compatible points in \( E_\sigma \), the equivariant spectra of two different \( H \)-compatible paths connecting them are the same.

**Remark 2.6.** Applying the method above, it is not hard to extend the notion of higher spectral flow in the sense of Dai and Zhang [1998] to the equivariant setting. We leave the details to the reader.

For the rest of this paper, we pick a particular family \( \{D_u\} \):
\[
(2–2) \quad D_u = (1 - u) D^5 \otimes \text{Id}_{\mathbb{C}^N} + u g^{-1} (D^5 \otimes \text{Id}_{\mathbb{C}^N}) g,
\]
where $D^S$ is the Dirac operator defined in (1–3) and $D^S \otimes \text{Id}_{C^N}$ is its extension to the Hilbert space $\mathcal{H} = L^2(S \otimes C^N)$. This family $\{D_u\}$ satisfy the conditions in Definitions 2.1 and 2.3. All the $D_u$'s have the same symbol, denoted by $\sigma$.

The next theorem clarifies the relation between the equivariant spectral flow and our original index problem. The method used here is from [Booß-Bavnbek and Wojciechowski 1993].

**Theorem 2.7.** $\text{Ind}(h, T_g) = -\text{sf}(h, \{D_u\})$.

**Proof.** Set $P_u = (1 - u)P + ug^{-1}Pg$, where $P$ is defined in (1–4). Apply the same argument used in the proof of [Booß-Bavnbek and Wojciechowski 1993, Theorem 17.17], which is compatible with our equivariant setting, to conclude that

$$\text{sf}_H(\{D_u\}) = \text{sf}_H(\{P_u\}).$$

Straightforward calculation gives

$$\text{ker} T_g = \{u \in P_+\mathcal{H}, gu \in P_-\mathcal{H}\}, \quad \text{coker} T_g = \{u \in P_-\mathcal{H}, gu \in P_+\mathcal{H}\}.$$

Then it is easy to see that

$$P_u(v) = \begin{cases} v & \text{if } v \in P_+\mathcal{H}, \ g v \in P_+\mathcal{H}, \\ -v & \text{if } v \in P_-\mathcal{H}, \ g v \in P_-\mathcal{H}, \\ (1 - 2u)v & \text{if } v \in \text{ker} T_g, \\ (2u - 1)v & \text{if } v \in \text{coker} T_g. \end{cases}$$

The equality in Theorem 2.7 follows from this and (2–3). \qed

### 3. Equivariant spectral flow and equivariant eta functions

Eta invariants first appeared in [Atiyah et al. 1975] and have a close relation with the spectral flow; see [Bismut and Freed 1986; Getzler 1993]. In this section we extend this relation to the equivariant case.

**Definition 3.1.** Let $D$ be a self-adjoint operator on the Hilbert space $\mathcal{H}$. The eta function associated to $D$ is defined as

$$\eta(s, D) = \sum_{\lambda \neq 0} \text{sign } \lambda \frac{\dim \Gamma_\lambda}{|\lambda|^s},$$

where $\text{Re } s$ is large enough, $\lambda$ runs over the nonzero eigenvalues of $D$ and $\Gamma_\lambda$ is the eigenspace of $D$ with eigenvalue $\lambda$.

It is then clear that

$$\eta(s, D) = \frac{1}{\Gamma\left(\frac{1}{2}(s+1)\right)} \int_0^\infty \text{Tr}(De^{-tD^2}) t^{(s-1)/2} dt.$$
By a result from [Bismut and Freed 1986], the eta function of $D$ is analytic for $\text{Re} \ s > -\frac{1}{2}$; in particular, we write

$$\eta(D) = \eta(0, D).$$

Define the truncated $\eta$ function, for $\epsilon > 0$, to be

$$\eta_{\epsilon}(s, D) = \frac{1}{\Gamma\left(\frac{1}{2}(s+1)\right)} \int_{\epsilon}^{\infty} \text{Tr}(De^{-tD^2}) t^{(s-1)/2} dt$$

and write

$$\eta_{\epsilon}(D) = \eta_{\epsilon}(0, D).$$

The equivariant eta function can be defined similarly:

**Definition 3.2.** Let $D$ be as in Definition 3.1. If there is a compact group $H$ acting on $\mathcal{H}$ and $D$ commutes with the action of $H$, the equivariant eta function associated to $D$ is defined as

$$\eta(h, s, D) = \frac{1}{\Gamma\left(\frac{1}{2}(s+1)\right)} \int_{0}^{\infty} \text{Tr}(hDe^{-tD^2}) t^{(s-1)/2} dt$$

for $\text{Re} \ s$ large enough.

A regularity result [Zhang 1990] allows us to write

$$\eta(h, D) = \eta(h, 0, D).$$

We also define the truncated equivariant eta function, for an $\epsilon > 0$, to be

$$\eta_{\epsilon}(h, s, D) = \frac{1}{\Gamma\left(\frac{1}{2}(s+1)\right)} \int_{\epsilon}^{\infty} \text{Tr}(hDe^{-tD^2}) t^{(s-1)/2} dt,$$

and we set

$$\eta_{\epsilon}(h, D) = \eta_{\epsilon}(h, 0, D) \quad \text{for any } h \in H.$$

We then consider the variation of equivariant eta functions.

Suppose $\mathcal{F}$ is the real Banach space of all bounded self-adjoint operators on $\mathcal{H}$. Let $\Phi$ be the affine space

$$\Phi = \{ D^S \otimes \text{Id}_{\mathbb{C}^N} + E : E \in \mathcal{F} \}.$$ 

It is clear that for any $u$, the $D_u$ defined in (2–2) lies in $\Phi$.

**Theorem 3.3.** For an $H$-invariant $D$ in $\Phi$ and any $h \in H$, define a one-form $\alpha_{\epsilon, h}$ on $\Phi$ such that

$$\alpha_{\epsilon, h}(X)(D) = (\epsilon/\pi)^{1/2} \text{Tr}(hXe^{-tD^2}) \quad \text{for } X \in T_D \Phi = \mathcal{F}.$$ 

Then $\alpha_{\epsilon, h}$ is closed and

$$d \eta_{\epsilon}(h, D) = 2\alpha_{\epsilon, h}(D).$$
Proof. The proof is almost the same of that of [Getzler 1993, Proposition 2.5], taking into account that $h$ and $D$ commute. □

We can now state the main result of this section:

**Theorem 3.4.** For any $H$-invariant path in $\Phi = \Phi(D_0)$ connecting $D_0$ and $D_1$ as in (2–2) and any $h \in H$,

$$sf(h, \{D_u\}) = -\int_\gamma \alpha_{e,h}.$$  \hspace{1cm} (3–2)

Proof. A similar formula for the scalar case is demonstrated in [Getzler 1993]; we imitate the method of proof.

By the closedness of $\alpha_{e,h}$ and Remark 2.5, both sides of (3–2) are independent of the choice of the $H$-invariant path $\gamma : I \rightarrow E_\sigma$ with $\gamma(0) = D_0$ and $\gamma(1) = D_1$. The union $\bigcup \text{Spec}_H(\{(\gamma(u))\})$ can be written as $\bigcup_j (f_j(u), R_j)$ as in Section 2, where $R_j \in R(H)$ and $f_j \in C(I)$ for $j \in \mathbb{N}$.

Using a standard transversality argument, we can choose an $H$-invariant path $\gamma$ such that the graph of each $f_j$ intersects $\{u = 0\}$ transversally. By Remark 2.4, there are only finitely many nonzero $\epsilon(f_j)$‘s; without loss of generality, let them be $f_1, \ldots, f_k$. It is easy to check that, for $h \in H$,

$$sf(h, \{D_u\}) = sf(h, \gamma) = \sum_{j=1}^{k} \epsilon(f_j) \text{Tr}(h, R_j).$$ \hspace{1cm} (3–3)

We calculate the truncated equivariant eta function. For any $j \in \{1, \ldots, k\}$, the contribution of the $(f_j(u), R_j)$ to $\eta_e(h, \gamma(u))$ for a certain $h \in H$, now denoted by $S_{u,j}$, is

$$\frac{1}{\sqrt{\pi}} \text{Tr}(h, R_j) \int_e^\infty f_j(u)e^{-tf_j(u)^2} t^{-1/2} dt.$$  \hspace{1cm} (3–4)

Now, $\frac{1}{\sqrt{\pi}} \int_e^\infty \lambda e^{-t\lambda^2} t^{-1/2} dt$ tends to $\pm 1$ as $\lambda \rightarrow 0\pm$. Hence, if $\tilde{u}$ is any zero of $f_j(u)$ and $\epsilon(\tilde{u})$ is the intersection number of $f_j(u)$ with $[0] \times I$ near $\tilde{u}$, we have

$$S_{\tilde{u}+,j} - S_{\tilde{u}-,j} = 2\epsilon(\tilde{u}) \text{Tr}(h, R_j).$$

Summing over all the zeros of $f_j$ and using the equality

$$\sum_{\{\tilde{u} \in I : f_j(\tilde{u}) = 0\}} \epsilon(\tilde{u}) = \epsilon(f_j),$$

we have

$$\epsilon(f_j) \text{Tr}(h, R_j) = \frac{1}{2} \sum_{\{\tilde{u} : f_j(\tilde{u}) = 0\}} (S_{\tilde{u}+,j} - S_{\tilde{u}-,j}) = \frac{1}{2} \left(-\int_\gamma dS_{u,j} + S_{1,j} - S_{0,j}\right).$$
Summing over all \( j \), we are led to
\[
\sum_{j=1}^{k} \epsilon(f_j(u)) \text{Tr}(h, R_j) = \frac{1}{2} \left( - \int d\eta_\epsilon(h, \cdot) + \eta_\epsilon(h, D_1) - \eta_\epsilon(h, D_0) \right).
\]

Combining this with (3–1) and (3–3) and noticing that \( \eta_\epsilon(h, D_1) = \eta_\epsilon(h, D_0) \), we have (3–2).

Combining Theorem 2.7 with Theorem 3.4, we have:

**Theorem 3.5.** \( \text{Ind}(h, T_\gamma) = \int_0^1 \sqrt{\frac{\epsilon}{\pi}} \text{Tr}(h \dot{D}_u e^{-\epsilon D_u^2}) \, du. \)

Here the dot denotes differentiation with respect to \( u \).

**Remark 3.6.** The right-hand side of this equality is independent of the choice of \( \epsilon \), so we can use local index technique to calculate the limit of the integrand of when \( \epsilon \) tends to 0. In this way we obtain a local version of Theorem 1.3.

**4. A Lefschetz theorem on odd spin manifolds**

We now apply the setting of [Lafferty et al. 1992] to compute the right-hand side of the equality in Theorem 3.5. We start with a Lichnerowicz-type formula for \( D_u^2 \).

**Lemma 4.1.** We have
\[
D_u^2 = -\Delta + \frac{K}{4} + u^2 c(\omega^2) + u(-\iota(\omega^*) \nabla^S + c(d\omega) + d^*\omega),
\]
where \( \omega = \dot{D}\_{u} = g^{-1} \, dg \) and \( K \) is the scalar curvature of \( M \).

**Proof.** This follows easily from [Berline et al. 1992, Proposition 3.45] and the standard Lichnerowicz formula.

For a fixed \( h \in H \), let \( F = \{ x \in M : hx = x \} \) be the fixed-point set of \( h \). Without loss of generality we assume \( F \) is a connected odd-dimensional totally geodesic submanifold and define its dimension to be \( k \). Let \( \nu \) be the normal bundle of \( F \) in \( TM \), with dimension \( 2s \), and set \( \nu(\delta) = \{ x \in \nu : \| x \| < \delta \} \). Thus \( \nu \) is invariant with respect to \( h_{TM} \); moreover \( h_{TM}|_{\nu} \) is nondegenerate.

If \( P_\epsilon(x, y) : (S \otimes \mathbb{C}^n) \to (S \otimes \mathbb{C}^n) \) is the kernel for the operator
\[
O_\epsilon = (\epsilon/\pi)^{1/2} \int_0^1 \dot{D}_u e^{-\epsilon D_u^2},
\]
by the standard heat equation argument, we have
\[
\text{Tr}(h O_\epsilon) = \int_M \text{Tr}(h P_\epsilon(hx, x)) \, d\text{vol}.
\]
A routine argument using pseudodifferential operators shows that
\[
\lim_{\epsilon \to 0} \text{Tr}(h P_\epsilon (hx, x)) = 0 \quad \text{if } hx \neq x.
\]
As a result, we may localize the computation to \( F \).

Both \( \dot{D}_u \) and \( h \) are bounded operators, so working in a local trivialization of \( \nu(\delta) \) yields the following:

**Lemma 4.2.** Define, for \( x \in F \),
\[
L_{\text{loc}}(x) = \lim_{\epsilon \to 0} \int_{\nu|_x} \text{Tr}(h P_\epsilon (y, hy)) \, dy \, d\text{vol}_F.
\]
This is well-defined and independent of \( \delta \). Furthermore,
\[
\text{Ind}(h, T_k) = \int_F L_{\text{loc}}(x).
\]

We now calculate \( L_{\text{loc}}(x) \) for \( x \in F \). Fix any \( x_0 \in F \), let \( e_1, \ldots, e_n \in TM \) be a local coordinate system in a neighborhood \( \mathcal{N} \) of \( x_0 \) such that the \( e_i \) are orthonormal at \( x_0 \) and are parallel along geodesics through \( x_0 \); moreover, assume \( e_1, \ldots, e_k \in TF \) and \( e_{k+1}, \ldots, e_n \in \nu(F) \). For any \( x \in \mathcal{N} \) such that \( hx \in \mathcal{N} \), there is an \( n \times n \)-matrix \( \mathcal{J}(x) \) satisfying
\[
h|TM(e_1(x), \ldots, e_n(x)) = (e_1(hx), \ldots, e_n(hx)) \mathcal{J}(x),
\]
while
\[
\mathcal{J}(x) = \begin{pmatrix}
1 & & \\
& \ddots & \\
& & 1
\end{pmatrix}
\]
with \( \Theta(x) \in so(2s) \).

Denote by \( R^{TM} \) the curvature matrix of the Levi-Civita connection on \( TM \) with respect to the chosen \( \{e_i\} \):
\[
(R^{TM})_{ij} = -\frac{1}{2} \sum_{p,q=1}^{n} R_{ijpq} e^k e^l
\]
for \( 1 \leq i, j \leq n \), where \( e^k \) is the dual vector of \( e_k \). If we choose the metrics and connections on \( TF \) and \( \nu(F) \) to be the restrictions of those of \( TM \), respectively,
we have the curvature matrices

\[(R^F)_{ij} = -\frac{1}{2} \sum_{p,q=1}^{k} R_{ijpq} e^p e^q \quad \text{for } 1 \leq i, j \leq k \text{ and}\]

\[(R^\nu(F))_{ij} = -\frac{1}{2} \sum_{p,q=1}^{k} R_{ijpq} e^p e^q \quad \text{for } k + 1 \leq i, j \leq n.\]

It is known that \(\mathcal{J}(x)\) is invariant along the fiber of \(\nu\) [Berline et al. 1992]. Hence, using the abbreviation \(Y(a) = (\begin{array}{cc} 0 & a \\ -a & 0 \end{array})\), we can fix the \(e_i\) in such a way that

\[
\Theta(x_0) = \begin{pmatrix} Y(\theta_1) & \cdots & \cdots \\ & \ddots & \cdots \\ & & Y(\theta_s) \end{pmatrix} \quad \text{with } 0 < \theta_i < 2\pi,
\]

\[
R^\nu(x_0) = \begin{pmatrix} Y(v_1) & \cdots \\ & \ddots \\ & & Y(v_s) \end{pmatrix},
\]

\[
R^F(x_0) = \begin{pmatrix} 0 & Y(u_1) & \cdots \\ & \ddots \\ & & Y(u_{(k-1)/2}) \end{pmatrix},
\]

where the \(u_i\)’s and \(v_i\)’s are two-forms representing Chern roots.

It is easy to see that the kernel of \(\sigma(hO_\epsilon)\) is \(\sigma(hP_\epsilon)\), where the symbol map \(\sigma\) is defined in (1–1).

Now we rescale \(T^*M\) as in [Berline et al. 1992] to get, for \(\epsilon \to 0\),

\[
L_0 = \lim_{\epsilon \to 0} \sigma(hO_\epsilon)
= \int_0^1 \frac{h}{\sqrt{\lambda}} \omega \exp \left( \sum_i \left( \frac{1}{4} \sum_j R^TM_{ij} b_j \right)^2 + u(1-u)\omega^2 \right) du,
\]

where the \(b_i\) are local coordinate functions on \(TM\) with respect to the chosen local charts.

We proceed as in [Lafferty et al. 1992] to get the following expression for \(Q_0(x_0, b) = \lim_{\epsilon \to 0} \sigma(hP_\epsilon(x, hx))\):
Let notations be as above

The method we have applied can also be used to prove similar local

Remark 4.4. The method we have applied can also be used to prove similar local

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References


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