REPRESENTATIONS OF THE BRAID GROUP
BY AUTOMORPHISMS OF GROUPS, INVARIANTS OF LINKS,
AND GARSIDE GROUPS

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From a group $H$ and $h \in H$, we define a representation $\rho : B_n \to \text{Aut}(H'^n)$, where $B_n$ denotes the braid group on $n$ strands, and $H'^n$ denotes the free product of $n$ copies of $H$. We call $\rho$ the Artin type representation associated to the pair $(H, h)$. Here we study various aspects of such representations.

Firstly, we associate to each braid $\beta$ a group $\Gamma(\beta)$ and prove that the operator $\Gamma(\beta)$ determines a group invariant of oriented links. We then give a topological construction of the Artin type representations and of the link invariant $\Gamma(\beta)$, and we prove that the Artin type representations are faithful if and only if $h$ is nontrivial. The last part of the paper is devoted to the study of some semidirect products $H'^n \rtimes_B B_n$, where $\rho : B_n \to \text{Aut}(H'^n)$ is an Artin type representation. In particular, we show that $H'^n \rtimes_B B_n$ is a Garside group if $H$ is a Garside group and $h$ is a Garside element of $H$.

1. Introduction

Throughout the paper, we shall denote by $B_n$ the braid group on $n$ strands, and by $\sigma_1, \ldots, \sigma_{n-1}$ the standard generators of $B_n$.

Let $H$ be a group and fix $h \in H$. Take $n$ copies $H_1, \ldots, H_n$ of $H$ and consider the group $H'^n = H_1 \ast \cdots \ast H_n$. We denote by $\phi_i : H \to H_i$ the natural isomorphism and we write $h_i = \phi_i(h) \in H_i$, for all $i = 1, \ldots, n$. For $k = 1, \ldots, n-1$, let $\tau_k : H'^n \to H'^n$ be the automorphism determined by

$$
\tau_k : \begin{cases}
\phi_k(y) &\mapsto \ h_k^{-1} \phi_{k+1}(y) \ h_k,
\phi_{k+1}(y) &\mapsto \ h_k \phi_k(y) \ h_k^{-1},
\phi_j(y) &\mapsto \ \phi_j(y) \quad \text{if } j \neq k, k+1
\end{cases}
$$

for $y \in H$. One can easily show the following.

**Proposition 1.1.** The mapping $\sigma_k \mapsto \tau_k$, $k = 1, \ldots, n-1$, determines a representation $\rho : B_n \to \text{Aut}(H'^n)$.

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Proof: This involves checking, case by case, that the usual braid group relations are satisfied by the automorphisms \(\tau_k\). For example, both \(\tau_k \tau_{k+1} \tau_k\) and \(\tau_{k+1} \tau_k \tau_{k+1}\) map \(\phi_k(y)\) to \(h_k^{-1} h_{k+1}^{-1} \phi_{k+2}(y) h_{k+1} h_k\), \(\phi_{k+1}(y)\) to \(h_k^{-1} h_{k+1} \phi_{k+1}(y) h_{k+1}^{-1} h_k\), etc. Similarly, one checks that \(\tau_k \tau_j = \tau_j \tau_k\) if \(k < j - 1\). We leave the details to the reader. \(\square\)

**Definition 1.2.** The representation of Proposition 1.1 shall be called the Artin type representation of \(B_n\) associated to the pair \((H, h)\).

The special case where \(h\) is taken to be the identity, \(h = \text{Id}_H\), gives a representation of \(B_n\) by permutations of the free factors of \(H^{\ast n}\). This representation has image the full symmetric group \(S_n\) and kernel the pure braid group. All other Artin type representations will be shown to be faithful (see Proposition 4.1).

If \(H = \mathbb{Z}\) and \(h = 1\) (a generator of \(\mathbb{Z}\) in the additive notation), then \(H^{\ast n} = F_n\) is the free group of rank \(n\) and \(\rho\) is the classical representation introduced by Artin [1925; 1947]. Another example which appears in the literature is the case where \(H = \mathbb{Z}\) and \(h\) is an arbitrary nonzero integer. This case was introduced by Wada [1992] in his construction of group invariants of links. Sections 2 and 3 of the present paper are inspired by [Wada 1992].

Our purpose in this paper is to study different aspects of the Artin type representations.

**Definition 1.3.** Let \(\rho : B_n \to \text{Aut}(H^{\ast n})\) be the Artin type representation associated to a pair \((H, h)\). Let \(\beta \in B_n\). Then we denote by \(\Gamma(\beta) = \Gamma(H, h)(\beta)\) the quotient of \(H^{\ast n}\) by the relations

\[g = \rho(\beta)g, \quad g \in H^{\ast n}.\]

For a braid \(\beta\), we denote by \(\hat{\beta}\) the oriented link (or more precisely the equivalence class of oriented links) represented by the closed braid of \(\beta\) as defined in [Birman 1974]. Given two braids \(\beta_1\) and \(\beta_2\) (not necessarily with the same number of strands), we prove in Section 2 that \(\Gamma(\beta_1) \simeq \Gamma(\beta_2)\) if \(\hat{\beta}_1 = \hat{\beta}_2\). This allows us to define a group invariant of oriented links, \(\Gamma_{(H, h)}(L)\), by setting \(\Gamma_{(H, h)}(L)\) to be the group \(\Gamma_{(H, h)}(\beta)\) for any braid \(\beta\) such that \(L = \hat{\beta}\). Note that, in the case \(H = \mathbb{Z}\) and \(h = 1\), the invariant \(\Gamma_{(\mathbb{Z}, 1)}(L)\) computes the link group, namely \(\Gamma_{(\mathbb{Z}, 1)}(L) \cong \pi_1(S^3 \setminus L)\) for any link \(L\) in \(S^3\).

The goal of Section 3 is to give topological constructions of the Artin type representations and of the groups \(\Gamma_{(H, h)}(\beta)\), for \(\beta \in B_n\). If \(H = \mathbb{Z}\) and \(h\) is a nonzero integer, then our constructions coincide with Wada’s constructions [1992, Section 3]. In fact, our constructions are straightforward extensions of Wada’s constructions to all Artin type representations.

In Section 4, we prove that Artin type representations are faithful whenever \(h\) is chosen nontrivial (Proposition 4.1). If \(h\) has infinite order, then the Artin type
representation $\rho : B_n \to \text{Aut}(H^*)$ contains the classical Artin representation and, therefore, is faithful by [Artin 1925; 1947]. So, Proposition 4.1 is mostly of interest in the case where $h$ has finite order. In fact the proof may be easily reduced to the case $H = \mathbb{Z}/k\mathbb{Z}$ and $h = 1$, however we will not need to use any such reduction, as our method applies just as easily in all cases. We note also that the case where $H$ is cyclic of order 2 follows (by somewhat different methods) from [Crisp and Paris 2005, Section 2.3]. The proof of Proposition 4.1 is inspired by the proof of [Shpilrain 2001, Theorem A], and it is based on Dehornoy’s work [1994; 1997a] on orderings of the braid group.

The remaining sections (Sections 5 and 6) are dedicated to the study of semidirect products $H^* \rtimes_{\rho} B_n$, where $\rho : B_n \to \text{Aut}(H^*)$ is the Artin type representation associated to a pair $(H, h)$. If $H = \mathbb{Z}$ and $h = 1$, then $H^* \rtimes_{\rho} B_n$ is the Artin group $A(B_n)$ associated to the Coxeter graph $B_n$ (not to be confused with the braid group $B_n$, which is itself an Artin group, of type $A_{n-1}$). This result is implicit in [Lambropoulou 1994; Crisp 1999], and is described explicitly in [Crisp and Paris 2005]. The group $A(B_n)$ is well-understood. In particular, solutions to the word and conjugacy problems in this group are known [Deligne 1972; Brieskorn and Saito 1972], it is torsion free [Brieskorn 1973; Deligne 1972], its center is an infinite cyclic group [Deligne 1972; Brieskorn and Saito 1972], it is biautomatic [Charney 1992; 1995], and it has an explicit finite dimensional classifying space [Deligne 1972; Bestvina 1999].

A natural next step is to understand the groups $H^* \rtimes_{\rho} B_n$ in the case where $\rho$ is a Wada representation (of type 4), namely, when $H = \mathbb{Z}$ and $h \in \mathbb{Z} \setminus \{0\}$. One can readily establish that, for these representations, the group $H^* \rtimes_{\rho} B_n$ fails to be an Artin group unless $h = \pm 1$. It turns out, however, that these groups do have quite a lot in common with Artin groups: like the Artin groups, they belong to a family of groups known as Garside groups.

Briefly, a Garside group is a group $G$ which admits a left invariant lattice order and contains a so-called Garside element, a positive element $\Delta$ whose positive divisors generate $G$ and such that conjugation by $\Delta$ leaves the lattice structure invariant (there are also conditions placed on the positive cone of $G$, that it be a finitely generated atomic monoid; see Section 5 for details). The notion of a Garside group was introduced in [Dehornoy and Paris 1999] in a slightly restricted sense, and in [Dehornoy 2002] in the larger sense in which it is now generally used. Their theory is largely inspired by [Garside 1969], which treated the case of braid groups, and [Brieskorn and Saito 1972], which generalized Garside’s work to Artin groups. The Artin groups of spherical (or finite) type which include, notably, the braid groups as well as the groups $A(B_n)$ mentioned above, are motivating examples. Other interesting examples of Garside groups include all torus link groups.
John Crisp and Luis Paris

[Picantin 2003] and some generalized braid groups associated to finite complex reflection groups [Bessis and Corran 2004].

Garside groups have many attractive properties. Solutions to the word and conjugacy problems in these groups are known [Dehornoy 2002; Picantin 2001b; Franco and González-Meneses 2003], they are torsion free [Dehornoy 1998], they admit canonical decompositions as iterated direct products of “irreducible” components, and the center of each component is an infinite cyclic group [Picantin 2001a], they are biautomatic [Dehornoy 2002], and they admit finite dimensional classifying spaces [Dehornoy and Lafont 2003; Charney et al. 2004]. Another important property of the Garside groups is that there exist criteria in terms of presentations to detect them [Dehornoy and Paris 1999; Dehornoy 2002].

In Section 6, we prove that, if \( H \) is a Garside group, \( h \) a Garside element of \( H \), and \( \rho \) the Artin type representation associated to \((H, h)\), then \( H^\ast \cong \rho B_n \) is also a Garside group (Theorem 6.1). This result applies in particular to the case \( H = \mathbb{Z} \) and \( h \in \mathbb{Z} \setminus \{0\} \), but also applies, for example, to the case where \( H \) is another braid group, say \( H = B_1 \), and \( h = \Delta^k \) is a nontrivial power of the fundamental element of \( B_1 \).

The proof of Theorem 6.1 is based on a necessary and sufficient criterion, explained in Section 5, for a group to be Garside. This criterion rests largely on the “coherence” condition of [Dehornoy and Paris 1999] and is essentially a variation on [Dehornoy 2002, Proposition 6.14]. Our version differs from Dehornoy’s [2002] in that it is not algorithmic. In particular, we do not give any method for finding a Garside element. However, our Criterion 5.9 is relatively easy to apply once one has an appropriate presentation and an expression for a Garside element to hand.

Finally, in the Appendix we answer a question posed by Shpilrain [2001] in his study of Wada’s representations.

Definition 1.4. Let \( G \) be a group. Two representations \( \rho, \rho' : B_n \to Aut(G) \) are called equivalent if there exist automorphisms \( \phi : G \to G \) and \( \mu : B_n \to B_n \) such that \( \rho'(\mu(\beta)) = \phi^{-1} \circ \rho(\beta) \circ \phi \) for all \( \beta \in B_n \).

Remark. If two representations \( \rho, \rho' : B_n \to Aut(G) \) are equivalent, then the groups \( G \rtimes_\rho B_n \) and \( G \rtimes_{\rho'} B_n \) are isomorphic.

Shpilrain’s question was simply to give a classification of Wada’s representations up to equivalence. This classification is given in Proposition A.1.

2. Link invariants

Let \( H \) be a group, \( h \in H \), and \( \rho : B_n \to Aut(H^\ast) \) be the Artin type representation associated to \((H, h)\). Recall that the group \( H^\ast \) is defined as \( H^\ast = H_1 \ast \cdots \ast H_n \),
where group isomorphisms $\phi_i : H_i \to H$ are given for $i = 1, 2, \ldots, n$. The goal of this section is to prove the following.

**Proposition 2.1.** Let $n, m \in \mathbb{N}$, and let $\beta_1 \in B_n$ and $\beta_2 \in B_m$. If $\hat{\beta}_1 = \hat{\beta}_2$, then $\Gamma_{(H,h)}(\beta_1) \simeq \Gamma_{(H,h)}(\beta_2)$.

**Definition 2.2** (Link invariant). Let $L$ be an oriented link. We set $\Gamma_{(H,h)}(L) := \Gamma_{(H,h)}(\beta)$, where $\beta$ is any braid (on any number of strings) such that $L = \hat{\beta}$. By Proposition 2.1, $\Gamma_{(H,h)}$ is a well-defined group invariant of oriented links.

**Proof of Proposition 2.1.** Let $n \in \mathbb{N}$ and let $\beta \in B_n$. We write $\Gamma$ for $\Gamma_{(H,h)}$. By Markov’s theorem [Birman 1974, Theorem 2.3], it suffices to show that

1. $\Gamma(\alpha^{-1} \beta \alpha) \simeq \Gamma(\beta)$ for all $\alpha \in B_n$,
2. $\Gamma(\beta \sigma_n) \simeq \Gamma(\beta)$, and
3. $\Gamma(\beta \sigma_n^{-1}) \simeq \Gamma(\beta)$,

where $\beta \sigma_n$ and $\beta \sigma_n^{-1}$ are viewed as braids on $n + 1$ strands.

Note that, if $\beta \in B_n$ and $n \leq m$, then the action of $\beta$ via $\rho$ on $H^{*m}$ agrees with the action via $\rho$ on $H^{*n} < H^{*m}$, and is trivial on the free factors $H_{n+1}, \ldots, H_m$. We suppress $\rho$ from our notation, writing simply $\beta(g)$ to mean $\rho(\beta)g$, for any $\beta \in B_n$ and $g \in H^{*m}$. This also amounts to writing $\sigma_k$ instead of $\tau_k$.

We now prove conditions (1), (2) and (3) above.

1. For $\beta \in B_n$, the group $\Gamma(\beta)$ is defined as the quotient of $H^{*n}$ by the relations $g = \beta(g)$ for all $g \in H^{*n}$. Since, for $\alpha \in B_n$, the relation $g = \alpha^{-1} \beta \alpha(g)$ is equivalent to the relation $\alpha(g) = \beta(\alpha(g))$, and $\alpha$ is an automorphism of $H^{*n}$, it is clear that $\Gamma(\alpha^{-1} \beta \alpha)$ is defined by the same set of relations as $\Gamma(\beta)$.

2. The group $\Gamma(\beta \sigma_n)$ may be defined as the quotient of $H^{*n+1}$ by the family of relations $R(i, x) : \phi_i(x) = \beta \sigma_n(\phi_i(x))$ for $i = 1, 2, \ldots, n + 1$ and $x \in H$. Note that $\sigma_n(\phi_{n+1}(x)) = h_n \phi_n(x) h_n^{-1}$. Therefore the relation $R(n + 1, x)$ is equivalent to the relation $R'(n + 1, x) : \phi_{n+1}(x) = \beta(h_n \phi_n(x) h_n^{-1})$, where the right hand side is actually an element of $H^{*n}$. In particular $\Gamma(\beta \sigma_n)$ is generated by the image of $H^{*n}$. Also,

$$\beta \sigma_n(\phi_n(x)) = \beta(h_n^{-1} \phi_{n+1}(x) h_n) = \beta(h_n^{-1}) \phi_{n+1}(x) \beta(h_n).$$

So, in view of $R'(n + 1, x)$, the relation $R(n, x)$ is now equivalent to the relation $R'(n, x) : \phi_n(x) = \beta(\phi_n(x))$. Finally, since $\sigma_n(\phi_i(x)) = \phi_i(x)$ for all $i < n$, the remaining relations $R(i, x)$ are equivalent to $R'(i, x) : \phi_i(x) = \beta(\phi_i(x))$ for all $i = 1, 2, \ldots, n - 1$, and all $x \in H$. It now follows that $\Gamma(\beta \sigma_n) \simeq \Gamma(\beta)$. 


(3) Observe that $\Gamma(\beta^{-1}) \simeq \Gamma(\beta)$, since the relation $g = \beta(g)$ is equivalent to $\beta^{-1}(g) = g$, for all $g \in \mathcal{H}^n$. Then

\[
\Gamma(\beta \sigma_n^{-1}) \simeq \Gamma(\sigma_n \beta^{-1}) \\
\simeq \Gamma(\beta^{-1} \sigma_n) \quad \text{by the proof of (1),} \\
\simeq \Gamma(\beta^{-1}) \quad \text{by the proof of (2),} \\
\simeq \Gamma(\beta). \quad \square
\]

3. Topological construction of the link invariants

Let $X$ be a CW-complex, let $P_0 \in X$ be a basepoint, and let $\alpha : [0, 1] \to X$ be a loop based at $P_0$. In this section we give a topological realization of the Artin type representation of $B_n$ associated to the pair $(\mathcal{H}, h) = (\pi_1(X, P_0), [\alpha])$, and we deduce a topological construction of the link invariant $\Gamma_{(\mathcal{H}, h)}$ of the previous section.

Let $D = D(\frac{n+1}{2}, \frac{n+1}{2})$ denote the disk in $\mathbb{C}$ centered at $\frac{n+1}{2}$ of radius $\frac{n+1}{2}$. Now, we construct a space $Y$ obtained from $D$ by making $n$ holes in $D$ and gluing a copy of $X$ into each hole by identifying the circular boundary of the hole to the loop $\alpha$ in $X$. Choose some small $\varepsilon > 0$ (we require only that $\varepsilon < \frac{1}{8}$). Let

\[
Y' = D \setminus \left( \bigcup_{k=1}^{n} \hat{D}(k, \varepsilon) \right),
\]

where $\hat{D}(k, \varepsilon)$ denotes the open disk centered at $k$ of radius $\varepsilon$. Take $n$ copies $X_1, \ldots, X_n$ of $X$, denote by $f_k : X \to X_k$ the natural homeomorphism, and write $\alpha_k = f_k \circ \alpha$ for all $k = 1, \ldots, n$. Then

\[
Y = \left( Y' \cup \left( \bigcup_{k=1}^{n} X_k \right) \right) / \sim,
\]

where $\sim$ is the identification defined by

\[
\alpha_k(t) \sim k + \varepsilon e^{2\sqrt{-1}\pi t}, \quad k = 1, \ldots, n, \quad t \in [0, 1].
\]

Finally, choose a basepoint $Q_0 \in \partial D$ for $Y$. The following result is a direct consequence of the above construction.

Lemma 3.1. Let $H = \pi_1(X, P_0)$, and let $H_1, \ldots, H_n$ be $n$ copies of $H$. Then $\pi_1(Y, Q_0) \simeq H_1 \ast \cdots \ast H_n$.

We now show that the braid group $B_n$ acts on $Y$ up to isotopy relative to the boundary of $D$ in such a way that the induced action on $\pi_1(Y)$ is the Artin type representation associated to $(H, h)$, where $h$ is the element of $H = \pi_1(X, P_0)$ represented by $\alpha$. 
Let $\xi \in \mathbb{C}$ and $0 < r < R$. Define the half Dehn twist $T = T(\xi, r, R)$ by

$$T(\xi + \rho e^{\sqrt{-1} \theta}) = \begin{cases} 
\xi + \rho e^{\sqrt{-1}(\theta - \pi)} & \text{if } 0 \leq \rho \leq r, \\
\xi + \rho e^{\sqrt{-1}(\theta - t \pi)} & \text{if } r \leq \rho \leq R \text{ and } t = \frac{R - \rho}{R - r}, \\
\xi + \rho e^{\sqrt{-1} \theta} & \text{if } \rho \geq R
\end{cases}$$

(see Figure 1).

Let $T^D_k : D \to D$ be the homeomorphism defined by

$$T^D_k = T(k, \varepsilon, 2\varepsilon)^{-3} \circ T(k + 1, \varepsilon, 2\varepsilon)^{-1} \circ T\left(k + \frac{1}{2}, \frac{1}{2} + \varepsilon, \frac{1}{2} + 2\varepsilon\right).$$

Note that $T^D_k$ leaves invariant the set $\bigcup_{j=1}^n D(j, \varepsilon)$, and therefore restricts to a homeomorphism $T'_k : Y' \to Y'$. See Figure 2.

One can verify (with a little effort) that $T'_k T'_{k+1} T'_k$ is isotopic to $T'_{k+1} T'_k T'_{k+1}$ relative to $\partial Y'$ for $k = 1, \ldots, n - 2$, and that $T'_k T'_l$ is isotopic to $T'_l T'_k$ relative to $\gamma_k$.

Figure 1. A half Dehn twist.

Figure 2. The homeomorphism $T'_k : Y' \to Y'$. 
\[ \partial Y \] for \( |k-l| \geq 2 \). Moreover, \( T'_k \) fixes \( \partial D \) and transforms the rest of \( \partial Y' \) as follows:

\[
T'_k(j + \varepsilon e^{\sqrt{-1} \theta}) = \begin{cases} 
  j + \varepsilon e^{\sqrt{-1} \theta} & \text{if } j \neq k, k+1, \\
  k + 1 + \varepsilon e^{\sqrt{-1} \theta} & \text{if } j = k, \\
  k + \varepsilon e^{\sqrt{-1} \theta} & \text{if } j = k+1.
\end{cases}
\]

Therefore, \( T'_k \) extends to a homeomorphism \( T_k: Y \to Y \) by setting, for all \( x \in X \),

\[
T_k(f_j(x)) = \begin{cases} 
  f_j(x) & \text{if } j \neq k, k+1, \\
  f_{k+1}(x) & \text{if } j = k, \\
  f_k(x) & \text{if } j = k+1.
\end{cases}
\]

The homeomorphism \( T_k \) is the identity on \( \partial D \). \( T_kT_{k+1}T_k \) is isotopic to \( T_{k+1}T_kT_{k+1} \) relatively to \( \partial D \) for \( k = 1, \ldots, n-2 \), and \( T_kT_j \) is isotopic to \( T_jT_k \) relatively to \( \partial D \) for \( |k-l| \geq 2 \).

These observations show that \( T_k \) determines an automorphism \( \tau_k: \pi_1(Y, Q_0) \to \pi_1(Y, Q_0) \). Moreover,

\[
\tau_k \tau_{k+1} \tau_k = \tau_{k+1} \tau_k \tau_{k+1} \quad \text{for } k = 1, \ldots, n-2,
\]

\[
\tau_k \tau_l = \tau_l \tau_k \quad \text{for } |k-l| \geq 2.
\]

Thus the mapping \( \sigma_k \to \tau_k \) determines a representation \( \rho: B_n \to \text{Aut}(\pi_1(Y, Q_0)) \).

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**Figure 3.** The path \( \gamma_k \).

Set \( Q_0 = \frac{n+1}{2} - \sqrt{-1} \frac{n+1}{2} \). Let \( \gamma_k: [0, 1] \to Y \) be the path from \( Q_0 \) to \( f_k(P_0) \) shown in **Figure 3**. We identify \( \pi_1(Y, Q_0) \) with \( H^{an} = H_1 \ast \cdots \ast H_n \) in such a way that the \( k \)-th embedding \( \phi_k: H = \pi_1(X, P_0) \to H_k \subset H^{an} \) is defined by

\[
\phi_k([\beta]) = [\gamma_k f_k(\beta) \gamma_k^{-1}].
\]

With this assumption, one can easily show the following.
Proposition 3.2. The representation \( \rho : B_n \to \text{Aut}(\pi_1(Y, Q_0)) \) described above coincides with the Artin type representation of \( B_n \) associated to \( (H, h) \), where \( H = \pi_1(X, P_0) \) and \( h \) is the element of \( H \) represented by \( \alpha \).

Proof. It suffices to observe, with the aid of Figure 2, that, for all \( k = 1, \ldots, n - 1 \) and all loops \( \beta \) at \( P_0 \) in \( X \),

(i) \( T_k(\gamma_k f_k(\beta)\gamma_k^{-1}) \) is homotopic to \( \gamma_k \alpha_k^{-1} f_{k+1}(\beta) \gamma_{k+1} \gamma_k \alpha_k \gamma_k^{-1} \);

(ii) \( T_k(\gamma_{k+1} f_{k+1}(\beta)\gamma_{k+1}^{-1}) \) is homotopic to \( \gamma_k \alpha_k f_k(\beta) \alpha_k^{-1} \gamma_k^{-1} \); and

(iii) \( T_k(\gamma_j f_j(\beta)\gamma_j^{-1}) \) is homotopic to \( \gamma_j f_j(\beta) \gamma_j^{-1} \), for all \( j \neq k, k+1 \). \( \square \)

We now introduce some standard notions and facts concerning framings of links and linking numbers. We refer the reader to [Rolfsen 1990], or any similar introductory text on knot theory, for further details.

Consider an oriented \( m \)-component link \( L = K_1 \cup \cdots \cup K_m \) in \( S^3 \). The knot \( K_i \) is an embedding \( K_i : S^1 \to S^3 \), and \( K_i(S^1 \cap K_j(S^1) = \emptyset \) for \( i \neq j \). Define a tubular neighborhood of \( K_i \) to be an embedding \( T_i : D^2 \times S^1 \to S^3 \) such that \( T_i(0, \xi) = K_i(\xi) \) for all \( \xi \in S^1 \). Here, \( D^2 \) denotes the disk centered at 0 of radius 1 in \( \mathbb{C} \). A framing of \( L \) is a collection \( \{ T_i : D^2 \times S^1 \to S^3 \}_{i=1}^m \) of embeddings such that \( T_i \) is a tubular neighborhood of \( K_i \), for \( i = 1, \ldots, m \), and \( T_i(D^2 \times S^1) \cap T_j(D^2 \times S^1) = \emptyset \) for \( i \neq j \). The longitude of the component \( K_i \) is the (oriented) embedding \( \lambda_i : S^1 \to S^3 \) such that \( \lambda_i(\xi) = T_i(1, \xi) \) for all \( \xi \in S^1 \). The framing of each component \( K_i \) is determined up to isotopy by the homology class of its longitude \( \lambda_i \) in the knot complement \( S^3 \setminus K_i \).

Given an oriented knot \( K \), we identify \( H_1(K) := H_1(S^3 \setminus K) \) with \( \mathbb{Z} \) in such a way that \( 1 \in \mathbb{Z} \) is represented by the 1-cycle depicted in Figure 4(a). Let \( K_1, K_2 \) denote disjoint oriented knots in \( S^3 \). One defines the linking number \( \text{lk}(K_1, K_2) \in \mathbb{Z} \) to be the class \( [K_1] \in H_1(K_2) = \mathbb{Z} \). The linking number \( \text{lk}(K_1, K_2) \) may be measured from any regular projection of the link \( K_1 \cup K_2 \) by counting with sign the crossings where \( K_1 \) passes over \( K_2 \), as indicated in Figure 4(b). (Equally one

![Figure 4](image-url)
may choose to count undercrossings with the appropriate sign, and one quickly sees that $lk(K_1, K_2) = lk(K_2, K_1)$.

**Notation** (Preferred framing). Let $L = K_1 \cup \cdots \cup K_m$ be an $m$-component oriented link in $S^3$. Up to isotopy, there is a unique framing in which the longitude $\lambda_i$ for each component $K_i$ satisfies the condition

$$\sum_{j=1}^{m} lk(\lambda_i, K_j) = 0.$$ 

Note that, for $j \neq i$, $lk(\lambda_i, K_j) = lk(K_i, K_j)$ and is determined by the oriented link $L$. We shall refer to the above framing as the preferred framing of $L$.

We now wish to associate to an oriented link $L$ the space $\Omega_1(L, X)$ obtained by performing a 'generalized' surgery on the link $L$ according to the preferred framing just described. More precisely, let $L = K_1 \cup \cdots \cup K_m$ and let $\{T_i : D^2 \times S^1 \to S^3\}_{i=1}^m$ be the preferred framing. Let $\tilde{T}_i$ denote the interior of $T_i(D^2 \times S^1)$ for $i = 1, \ldots, m$, and set

$$\Omega'(L) = S^3 \setminus \left( \bigcup_{i=1}^{m} \tilde{T}_i \right).$$

Take $m$ copies $X_1, \ldots, X_m$ of $X$, denote by $f_i : X \to X_i$ the natural homeomorphism, and write $\alpha_i = f_i \circ \alpha$. Then

$$\Omega(L, X) = \left( \Omega'(L) \cup \left( \bigcup_{i=1}^{m} (X_i \times S^1) \right) \right) / \sim,$$

where $\sim$ is the identification defined by putting

$$(\alpha_i(t), \eta) \sim T_i(e^{2\sqrt{-1}\pi t}, \eta), \quad i = 1, \ldots, m, \quad t \in [0, 1], \quad \eta \in S^1.$$ 

The following proposition yields a second proof of the fact that $\Gamma_{(H,h)}$ is a link invariant for any finitely generated group $H$ and any element $h \in H$.

**Proposition 3.3.** Let $\beta$ be a braid, and let $\hat{\beta}$ denote the closed braid of $\beta$. Let $X$ be a CW-complex with basepoint $P_0$ and let $\alpha$ be a loop in $X$. Then $\pi_1(\Omega(\hat{\beta}, X))$ is isomorphic to $\Gamma_{(H,h)}(\beta)$, where $H = \pi_1(X, P_0)$ and $h$ is the element of $H$ represented by $\alpha$.

**Proof:** We first remind the reader of the standard construction of the closed braid $\hat{\beta}$ from a braid $\beta$ [Birman 1974]. The notation used to describe this construction will be needed for the completion of the proof. Firstly, decompose $S^3$ as follows: let $T_1, T_2$ be two copies of the solid torus $D \times S^1$ and write

$$S^3 = T_1 \bigcup_{\kappa: \partial T_1 \to \partial T_2} T_2.$$
where the identifying map \( \kappa \) is a homeomorphism carrying \( \partial D \to S^1 \) and \( S^1 \) to \( \partial D \). Let \( g \) denote the inclusion of \( T_1 \) in \( S^3 \), and let \( f : D \times [0, 1] \to T_1 = D \times S^1 \) be the identification map defined by
\[
f(p, t) = (p, e^{2\sqrt{-1} \pi t}), \quad \text{for } p \in D \text{ and } t \in [0, 1].
\]
The closed braid \( \hat{\beta} \) is the oriented link which is induced by composing the braid \( \beta : \{1, \ldots, n\} \times [0, 1] \to D \times [0, 1] \) with the map \( g \circ f : D \times [0, 1] \to S^3 \). The orientation on \( \hat{\beta} \) is naturally induced from a choice of orientation of the interval \([0, 1]\).

Given a standard projection of a braid \( \beta \) we may describe a projection of the closed braid \( \hat{\beta} \) with the same number of crossings, as indicated in Figure 5. We now produce a framing \( \Lambda \) of \( \hat{\beta} \) by choosing a longitude \( \lambda_i \) for each component \( K_i \) of \( \hat{\beta} \) whose projections are as indicated in Figure 6 in the vicinity of a crossing, and otherwise parallel to the link projection. It is easily enough verified, by counting overcrossings, that this framing is exactly the preferred framing of \( \hat{\beta} \).

\[
\sigma_k^{-1} \sigma_k \kappa \sigma_k \kappa^{-1} \lambda_i \kappa \sigma_k \kappa^{-1} \kappa^{-1} \lambda_i \kappa^{-1}
\]

**Figure 6.** Choosing a framing for \( \hat{\beta} = K_1 \cup \cdots \cup K_m \).
Write $\beta = \sigma_{k_1}^{\epsilon_1} \sigma_{k_2}^{\epsilon_2} \cdots \sigma_{k_r}^{\epsilon_r}$ and define $T_{\beta}^D : D \to D$ as the composition of the homeomorphisms $(T_{k_j}^D)^{\epsilon_j}$ for $j = 1, \ldots, r$. Similarly, define $T_{\beta} = T_{k_1}^{\epsilon_1} T_{k_2}^{\epsilon_2} \cdots T_{k_r}^{\epsilon_r} : Y \to Y$. For $j = 1, \ldots, n$, denote by $b_j$ the point $j + \varepsilon$ on $\partial D(j, \varepsilon)$. This is the point on $\partial Y'$ to which the basepoint of $X_j$ is attached when forming $Y$. Since $T_{\beta}^D$ is isotopic to $\text{Id}_D$ relative to $\partial D$, there is a homeomorphism $U : D \times [0, 1] \to D \times [0, 1]$ such that $U(x, 0) = (x, 0)$, $U(x, 1) = (T_{\beta}^D(x), 1)$, for all $x \in D$, and $U$ fixes $\partial D \times [0, 1]$ pointwise. Moreover, by construction, $U$ carries
to a tubular neighborhood of (a representative of) the braid $\beta$, and $g \circ f \circ U$ carries
the arcs $\{b_j \times [0, 1] : j = 1, \ldots, n\}$ to a framing of $\hat{\beta}$ equivalent to that described
in Figure 6, namely the preferred framing. Consequently the space $\Omega(\hat{\beta}, X)$ is
homeomorphic to $T_1' \cup T_2$ where

$$T_1' = Y \times [0, 1]/((y, 0) \sim (T_{\beta}(y), 1)).$$

We therefore have $\pi_1(T_1') \cong H^a \ast \langle t \rangle / \langle t x t^{-1} = \rho(\beta)x \text{ for } x \in H^a \rangle$, an HNN-extension. Attaching $T_2$ to $T_1'$ has the effect of simply killing the stable letter $t$. Consequently

$$\pi_1(\Omega(\hat{\beta}, X)) \cong H^a / \langle x = \rho(\beta)x \text{ for } x \in H^a \rangle = \Gamma(H, h)(\beta). \quad \square$$

4. Faithfulness

Recall that, for any group $H$ and for $n \in \mathbb{N}$, we write $H^a$ for the free product
$H_1 \ast \cdots \ast H_n$, where each free factor $H_i$ is isomorphic to $H$ by an isomorphism
$\phi_i : H \to H_i$. The aim of this section is to prove the following.

**Proposition 4.1.** Let $\rho : B_n \to \text{Aut}(H^a)$ be the Artin type representation of $B_n$
associated to the pair $(H, h)$ where $H$ is a group and $h \in H$.

(i) If $h \neq \text{Id}_H$, then $\rho$ is faithful.

(ii) If $h = \text{Id}_H$, then $\ker(\rho)$ is the pure braid group and $B_n/\ker(\rho) \cong S_n$, the
symmetric group, acts by permutations of the free factors of $H^a$ (respecting
the isomorphisms $\{\phi_1, \ldots, \phi_n\}$).

**Remark.** Part (ii) of this proposition requires no proof but is included here for
completeness. We concern ourselves below with the case of $h$ nontrivial.

As pointed out in the introduction, the proof of Proposition 4.1(i) is strongly
inspired by the proof of [Shpilrain 2001, Theorem A], and its main ingredient is the following:
Proposition 4.2 [Dehornoy 1994; 1997a]. Let $B_{[2,n]}$ denote the subgroup of $B_n$ generated by $\sigma_2, \ldots, \sigma_{n-1}$ (namely the braid group on the second through nth strings). Let $\beta \in B_n$. Then either

1. $\beta \in B_{[2,n]}$, or
2. one of $\beta$ or $\beta^{-1}$ can be written as $\alpha_0 \sigma_1 \sigma_1 \alpha_2 \ldots \sigma_1 \alpha_l$, where $l \geq 1$ and $\alpha_0, \ldots, \alpha_l \in B_{[2,n]}$.

The following lemma is preliminary to the proof of Proposition 4.1. We first fix a nontrivial $h \in H$, and write $h_i = \phi_i(h)$ for $i = 1, \ldots, n$.

Lemma 4.3. Let $K = H_2 \ast \cdots \ast H_n$. Let $u \in H^{*n}$ such that the normal form of $u$ with respect to the decomposition $H^{*n} = H_1 \ast K$ starts with $h_1^{-1}$ and ends with $h_1$.

1. The normal form of $\rho(\sigma_0)(u)$ with respect to the decomposition $H^{*n} = H_1 \ast K$ also starts with $h_1^{-1}$ and ends with $h_1$.
2. Let $k \in \{2, \ldots, n-1\}$ and $\varepsilon \in \{-1\}$. The normal form of $\rho(\sigma_k^\varepsilon)(u)$ with respect to the decomposition $H^{*n} = H_1 \ast K$ also starts with $h_1^{-1}$ and ends with $h_1$.

Proof. Let $v \in H_1 \ast H_2$. Suppose that the normal form of $v$ is

$$v = \phi_1(x_1) \phi_2(y_1) \ldots \phi_1(x_l) \phi_2(y_l),$$

where $x_1, \ldots, x_l, y_1, \ldots, y_{l-1} \in H \setminus \{\text{Id}\}$, and $y_l \in H$. Then

$$\rho(\sigma_1)(v) = h_1^{-1} \cdot \phi_2(x_1) \cdot h_1^2 \phi_1(y_1) h_1^{-2} \cdot \ldots \cdot \phi_2(x_l) \cdot h_1^2 \phi_1(y_l) h_1^{-1};$$

thus the normal form of $\rho(\sigma_1)(v)$ starts with $h_1^{-1}$.

Similarly, if the normal form of $v$ is

$$v = \phi_2(y_1) \phi_1(x_1) \ldots \phi_2(y_l) \phi_1(x_l),$$

where $x_1, \ldots, x_l, y_2, \ldots, y_{l-1} \in H \setminus \{\text{Id}\}$ and $y_1 \in H$, then the normal form of $\rho(\sigma_1)(v)$ ends with $h_1$.

Now, write

$$u = u_0 w_1 u_1 \ldots w_l u_l,$$

where $u_i \in (H_1 \ast H_2) \setminus \{\text{Id}\}$ and $w_j \in (H_3 \ast \cdots \ast H_n) \setminus \{\text{Id}\}$, and $l \geq 0$. The hypothesis that $u$ starts with $h_1^{-1}$ implies that $u_0$ starts with $h_1^{-1}$, and the hypothesis that $u$ ends with $h_1$ implies that $u_l$ ends with $h_1$. Both groups, $H_1 \ast H_2$ and $H_3 \ast \cdots \ast H_n$, are invariant by $\rho(\sigma_1)$, and $\rho(\sigma_1)$ is the identity on $H_3 \ast \cdots \ast H_n$. So,

$$\rho(\sigma_1)(u) = \rho(\sigma_1)(u_0) \cdot w_1 \cdot \rho(\sigma_1)(u_1) \cdot \ldots \cdot w_l \cdot \rho(\sigma_1)(u_l).$$

By the observations above, $\rho(\sigma_1)(u_0)$ starts with $h_1^{-1}$ and $\rho(\sigma_1)(u_l)$ ends with $h_1$; thus $\rho(\sigma_1)(u)$ starts with $h_1^{-1}$ and ends with $h_1$. 


Let $k \in \{2, \ldots, n - 1\}$ and $\varepsilon \in \{-1, 1\}$. Write
\[
u = h_1^{-1} v_1 \ldots v_{l-1} w_l h_1,
\]
where $v_1, \ldots, v_{l-1} \in H_1 \setminus \{\text{Id}\}$ and $w_1, \ldots, w_l \in K \setminus \{\text{Id}\}$. Both groups, $H_1$ and $K$, are invariant by $\rho(\sigma_k^\varepsilon)$, and $\rho(\sigma_k^\varepsilon)$ is the identity on $H_1$. So
\[
\rho(\sigma_k^\varepsilon)(u) = h_1^{-1} \cdot \rho(\sigma_k^\varepsilon)(w_1) \cdot v_1 \cdot \ldots \cdot v_{l-1} \cdot \rho(\sigma_k^\varepsilon)(w_l) \cdot h_1;
\]
thus the normal form of $\rho(\sigma_k^\varepsilon)(u)$ starts with $h_1^{-1}$ and ends with $h_1$.

**Proof of Proposition 4.1(i).** We argue by induction on $n$. Assume $n = 2$. We have
\[
\rho(\sigma_1^n)(h_1) = (h_2 h_1)^{-l} h_1 (h_2 h_1)^l \neq h_1, \quad \text{for } l \in \mathbb{Z} \setminus \{0\}
\]
\[
\rho(\sigma_1^{2l+1})(h_1) = (h_2 h_1)^{-l} h_1^{-1} h_2 h_1 (h_2 h_1)^l \neq h_1, \quad \text{for } l \in \mathbb{Z};
\]
thus the representation $\rho : B_2 \to \text{Aut}(H_1 \ast H_2)$ is faithful.

Now, assume $n \geq 3$. Let $\beta \in B_n \setminus \{\text{Id}\}$. By Proposition 4.2, either $\beta \in B_{[2,n]}$, or one of $\beta$ or $\beta^{-1}$ is written $\alpha_0 \sigma_1 \ldots \sigma_l \alpha_l$, where $l \geq 1$ and $\alpha_0, \ldots, \alpha_l \in B_{[2,n]}$.

Suppose $\beta \in B_{[2,n]}$. By induction, $\rho(\beta)$ acts nontrivially on $K = H_2 \ast \ldots \ast H_n$; thus $\rho(\beta)$ acts nontrivially on $H^{*n} = H_1 \ast K$.

Suppose $\beta = \alpha_0 \sigma_1 \ldots \sigma_l \alpha_l$, where $l \geq 1$ and $\alpha_0, \ldots, \alpha_l \in B_{[2,n]}$. Let
\[
u = \rho(\sigma_1 \alpha_l)(h_1) = \rho(\sigma_1)(h_1) = h_1^{-1} h_2 h_1.
\]
By Lemma 4.3, the normal form of $\rho(\alpha_0 \sigma_1 \ldots \sigma_l \alpha_{l-1})(u) = \rho(\beta)(h_1)$ starts with $h_1^{-1}$ and ends with $h_1$. In particular, $\rho(\beta)(h_1) \neq h_1$; thus $\rho(\beta) \neq \text{Id}$.

Finally, suppose $\beta^{-1} = \alpha_0 \sigma_1 \ldots \sigma_l \alpha_l$, where $l \geq 1$ and $\alpha_0, \ldots, \alpha_l \in B_{[2,n]}$. By the previous case, $\rho(\beta^{-1}) \neq \text{Id}$; thus $\rho(\beta) \neq \text{Id}$.

5. Garside groups

In this section we give a brief presentation of the definition and salient properties of a Garside group, and then establish the necessary and sufficient criteria for a group to be a Garside group which we shall use in the subsequent section. Our Criterion 5.9 is essentially a variation on [Dehornoy 2002, Proposition 2.1]. The theory of Garside groups, as developed in [Dehornoy and Paris 1999; Dehornoy 1997b; 2002], provides the most natural general setting for the combinatorial arguments contained in Garside’s original treatment [1969] of the braid groups, and its generalization to Artin groups in [Brieskorn and Saito 1972].

**Definition 5.1.** Let $M$ be an arbitrary monoid. We say that $M$ is *atomic* if there exists a function $\nu : M \to \mathbb{N}$ such that
\[
\begin{align*}
\nu(a) &= 0 \text{ if and only if } a = 1; \\
\nu(ab) &\geq \nu(a) + \nu(b) \text{ for all } a, b \in M.
\end{align*}
\]
Such a function \( \nu : M \to \mathbb{N} \) is called a \textit{norm} on \( M \). An element \( a \in M \) is called an \textit{atom} if it is indecomposable, namely, if \( a = bc \) then either \( b = 1 \) or \( c = 1 \).

We note that any generating set of \( M \) contains the set of all atoms. In particular, \( M \) is finitely generated if and only if it has only finitely many atoms. For details see [Dehornoy and Paris 1999].

Given that a monoid \( M \) is atomic, we may define left and right invariant partial orders \( \leq_L \) and \( \leq_R \) on \( M \) as follows:

- set \( a \leq_L b \) if there exists \( c \in M \) such that \( ac = b \);
- set \( a \leq_R b \) if there exists \( c \in M \) such that \( ca = b \).

We shall call these the \textit{left} and \textit{right divisibility orders} on \( M \).

**Definition 5.2.** A \textit{Garside monoid} is a monoid \( M \) such that

(i) \( M \) is atomic and finitely generated;

(ii) \( M \) is (left and right) cancellative, i.e. \( abc = ab'c \) implies \( b = b' \);

(iii) \((M, \leq_L)\) and \((M, \leq_R)\) are lattices;

(iv) there exists an element \( \Delta \in M \), which we call a \textit{Garside element}, such that

(a) the set \( L(\Delta) := \{ x \in M : x \leq_L \Delta \} \) generates \( M \), and

(b) the sets \( L(\Delta) \) and \( R(\Delta) := \{ x \in M : x \leq_R \Delta \} \) are equal.

**Definition 5.3.** For any monoid \( M \) one can define the group \( G(M) \) which is presented by the generating set \( M \) and relations \( ab = c \) whenever \( ab = c \) in \( M \). There is an obvious canonical homomorphism \( M \to G(M) \). This homomorphism is not injective in general. The group \( G(M) \) is known as the \textit{group of fractions} of \( M \). Define a \textit{Garside group} to be the group of fractions of a Garside monoid.

**Remark.** (1) A Garside monoid \( M \) satisfies ôre’s conditions (left and right cancellativity and the existence of common upper bounds in \((M, \leq_L)\)); thus the canonical homomorphism \( M \to G(M) \) is injective. Moreover the partial orders \( \leq_L \) and \( \leq_R \) extend respectively to left- and right-invariant lattice orders on \( G(M) \) with positive cone \( M \).

(2) A Garside element is never unique. For example, if \( \Delta \) is a Garside element, then \( \Delta^k \) is also a Garside element for all \( k \geq 1 \) [Dehornoy 2002, Lemma 2.2].

(3) Elsewhere in the literature the condition that \( M \) is finitely generated is often incorporated into condition (iv) of the definition by saying that the set \( L(\Delta) \) is finite. It seems more natural to state this condition separately. Note that, if \( M \) is finitely generated and atomic, then \( L(a) = \{ x \in M : x \leq_L a \} \) is finite for all \( a \in M \).
We now introduce some terminology needed in order to state Criterion 5.9.

For a finite set $S$, we denote by $S^*$ the free monoid on $S$. The elements of $S^*$ are called words on $S$. The empty word is denoted by $\epsilon$. Let $\equiv$ be a congruence relation on $S^*$, and let $M = (S^*/\equiv)$. For $w \in S^*$, we denote by $\overline{w}$ the element of $M$ represented by $w$, and we call $w$ an expression of $\overline{w}$.

**Definition 5.4.** A complement is a function $f : S \times S \to S^*$ such that $f(x,x) = \epsilon$ for all $x \in S$. To a complement $f : S \times S \to S^*$ we associate the two monoids

$$M^f_L = \{ S \mid xf(x,y) = yf(y,x) \text{ for } x, y \in S \}^+, \quad M^f_R = \{ S \mid f(y,x)x = f(x,y)y \text{ for } x, y \in S \}^+.$$

For $u, v \in S^*$, we write $u \equiv^f_L v$ if $u$ and $v$ are expressions of the same element of $M^f_L$, and we write $u \equiv^f_R v$ if $u$ and $v$ are expressions of the same element of $M^f_R$.

**Definition 5.5.** A word $w$ in $(S \cup S^{-1})^*$ is $f$-reversible on the left in one step to a word $w'$ if $w'$ is obtained from $w$ by replacing some subword $x^{-1}y$ (with $x, y \in S$) by the corresponding word $f(x,y)f(y,x)^{-1}$. Let $p \geq 0$. We say that $w$ is $f$-reversible on the left in $p$ steps to a word $w'$ if there exists a sequence $w_0 = w, w_1, \ldots, w_p = w'$ in $(S \cup S^{-1})^*$ such that $w_{i-1}$ is $f$-reversible on the left in one step to $w_i$ for all $i = 1, \ldots, p$. The property “$w$ is $f$-reversible on the left to $w'$” is denoted by $w \rightarrow^f_L w'$.

We define $f$-reversibility on the right in a similar way, replacing subwords $yx^{-1}$ (with $x, y \in S$) by the corresponding words $f(x,y)^{-1}f(y,x)$. The property “$w$ is $f$-reversible on the right to $w'$” is denoted by $w \rightarrow^f_R w'$.

It is shown in [Dehornoy 1997b] that a reversing process is confluent, namely:

**Proposition 5.6** [Dehornoy 1997b, Lemma 1]. Let $f : S \times S \to S^*$ be a complement, and let $w \in (S \cup S^{-1})^*$. Suppose that the word $w$ is $f$-reversible on the left in $p$ steps to a word $uv^{-1}$, with $u, v \in S^*$. Then any sequence of left $f$-reversing transformations starting from $w$ leads in $p$ steps to $uv^{-1}$.

**Definition 5.7.** Let $f : S \times S \to S^*$ be a complement and let $u, v \in S^*$. Assume that there exist $u', v' \in S^*$ such that $u^{-1}v \rightarrow^f_L u'(v')^{-1}$. By Proposition 5.6, $u'$ and $v'$ are unique (if they exist). Then we write $u' = C^f_L(u,v)$ and $v' = C^f_L(v,u)$. One has, by [Dehornoy 1997b, Lemma 2],

$$uC^f_L(u,v) \equiv^f_L vC^f_L(v,u).$$

If no such words $u'$, $v'$ exist, we write $C^f_L(u,v) = C^f_L(v,u) = \infty$.

Similarly, define the words $C^f_R(u,v)$ and $C^f_R(v,u)$ to be the unique elements of $S^*$ which satisfy $vu^{-1} \rightarrow^f_R C^f_R(v,u)^{-1}C^f_R(v,u)$, or write $C^f_R(u,v) = C^f_R(v,u) = \infty$ if no such words exist.
Definition 5.8 [Dehornoy 1997b, p. 120]. Let \( f : S \times S \to S^* \) be a complement. We say that \( f \) is coherent on the left if, for all \( x, y, z \in S \) such that \( C_L^f(f(x, y), f(x, z)) \neq \infty \) we have

\[
C_L^f(f(x, y), f(x, z)) \equiv \max_{L} C_L^f(f(y, x), f(y, z)).
\]

Similarly, we say that \( f \) is coherent on the right if, for all \( x, y, z \in S \) such that \( C_R^f(f(z, x), f(y, x)) \neq \infty \) we have

\[
C_R^f(f(z, x), f(y, x)) \equiv \max_{R} C_R^f(f(z, y), f(x, y)).
\]

It can be shown [Dehornoy 1997b, Lemma 4] that if an atomic monoid \( M \) can be written \( M = M_L^f \) where the complement \( f \) is coherent on the left, then \( M \) is left cancellative and \( (M, \leq_L) \) is a quasi-lattice: every pair of elements \( x, y \in M \) which has a common upper bound \( z \) such that \( x \leq_L z \) and \( y \leq_L z \) has a least upper bound, written \( x \lor_L y \). This argument is based on Garside’s [1969] original argument (see also [Brieskorn and Saito 1972]), and forms the cornerstone of the theory of Garside groups. (The analogous statement when \( M = M_R^g \) is atomic and \( g \) is coherent on the right obviously holds as well.)

We are now ready to state a criterion for a monoid \( M \) to be a Garside monoid:

Criterion 5.9. Let \( M \) be a monoid. Then \( M \) is a Garside monoid if and only if it satisfies the following properties:

- (C1) \( M \) is finitely generated and atomic.
- (C2) There exist complements \( f : S_1 \times S_1 \to S_1^* \), coherent on the left, and \( g : S_2 \times S_2 \to S_2^* \), coherent on the right, such that \( M \cong M_L^f \) and \( M \cong M_R^g \).
- (C3) \( M \) possesses a Garside element, namely an element \( \Delta \in M \) such that the sets \( L(\Delta) = \{ x \in M : x \leq_L \Delta \} \) and \( R(\Delta) = \{ x \in M : x \leq_R \Delta \} \) are equal and generate \( M \).

Proof. Suppose first that \( M \) satisfies (C1), (C2), and (C3). It follows from [Dehornoy 1997b, Lemma 4] (see the remark above) that \( M \) is left and right cancellative and \( (M, \leq_L) \) is a quasi-lattice. In this situation we may define an operation \( \setminus_L : M \times M \to M \cup \{ \infty \} \) such that \( a(a \setminus_L b) = a \lor_L b \) if \( a \) and \( b \) have a common upper bound, and \( a \setminus_L b = \infty \) otherwise. According to [Dehornoy 2002, Proposition 2.1], the above conditions together with the following condition (D) are sufficient to show that \( M \) is a Garside monoid:

- (D) There exists a finite subset \( P \subseteq M \) which generates \( M \) and which is closed under the operation \( \setminus_L \) (namely, if \( a, b \in P \) then \( a \setminus_L b \in P \)).

We show that \( M \) satisfies (D). Let \( P = \Delta(\Delta) = R(\Delta) \). Note that, by (C3), \( P \) generates \( M \). Let \( a, b \in P \). Since \( a \leq_L \Delta \) and \( b \leq_L \Delta \), we have \( a \lor_L b \leq_L \Delta \).
Let \( c \in M \) such that \( \Delta = (a \lor_{L} b) c = a (a \setminus_{L} b) c \). Then \((a \setminus_{L} b) c \leq_{R} \Delta \); thus \((a \setminus_{L} b) c \leq_{L} \Delta \) (since, by (C3), \( L(\Delta) = R(\Delta) \)); therefore \((a \setminus_{L} b) \leq_{L} \Delta \), that is \((a \setminus_{L} b) \in P \).

Now suppose that \( M \) is a Garside monoid. Clearly, \( M \) satisfies (C1) and (C3). So, we just need to show that \( M \) satisfies (C2). Choose some finite generating set \( S \) for \( M \), and consider complements \( f : S \times S \to S^{\ast} \) and \( g : S \times S \to S^{\ast} \) such that

\[
\overline{f}(x, y) = x \lor_{L} y, \quad \overline{g}(x, y)x = y \lor_{R} x,
\]

for all \( x, y \in S \). Then, by [Dehornoy and Paris 1999, Theorem 4.1], \( M = M_{L}^{f} = M_{R}^{g} \), and, by [Dehornoy 2002, Lemma 5.2], \( f \) is coherent on the left and \( g \) is coherent on the right.

It will be convenient, in Section 6, to have the following characterization of a Garside element.

**Lemma 5.10** (Garside elements). Let \( M \) be a (left and right) cancellative monoid. Then \( \Delta \) is a Garside element (meaning that \( L(\Delta) \) coincides with \( R(\Delta) \) and generates \( M \)) if and only if the following condition holds:

(C4) \( L(\Delta) := \{ x \in M : x \leq_{L} \Delta \} \) generates \( M \) and there exists a (necessarily unique) monoid automorphism \( \tau : M \to M \) such that \( w \Delta = \Delta \tau(w) \) for all \( w \in M \).

Consequently, we may replace condition (C3) in Criterion 5.9 with condition (C4).

**Proof.** We first show sufficiency. Suppose that (C4) is satisfied. In particular, we have \( \tau(\Delta) = \Delta \) and therefore \( \tau(L(\Delta)) = L(\Delta) \) (since \( \tau \) is a monoid automorphism). On the other hand, by using left and right cancellation one easily obtains from the equation \( x \Delta = \Delta \tau(x) \) that \( \tau(L(\Delta)) = R(\Delta) \). But then \( L(\Delta) = \tau(L(\Delta)) = R(\Delta) \) and, by hypothesis (C4), \( L(\Delta) \) also generates \( M \). Thus \( \Delta \) is a Garside element.

Now suppose that \( \Delta \) is a Garside element. By cancellativity and the fact that \( L(\Delta) = R(\Delta) \), one has a well-defined bijection \( c : L(\Delta) \to L(\Delta) \) such that \( x c(x) = \Delta \) for all \( x \in L(\Delta) \). Note that, if \( x \in L(\Delta) \) then so is \( c(x) \) and \( \Delta \) may be written either \( x c(x) \) or \( c(x) c^{\ast}(x) \). Thus \( x \Delta = x c(x) c^{\ast}(x) = \Delta c^{\ast}(x) \), for all \( x \in L(\Delta) \). Since \( L(\Delta) \) generates \( M \), it follows by cancellativity that the bijection \( c^{\ast} \) extends uniquely to a monoid automorphism \( \tau \) satisfying (C4). \( \Box \)

### 6. Semidirect products

We now turn back to the Artin type representations. Given an Artin type representation \( \rho : B_{n} \to \text{Aut}(H^{\ast n}) \) associated to a group \( H \) and an element \( h \in H \), we may form the semidirect product \( H^{\ast n} \rtimes_{\rho} B_{n} \). The aim of this section is to prove the following.
Theorem 6.1. Assume that $H$ is the group of fractions of a Garside monoid $M$ and that $h \in M$ is a Garside element. Let $G = H^n \rtimes \rho B_n$, where $\rho : B_n \to \text{Aut}(H^n)$ denotes the Artin type representation associated to $(H, h)$ (as defined in the Introduction), and let $P$ be the submonoid of $G$ generated by $M_1 = \phi_1(M)$ and the monoid $B_n^+$ of positive braids. Then $P$ is a Garside monoid, $\Delta = (h_1 \sigma_1 \sigma_2 \ldots \sigma_{n-1})^n$ is a Garside element of $P$, and $G$ is the group of fractions of $P$.

The first step in the proof is to find a presentation for $H^n \rtimes \rho B_n$:

Proposition 6.2. Let $H = \langle S \mid \mathbb{R} \rangle$ be a presentation for $H$, and let $D \in S^*$ be an expression for $h$. Then $G = H^n \rtimes \rho B_n$ has a presentation with generators

$$S \cup \{\sigma_1, \ldots, \sigma_{n-1}\}$$

and relations

$$r \quad \text{for } r \in \mathbb{R},$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{for } i = 1, \ldots, n-2,$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for } |i-j| \geq 2,$$

$$\sigma_i x = x \sigma_i \quad \text{for } x \in S \text{ and } i = 2, \ldots, n-1,$$

$$x \sigma_i D \sigma_1 = \sigma_1 D \sigma_1 D^{-1} x D \quad \text{for } x \in S.$$

Proof of the proposition. Let $G_0$ denote the abstract group generated by the union $S \cup \{\sigma_1, \ldots, \sigma_{n-1}\}$, subject to the relations given in the statement of the proposition. Set $X = (\bigcup_{i=1}^n \phi_i(S)) \cup \{\sigma_1, \ldots, \sigma_{n-1}\}$. With a little effort one can verify that the mapping $\varphi : X \to G_0$ defined by

$$\varphi(\phi_i(x)) = \sigma_i^{-1} \ldots \sigma_{i-1}^{-1} D^{i-1} x D^{i-1} \sigma_i \ldots \sigma_{i-1} \quad \text{for } i = 1, \ldots, n \text{ and } x \in S,$$

$$\varphi(\sigma_i) = \sigma_i \quad \text{for } i = 1, \ldots, n-1$$

determines a homomorphism $\varphi : G \to G_0$, and somewhat more easily that the mapping $\psi : S \cup \{\sigma_1, \ldots, \sigma_{n-1}\} \to G$ defined by

$$\psi(x) = \phi_i(x) \quad \text{for } x \in S,$$

$$\psi(\sigma_i) = \sigma_i \quad \text{for } i = 1, \ldots, n-1$$

determines a homomorphism $\psi : G_0 \to G$. One checks without much difficulty that $(\psi \circ \varphi)(a) = a$ for all $a \in X$, and $(\varphi \circ \psi)(b) = b$ for all $b \in S \cup \{\sigma_1, \ldots, \sigma_{n-1}\}$; thus $\psi \circ \varphi = \text{Id}_G$ and $\varphi \circ \psi = \text{Id}_{G_0}$. \[\square\]

Proof of Theorem 6.1. Let $\tau : M \to M$ denote the automorphism of $M$ induced by conjugation by $h^{-1}$, so that $xh = h \tau(x)$ for all $x \in M$ (see Lemma 5.10). Let $S$ be a finite generating set for $M$. We may, and do, choose $S$ so that $\tau(S) = S$ (for instance we may simply choose $S$ to be the set of atoms of $M$). Define $f : S \times S \to S^*$ so that

$$xf(x, y) = yf(y, x) = x \vee_L y \quad \text{for all pairs } x, y \in S.$$

Similarly define $g : S \times S \to S^*$.
so that \( g(x, y) = \frac{g(y, x)}{x \lor y} \) for all pairs \( x, y \in S \). As pointed out in the proof of Criterion 5.9, one has \( M = M_L^f = M_R^f \), \( f \) is coherent on the left, and \( g \) is coherent on the right. We simply write \( \sim \) for the congruence relation on \( S^* \) defined by the relations in \( M \) (namely, \( \equiv_L^f \), or equally \( \equiv_R^f \)). Let \( D \in S^* \) be an expression of \( h \). Note that for \( x \in S \) we have \( xD \sim D \tau(x) \) and \( \tau^{-1}(x)D \sim Dx \), where \( \tau(x) \) and \( \tau^{-1}(x) \) also denote elements of the generating set \( S \). The last family of relations appearing in Proposition 6.2 may be replaced with \( x\sigma_1D\sigma_1 = \sigma_1D\sigma_1\tau(x) \) for all \( x \in S \), or equivalently with \( \tau^{-1}(x)\sigma_1D\sigma_1 = \sigma_1D\sigma_1x \) for all \( x \in S \).

Let \( X = S \cup \{\sigma_1, \ldots, \sigma_n\} \). Let \( \lambda : X \times X \to X^* \) be the complement defined by

- \( \lambda(x, y) = f(x, y) \) for \( x, y \in S \), \( \lambda(\sigma_1, x) = x \) for \( x \in S \) and \( i \geq 2 \),
- \( \lambda(x, \sigma_1) = \sigma_1D\sigma_1 \) for \( x \in S \), \( \lambda(\sigma_1, \sigma_j) = \sigma_j\sigma_1 \) for \( |i - j| = 1 \),
- \( \lambda(\sigma_1, x) = D\sigma_1\tau(x) \) for \( x \in S \), \( \lambda(\sigma_1, \sigma_j) = \sigma_j \) for \( |i - j| \geq 2 \),
- \( \lambda(x, \sigma_i) = \sigma_i \) for \( x \in S \) and \( i \geq 2 \).

and let \( \delta : X \times X \to X^* \) be the complement defined by

- \( \delta(x, y) = g(x, y) \) for \( x, y \in S \), \( \delta(x, \sigma_j) = x \) for \( x \in S \) and \( i \geq 2 \),
- \( \delta(\sigma_1, x) = \sigma_1D\sigma_1 \) for \( x \in S \), \( \delta(\sigma_j, \sigma_i) = \sigma_i\sigma_j \) for \( |i - j| = 1 \),
- \( \delta(x, \sigma_1) = \tau^{-1}(x)\sigma_1D \) for \( x \in S \), \( \delta(\sigma_j, \sigma_i) = \sigma_j \) for \( |i - j| \geq 2 \),
- \( \delta(\sigma_i, x) = \sigma_i \) for \( x \in S \) and \( i \geq 2 \).

Let \( P_0 \) denote the monoid defined by the presentation with generators \( X \) and relations as laid out in Proposition 6.2. Then clearly \( P_0 \cong M_L^f \cong M_R^f \). We denote by \( \approx \) the congruence relation on \( X^* \) defined by the relations of \( P_0 \). (So \( \approx \) is the same congruence relation as \( \equiv_L^f \) and \( \equiv_R^f \).) We now show that \( P_0 \) satisfies Criterion 5.9 with complements \( \lambda \) and \( \delta \) and Garside element \( \Delta = (D\sigma_1\sigma_2 \ldots \sigma_{n-1})^n \). It will follow that \( P_0 \) is a Garside monoid with group of fractions \( G \) and is canonically isomorphic to the submonoid \( P \subset G \) in the statement of the Theorem.

Clearly \( P_0 \) is finitely generated. We check that \( P_0 \) is atomic. Let \( v : M \to \mathbb{N} \) be a norm for \( M \). Let \( \Sigma = \{\sigma_1, \ldots, \sigma_n\} \) and define the function \( \ell : \Sigma^* \to \mathbb{N} \) by \( \ell(\sigma_1 \ldots \sigma_l) = l \). We define a function \( v_P : X^* \to \mathbb{N} \) as follows. Let \( w \in X^* \). Write \( w = u_1v_1 \ldots u_lv_l \), where \( u_1 \in S^*, u_2, \ldots, u_l \in S^* \setminus \{\epsilon\} \), \( v_1, \ldots, v_{l-1} \in \Sigma^* \setminus \{\epsilon\} \), and \( v_l \in \Sigma^* \). Then

\[ v_P(w) = v(u_1u_2 \ldots u_l) + \ell(v_1v_2 \ldots v_l). \]

One can easily verify that \( v_P \) is invariant with respect to all of the relations given in Proposition 6.2, and therefore defines a function \( v_P : P_0 \to \mathbb{N} \). Moreover, it is easily seen that \( v_P \) is a norm, and therefore \( P_0 \) is atomic.
The proof that \( \lambda \) is coherent on the left may be deduced from the existence, for each triple \( \alpha, \beta, \gamma \in X \), of a certain tiling of the 2-sphere by relations from \( M^2_L \) (i.e., relations of the form \( \alpha \lambda(\alpha, \beta) \approx \beta \lambda(\beta, \alpha) \) for \( \alpha, \beta \in X \)). We illustrate the two most difficult cases, namely when \( \{\alpha, \beta, \gamma\} = \{\sigma_1, \sigma_2, x\} \) for some \( x \in S \) (Figure 7), and when \( \{\alpha, \beta, \gamma\} = \{\sigma_1, x, y\} \) for some \( x \neq y \in S \) (Figure 8). In the latter case note that, if \( f(x, y) \) is written \( a_1 a_2 \ldots a_k \) as a product of generators \( a_i \in S \) then \( \tau(f(x, y)) \approx \tau(a_1) \tau(a_2) \ldots \tau(a_k) \) and the face containing \( f(x, y) \) and \( \tau(f(x, y)) \) in Figure 8 decomposes into \( k \) faces corresponding to the relations
Lemma 5.10. Consider the
Theorem 6.1 of
In closing, we remark that both the above proof and the formulation
were strongly inspired by the example of the Artin group $A(B_n)$
which, as noted in the introduction, is isomorphic to the semidirect product $F_n \rtimes B_n$
associated to Artin’s 1925 representation [Artin 1925; 1947], namely the Artin type
representation associated to $(\mathbb{Z}, 1)$ (using additive notation). This is evident in both

$$a_i \sigma_1 D \sigma_1 \approx \sigma_1 D \sigma_1 \tau(a_i).$$
Similarly for $f(y, x)$. The remaining cases are easily handled since in these cases at least one of $\alpha, \beta, \gamma$ satisfies a commuting relation
(explicit in the presentation $M^+_2$) with each of the others.

The proof that $\delta$ is coherent on the right is similar.

Finally we show that the word $\Delta = (D \sigma_1 \sigma_2 \ldots \sigma_{n-1})^n$ represents a Garside element of $P_0$. We shall employ condition (C4) of Lemma 5.10. Consider the
Artin monoid presentation

$$A^+(B_n) = \langle \beta_1, \beta_2, \ldots, \beta_n \mid \beta_1 \beta_2 \beta_1 \beta_2 = \beta_2 \beta_1 \beta_2 \beta_1, \beta_i \beta_{i+1} \beta_i = \beta_{i+1} \beta_i \beta_{i+1} \text{ for } 2 \leq i \leq n-1, \beta_i \beta_j = \beta_j \beta_i \text{ for } |i - j| \geq 2 \rangle^+.$$  

This monoid $A^+(B_n)$ is well-known as the Artin monoid of type $B_n$, and has Garside element $\Delta_B = (\beta_1 \beta_2 \ldots \beta_n)^n$. Clearly there exists a monoid homomorphism
$A^+(B_n) \to P_0$ such that $\beta_1 \mapsto D$ and $\beta_i \mapsto \sigma_{i-1}$ for $i = 2, 3, \ldots, n$. Thus any relation which is observed in $A^+(B_n)$ may be deduced in $P_0$. In particular, the fact that $\Delta_B$ is a Garside element in $A^+(B_n)$ implies that $\Delta$ is left divisible by $D, \sigma_1, \ldots, \sigma_{n-1}$ and hence is left divisible by every element of $X$. It remains to verify that there exists an automorphism $\tau_P : P_0 \to P_0$ such that $w \Delta = \Delta \tau_P(w)$
for all $w \in P_0$.

We already know that $\Delta_B$ is central in $A^+(B_n)$. Thus we have $\sigma_i \Delta = \Delta \sigma_i$ for all $i = 1, 2, \ldots, n - 1$. We may also check (by performing the calculation in $A^+(B_n)$) that

$$\Delta \approx D U^{n-1} \quad \text{where } U := \sigma_1 D \sigma_1 \sigma_2 \sigma_3 \cdots \sigma_{n-1}.$$  

Recall that $\tau$ denotes the automorphism of $M$ such that, at the level of words,
$x D \sim D \tau(x)$ for all $x \in S^n$. Observe also that $x U \approx U \tau(x)$ for all $x \in S^n$ (or more loosely speaking, for all $x \in M$). We now define $\tau_P : P_0 \to P_0$ such that

$$\tau_P(\sigma_i) = \sigma_i \quad \text{for } i = 1, 2, \ldots, n - 1,$$
$$\tau_P(x) = \tau^n(x) \quad \text{for all } x \in M.$$  

It is easily seen that $\tau_P$ is a monoid isomorphism. Moreover, for all $x \in M$,

$$x \Delta \approx x D U^{n-1} \approx D \tau(x) U^{n-1} \approx DU^{n-1} \tau^n(x) \approx \Delta \tau_P(x),$$  

and $\sigma_i \Delta \approx \Delta \sigma_i$ for all $i = 1, 2, \ldots, n - 1$. Thus condition (C4) of Lemma 5.10 is satisfied, and $\Delta$ is a Garside element.

Remark. In closing, we remark that both the above proof and the formulation
of Theorem 6.1 were strongly inspired by the example of the Artin group $A(B_n)$
which, as noted in the introduction, is isomorphic to the semidirect product $F_n \rtimes B_n$
associated to Artin’s 1925 representation [Artin 1925; 1947], namely the Artin type
representation associated to $(\mathbb{Z}, 1)$ (using additive notation). This is evident in both
the description of the fundamental element, and the checking of coherence (see Figures 7 and 8) which follow closely the proof that $A(B_n)$ has a Garside structure. Note, in particular, that the diagram shown in Figure 7 depicts the Cayley graph for the Coxeter group of type $B_3$, once the labels $D, x, \tau(x)$ and $\tau^2(x)$ are replaced with a single generator.

In response to a question posed by the referee, we are not aware of any other general constructions of Garside groups obtained in a similar fashion by studying other Artin groups of finite type. We note however that the Artin group of type $D_n$ is isomorphic to the index 2 torsion free subgroup of the semidirect product $(C_2)^{2n} \rtimes B_n$ associated to the Artin type representation determined by the nontrivial element of $C_2$. However, the group $C_2$ of order 2 is clearly not Garside (it has torsion!) so that while $A(D_n)$ admits a Garside structure, this does not arise by virtue of Theorem 6.1 just proved. The Artin groups of type $B_n$, $n \geq 2$, would appear to be the only Artin groups of irreducible finite type which are covered in this way by Theorem 6.1.

Appendix

We denote by $F_n$ the free group of rank $n$, and fix a basis $x_1, \ldots, x_n$ for $F_n$.

**Definition.** According to Shpilrain’s terminology [2001], a *Wada representation of type* (1) is an Artin type representation associated to $(\mathbb{Z}, h)$, where $h$ is a nonzero integer. Such a representation will be denoted by $\rho_h^{(1)} : B_n \to \text{Aut}(F_n)$. It is determined by

$$\rho_h^{(1)}(\sigma_k)(x_i) = \begin{cases} x_i & \text{if } i \neq k, k + 1, \\ x_k^{-h}x_{k+1}^hx_k & \text{if } i = k, \\ x_k & \text{if } i = k + 1. \end{cases}$$

The *Wada representation of type* (2) is the representation $\rho^{(2)} : B_n \to \text{Aut}(F_n)$ determined by

$$\rho^{(2)}(\sigma_k)(x_i) = \begin{cases} x_i & \text{if } i \neq k, k + 1, \\ x_kx_{k+1}^{-1}x_k & \text{if } i = k, \\ x_k & \text{if } i = k + 1. \end{cases}$$

and the *Wada representation of type* (3) is the representation $\rho^{(3)} : B_n \to \text{Aut}(F_n)$ determined by

$$\rho^{(3)}(\sigma_k)(x_i) = \begin{cases} x_i & \text{if } i \neq k, k + 1, \\ x_k^2x_{k+1} & \text{if } i = k, \\ x_{k+1}^{-1}x_k^{-1}x_{k+1} & \text{if } i = k + 1. \end{cases}$$
Proposition A.1. (1) Let $k, l \in \mathbb{Z} \setminus \{0\}$. Then $\rho_k^{(1)}$ and $\rho_l^{(1)}$ are equivalent if and only if $l = \pm k$.

(2) $\rho^{(2)}$ and $\rho^{(3)}$ are equivalent.

(3) Let $k \in \mathbb{Z} \setminus \{0\}$. Then $\rho^{(2)}$ and $\rho_k^{(1)}$ are not equivalent.

The following lemmas are preliminary to the proof of this proposition.

Lemma A.2. Consider the action of $B_n$ on $F_n$ via the representation $\rho_h^{(1)}$. For all $i = 1, \ldots, n-1$, the subgroup of $F_n$ left fixed by $\langle \sigma_i \rangle$, and written $F_n^{\langle \sigma_i \rangle}$, is freely generated by the elements

$$x_1, \ldots, x_{i-1}, x_i^{h} x_i^{h+1} x_i^{h+2}, \ldots, x_n.$$

Proof: Write $F_n = C * D$, where $C = \langle x_j, x_{j+1} \rangle$, $D = \langle x_1, \ldots, x_{i-1}, x_{i+2}, \ldots, x_n \rangle$. Both groups, $C$ and $D$, are invariant by the action of $\sigma_i$. Moreover, $\sigma_i$ is the identity on $D$ and acts on $C$ by $x_i \mapsto x_i^{-h}$, $x_{i+1} \mapsto x_i$, $x_{i+2} \mapsto x_i$. In particular, $F_n^{\langle \sigma_i \rangle} = C^{\langle \sigma_i \rangle} * D$.

Let $u \in C^{\langle \sigma_i \rangle}$. Write $$u = x_i^{n_1} x_i^{n_2} \cdots x_i^{n_r} x_i^{n_1},$$ where $r \geq 1, m_1, \ldots, m_{r-1}, n_2, \ldots, n_r \in \mathbb{Z} \setminus \{0\}$, and $m_r, n_1 \in \mathbb{Z}$. First, suppose $n_1 \neq 0$. Then $$\sigma_i(u) = x_i^{-h} x_i^{n_1} x_i^{m_1} \cdots x_i^{n_r} x_i^{m_1} x_i^{n_1} = u.$$ Thus $$-h = n_1, n_1 = m_1, \ldots, n_r = m_r,$$ hence $u = (x_i^{n_1} x_i^{m_1})^{-r}$. Now, suppose $n_1 = 0$. Then $$\sigma_i(u) = x_i^{m_1} x_i^{m_2} \cdots x_i^{n_r} x_i^{m_1} x_i^{m_1} x_i^{n_1}.$$ Thus $$m_1 - h = 0, m_1 = n_2, n_2 = m_2, \ldots, n_r = m_r + h,$$ hence $u = (x_i^{n_1} x_i^{m_1})^{-r}$.

Lemma A.3. Consider the action of $B_n$ on $F_n$ via $\rho_h^{(1)}$. Then the fixed subgroup $F_n^{B_n}$ is the cyclic subgroup of $F_n$ generated by $x_1^h \ldots x_2^h x_1^h$.

Proof: Let $u \in F_n^{B_n}$. We have $u \in F_n^{\langle \sigma_i \rangle}$ for all $i = 1, \ldots, n-1$. Thus, by Lemma A.2, the reduced form of $u$ satisfies the following properties:

- All the exponents are either equal to $h$ or equal to $-h$.
- If $i \neq 1$, then $x_i^h$ is followed by $x_{i-1}^h$, and, if $i \neq n$, then $x_i^h$ is preceded by $x_{i+1}^h$.
- If $i \neq n$, then $x_i^{-h}$ is followed by $x_{i+1}^{-h}$, and, if $i \neq 1$, then $x_i^{-h}$ is preceded by $x_{i-1}^{-h}$. 

Proposition A.1, we have Lemma A.3.

Proof of Proposition A.1. (1) Let \( k \in \mathbb{Z} \setminus \{0\} \). Let \( \phi : F_n \to F_n \) be the automorphism determined by \( \phi(x_i) = x_i^{-1} \) for all \( i = 1, \ldots, n \). One can easily verify that

\[
\phi^{-1} \circ \rho_k^{(1)}(\sigma_i) \circ \phi = \rho_{-k}^{(1)}(\sigma_i)
\]

for all \( i = 1, \ldots, n-1 \); thus \( \rho_k \) and \( \rho_{-k} \) are equivalent.

Let \( k, l > 0 \). For a group \( G \), we denote by \( H_1(G) \) the abelianization of \( G \), and, for a subgroup \( H \) of \( G \), we denote by \( \langle \langle H \rangle \rangle \) the normal subgroup of \( G \) generated by \( H \). By Lemma A.3, we have

\[
F_n / \langle \langle F_n^\rho_k^{(1)}(B_n) \rangle \rangle \simeq \langle x_1, \ldots, x_n \mid x_1^k, \ldots, x_n^k x_1 x_n^{-1} = 1 \rangle;
\]

hence

\[
H_1(F_n / \langle \langle F_n^\rho_k^{(1)}(B_n) \rangle \rangle) \simeq (\mathbb{Z} / k\mathbb{Z}) \times \mathbb{Z}^{n-1}.
\]

So, if \( \rho_k^{(1)} \) and \( \rho_l^{(1)} \) are equivalent, then \( (\mathbb{Z} / k\mathbb{Z}) \times \mathbb{Z}^{n-1} \simeq (\mathbb{Z} / l\mathbb{Z}) \times \mathbb{Z}^{n-1} \); thus \( k = l \).

(2) Write

\[
y_i = x_1^2 \ldots x_{i-1}^2 x_i \quad \text{for} \quad i = 1, \ldots, n.
\]

One can easily verify that

\[
\rho^{(3)}(\sigma_k)(y_i) = \begin{cases} y_i & \text{if} \ i \neq k, k+1, \\ y_{k+1} & \text{if} \ i = k, \\ y_{k+1} y_k^{-1} y_{k+1} & \text{if} \ i = k + 1. 
\end{cases}
\]

Let \( \phi : F_n \to F_n \) be the automorphism determined by \( \phi(x_i) = y_i \) for \( i = 1, \ldots, n \), and let \( \mu : B_n \to B_n \) be the automorphism determined by \( \mu(\sigma_i) = \sigma_i^{-1} \) for \( i = 1, \ldots, n-1 \). From the expression of \( \rho^{(3)}(\sigma_k)(y_i) \) given above, there follows

\[
\phi^{-1} \circ \rho^{(3)}(\sigma_i) \circ \phi = \rho^{(2)}(\mu(\sigma_i))
\]

for all \( i = 1, \ldots, n-1 \); thus \( \rho^{(2)} \) and \( \rho^{(3)} \) are equivalent.

(3) Let \( k > 0 \). For \( u \in F_n \), we denote by \( [u] \) the element of \( H_1(F_n) \simeq \mathbb{Z}^n \) represented by \( u \). We have

\[
\rho^{(2)}(\sigma_i^t)(x_1) = (t + 1)[x_1] - t[x_2]
\]

for all \( t \in \mathbb{N} \). On the other hand, \( \rho_k^{(1)}(\beta) \) has finite order as an automorphism of \( H_1(F_n) \), for all \( \beta \in B_n \). This shows that \( \rho^{(2)} \) and \( \rho_k^{(1)} \) are not equivalent. \( \square \)
References


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