Pacific Journal of Mathematics

CLUSTERS WITH MULTIPLICITIES IN $\mathbb{R}^2$

FRANK MORGAN

Volume 221 No. 1 September 2005

PACIFIC JOURNAL OF MATHEMATICS
Vol. 221, No. 1, 2005

CLUSTERS WITH MULTIPLICITIES IN $\mathbb{R}^2$

FRANK MORGAN

Perimeter-minimizing planar double soap bubbles in which regions are allowed to overlap with multiplicities meet in fours, fives, and sixes as well as threes. We further provide certain generalizations to immiscible fluids and higher dimensions, and an associated theory of calibrations. We work in the category of flat chains with coefficients in a normed group.

1. Introduction

Physically, regions of soap bubble clusters or immiscible fluids cannot overlap, but it can be of theoretical interest and use to consider such overlap, as in the simplest proof of the planar Double Bubble Theorem [Morgan 2001, §3] or in studies of planar triple bubbles [Cox et al. 1994/95, §7]. Our Regularity Theorem 5.6 below considers planar double bubbles which can overlap with multiplicity, and shows how perimeter minimizers with prescribed boundaries and areas meet:

(a) in threes, at 120 degrees, forming a Y;

(b) in fours, as two arcs tangent or crossing or a Y with stem extended;

(c) in fives, a Y with two arms extended;

(d) in sixes, a Y with all three arms extended.

In classical clusters without multiplicities, only (a) occurs. Similar questions about clusters of more than two regions remain open.

Immiscible fluids. More generally, Section 3 considers clusters of m immiscible fluids, in which the cost of an interface depends on the fluids (with multiplicities) it separates, as determined by a norm on $\mathbb{R}^m$. Our fundamental Regularity Theorem 3.7 shows that *for certain "simplicial" norms, a planar minimizer with finite boundary and prescribed areas consists of finitely many constant-curvature arcs*. The admissible norms include double soap bubbles but not clusters of three or more soap bubbles. Extensions from $\mathbb{R}^2$ to $\mathbb{R}^n$ remain conjectural and would require more sophisticated methods.

MSC2000: 49Q20.

Keywords: clusters with multiplicities, soap bubbles, immiscible fluids, paired calibrations, flat chains with coefficients in a normed group.

Our results are generalizations of earlier results in [White 1986; White 1996, Theorem 3; Morgan 1998] that do not allow multiplicities, and are special cases of more general conjectures of B. White [2001] (see Remark 3.6).

Chains with coefficients. Section 2 describes the technical formulation of clusters as chains with coefficients in a normed group G representing the various immiscible fluids. It follows [Fleming 1966] and [White 1996].

Calibrations. Section 4 provides an appropriate theory of calibrations for proving minimization. Classical calibrations [Morgan 2000, §§6.4, 6.5] are closed differential forms used to prove chains with coefficients in $\mathbb{R}$ minimizing. The Calibration Lemma 4.1 provides an extension from $\mathbb{R}$ to any complete normed group G by defining a calibration as a homomorphism from G into the space of closed differential forms. This extension includes paired calibrations [Lawlor and Morgan 1994].

One consequence is Corollary 4.3, which is Choe's Theorem [1996] that a stationary polyhedral chain is area minimizing under diffeomorphisms. Another is a useful characterization (Proposition 4.8) of when two lines through the origin with coefficients in certain normed groups are minimizing.

Proof of regularity. The main Regularity Lemma 3.5 for proving the Regularity Theorem 3.7 for certain planar minimizing immiscible fluid clusters shows that if in a small ball the cluster S is weakly close to a horizontal diameter with coefficient g, then in a shrunken ball it consists of constant-coefficient arcs. By the simplicial hypothesis on the norm, one may assume as in Propositions 3.2 and 3.3 that it connects points with coefficients $-f_i$ on the left to corresponding points with coefficients $+f_i$ on the right, where the f_i correspond to the (unique) norm decomposition of g, perhaps occurring with multiplicity. Also, S has a decomposition of the form $S = \sum f_i S_i$, with some cancellation only when some S_i and S_j have opposite orientation, which we must show cannot occur. Each S_i consists of curves from a point with coefficient $-f_i$ to a point with coefficient $+f_i$ (and cycles which turn out to be negligible). In comparison, we consider constant-curvature arcs $C_{i,\alpha}$ with the same boundaries and area constraint, all nearly horizontal. For the S_i's to have opposite orientation, they must stray far from horizontal, which entails too much extra cost. Hence there is no cancellation and

$$\boldsymbol{M}(S) \geq \sum \boldsymbol{M}(f_i S_i) \geq \sum \boldsymbol{M}(f_i C_{i,\alpha}),$$

with equality only if $S = \sum f_i C_{i,\alpha}$ consists of constant-curvature arcs as desired.

The stronger Regularity Theorem 5.6 for the special case of double soap bubbles follows from the analysis in Section 5 of minimizing tangent cones, using results from the calibration theory of Section 4. Locally a double bubble can be decomposed into three basic parts: the exterior boundary of the first region, the exterior boundary of the second region, and the interior interface between the two regions. These parts meet at angles of at least 120 degrees (suitably oriented), leading eventually to possibilities (a)–(d) above.

2. Clusters with multiplicities as flat chains over a group *G*

We follow White's [1996] treatment after Fleming [1966] of soap bubble clusters or more generally immiscible fluid clusters as chains with coefficients representing the various regions or fluids f_i. To model overlapping clusters with multiplicities, we allow integer combinations of the f_i.

Integral flat chains are generalized curves, surfaces, and regions of geometric measure theory [Morgan 2000], with integer coefficients to allow for multiplicities. Fleming's [1966] development of flat chains as limits of polyhedral chains admits groups G of coefficients other than the integers. Following White [1996] we use $G = \mathbb{Z}^m \subset \mathbb{R}^m$, with components representing different regions or immiscible fluids, possibly overlapping with multiplicities. A norm $\| \ \|$ on $\mathbb{Z}^m$ describes the energetic cost of various interfaces. For example, the unit cost of the exterior boundary of the first region would be $\|(1, 0, 0, \dots)\|$, while the unit cost of an interface between the first two regions would be $\|(1, -1, 0, 0, \dots)\|$. More complicated interfaces can be decomposed into these two types. For double soap bubbles for example,

$$\|(1, 0)\| = \|(0, 1)\| = \|(1, -1)\| = 1,$$

and otherwise $\|(a, b)\|$ is as large as possible subject to the triangle inequality:

$$\begin{aligned}\|(a, b)\| &= \inf\left\{|\lambda_1| + |\lambda_2| + |\lambda_3| : (a, b) = \lambda_1(1, 0) + \lambda_2(0, 1) + \lambda_3(1, -1)\right\} \\ &= \tfrac{1}{2}(|a| + |b| + |a + b|).\end{aligned}$$

Definitions 2.1. A *cluster S with multiplicities* of m immiscible fluids f_i in $\mathbb{R}^n$ is an $(n-1)$-dimensional flat chain (of compact support) with coefficients in the free abelian group $G \simeq \mathbb{Z}^m$ with generators f_i, contained in $\mathbb{R} \otimes G \simeq \mathbb{R}^m$. If S itself as a flat chain has no boundary ($\partial S = 0$), then S is the boundary of some n-dimensional flat chain (the fluids themselves); but we will consider more general S with prescribed boundary as well as volume constraints.

The mass norm $\boldsymbol{M}$ on chains is induced by a given norm on $\mathbb{R} \otimes G$. For soap bubble clusters, it is the largest norm such that

$$\|f_i| = \|f_i - f_j\| = 1$$

(exterior boundaries and interfaces between regions with multiplicity one have unit cost). For immiscible fluids, it is the largest norm such that

$$\|f_i\| = a_i, \quad \|f_i - f_j\| = a_{ij}$$

(the cost depends on the fluids). We sometimes consider more general norms and chains of higher codimension.

These spaces are variations on the classical $G = \mathbb{Z}$ cases of geometric measure theory [Morgan 2000] generalized to any complete normed abelian group G by Fleming [1966], with further recent improvements by White [1996; 1999a; 1999b; 2001]. ([White 1999b] shows for which groups flat chains of finite mass are rectifiable, as is well known to hold for our case of boundaries of regions of finite perimeter; see [Morgan 2001, Lemma 2.1] or [Federer 1969, 4.5.12, 2.10.6]).

Monotonicity and tangent cones. Alternatively, a rectifiable chain S with variable coefficient g in G may be viewed as a rectifiable varifold with multiplicity $\|g\|$. If S is mass minimizing, perhaps with volume constraints, then away from ∂S the varifold has weakly bounded mean curvature by a lemma of Almgren (see [Morgan 2000, Lemma 13.5]). Assume that ∂S is smooth and compact. Then for some $M > 0$, for any point a in S,

$$\boldsymbol{M}(S \llcorner \boldsymbol{B}(a, r))\, e^{Mr}$$

is a monotonically nondecreasing function of r; see [Allard 1972, Corollary 5.1(3), p. 446; Allard 1975, Theorem 3.3(2), p. 426]. It follows that a minimizer has compact support. It also follows that a minimizer has at least one oriented tangent cone T at a. The truncated cone $T \llcorner \boldsymbol{B}(0, 1)$ is minimizing (without volume constraints) among chains with coefficients in $G_1 = G \cap V$, where V is the real vector space spanned by the coefficients occurring in S. The proof, a standard limit argument, considers $S + \partial W$ and requires small adjustments v in $\int_W dV$, which by the lemma of Almgren [Morgan 2000, Lemma 13.5] may be accomplished at cost at most $K\|v\|$, provided that $v \in V$.

Existence. For a nice compact domain in $\mathbb{R}^n$, the existence of a mass-minimizing cluster with prescribed volumes and boundary follows by compactness [Fleming 1966, Corollary 7.5] and the lower semicontinuity of mass [Fleming 1966, Theorem 2.3 and §3]. In $\mathbb{R}^2$ (our main concern), the reduction to the case of a compact subdomain is an easy argument: any cluster of bounded mass is contained in disjoint balls of radius r_i with $\sum r_i$ bounded, and distant balls may be translated inside a single ball of bounded radius. In $\mathbb{R}^n$, the existence of a mass-minimizing cluster requires the arguments of [Morgan 1994, §4]; see [Morgan 2000, Chapter 13].

3. Regularity for planar minimizing flat chains with area constraints

The main Regularity Theorem 3.7 says that planar minimizers for given boundary and areas among chains with coefficients in $\mathbb{Z}^m$ with a certain "simplicial" norm consist of finitely many circular arcs. The Regularity Lemma 3.5 says that a minimizer weakly close to a horizontal diameter in a small ball consists of nearly horizontal circular arcs in a shrunken ball. It depends on Proposition 3.2, which says that certain collections of nearly horizontal circular arcs are uniquely minimizing. We begin with an easy calculus lemma.

Lemma 3.1. *Let C be a graph over the unit ball $\boldsymbol{B}^{n-1}(0,1)$ with constant mean curvature* (*sum of principal curvatures*) *H. Let S be an $(n-1)$-dimensional integral current in $\boldsymbol{B}^{n-1}(0,1)\times\mathbb{R}$ with the same boundary. Almost everywhere on S, let θ denote the angle between the oriented normal to S and the normal to C at the point directly below* (*or above*). *Then*

$$\int_S \cos\theta = \text{measure } C + (-1)^n H\left(\int_S y\,dx - \int_C y\,dx\right).$$

If instead S has no boundary, then

$$\int_S \cos\theta = (-1)^n H \int_S y\,dx.$$

Proof. Define a vector field V on $\{(x,y)\in\boldsymbol{B}^{n-1}(0,1)\times\mathbb{R}\}$ by translating the unit normal to C vertically. Then $\operatorname{div} V = -H$. Let R be the n-current with $\partial R = S - C$. By the theorems of Gauss and Stokes,

$$\begin{aligned}\int_S \cos\theta - \text{measure } C &= \int_{\partial R} V\cdot n = \int_R \operatorname{div} V = -H\int_R dx\,dy\\ &= (-1)^n H\left(\int_S y\,dx - \int_C y\,dx\right).\end{aligned}$$

Similarly, if S has no boundary, $S=\partial R$ and

$$\int_S \cos\theta = \int_{\partial R} V\cdot n = (-1)^n H\int_S y\,dx. \qquad\square$$

The following proposition is the technical heart of the regularity theory, as described in the Introduction.

Proposition 3.2. *Let $G=\mathbb{Z}^k$ with generators $f_1,\ldots,f_k$ and largest norm such that*

$$\|f_i\| = a_i, \quad \|\hat{f}_i - \hat{f}_j\| = a_{ij} > 0.$$

(*Here $\hat{f}_i = f_i/\|f_i\|$ and the second equality refers to the equivalent norm on $\mathbb{R}\otimes G$.*) *Then there is an $\varepsilon>0$ such that the following holds.*

For $1\le i\le k$, let $P_{\pm i}$ be a sequence of $k_i>0$ points in $\{\pm1\}\times(-\varepsilon,\varepsilon)$. Let $|A_i|<\varepsilon$. Let S minimize mass among rectifiable currents in $[-1,1]\times\mathbb{R}$ with *coefficients in G, boundary $\sum f_i(P_i - P_{-i})$, and area constraint*

$$\int_S y\,dx = \sum A_i f_i \in \mathbb{R} \otimes G. \tag{3–1}$$

Then S consists of constant-curvature arcs with coefficients f_i from points of P_{-i} to points of P_i.

Moreover, S remains minimizing among flat chains with coefficients in $\mathbb{R} \otimes G$.

Proof. Choose ε small with respect to the a_{ij}. Any minimizing rectifiable current is of the form $S = \sum_{i=1}^{k} f_i S_i$. By the definition of the norm there is cancellation $\big(\boldsymbol{M}(S) < \sum \boldsymbol{M}(f_i S_i)\big)$ only when some S_i and S_j have opposite orientation along some intersection of positive length, which we will show cannot occur.

Let $S_{i,\alpha}$ be the components of S_i. We may assume that for $1 \le \alpha \le k_i$ the boundary of $S_{i,\alpha}$ is

$$\partial S_{i,\alpha} = p_{i,\alpha} - p_{-i,\alpha}, \quad P_{\pm i} = \{p_{\pm i,\alpha}\},$$

and that $\partial S_{i,\alpha} = 0$ for $\alpha > k_i$.

Let $C_{i,\alpha}$ denote the arc from $p_{-i,\alpha}$ to $p_{i,\alpha}$, or if $S_{i,\alpha}$ has no boundary from $(-1, 0)$ to $(1, 0)$, of (small) constant curvature κ_i, chosen so that the $C_{i,\alpha}$ preserve the area constraint:

$$\sum_\alpha \int_{C_{i,\alpha}} y\,dx = A_i f_i. \tag{3–2}$$

Along $S_{i,\alpha}$, let $\theta_{i,\alpha}$ denote the angle with $C_{i,\alpha}$ at the same x-coordinate. Note that all the unit tangents to all the $C_{i,\alpha}$ remain close to each other.

For this paragraph we consider a typical point of S. The coefficient g has a mass decomposition

$$g = \sum_{i<j} \alpha_{ij}(\hat{f}_i - \hat{f}_j) + \sum \beta_i \hat{f}_i = \frac{1}{2} \sum \alpha_{ij}(\hat{f}_i - \hat{f}_j) + \sum \beta_i \hat{f}_i \tag{3–3}$$

with $\alpha_{ji} = -\alpha_{ij}$; all coefficients of $\hat{f}_i$ have the same sign. Let n_i denote the number of components $S_{i,\alpha}$ of S_i present, necessarily all with the same orientation. The coefficient of $\hat{f}_i$ satisfies

$$\sum_j \alpha_{ij} + \sum \beta_i = \pm n_i \|f_i\|. \tag{3–4}$$

If $|\alpha_{ij}| > 0$, then S_i and S_j have opposite orientations. Because ε is small, each $\theta_{i,\alpha}$ and $\theta_{j,\beta}$ differ by approximately π, so $|\cos\theta_{i,\alpha} + \cos\theta_{j,\beta}|$ is small, and we may assume that

$$\|\hat{f}_i - \hat{f}_j\| > |\cos\theta_{i,\alpha} + \cos\theta_{j,\beta}|.$$

Therefore

$$\begin{aligned}\|g\| &\geq \frac{1}{2}\sum \frac{1}{n_i}\frac{1}{n_j}\,|\alpha_{ij}|\,(\cos\theta_{i,\alpha}+\cos\theta_{j,\beta}) + \sum_i \frac{1}{n_i}\,|\beta_i|\,|\cos\theta_{i,\alpha}|\\ &= \sum_{i,\alpha}\frac{1}{n_i}\Big(\sum_j |\alpha_{ij}|\,\cos\theta_{i,\alpha} + |\beta_i|\,|\cos\theta_{i,\alpha}|\Big)\\ &= \sum_{i,\alpha}\frac{1}{n_i}\Big(\Big|\sum_j \alpha_{ij}\Big|\,\cos\theta_{i,\alpha} + |\beta_i|\,|\cos\theta_{i,\alpha}|\Big),\end{aligned}$$

because for fixed i, each α_{ij} has the same sign. Now by the general inequality $|A|\,C+|B|\,|C| \geq |A+B|\,C$ (which holds whether C is positive or negative by the triangle inequality),

$$\|g\| \geq \sum_{i,\alpha}\frac{1}{n_i}\left|\sum_j \alpha_{ij}+\beta_i\right|\cos\theta_{i,\alpha} = \sum_{i,\alpha}\|f_i\|\cos\theta_{i,\alpha}$$

by (3–4).

Now by Lemma 3.1 the global mass of S satisfies

$$\begin{aligned}\boldsymbol{M}(S) &\geq \sum \int_{S_{i,\alpha}} \|f_i\|\cos\theta_{i,\alpha}\\ &= \sum_i\bigg(\sum_{1\leq\alpha\leq k_i}\Big(\boldsymbol{M}(f_iC_{i,\alpha}) + \kappa_i\int_{S_{i,\alpha}} y\,dx - \kappa_i\int_{C_{i,\alpha}} y\,dx\Big) + \sum_{\alpha>k_i}\kappa_i\int_{S_{i,\alpha}} y\,dx\bigg)\\ &= \sum \boldsymbol{M}(f_iC_{i,\alpha}),\end{aligned}$$

by (3–1) and (3–2). Moreover, equality holds only if there is no cancellation (every α_{ij} vanishes) and each $\cos\theta_{i,\alpha}$ is almost always 1, i.e., $S=\sum_{1\leq\alpha\leq k_i} f_iC_{i,\alpha}$, as was to be proved. □

Remark. The proof of Proposition 3.2 does not provide a single, generalized calibration as described by the Calibration Lemma 4.1. Pieces are indeed calibrated by vertical translations of the duals to circular arcs, but even in the case $k=1$, two circular arcs with nonparallel chords are not so calibrated simultaneously. Incidentally, although the proof for $k>1$ requires the arcs to be small, the calibrations work on arcs up to π radians individually.

The next result generalizes the hypotheses of Proposition 3.2.

Proposition 3.3. *Let $G=\mathbb{Z}^m$ with a norm. Let F be a simplicial* (*polyhedral*) *face of the unit norm ball in $\mathbb{R}\otimes G$. Suppose that positive real multiples $f_1,\ldots,f_k$ of the vertices of F generate $G_0=G\cap\operatorname{span}_{\mathbb{R}}F$.*

There is an $\varepsilon>0$ such that the following holds. For $1\leq i\leq m$, let $P_{\pm i}$ be a sequence of $k_i>0$ points in $\{\pm1\}\times(-\varepsilon,\varepsilon)$. Let $|A_i|<\varepsilon$. Let S minimize *mass among rectifiable currents in* $[-1, 1] \times \mathbb{R}$ *with coefficients in* G, *boundary* $\sum f_i(P_i - P_{-i})$, *and perhaps area constraints*

$$\int_S y\,dx = \sum A_i f_i \in \operatorname{span}_{\mathbb{R}} F. \tag{3–5}$$

Then S *consists of constant-curvature arcs* (*always straight lines in the case of no area constraints*) *with coefficients* f_i *from points of* P_{-i} *to points of* P_i.

Moreover, S *remains minimizing among flat chains with coefficients in* $\mathbb{R} \otimes G$.

Proof. Since F is polyhedral, the norm on $\operatorname{span}_{\mathbb{R}} F = \mathbb{R} \otimes G_0$ is greater than a norm as in Proposition 3.2, with equality on the f_i, so that a minimizer S_0 over G_0 is of the asserted form and remains minimizing over $\mathbb{R} \otimes G_0$.

There is a (linear) retraction $\rho : \mathbb{R} \otimes G \to \mathbb{R} \otimes G_0 = \operatorname{span}_{\mathbb{R}} F$ which is strictly norm decreasing off $\mathbb{R} \otimes G_0$. A minimizer S over $\mathbb{R} \otimes G$, with boundary coefficients in $\mathbb{R} \otimes G_0$, must have coefficients in $\mathbb{R} \otimes G_0$, or

$$\boldsymbol{M}(S) > \boldsymbol{M}(\rho(S)) \geq \boldsymbol{M}(S_0),$$

a contradiction. Therefore a minimizer over G has coefficients in G_0, as above has the asserted form, and remains minimizing over $\mathbb{R} \otimes G$. □

Proposition 3.4. *In Proposition 3.3*, *it is necessary to assume that the* $\hat{f}_i$ *are vertices of a simplicial face* F *of the unit norm ball*, *even when* $k = m$.

Proof. If the $\hat{f}_i$ do not lie on a common face of the unit norm ball, then for some positive integers m_i,

$$\left\| \sum m_i f_i \right\| < \sum m_i \| f_i \|,$$

and nearly coincident horizontal lines with coefficients $m_i f_i$ are not minimizing, contrary to the first conclusion of Proposition 3.3.

If the face is not simplicial or if the $\hat{f}_i$ are not vertices, then some element of the face is a real linear combination of the $\hat{f}_i$ with both positive and negative coefficients. Moreover, some $g_1 \in G \cap (\mathbb{R}^+)F$ is an integer linear combination of the f_i with both positive and negative coefficients; i.e., we may assume that there are nonnegative integers m_i and $0 < l < k$, with $m_i > 0$ for some $i \leq k$ and for some $i > k$, such that

$$g_1 = g_3 - g_2,$$

with

$$g_2 = \sum_{i=1}^{l} m_i f_i, \quad g_3 = \sum_{i=l+1}^{k} m_i f_i.$$

Since all the g_i lie in $(\mathbb{R}^+)F$,

$$\|g_3\| = \|g_1\| + \|g_2\|.$$

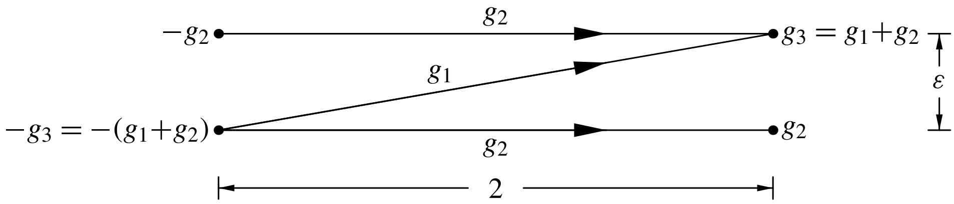

Figure 1. If $\|g_1 + g_2\| = \|g_1\| + \|g_2\|$, the pictured Z has less mass than an X with coefficients g_2, g_3.

Hence the Z-shaped network of Figure 1, with mass

$$\lambda\|g_1\| + 4\|g_2\|,$$

where $\lambda = \sqrt{4+\varepsilon^2} > 2$, has less mass than the straight lines with coefficients $m_i f_i$ (i.e., an X with coefficients g_2, g_3) with mass

$$\lambda\|g_2\| + \lambda\|g_3\| = \lambda\|g_1\| + 2\lambda\|g_2\|.$$

Both networks satisfy the same area constraint. Thus the first conclusion of Proposition 3.3 fails again. □

Regularity Lemma 3.5. *Let $g_0 \in G = \mathbb{Z}^m$ with a polyhedral norm. Suppose that $\hat{g}_o = g_0/\|g_0\| \in \mathbb{R} \otimes G$ lies in a simplicial face F of the unit norm ball. Suppose that positive real multiples $f_i, \dots, f_k$ of the vertices of F generate*

$$G_0 = G \cap \operatorname{span}_{\mathbb{R}} F.$$

Let S minimize mass among 1-dimensional rectifiable chains in $\mathbb{R}^2$ with coefficients in G_0 of the form $S + \partial W$, with $\int_W dA = 0$. Given $\delta > 0$, there is an $\varepsilon > 0$ such that if, in a small ball $\boldsymbol{B}(a, r)$ away from ∂S, S is ε weakly close to a diameter with coefficient g_0 (meaning that its homothetic expansion to a unit ball is within ε in the flat norm of a diameter with coefficient g_0), then in a shrunken ball $\boldsymbol{B}(a, \delta r)$, S consists of constant-curvature arcs with coefficients f_i. At all such points, each arc has the same curvature κ_i.

White [2001] has conjectured for example that this lemma holds for $(n-1)$-chains in $\mathbb{R}^n$ whenever g_0 has a unique norm decomposition, a hypothesis which follows from our stronger hypotheses on g_0 and F.

Proof. The hypotheses imply that g_0 has a unique irreducible norm decomposition, of the form

$$g_0 = \sum m_i f_i. \tag{3–6}$$

By replacing F by a subface, we may assume that each $m_i > 0$. There is an $c_1 > 0$ such that any other decomposition $g_0 = \sum g_i$ which does not reduce to

(3–6) satisfies

$$(1+c_1)\|g_0\| \le \sum \|g_i\|. \tag{3–7}$$

Real linear combinations of the coefficients that occur in S away from ∂S constitute a vector subspace $V \subset \mathbb{R}\otimes G$. By a lemma of Almgren [Morgan 2000, Lemma 13.5], there exist $C_1 > 0$ and $0 < r_1 < 1$ such that, outside any ball of radius less than r_1, arbitrary small adjustments $\int_W dA$ contained in V can be made at cost at most

$$C_1 \left\| \int_W dA \right\|. \tag{3–8}$$

By scaling, in λS, outside any ball of radius less than λr_1, such adjustments can be made at cost at most

$$\lambda^{-1} C_1 \left\| \int_W dA \right\|. \tag{3–9}$$

Since F is a face of a polyhedral unit norm ball, there are $c_2 > 0$ and a linear retraction π_1 of $\mathbb{R}\otimes G$ onto $\mathbb{R}\otimes G_0$ such that

$$\|\pi_1 x\| \le \|x\| - c_2\|\pi_2 x\| \tag{3–10}$$

where $\pi_2 x = x - \pi_1 x$.

Let ε_1 be the ε provided by Proposition 3.3. We may assume that ε_1 is small and in particular less than c_2/C_1.

Choose $\varepsilon > 0$ such that if for $0 < r < r_1$, inside any $\boldsymbol{B}(a, r)$ away from ∂S, S is ε weakly close to a diameter with coefficient g_0, then inside $\boldsymbol{B}(a, \delta^{1/3} r)$, the four conditions below hold. (For convenience of statement we assume that $a = 0$ and that the diameter is the x-axis.)

(i) S projects onto $\{y = 0,\ |x| \le \delta^{1/2} r\}$ with coefficient g_0.

(ii) S lies inside $\{|y| \le \varepsilon_1 \delta r\}$ (possible by monotonicity).

(iii) For some $\delta r \le s \le \delta^{1/2} r$, the slices by $\{r = \pm s\}$ are norm decompositions of $\pm g_0$ (possible by (3–7) and slicing).

(iv) Area constraints are small relative to ε_1.

We now restrict attention to $\{|x| \le s,\ |y| \le \varepsilon_1 \delta r\}$. By (3–10),

$$\boldsymbol{M}(\pi_1 S) \le \boldsymbol{M}(S) - c_2 \boldsymbol{M}(\pi_2 S) \tag{3–11}$$

because $|y| \le \varepsilon_1$ and $\varepsilon_1 < c_2/C_1$. Since S is minimizing, $\pi_2 S$ must be 0. Hence the area constraints lie in $\mathrm{span}_{\mathbb{R}} F$. It now follows by Proposition 3.3 and conditions (ii)–(iv) that S consists of constant-curvature arcs with coefficients f_i. A simple variational argument shows that each such arc has the same curvature κ_i. $\square$

Remark 3.6. Because the unit norm ball was polyhedral, this proof was able to use the projection argument of [Morgan 1998, Proposition 4.2] to handle area constraints outside $\operatorname{span}_{\mathbb{R}} F$. The more general alternative argument of White [Morgan 1998, p. 446] does not apply, because a minimizer without area constraints close to a diameter with decomposable coefficient g_0 need not be a diameter inside a shrunken ball: it can be a sum of lines corresponding to a decomposition of g_0.

Here is one of the main results of this paper.

Regularity Theorem 3.7. *Let $G = \mathbb{Z}^m$ with a norm. Suppose that every maximal face F of the unit norm ball in $\mathbb{R} \otimes G$ is simplicial and that real multiples of the vertices of F generate G.*

Let S minimize mass among rectifiable chains in $\mathbb{R}^2$ with coefficients in G of the form $S + \partial W$, with $\int_W dA = 0$. Suppose that S has finite boundary mass. Then S consists of finitely many constant-curvature arcs with coefficients in G.

Proof. Note that the unit norm ball is polyhedral, that every face F' is simplicial, and that real multiples of the vertices of F' generate $G \cap \operatorname{span}_{\mathbb{R}} F'$.

Let a be any point in S. Any tangent cone to S at a consists of rays from a with coefficients in G. By the Regularity Lemma 3.5, in any sufficiently small annulus $\boldsymbol{B}(a, r) - \boldsymbol{B}(a, r/2)$, S consists of nearly radial constant-curvature arcs. Coherence implies that at a S consists of finitely many constant-curvature arcs. □

Examples. For $m = 2$, Regularity Theorem 3.7 applies to any polyhedral norm. For $m = 3$, the unit ball could be for example the regular octahedron with vertices $\pm f_i$ or the polyhedron with vertices $\pm f_i$, $\pm f_i \pm f_j$ and 32 faces. Unfortunately the "soap film" unit norm ball, with vertices $\pm f_i$, $f_i - f_j$, has some rectangular faces, for example with vertices f_1, f_2, $f_1 - f_3$, $f_2 - f_3$, with corresponding nonunique norm decomposition

$$f_1 + (f_2 - f_3) = f_2 + (f_1 - f_3).$$

Surfaces. The results of Section 3 generalize locally to a smooth Riemannian surface M.

To extend the preliminary geometric measure theory, including for example modified monotonicity, one usually uses Nash's theorem to embed M isometrically in some Euclidean space; see [Allard 1972, Remark 4.4].

To extend Lemma 3.1 to an arc C of small constant curvature κ, take a geodesic γ normal to C and foliate the unit disc by arcs of curvature κ normal to γ. For any curve S, let θ denote the angle with the foliation. Choose a smooth 1-form φ with $d\varphi = dA$ (such as $y\,dx$ in Euclidean space). If $\partial S = \partial C$ so that $S - C = \partial R$,

$$\int_{S-C} \cos\theta = \kappa \int_R dA = \int_{S-C} \varphi.$$

For another arc C' of small constant curvature κ' at a small angle to the foliation of C, the whole foliation is at a small angle to the foliation of C.

To extend Proposition 3.2, Proposition 3.3, and the Regularity Lemma 3.5, use normal coordinates in a small ball. Proposition 3.2 extends without difficulty, using the foliations of the extension of Lemma 3.1. Proposition 3.3 extends immediately. The slicing argument in the proof of the Regularity Lemma 3.5, which has some leeway, extends because the coordinates are approximately Euclidean.

The Regularity Theorem 3.7 now extends immediately to surfaces.

4. Calibrations over a group G

The next result extends the theory of calibrations from real coefficients to coefficients over any complete normed group G. as described in the Introduction.

Calibration Lemma 4.1. *Let G be a complete normed group. Let S be a rectifiable m-chain in $\mathbb{R}^n$ with coefficients in G. Suppose there is a calibration of S: namely, a homomorphism*

$$F : G \to \textit{closed differential } m\textit{-forms (with comass norm)}$$

satisfying

$$\|F(g)\| \leq \|g\| \tag{4–1}$$

and

$$\langle \vec{S}(x), F(g)(x) \rangle = \|g\| \quad \textit{for almost all } x \in S. \tag{4–2}$$

Then S is minimizing among flat chains with coefficients in G homologous to S. Moreover (4–2) *holds for any rectifiable minimizer S'.*

Proof. For any rectifiable chain T, let

$$F(T) = \int \langle \vec{T}(x), F(g) \rangle.$$

For an m-polyhedron P_1 with coefficient g_1,

$$F(P_1) = \int_{P_1} F(g_1) \quad \text{and} \quad |\mathrm{F(P_1)}| \leq \boldsymbol{M}(\mathrm{P_1}).$$

For an $(m+1)$-polyhedron P_2 with coefficient g_2,

$$F(\partial P_2) = \int_{\partial P_2} F(g_2) = 0$$

because $F(g_2)$ is a closed form. F extends to polyhedral chains and then by approximation to all flat chains, with similar properties.

Let S' be homologous to S, so that $S' - S = \partial R$ and $F(S') = F(S)$. Then

$$\boldsymbol{M}(S') \geq F(S') = F(S) = \boldsymbol{M}(S).$$

Moreover for any rectifiable minimizer $\boldsymbol{M}(S') = F(S')$, and hence (4–1) holds. □

Remark. When G is a real vector space, $m = 1$, and

$$\text{range } F = \{\text{constant-coefficient differential 1-forms}\} \simeq \mathbb{R}^n ,$$

condition (4–1) just says that $|F(g)| \leq \|g\|$, i.e., that the ellipsoid or elliptical cylinder

$$F^{-1}\{|x| \leq 1\} \subset G$$

contains the unit ball $\{\|g\| \leq 1\}$. If the unit ball is polyhedral, it suffices to check that its vertices g_i satisfy $|F(g_i)| \leq 1$.

Corollary 4.2. *Let $P = \vec{P} \wedge \|P\|$ be an m-dimensional rectifiable chain with real coefficients in $\mathbb{R}^n$. Consider $\vec{P} \otimes P$, an associated flat chain in $\mathbb{R}^n$ with coefficients in $\Lambda_m \mathbb{R}^n$. Then $\vec{P} \otimes P$ is minimizing.*

Proof. Apply the Calibration Lemma 4.1, with $F(g)$ the dual covector. □

Example. Consider 1-dimensional flat chains in $\mathbb{R}^n$ ($n \geq 2$) with coefficients in $\mathbb{R}^n$. Every cone of the form $\sum v_i \otimes v_i$ is minimizing. In particular, arbitrarily many curves can meet at a point in a minimizer.

Corollary 4.3 [Choe 1996]. *A stationary polyhedral m-chain in $\mathbb{R}^n$ with real coefficients is area minimizing under diffeomorphisms of $\mathbb{R}^n$ fixing the boundary.*

Proof. By Corollary 4.2, the associated $P^* = \vec{P} \otimes P$ is minimizing. Note that $\boldsymbol{M}(P^*) = \boldsymbol{M}(P)$. Since P is stationary, P^* has no interior boundary. For an image S of P under a diffeomorphism, let S^* denote the associated flat chain with the same coefficients as P^*. Since P^* has no interior boundary, S^* is homologous to P^*. Therefore

$$\boldsymbol{M}(S) = \boldsymbol{M}(S^*) \geq \boldsymbol{M}(P^*) = \boldsymbol{M}(P),$$

so that P is area minimizing. □

Variational Lemma 4.4. *For unit vectors v_i in $\mathbb{R}^n$ and coefficients g_i in a real vector space G, consider the 1-dimensional cone $C = \sum f_i v_i \in \mathbb{R}^n$ with coefficients in G. If C is minimizing, then the following variational conditions hold: for $\lambda_i \geq 0$,*

$$\left|\sum \lambda_i \|g_i\| v_i\right| \leq \left\|\sum \lambda_i g_i\right\|. \tag{4–3}$$

If C has no boundary at the origin $\big($i.e., if $\sum g_i = 0\big)$, (4–3) holds for all real λ_i.

Proof. Let $\lambda_i \geq 0$. We may assume that $0 \leq \lambda_i \leq 1$. Given a vector u, move the vertex of $\sum \lambda_i g_i v_i$ by tu and add $\bigl(\sum \lambda_i g_i\bigr)tu$. The initial derivative of mass is

$$\Bigl\|\sum \lambda_i g_i\Bigr\| |u| - \Bigl(\sum \lambda_i \|g_i\| v_i\Bigr) \cdot u.$$

Choose $u = \sum \lambda_i \|g_i\| v_i$ (to make the derivative as small as possible). Since by minimization the derivative must be nonnegative, we get

$$0 \leq \Bigl\|\sum \lambda_i g_i\Bigr\| |u| - |u|^2, \quad \text{hence} \quad |u| \leq \Bigl\|\sum \lambda_i g_i\Bigr\|,$$

which is (4–3).

Now suppose that $\sum g_i = 0$ and let $\lambda_i \in \mathbb{R}$. Choose $M \geq 0$ such that $\lambda_i' = \lambda_i + M \geq 0$. Using (4–3) and $\sum g_i = 0$, we get the desired inequality:

$$\Bigl|\sum \lambda_i \|g_i\| v_i\Bigr| = \Bigl|\sum \lambda_i' \|g_i\| v_i\Bigr| \leq \Bigl\|\sum \lambda_i' g_i\Bigr\| = \Bigl\|\sum \lambda_i g_i\Bigr\|. \qquad \square$$

Proposition 4.5. *Consider a cone $C = \sum_{i=1}^k g_i v_i$ in $\mathbb{R}^n$ of unit vectors v_i with coefficients in the real vector space $G = \operatorname{span}\{g_i\}$. Suppose the variational conditions* (4–3) — *which we repeat for convenience*:

$$\Bigl|\sum \lambda_i \|g_i\| v_i\Bigr| \leq \Bigl\|\sum \lambda_i g_i\Bigr\| \tag{4–4}$$

hold for all real λ_i. Then C is minimizing among flat chains with coefficients in G (*indeed*, *calibrated by a constant-coefficient calibration*). *If equality in* (4–4) *holds only for multiples of g_i, then any rectifiable minimizer consists of multiples of line segments with tangent v_i and coefficient g_i.*

Proof. We apply the Calibration Lemma 4.1. By (4–4), whenever $\sum \lambda_i g_i = 0$, $\sum \lambda_i \|g_i\| v_i = 0$. Therefore there is a linear map F from G to $\mathbb{R}^n$ (or its dual space) such that $F(g_i) = \|g_i\| v_i$. Moreover for any $g = \sum \lambda_i g_i$ in G, by (4–4)

$$\bigl|F(g)\bigr| = \Bigl|\sum \lambda_i \|g_i\| v_i\Bigr| \leq \|g\|, \tag{4–5}$$

with equality for $g \in \{g_i\}$. By the Calibration Lemma 4.1, C is minimizing. Moreover, at almost every point of a rectifiable minimizer S, the image of the coefficient is a multiple of the tangent covector.

If equality in (4–4) holds only for multiples of g_i, then at almost every point of S, the coefficient is a multiple of some g_i; hence S consists of multiples of lines with tangent v_i and coefficient g_i. $\square$

Remarks. (1) Even if equality in (4–4) holds only for multiples of g_i, C need not be unique. For example, consider the cone $C = \sum_{i=1}^6 g_i v_i$ in $\mathbb{R}^2 \simeq \mathbb{C}$ with coefficients in $G = \mathbb{R}^2$, with v_i the sixth roots of unity and $g_i = v_i$. C and another minimizer are pictured in Figure 2.

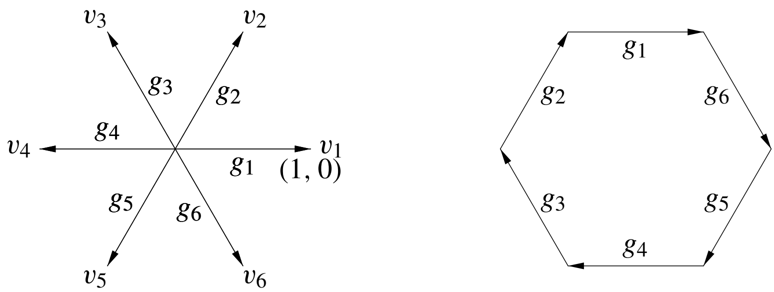

Figure 2. Two calibrated minimizers with the same boundary. In the boundary, $(1, 0)$ (for example) occurs with coefficient $g_1 = g_6 - g_5$.

(2) If C has no boundary at the origin, then condition (4–4) is necessary by the Variational Lemma 4.4. On the other hand, the cone $C_1 = e_1 + e_2$ in $\mathbb{R}^2$ with coefficients in $\mathbb{R}$ is uniquely minimizing, but (4–4) fails because

$$|e_1 - e_2| > 0.$$

C_1 has no constant-coefficient calibration as in the proof of Proposition 4.5, but it does have a singular variable-coefficient calibration, dr. We conjecture that the necessary variational condition with $\lambda_i \geq 0$, (4–3), satisfied of course by C_1 for example, is in general sufficient for minimization.

(3) Even if C has no boundary at the origin, the hypothesis that the g_i span G is necessary, or it could save cost to introduce other coefficients uncontrolled by (4–4), as to introduce a fourth cheap fluid into a junction of three expensive immiscible fluids. For the more general case $G \supset \operatorname{span}\{g_i\}$, there is a natural sufficient extension of (4–4), a "point-placing" condition equivalent to the existence of a constant-coefficient calibration, probably necessary for a minimizer over $G \simeq \mathbb{R}^n$ (open question), though not for a minimizer among integer chains without multiplicity ("immiscible fluids"), by counterexample. These issues are treated in [Futer et al. 2000]. (Lemma 4.7 therein assumes that regions have no overlap or multiplicity.)

Conjecture 4.6. *Suppose $G = \mathbb{Z}^2$ with generators f_1, f_2.*

The cone $C = m_1 f_1 v_1 + m_2 f_2 v_2$, with $m_i \in Z^+$, is (uniquely) minimizing among flat chains with coefficients in G if and only if

$$\big|\lambda_1 \|f_1\| v_1 + \lambda_2 \|f_2\| v_2\big| \leq \big\|\lambda_1 f_1 + \lambda_2 f_2\big\| \tag{4–6}$$

for all positive integers $\lambda_i \leq m_i$.

In particular, two lines crossing in directions v_1, v_2 with coefficients $m_1 f_1$ and $m_2 f_2$ are (uniquely) minimizing if and only if

$$\big|\lambda_1 \|f_1\| v_1 + \lambda_2 \|f_2\| v_2\big| \leq \big\|\lambda_1 f_1 + \lambda_2 f_2\big\| \tag{4–7}$$

for all integers $-m \le \lambda_i \le m_i$.

Remark. One may assume that $\|f_1\| = \|f_2\| = 1$. The condition then becomes

$$|\lambda_1 v_1 + \lambda_2 v_2| \le \|\lambda_1 + \lambda_2\|.$$

Variational Lemma 4.7. *Two vectors from the origin at angle* θ *with coefficients* g_1, g_2 *have negative first variation under moving the vertex and inserting a segment with coefficient* $g_1 + g_2$ *if*

$$2\|g_1\|\|g_2\|\cos\theta > \|g_1 + g_2\|^2 - \|g_1\|^2 - \|g_2\|^2. \tag{4–8}$$

If $\|g_1\| = \|g_2\| = 1$, *this condition becomes*

$$\cos(\theta/2) > \|g_1 + g_2\|/2.$$

If furthermore $\|g_1 + g_2\| = 1$, *the condition becomes the familiar* $\theta < 120$ *degrees.*

Remark. For k vectors, positive multiples of unit vectors v_i, the condition (4–8) takes the form

$$\left\|\sum g_i\right\| < \left|\sum \|g_i\| v_i\right|, \tag{4–9}$$

generalizing [Morgan 1998, 4.4].

Proof. Let v_1, v_2 be unit vectors in the two directions, and insert a vector tu. Then the initial derivative of mass is

$$\|g_1 + g_2\|\,|u| - \big(\|g_1\|v_1 + \|g_2\|v_2\big)\cdot u.$$

Choose $u = \|g_1\|v_1 + \|g_2\|v_2$ (to make the derivative as negative as possible). The derivative is negative if

$$\|g_1 + g_2\|^2 < |u|^2 = \|g_1\|^2 + \|g_2\|^2 + 2\|g_1\|\|g_2\|\cos\theta$$

as desired. The equivalent conditions follow trivially. □

Alternatively, the lemma could be derived from the Variational Lemma 4.4.

A special case of the following proposition (Proposition 5.2) will be used in the regularity theorem for planar double bubbles (Regularity Theorem 5.6).

Proposition 4.8. *Let* $G \simeq \mathbb{R}^2$ *be a real vector space with basis* f_1, f_2, *representing two fluids or regions. For* $0 < \delta_i \le 1$, *give* G *the largest norm such that*

$$\|f_1\| = \|f_2\| = 1, \quad \|f_1 + f_2\| = 2\delta_1, \quad \|f_1 - f_2\| = 2\delta_2.$$

Let S *consist of two oriented diameters of the unit disc in* $\mathbb{R}^2$ *with coefficients* $m_i f_i$ $(m_i > 0)$ *at an angle* $0 \le \theta_1 \le \pi$ *with supplementary angle* $\theta_2 = \pi - \theta_1$. *If each* $\cos(\theta_i/2) \le \delta_i$ (*so that* $\delta_1^2 + \delta_2^2 \ge 1$), *then* S *is minimizing among rectifiable chains over* G, *uniquely if strict inequality holds. In particular, if* $\delta_1 = 1$ *and* $\delta_2 = 1/2$, *the lines crossing at angle* $0 \le \theta_1 \le \pi/3$ *are minimizing. Conversely, if either* $\cos(\theta_i/2) > \delta_i$, *then* S *is not minimizing, not even over the subgroup* $\mathbb{Z}^2 \subset G$ *(assuming that the* m_i *are positive integers).*

Proof. If either $\cos(\theta_i/2) > \delta_i$, S is not minimizing by the Variational Lemma 4.7.

Assume that each $\cos(\theta_i/2) \le \delta_i$. We apply the Calibration Lemma 4.1. Let $F(f_1)$ be a unit covector along the first line, $F(f_2)$ a unit covector along the second. Then $\left|F(f_i)\right| = 1 = \|f_i\|$,

$$\begin{aligned}\left|F(f_1+f_2)\right| &= 2\cos(\theta_1/2) \le 2\delta_1 = \|f_1+f_2\|,\\ \left|F(f_1-f_2)\right| &= 2\cos(\theta_2/2) \le 2\delta_2 = \|f_1-f_2\|.\end{aligned}$$

Hence, for any unit vector v and coefficient g,

$$\left\langle g, F(v)\right\rangle \le \|g\|,$$

with equality precisely when v is $\pm F(f_1)$, $\pm F(f_2)$ and g is an appropriately signed multiple of f_1 or f_2, respectively. It follows that for fixed boundary, S is uniquely minimizing. □

Remark. If each $\cos(\theta_i/2) = \delta_i$, then the rectangle with coefficients $\pm f_1 \pm f_2$ is another minimizer with the same boundary.

5. Planar double soap bubbles with multiplicities

The main Regularity Theorem 5.6 deduces the description of planar double bubbles with multiplicities (see Introduction) from the more general Regularity Theorem 3.7. Propositions 5.2–5.5 provide the requisite characterization of minimizing tangent cones.

Definitions 5.1. A *double soap bubble with multiplicities* in $\mathbb{R}^n$ (or any dimension-n submanifold of $\mathbb{R}^N$) is a rectifiable $(n-1)$-chain with coefficients in the free group G_2 on two generators f_1, f_2. Let R_1 be the region with coefficient f_1, R_2 the region with coefficient f_2. If a region has coefficient $m_1 f_1 + m_2 f_2$, we say that R_i has multiplicity m_i. Interfaces from the exterior to R_1 to R_2 to the exterior have coefficients $h_1 = f_1$, $h_2 = f_2 - f_1$, and $h_3 = -f_2$, respectively. Any two of the h_i generate G_2. Give G_2 the largest norm such that

$$\|h_1\| = \|h_2\| = \|h_3\| = 1.$$

More generally we consider m-chains in $\mathbb{R}^n$ with coefficients in G_2, without any interpretation as clusters. We focus on 1-chains in $\mathbb{R}^n$.

Any $g \in G_2$, except for multiples of the h_i, has a unique maximal norm decomposition in terms of two of the h_i:

$$g = m_i h_i - m_j h_j, \quad \|g\| = m_i + m_j, \quad m_i, m_j > 0.$$

Proposition 5.2. *Two crossing lines with coefficients* m_1h_1, m_2h_2 $(m_i \geq 1)$ *crossing at an angle* θ *are minimizing over* G_2 *if and only if* $\theta \geq 120$ *degrees.*

Proof. This is a special case of Proposition 4.8, when $\delta_1 = 1$ and $\delta_1 = \frac{1}{2}$. □

Proposition 5.3. *The standard Y*

$$Y = h_1 v_1 + h_2 v_2 + h_3 v_3$$

with v_i *planar unit vectors at* 120*-degree angles, and multiples* mY *thereof, are uniquely minimizing over* G_2.

More generally, any nonnegative linear combination

$$C = \sum \left\{ m_{\pm i}(\pm h_i)(\pm v_i) : \pm i \in \pm\{1, 2, 3\} \right\}$$

is minimizing, and uniquely so if and only if some $m_{\pm i}$ *vanishes.*

Proof. Apply the Calibration Lemma 4.1 with $F(h_i)$ the covector dual to v_i, which defines a linear map because $\sum v_i = 0$. Since $\left|F(h_i)\right| = 1$, therefore by the definition of the norm on G_2 as the largest norm with $\|h_i\| = 1$, for all $g \in G_2$,

$$\left|F(g)\right| \leq \|g\|,$$

with equality only for multiples of some h_i. By the Calibration Lemma 4.1, C is minimizing.

To prove uniqueness for mY, note that by (4–2), any minimizer consists of collections $\mathcal{S}_i$ of segments in the direction v_i with coefficient a positive multiple of h_i. Let $-\mathcal{S}_i$ denote the segments of $\mathcal{S}_i$ with orientations reversed. Let p_i denote the point at the end of v_i. Then $-\mathcal{S}_1 \cup \mathcal{S}_2$ provides m paths from p_1 to p_2; $-\mathcal{S}_2 \cup \mathcal{S}_3$ provides m paths from p_2 to p_3; $-\mathcal{S}_3 \cup \mathcal{S}_1$ provides m paths from p_3 to p_1. mY is the only consistent possibility.

More generally, to prove uniqueness when say $m_{-3} = 0$, it suffices to consider multiples mC_5 of

$$C_5 = h_1 v_1 + h_2 v_2 + h_3 v_3 + (-h_1)(-v_1) + (-h_2)(-v_2),$$

since every case is a subset of mC_5. Now $-\mathcal{S}_1 \cup \mathcal{S}_2$ provides m paths from p_1 and $-p_2$ to p_2 and $-p_1$; $-\mathcal{S}_2 \cup \mathcal{S}_3$ provides m paths from p_2 and 0 to p_3 and $-p_2$; $-\mathcal{S}_1 \cup \mathcal{S}_2$ provides m paths from p_3 and $-p_1$ to p_1 and 0. mC_5 is the only consistent possibility.

To prove nonuniqueness when every $m_{\pm i} \geq 1$, it suffices to consider $m_{\pm i} = 1$. Another minimizer with the same boundary is the hexagon of Figure 2. □

Because every $g \in G_2$ has a norm decomposition in terms of the h_i, every sum of unit vectors takes the form

$$\sum \left\{ \sum_k h_i v_{ik} : i \in \pm\{1, 2, 3\} \right\}$$

with the convention $h_{-i} = -h_i$. When vectors $v_{ik} = v_{jl}$ coincide, we assume that $\|h_i + h_j\| = 2$ (no cancellation); e.g., if $i = 1$, then $j \in \{1, -2, -3\}$.

Proposition 5.4. *Consider a minimizing sum*

$$\sum \Big\{ \sum_k h_i v_{ik} : i \in \pm\{1, 2, 3\} \Big\} \qquad (h_{-i} = -h_i)$$

of unit vectors in $\mathbb{R}^n$ with coefficients in G_2 (without cancellation).

(1) *If h_1 and $h_{-1} = -h_1$ both occur, then $v_{-11} = -v_{11}$, the v_{1k} are all equal, and the v_{-1k} are all equal. (Of course a similar statement holds for h_2 and for h_3.)*

(2) *The angle between v_{ik} and v_{jl} is at least 120 degrees for $ij > 0$ and $i \neq j$.*

(3) *If h_1, h_2, h_3 all occur, then v_{11}, v_{21}, v_{31} are at 120 degrees and the v_{1k} are all equal (as are the v_{2k} and the v_{3k}).*

Remark. These conditions are probably sufficient as well as necessary.

Proof. Every v_{-1k} must be opposite to every v_{1k}, or the straight line would be shorter, proving (1). To prove (2), note that $h_i + h_j = -h_k$ has norm 1, so that (2) is just the well-known fact about classical networks. Finally, (3) follows immediately from (2). $\square$

Proposition 5.5. *Nonlinear 1-dimensional minimizing cones over G_2 in $\mathbb{R}^n$ without interior boundary are precisely pairs of lines with coefficients $m_i h_i$, $-m_j h_j$ (m_i, m_j positive) at an angle of at most 60 degrees and nonnegative linear combinations*

$$\sum \big\{ m_i h_i v_i : i \in \pm\{1, 2, 3\} \big\} \quad (h_{-i} = -h_i) \tag{5–1}$$

of three vectors v_i at 120 degrees with coefficients h_i and their opposites $v_{-i} = -v_i$ with coefficients $-h_i$, with

$$m_1 - m_{-1} = m_2 - m_{-2} = m_3 - m_{-3}. \tag{5–2}$$

In any case, C is planar.

Proof. By Propositions 5.2 and 5.3, all such cones are minimizing.

Recall the well-known classical variational fact that two vectors in equilibrium meet at 180 degrees and that three vectors in equilibrium meet at 120 degrees (see the Variational Lemma 4.7). Recall also that every $g \in G_2$ has a norm decomposition of the form $g = m_i h_i - m_j h_j$ with m_i, m_j nonnegative integers.

If C is indecomposable, then the coefficients that occur are either $\{h_i, -h_i\}$ or $\pm\{h_1, h_2, h_3\}$. In the first case, C is a line; in the second, $C = \pm Y$, a standard Y (possibly with orientation reversed).

A sum $Y_1 + Y_2$ of two standard Y's is minimizing only if $Y_2 = Y_1$. Otherwise the sum can be decomposed into two triples with angles not 120 degrees. A sum

$Y_1 - Y_2$ is minimizing only if $Y_2 = Y_1'$, where $(\sum h_i v_i)' = \sum(-h_i)(-v_i)$. Otherwise vectors $h_i v$, $-h_i w$ could be replaced with a straight line. Similarly a minimizing sum of a Y and a line with coefficient $\pm h_i$ must be of form (5–1). We conclude that a C containing a $\pm$Y must be of the asserted form (5–1).

A sum of two distinct lines, oriented to have coefficients in $\{h_1, h_2, h_3\}$, must have distinct coefficients and meet at an angle of at least 120 degrees (Proposition 5.2). With one of them reoriented, they have coefficients h_i, $-h_j$ and meet at an angle of at most 60 degrees, as asserted.

Finally consider a sum of three distinct lines, oriented to have coefficients in $\{h_1, h_2, h_3\}$. As before, they meet pairwise at angles of at least 120 degrees. Since there are three of them, they meet at exactly 120 degrees. Thus C has a Y component, a case already established.

The final condition $m_1 - m_{-1} = m_2 - m_{-2} = m_3 - m_{-3}$ just says that there is no interior boundary. $\square$

The next theorem is one of the main results of this paper (see the Introduction).

Regularity Theorem 5.6 (for double soap bubbles with multiplicity). *Let S minimize mass among rectifiable chains with coefficients in G_2 in a smooth Riemannian surface of the form $S + \partial W$ with $\int_W dA = 0$ (i.e., prescribed boundary and areas). Suppose that S has finite boundary mass. Then S consists of finitely many constant-coefficient arcs with coefficients in G_2. Away from the boundary, the arcs meet in threes at* 120 *degrees* (*a Y*), *fours* (*two arcs tangent or crossing at angle at most* 60 *degrees or a Y with stem extended*), *fives* (*a Y with two arms extended*), *or sixes* (*a Y with all three arms extended*), *as described by Proposition 5.5. By the same proposition, all such singularities can occur.*

Proof. This follows immediately from the Regularity Theorem 3.7 and the discussion at the end of Section 3 (page 133), except for the characterization of how arcs meet, which we will deduce with the help of Proposition 5.5.

At a point p where the tangent cone is linear, we may assume that its coefficient is $m_1 h_1 - m_2 h_2$, with $m_1 > 0$, $m_2 \geq 0$. S consists of m_1 arcs with coefficient h_1 and m_2 arcs with coefficient h_2. The arcs with coefficient h_1 must have the same curvature and hence coincide. Similarly the arcs with coefficient h_2 must coincide. Hence S is either two arcs tangent at p or (if they coincide or if $m_2 = 0$) one arc through p.

At a point p where the tangent cone is nonlinear, we will use Proposition 5.5. As described by Proposition 5.3, a Y is (a multiple of) $\sum h_i v_i$ or $\sum(-h_i)(-v_i)$. The extended Y's come from adding a line $h_i l_i$ in the direction v_i. Every possible cone from Proposition 5.5, other than lines and pairs of lines, so arises. Indeed, by subtracting a multiple of a Y, condition (2) of Proposition 5.5 becomes $m_i = m_{-i}$, representing a sum $\sum m_i h_i l_i$. Finally, since tangent arcs have positive multiples of the same h_i as coefficients and hence the same curvature, they must coincide. $\square$

Conjecture 5.7 (Double bubble conjecture for bubbles with multiplicity). *A mass-minimizing double bubble with multiplicities* (*rectifiable chain with coefficients in* G_2) *in* $\mathbb{R}^2$ *with prescribed areas* ($S = \partial W$ *with* $\int_W dA$ *prescribed*) *is standard* (*multiplicity one*).

Remark. The proofs for bubbles without multiplicity [Foisy et al. 1993; Morgan 2001, §3] do not eliminate possibilities as in Figure 3.

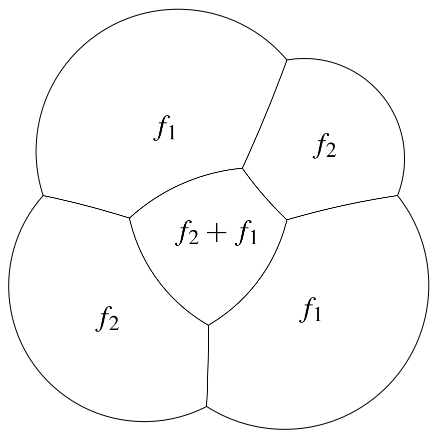

Figure 3. A sample challenger with multiplicities to the standard planar double bubble.

In a Riemannian surface, even single bubbles sometimes prefer multiplicity. For example, in $\mathbb{R}^2$ with a long skinny tentacle of area $A/2$ as in Figure 4, left, the tentacle with multiplicity 2 has less perimeter than any region of area A of multiplicity 1.

In a Riemannian surface, mass-minimizing double bubbles with multiplicity can apparently have interesting new singularities. For example, in a surface as in Figure 4, right, the minimizer for certain areas consists of two partially overlapping regions, redrawn in different coordinates in the top left diagram of Figure 5. The rest of Figure 5 suggests that all types of singularities allowed by Regularity Theorem 5.6 may well occur in minimizing double bubbles (without boundary) in Riemannian surfaces.

Clusters of k bubbles. To generalize from 2 bubbles to k bubbles, one replaces the coefficient group G_2 by the free group G_k on k generators $f_1, \dots, f_k$ with largest norm such that

$$\|f_i\| = \|f_i - f_j\| = 1.$$

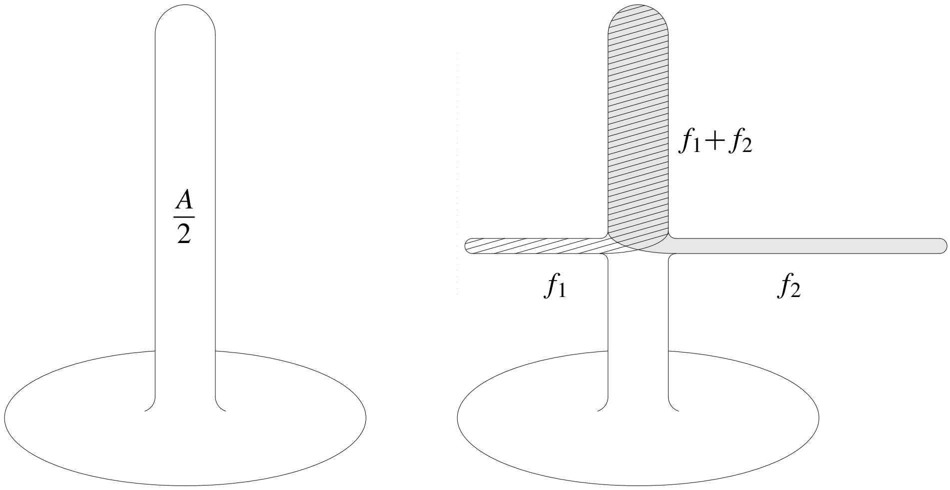

Figure 4. Left: A tentacle of multiplicity two can have less perimeter than any region of multiplicity one of the same area. Right: In a surface, a minimizing double bubble may consist of regions which partially overlap.

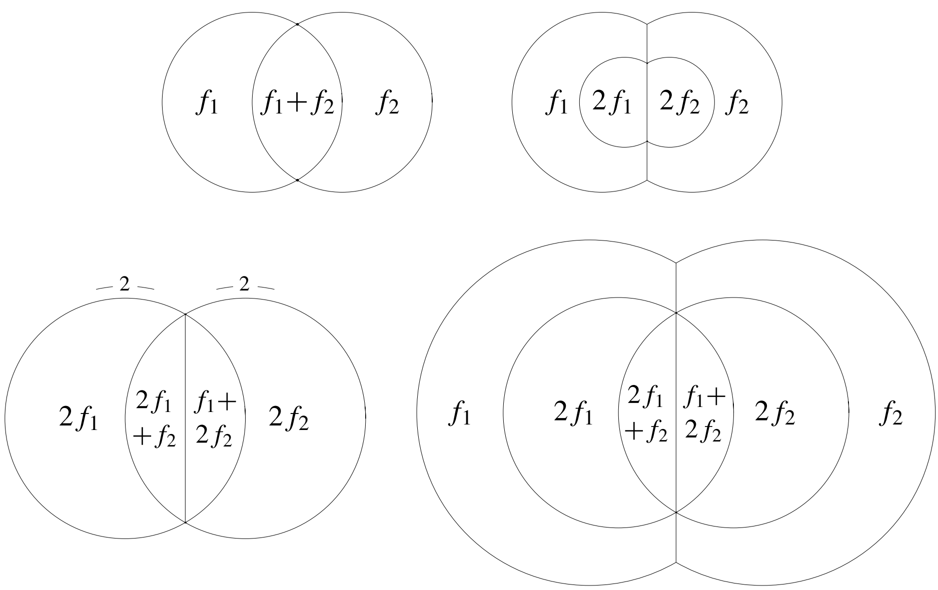

Figure 5. All types of singularities allowed by Regularity Theorem 5.6 may well occur in minimizing double bubbles under some Riemannian metric.

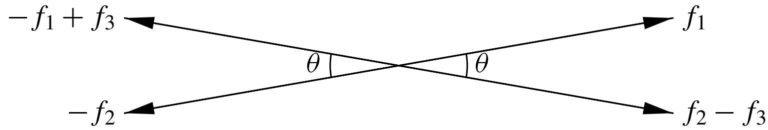

Figure 6. A minimizing cone over G_3 (with $0<\theta\le 60$ degrees). Rotating the left-point vectors about the x-axis in $\mathbb{R}^3$ yields nonplanar minimizing cones.

Over G_3, minimizing cones (without boundary at the vertex) are already more complicated and interesting, largely due to the rectangular face of the unit norm ball with vertices f_1, f_2, f_1-f_3, f_2-f_3. One family of planar examples is the cones of Figure 6 with $0<\theta\le 60$ degrees. Rotating the two vectors on the left by any amount about the x-axis in $\mathbb{R}^3$ yields nonplanar minimizing cones. All of these cones are minimizing (and calibrated) by Proposition 4.5.

Networks in higher dimensions. Without area constraints, the Regularity Theorem 5.6 holds in a smooth n-dimensional Riemannian manifold. A mass-minimizing rectifiable chain with coefficients in G_2 with given boundary of finite mass consists of finitely many geodesic arcs with coefficients meeting on the interior at most six at a time as described by Theorem 5.6. In particular, the interior tangent cones are planar.

Double bubbles in R^3. For two-dimensional bubble clusters in $\mathbb{R}^3$ without multiplicities, there are ten candidate minimizing cones; see [Morgan 2000, Figure 13.9.1]. Of these, only three correspond to double bubbles — plane, triple junction, cone over cube — and only the first two are minimizing. It would be interesting to classify the candidate cones for double bubbles with multiplicity, i.e., cones over geodesic nets satisfying the Regularity Theorem 5.6, and to identify the minimizers. One new candidate consists of two triple junctions meeting at right angles.

Acknowledgements

This work was inspired by the conjectures of B. White [2001] and a question from W. Wichiramala. Thanks to George Lee for helpful comments.

References

[Allard 1972] W. K. Allard, "On the first variation of a varifold", *Ann. of Math.* (2) **95** (1972), 417–491. MR 46 #6136 Zbl 0252.49028

[Allard 1975] W. K. Allard, "On the first variation of a varifold: boundary behavior", *Ann. of Math.* (2) **101** (1975), 418–446. MR 53 #1379 Zbl 0319.49026

[Choe 1996] J. Choe, "Every stationary polyhedral set in $\mathbf{R}^n$ is area minimizing under diffeomorphisms", *Pacific J. Math.* **175**:2 (1996), 439–446. MR 98e:49094 Zbl 0865.53010

[Cox et al. 1994/95] C. Cox, L. Harrison, M. Hutchings, and et al., “The shortest enclosure of three connected areas in $\mathbf{R}^2$”, *Real Anal. Exchange* **20**:1 (1994/95), 313–335. MR 95k:53009

[Federer 1969] H. Federer, *Geometric measure theory*, Grundlehren der math. Wiss. **153**, Springer, New York, 1969. MR 41 #1976 Zbl 0176.00801

[Fleming 1966] W. H. Fleming, “Flat chains over a finite coefficient group”, *Trans. Amer. Math. Soc.* **121** (1966), 160–186. MR 32 #2554 Zbl 0136.03602

[Foisy et al. 1993] J. Foisy, M. Alfaro, J. Brock, N. Hodges, and J. Zimba, “The standard double soap bubble in $\mathbf{R}^2$ uniquely minimizes perimeter”, *Pacific J. Math.* **159**:1 (1993), 47–59. MR 94b:53019 Zbl 0738.49023

[Futer et al. 2000] D. Futer, A. Gnepp, D. McMath, B. A. Munson, T. Ng, S.-H. Pahk, and C. Yoder, “Cost-minimizing networks among immiscible fluids in $\mathbf{R}^2$”, *Pacific J. Math.* **196**:2 (2000), 395–414. MR 2001m:53010 Zbl 0986.49024

[Lawlor and Morgan 1994] G. Lawlor and F. Morgan, “Paired calibrations applied to soap films, immiscible fluids, and surfaces or networks minimizing other norms”, *Pacific J. Math.* **166**:1 (1994), 55–83. MR 95i:58051 Zbl 0830.49028

[Morgan 1994] F. Morgan, “Clusters minimizing area plus length of singular curves”, *Math. Ann.* **299**:4 (1994), 697–714. MR 95g:49083 Zbl 0805.49025

[Morgan 1998] F. Morgan, “Immiscible fluid clusters in $\mathbf{R}^2$ and $\mathbf{R}^3$”, *Michigan Math. J.* **45**:3 (1998), 441–450. MR 99k:49093 Zbl 0967.49026

[Morgan 2000] F. Morgan, *Geometric measure theory*, third ed., Academic Press, San Diego, CA, 2000. MR 2001j:49001 Zbl 0974.49025

[Morgan 2001] F. Morgan, “Proof of the double bubble conjecture”, *Amer. Math. Monthly* **108**:3 (2001), 193–205. MR 2002i:53013 Zbl 0990.49029

[White 1986] B. White, “Regularity of the singular sets in immiscible fluid interfaces and solutions to other Plateau-type problems”, pp. 244–249 in *Miniconference on geometry and partial differential equations* (Canberra, 1985), Proc. Centre Math. Anal. Austral. Nat. Univ. **10**, Austral. Nat. Univ., Canberra, 1986. MR 857670 Zbl 0606.49029

[White 1996] B. White, “Existence of least-energy configurations of immiscible fluids”, *J. Geom. Anal.* **6**:1 (1996), 151–161. MR 97e:49038 Zbl 0853.49030

[White 1999a] B. White, “The deformation theorem for flat chains”, *Acta Math.* **183**:2 (1999), 255–271. MR 2000m:49060 Zbl 0980.49035

[White 1999b] B. White, “Rectifiability of flat chains”, *Ann. of Math.* (2) **150**:1 (1999), 165–184. MR 2000j:49065 Zbl 0965.49024

[White 2001] B. White, “Flat chains over general coefficient groups”, 2001. Lecture at the Clay Mathematical Institute Summer School (MSRI, Berkeley).

Received January 2, 2002. Revised July 22, 2002.

FRANK MORGAN
DEPARTMENT OF MATHEMATICS AND STATISTICS
WILLIAMS COLLEGE
WILLIAMSTOWN, MA 01267

Frank.Morgan@williams.edu