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UNIQUENESS AND A PRIORI ESTIMATES FOR SOME NONLINEAR ELLIPTIC NEUMANN EQUATIONS IN $\mathbb{R}^3$

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Under some conditions on $f(u)$, we show that for λ small and $\Omega \subset \mathbb{R}^3$ convex, the only solution to the elliptic equation $\Delta u - \lambda u + f(u) = 0$ in Ω, with $u > 0$ in Ω and $\partial u/\partial \nu = 0$ on $\partial\Omega$, is constant.

1. Introduction

We consider the nonlinear elliptic equation

$$\begin{cases} \Delta u - \lambda u + f(u) = 0 \ \text{ in } \Omega, \\ u > 0 \ \text{ in } \Omega, \ \dfrac{\partial u}{\partial \nu} = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth and bounded domain in $\mathbb{R}^3$, the function f lies in $C^{1+\theta}$ for some $0 < \theta < 1$, and ν is the unit outer normal vector field at $\partial\Omega$. We suppose that $f(u)$ can be written as $f(u) = u^5(1 + \rho(u))$, with $\rho'(u) \le 0$, $0 \le u^5\rho(u) \le C\sum_{i=1}^K u^{p_i}$ for some $1 < p_i \le \frac{13}{3}$ and some constant $C > 0$. A typical example for $f(u)$ is the function

$$f(u) = u^5 + \sum_{i=1}^K a_i u^{p_i}, \quad \text{with } 1 < p_i \le \tfrac{13}{3} \text{ and } a_i \ge 0 \text{ for } i = 1, \dots, K.$$

Our main result is the following.

Theorem 1. *Suppose that $\Omega \subset \mathbb{R}^3$ is convex. Then there exist $\lambda_0 > 0$ and $C > 0$ such that for $\lambda < \lambda_0$ we have*

$$u \le C\lambda^{1/4}$$

for all solutions u of (1.1), *where C is independent of λ.*

As a consequence, $u \equiv C_\lambda$ for some constant C_λ.

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Remark. The restriction that $u^5\rho(u) \le C\sum_{i=1}^{K} u^{p_i}$ for $p_i \le \frac{13}{3}$ just reflects a technical difficulty. We believe that Theorem 1 also holds under the weaker assumption $0 \le u^5\rho(u) \le C\sum_{i=1}^{K} u^{p_i}$ for $1 < p_i < 5$.

The proof of Theorem 1 is a direct consequence of integration by parts. It also gives a new and rather simple derivation of the following theorem.

Theorem 2. *Suppose that $\Omega \subset \mathbb{R}^3$ is convex and $f(u) = u^5$. Then for λ small, $u = \lambda^{1/4}$ is the unique solution of* (1.1).

Theorem 2 is a special case of Lin and Ni's conjecture [Lin and Ni 1988]. The conjecture says that for λ small and $f(u) = u^p$, $p > 1$, the problem (1.1) admits only constant solutions. (Problem (1.1) with $f(u) = u^p$ arises in the study of steady-state solutions of the model of Keller and Segel [1970] in chemotaxis and the Gierer–Meinhardt system [1972] in pattern formation. For more background on (1.1), see [Ni 1998].)

Lin and Ni proved their conjecture in the case when p is subcritical. Zhu [1999] proved the conjecture for three-dimensional convex domains (the same class of domains as Theorem 2) by using a very complicated blow-up analysis. Zhu's proof is by contradiction and thus indirect. It is unclear if Zhu's proof can be generalized to other functions $f(u)$. Our proof is much simpler and elementary. In fact since our proof is direct, it can yield an explicit value for the number λ_0 in Theorem 1. We remark that when $\Omega = B_R(0)$ and u is radial, similar results were proved by Adimurthi and Yadava [1993].

In higher dimensions, the Lin–Ni conjecture may be wrong. Counterexamples in the radial case are given in [Adimurthi and Yadava 1991; 1997]. For the nonradial case, see [Gui and Wei 2005].

We use an idea developed by Chang, Gursky and Yang in the study of three-dimensional prescribed scalar curvature problem [Chang et al. 1993]. Our starting point is to write equation (1.1) as

$$\Delta u - \lambda u + f(u) = \Delta u + R(u)u^5,$$

where $R(u) = 1 + \rho(u) - \lambda u^{-4}$. Thus if we introduce the conformal transformation $g = u^4 g_0$, where g_0 is the usual Euclid metric, R becomes the scalar curvature in the new metric.

We first make some preliminary notes. (Throughout this paper, C will denote different constants independent of λ.)

From equation (1.1), we see that

$$\int_\Omega u^5 \le \int_\Omega f(u) = \lambda \int_\Omega u \le C\lambda \left(\int_\Omega u^5\right)^{1/5},$$

which implies that

$$\int_\Omega u^5 \le C\lambda^{5/4}, \quad \int_\Omega u \le \lambda^{1/4} \tag{1.2}$$

and thus

$$u_{\min} := \min_{x\in\bar\Omega} u(x) \le C\lambda^{1/4}. \tag{1.3}$$

From (1.2), we also obtain that

$$\int_\Omega u^q \le C\lambda^{q/4} \quad \text{for all } 0 < q \le 5. \tag{1.4}$$

Next we need a well-known fact found in [Lions 1982, Appendix 1, Lemma 5], for example: if Ω is convex and $\partial u/\partial\nu = 0$ on $\partial\Omega$, then

$$\frac{\partial}{\partial\nu}|\nabla u|^2 \le 0 \quad \text{on } \partial\Omega. \tag{1.5}$$

2. Proof of Theorem 1

Let g_0 be the usual Euclidean metric and $g = u^4 g_0$. In the new metric, we consider the trace-free Ricci tensor B. In a local coordinate system, we may write

$$B_{ij} = -u^{-2}\big((u^2)_{g,ij} - \tfrac{1}{3}(\Delta_g u^2)g_{ij}\big).$$

(Here the covariant derivatives are taken with respect to g, not g_0.)

We first obtain an integral estimate for $\int_\Omega |B|^{3/2}\, dV_g$. (Here dV_g denotes integration with respect to the new metric g. It is easy to see that $dV_g = u^6\, dx$.) To this end, we consider

$$\begin{aligned}\int_\Omega |B|^2 u^2\, dV_g &= \int_\Omega g^{ik} g^{jl} B_{ij} B_{kl} u^2\, dV_g \\ &= -\int_\Omega (u^2)_{g,ij} B_{kl} g^{ik} g^{jl}\, dV_g + \frac{1}{3}\int_\Omega (\Delta_g u^2) g_{ij} g^{ik} g^{jl} B_{kl}\, dV_g,\end{aligned}$$

where the second term vanishes because B is trace-free.

Now using integration by parts, together with the Neumann boundary condition, we have

$$\int_\Omega |B|^2 u^2\, dV_g = \int_\Omega (u^2)_{g,i} B_{kl;j} g^{ik} g^{jl}\, dV_g + \int_{\partial\Omega} \frac{\partial}{\partial\nu}|\nabla u|^2 d\sigma_g,$$

where $d\sigma_g$ is the surface element in the new metric g.

Using the contracted second Bianchi identity and (1.5) we obtain

$$\begin{aligned}\int_\Omega |B|^2u^2\,dV_g &\le \frac{1}{6}\int_\Omega (u^2)_{g,i}R_{,k}g^{ijk}\,dV_g\\ &= \frac{1}{6}\int_\Omega \langle \nabla_g(u^2),\,\nabla_g R\rangle\,dV_g = \frac{1}{12}\int_\Omega \langle \nabla(u^4),\,\nabla R\rangle\,dx\\ &= \frac{1}{12}\int_\Omega \nabla u^4\nabla(1+\rho(u)-\lambda u^{-4})\\ &\le \tfrac{4}{3}\lambda\int_\Omega u^{-2}|\nabla u|^2\,dx \quad (\text{since } \rho'(u)\le 0)\\ &= \tfrac{4}{3}\lambda\int_\Omega u^{-1}\Delta u\,dx = \tfrac{4}{3}\lambda\int_\Omega u^{-1}(\lambda u - f(u)) \le C\lambda^2.\end{aligned}$$

Thus we obtain the following key estimate (here we need $\Omega\subset\mathbb{R}^3$)

$$\int_\Omega |B|^{3/2}\,dV_g \le \left(\int_\Omega |B|^2u^2\,dV_g\right)^{3/4}\left(\int_\Omega u^{-6}dV_g\right)^{1/4} \le C\lambda^{3/2}. \tag{2.6}$$

Next we estimate $\int_\Omega \left|\nabla u^{-2}\right|^6 dx$. Let $v=1/u$. Recall the Bochner identity:

$$\Delta_g|\nabla_g v|^2 = 2|\nabla_g^2 v|^2 + 2\left\langle \nabla_g v,\,\nabla_g(\Delta_g v)\right\rangle + 2\,\mathrm{Ric}_g(\nabla_g v,\,\nabla_g v). \tag{2.7}$$

If Ω is convex, $\int_\Omega \Delta_g|\nabla_g v|^2\,dV_g\le 0$ by (1.5). Integrating both sides of (2.7) and using the divergence theorem, we get

$$\int_\Omega |\nabla_g^2 v|^2\,dV_g \le \int_\Omega (\Delta_g v)^2\,dV_g + \int_\Omega |\mathrm{Ric}|\,|\nabla_g v|^2 dV_g.$$

Recall the Sobolev inequality in $\mathbb{R}^3$

$$\left(\int_\Omega \psi^6\,dV_g\right)^{1/3} \le C\int_\Omega |\nabla_g\psi|^2\,dV_g + C\int_\Omega \psi^2\,dV_g.$$

Take $\psi=|\nabla_g v|$. Then we obtain

$$\begin{aligned}&\left(\int_\Omega |\nabla_g v|^6\,dV_g\right)^{1/3}\\ &\quad\le C\int_\Omega \left|\nabla_g\,|\nabla_g v|^2\right| + C\int_\Omega |\nabla_g v|^2\\ &\quad\le C\int_\Omega |\nabla_g^2 v|^2\,dV_g + C\int_\Omega |\nabla_g v|^2\,dV_g \quad (\text{by Kato's inequality})\\ &\quad\le C\int_\Omega |\Delta_g v|^2\,dV_g + C\int_\Omega |\mathrm{Ric}|\,|\nabla_g v|^2\,dV_g + C\int_\Omega |\nabla_g v|^2\,dV_g\\ &\quad\le C\int_\Omega |\Delta_g v|^2\,dV_g + C\int_\Omega |B|\,|\nabla_g v|^2\,dV_g\\ &\qquad\qquad + C\int_\Omega |R|\,|\nabla_g v|^2\,dV_g + C\int_\Omega |\nabla_g v|^2\,dV_g,\end{aligned} \tag{2.8}$$

where we have used the fact that $|\operatorname{Ric}| \le |B| + \frac{1}{3}|R|$.

We estimate each term on the last two lines of (2.8).

Recall that $g_0 = u^{-4}g = v^4 g$. The scalar curvature R_0 with respect to g_0 is 0. So

$$\begin{aligned}\Delta_g v &= (\Delta_g + \tfrac{1}{8}R)v - \tfrac{1}{8}Rv = -\tfrac{1}{8}R_0 v^5 - \tfrac{1}{8}Rv\\ &= -\tfrac{1}{8}Rv = -\tfrac{1}{8}\big(1+\rho(u) - \lambda u^{-4}\big)u^{-1}.\end{aligned}$$

Therefore

$$(\Delta_g v)^2 \le C\big(u^{-2} + u^{-2}\rho^2(u) + \lambda^2 u^{-10}\big) \le C\Big(u^{-2} + \sum_{i=1}^K u^{2p_i-12} + \lambda^2 u^{-10}\Big)$$

and

$$\int_\Omega (\Delta_g v)^2\, dV_g \le C + C\sum_{i=1}^K \int_\Omega u^{2p_i-6}\, dx + C\lambda^2 \int_\Omega u^{-4}\, dx \le C\big(1 + u_{\min}^{-2\tau} + \lambda^2 u_{\min}^{-4}\big),$$

where $\tau = \min(3-p_1, \dots, 3-p_K, 0) < 2$ since $1 < p_i < 5$.

For the second term in the right-hand side of (2.8), we have, by (2.6),

$$\begin{aligned}\int_\Omega |B|\,|\nabla_g v|^2\, dV_g &\le C\Big(\int_\Omega |B|^{3/2}\, dV_g\Big)^{2/3}\Big(\int_\Omega |\nabla_g v|^6\, dV_g\Big)^{1/3}\\ &\le C\lambda\Big(\int_\Omega |\nabla_g v|^6\, dV_g\Big)^{1/3}.\end{aligned}$$

The terms on the last line of (2.8) can be estimated as follows:

$$\int_\Omega |\nabla_g v|^2\, dV_g = \int_\Omega |\nabla_0 u|^2 u^{-2}\, dx \le C\lambda,$$

$$\begin{aligned}\int_\Omega |R|\,|\nabla_g v|^2\, dV_g &\le \int_\Omega C\,(1+\rho(u)+\lambda u^{-4})\,|\nabla_g v|^2\, dV_g\\ &\le C\lambda + \Big(\int_\Omega \big|\rho(u) + \lambda u^{-4}\big|^{3/2}\, dV_g\Big)^{2/3}\Big(\int_\Omega |\nabla_g v|^6\, dV_g\Big)^{1/3}\\ &\le C\lambda + C\Big(\int_\Omega\Big(\sum_{i=1}^K u^{(3/2)(p_i-5)+6} + \lambda^{3/2}\Big)\, dx\Big)^{2/3}\\ &\qquad\qquad \times\Big(\int_\Omega |\nabla_g v|^6\, dV_g\Big)^{1/3}\\ &\le C\lambda + C\sum_{i=1}^K \lambda^{(1/4)(p_i-5)+1}\Big(\int_\Omega |\nabla_g v|^6\, dV_g\Big)^{1/3}\end{aligned}$$

by (1.4), since $0 < \frac{3}{2}(p_i - 5) + 6 \le 5$. (This is where we need the assumption $p_i \le \frac{13}{3}$.)

Combining all the previous estimates, we have

$$\left(\int_\Omega |\nabla_g v|^6\, dV_g\right)^{1/3} \le C\left(1+u_{\min}^{-2\tau}+\lambda^2 u_{\min}^{-4}\right).$$

Note that

$$\int_\Omega |\nabla_g v|^6\, dV_g = \int_\Omega |\nabla_0 u|^6 u^{-18}\, dx.$$

So

$$\left(\int_\Omega |\nabla_g u^{-2}|^6\, dx\right)^{1/3} \le C\left(1+u_{\min}^{-2\tau}+\lambda^2 u_{\min}^{-4}\right).$$

By the Sobolev embedding theorem, for any P, $Q \in \bar{\Omega}$, we have

$$\begin{aligned}\left|u^{-2}(P)-u^{-2}(Q)\right| &\le C\left\|\nabla(u^{-2})\right\|_{L^6(\Omega)}|P-Q|^{1/2}\\ &\le C\left\|\nabla(u^{-2})\right\|_{L^6(\Omega)} \le C\left(1+u_{\min}^{-\tau}+\lambda u_{\min}^{-2}\right).\end{aligned}$$

Therefore, for any P, $Q \in \bar{\Omega}$,

$$u^{-2}(P) \ge u^{-2}(Q) - C\left(1+u_{\min}^{-\tau}+\lambda u_{\min}^{-2}\right).$$

Choose Q so that $u(Q)=u_{\min}=\min_{\bar{\Omega}} u$. Since $u_{\min}\le C\lambda^{1/4}$ (by (1.3)) and $\tau<2$, we see that

$$u^{-2}(Q) - C\left(1+u_{\min}^{-\tau}+\lambda u_{\min}^{-2}\right) \ge \tfrac{1}{2}u_{\min}^{-2},$$

which implies that

$$u(P) \le C u_{\min} \le C\lambda^{1/4} \quad \text{for all } P \in \bar{\Omega}.$$

Now let $w=u-\bar{u}$, where $\bar{u}=(1/|\Omega|)\int_\Omega u$. Then w satisfies

$$\Delta w - \lambda w + f(w+\bar{u}) - f(\bar{u}) - \lambda\bar{u} + f(\bar{u}) = 0$$

Multiplying by w and integrating by parts, we get

$$\int_\Omega \left(|\nabla w|^2 + \lambda w^2 + c(w)w^2\right) = 0, \tag{2.9}$$

where

$$c(w) = -\frac{f(w+\bar{u})-f(\bar{u})}{w} \to 0 \quad \text{as } \lambda \to 0.$$

Since $\int_\Omega w = 0$, (2.9) implies that $w \equiv 0$ and hence $u \equiv \bar{u}$. Theorem 1 is thus proved.

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