FROBENIUS–SCHUR INDICATORS FOR A CLASS OF FUSION CATEGORIES

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We give an explicit description of group-theoretical quasi-Hopf algebras up to gauge equivalence. We then use it to compute the Frobenius–Schur indicators for group-theoretical fusion categories.

1. Introduction

There are strong analogies between the theory of finite groups and the theory of semisimple Hopf algebras; some of them, however, still remain conjectural. In particular, the problem of classifying semisimple Hopf algebras, say over the field of complex numbers, seems to be a difficult one, even in low dimensions. Perhaps the most important feature of these objects, which relates them to other branches of mathematics and physics, is that their category of representations is a special case of a so called *fusion category*. This fact leads to the consideration of the classification problem, not only modulo Hopf algebra isomorphisms, but modulo *gauge equivalences*: roughly, two finite-dimensional (quasi-)Hopf algebras H and H' give rise to the same fusion category of representations if and only if they are *gauge equivalent*, in the sense that $H = H'$ as algebras, and the comultiplication of H' is obtained by "twisting" that of H by means of $\Delta_{H'}(h) = F\Delta(h)F^{-1}$, for some *gauge transformation* $F \in (H \otimes H)^{\times}$.

An important class of examples of semisimple quasi-Hopf algebras was introduced by Ostrik [2002] and studied by Etingof, Nikshych and Ostrik [Etingof et al. 2002]. Called *group-theoretical*, these are, by definition, those for which the category of representations is a group-theoretical category $\mathscr{C}(G, \omega, F, \alpha)$, where G is a finite group, $F \subseteq G$ is a subgroup, $\omega : G \times G \times G \to k^{\times}$ is a normalized 3-cocycle and $\alpha : F \times F \to k^{\times}$ is a normalized 2-cochain, such that $\omega|_F = d\alpha$. More precisely, $\mathscr{C}(G, \omega, F, \alpha)$ is the category of $k_\alpha F$-bimodules in the tensor category Vec_ω^G of finite-dimensional G-graded vector spaces, with associativity constraint given by ω. All concrete known examples of semisimple Hopf algebras (and this

does not extend to the *quasi*-context), turn out to be group-theoretical; this can be seen as a consequence of our results in [Natale 2003]. The following question has been posed in [Etingof et al. 2002]. An answer to this question (even an affirmative one) would be of great significance in the classification program.

Question 1–1. Does there exist a semisimple Hopf algebra which is not group-theoretical?

The categorical nature of this question leads naturally to the problem of finding and computing *gauge invariants* of group-theoretical quasi-Hopf algebras, that is, invariants which depend on the gauge equivalence class of the object rather than on the isomorphism class itself.

Recently, Mason and Ng have constructed a gauge invariant, the *Frobenius–Schur* indicators, for semisimple quasi-Hopf algebras [Mason and Ng 2005]. They have proved a generalization of the Frobenius–Schur Theorem for finite groups; compare [Serre 1967]. Their construction extends results of Linchenko and Montgomery for semisimple Hopf algebras [Linchenko and Montgomery 2000]; it also extends results of Bantay on the Frobenius–Schur indicators for the Dijkgraaf–Pasquier–Roche quasi-Hopf algebra $D^\omega G$, after the definition in [Bantay 1997] of the indicators attached to conformal field theories.

Essentially, Frobenius–Schur indicators were defined in a categorical fashion for any semisimple rigid tensor category which is *pivotal*, *i.e.*, which admits a natural tensor isomorphism between the identity and the second left duality functors, in the work of Fuchs, Ganchev, Szlachányi and Vescernyés [Fuchs et al. 1999]. It is shown in [Etingof et al. 2002] that representation categories of semisimple quasi-Hopf algebras are in fact pivotal.

Other gauge invariants can be attached to a semisimple quasi-Hopf algebra. One of the most studied is the K_0-ring of its representation category. This invariant does not distinguish the group algebras of the two nonabelian groups of order 8: the dihedral group D_4 and the quaternionic group Q_2. However, by a result of Tambara and Yamagami [1998], these two groups are not gauge equivalent. In their paper, Mason and Ng have noted that the Frobenius–Schur indicators do distinguish the dihedral and quaternionic groups. As pointed out to us by the referee, in some cases it may happen that the Frobenius–Schur indicators contain less information than the K_0-ring, e.g., in the case of dual group algebras. So, in some sense, these two invariants are of very different nature.

In this paper we give an explicit description of group-theoretical Hopf algebras and use it to compute their irreducible characters and Frobenius–Schur indicators. Our main original contributions in the description of the quasi-structure are on the one hand the proof of the existence of a certain normalization of the 3-cocycle ω

(Proposition 4.2), and on the other hand, the construction of a quasi-antipode (Theorem 4.12) which is, of course, essential in the computation of the Frobenius–Schur invariants. We obtain the following formula for the indicator of the irreducible character χ:

$$(1\text{--}2) \qquad \chi(v_{A^{\mathrm{op}}}) = |F|^{-1} \sum_{(q \triangleleft x).q=e} \omega(xq, xq, xq) \quad \chi(\delta_q(xq)^2),$$

which involves a certain normalization of the 3-cocycle ω. See Corollary 5.4. Here, q runs over an appropriate choice of representatives of G modulo F.

One instance of these examples comes from an exact factorization $G = FQ$ of the group G into its subgroups F and Q. In this case, there is a group

$$\mathrm{Opext}(kF, k^Q)$$

that classifies the *abelian* Hopf algebra extensions of k^Q by kF: as a Hopf algebra, the extension corresponding to an element $[\sigma, \tau] \in \mathrm{Opext}(kF, k^Q)$ is a bicrossed product $k^Q \#_\sigma^\tau kF$, where $\sigma : F \times F \to (k^Q)^\times$ and $\tau : Q \times Q \to (k^F)^\times$ are a pair of compatible cocycles; see [Natale 2003, Theorem 1.2]. In this case, the 3-cocycle ω in our formula for the Frobenius–Schur indicators is the one associated to $[\sigma, \tau]$ in the Kac exact sequence. This gives an alternative compact expression for the formula found by Kashina, Mason and Montgomery in [Kashina et al. 2002].

We would like to point out that the description for the quasi-Hopf algebra structure for group-theoretical quasi-Hopf algebras generalizes the construction of the twisted quantum doubles $D^\omega G$ by Dijkgraaf, Pasquier and Roche. This agrees with the characterization given in [Natale 2003] in terms of quantum (or Drinfeld) doubles; so in some sense these quasi-Hopf algebras are all of *DPR-type*.

The paper is organized as follows. In Sections 2 and 3 we recall the definition of the indicators constructed in [Mason and Ng 2005] and the definition and main properties of group-theoretical categories as given in [Ostrik 2002; Etingof et al. 2002]. In Section 4 we give a description, up to gauge equivalence, of the structure of group-theoretical quasi-Hopf algebras, and finally in Section 5 we give an explicit formula for the Frobenius–Schur indicators of group-theoretical categories. We consider some examples in Section 6. Throughout this paper we work over an algebraically closed field k of characteristic zero.

2. Frobenius–Schur indicators

Let $(H, \Delta, \epsilon, \Phi, \mathscr{S}, \alpha, \beta)$ be a finite-dimensional semisimple quasi-Hopf algebra [Drinfeld 1989] (later on indicated (H, Φ) for short), that is, H is an associative unital algebra over k which is semisimple and finite-dimensional; $\epsilon : H \to k$ and

$\Delta : H \to H \otimes H$ are algebra maps; $\Phi \in H^{\otimes 3}$ is an invertible element such that

$$(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\Phi)(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\Phi) = (1 \otimes \Phi)(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\Phi)(\Phi \otimes 1),$$

$$(\mathrm{id} \otimes \epsilon \otimes \mathrm{id})(\Phi) = 1 \otimes 1,$$

$$(\epsilon \otimes \mathrm{id})\Delta(h) = h = (\mathrm{id} \otimes \epsilon)\Delta(h),$$

$$(2\text{--}1) \qquad \Phi(\Delta \otimes \mathrm{id})\Delta(h)\Phi^{-1} = (\mathrm{id} \otimes \Delta)\Delta(h),$$

for all $h \in H$. The map $\mathcal{S} : H \to H^{\mathrm{op}}$ is an algebra anti-automorphism of H; $\alpha, \beta \in H$ are such that

$$(2\text{--}2) \qquad \mathcal{S}(h_1)\alpha h_2 = \epsilon(h)\alpha \quad \text{and} \quad h_1 \beta \mathcal{S}(h_2) = \epsilon(h)\beta \quad \text{for all } h \in H$$

and

$$(2\text{--}3) \qquad \Phi^{(1)} \beta \mathcal{S}(\Phi^{(2)})\alpha \Phi^{(3)} = 1 = \mathcal{S}(\Phi^{(-1)})\alpha \Phi^{(-2)} \beta \mathcal{S}(\Phi^{(-3)}),$$

where we are using the abbreviated notation $\Phi = \Phi^{(1)} \otimes \Phi^{(2)} \otimes \Phi^{(3)}$ and $\Phi^{-1} = \Phi^{(-1)} \otimes \Phi^{(-2)} \otimes \Phi^{(-3)}$.

The category $\mathrm{Rep}\, H =: \mathrm{Rep}(H, \Phi)$ is a *fusion category*, in the terminology of [Etingof et al. 2002]. The associativity constraint is given by the natural action of Φ; the left dual of an object V of $\mathrm{Rep}\, H$ is the vector space $V^* = \mathrm{Hom}(V, k)$ with the H-action $\langle h.f, v \rangle = \langle f, \mathcal{S}(h)v \rangle$; and the evaluation and coevaluation maps are given, respectively, by

$$\mathrm{ev} : V^* \otimes V \to k, \qquad \mathrm{ev}(f \otimes v) = \langle f, \alpha.v \rangle,$$

$$\mathrm{coev} : k \to V \otimes V^*, \qquad 1 \mapsto \sum_i \beta.v_i \otimes v^i,$$

for all $f \in V^*$, $v \in V$, where (v_i) and (v^i) are dual bases of V.

Two quasi-Hopf algebras H_1 and H_2 are called *gauge equivalent* if there exists a *gauge transformation*, *i.e.*, an invertible normalized element $F \in H_1 \otimes H_1$, such that $(H_1)_F$ and H_2 are isomorphic as quasi-bialgebras, where $(H_1)_F$ is the quasi-Hopf algebra $(H_1, \Delta_F, \epsilon, \Phi_F, \mathcal{S}_F, \alpha_F, \beta_F)$ defined by

$$\Delta_F(h) = F\Delta(h)F^{-1}, \qquad h \in H,$$

$$\Phi_F = (1 \otimes F)(\mathrm{id} \otimes \Delta)(F)\Phi(\Delta \otimes \mathrm{id})(F^{-1})(F^{-1} \otimes 1),$$

$$\alpha_F = \mathcal{S}(F^{(-1)})\alpha F^{(-2)}, \qquad \beta_F = F^{(1)} \beta \mathcal{S}(F^{(2)}),$$

for $F = F^{(1)} \otimes F^{(2)}$ and $F^{-1} = F^{(-1)} \otimes F^{(-2)}$.

Two finite-dimensional quasi-Hopf algebras H_1 and H_2 are gauge equivalent if and only if $\mathrm{Rep}\, H_1$ is equivalent to $\mathrm{Rep}\, H_2$ as k-linear tensor categories. See [Etingof and Gelaki 2002].

Remark 2.1. It is shown in [Etingof et al. 2002] that the fusion categories of the form $\mathrm{Rep}(H, \Phi)$ are exactly those for which the Frobenius–Perron dimensions of simple objects are integers.

Now let (H, Φ) be a finite-dimensional quasi-Hopf algebra. A *normalized two-sided integral* of H is an element $\Lambda \in H$ such that

$$h\Lambda = \epsilon(h)\Lambda = \Lambda h \quad \text{for all } h \in H$$

and $\epsilon(\Lambda) = 1$. Suppose that (H, Φ) is semisimple. Then H contains a unique normalized two-sided integral [Hausser and Nill 1999].

The following definition is due to Mason and Ng [2005]. It generalizes a previous definition for semisimple Hopf algebras given by Linchenko and Montgomery [2000].

Definition 2.2. Let (H, Φ) be a finite-dimensional semisimple quasi-Hopf algebra and let $\Lambda \in H$ be a normalized two-sided integral. Let also $\chi \in H^*$ be an irreducible character of H. The *Frobenius–Schur indicator* of χ is the element $\chi(\nu_H)$, where ν_H is the canonical central element of H given by

$$\nu_H = m(q_L \Delta(\Lambda) p_L);$$

here, $m : H \otimes H \to H$ is the multiplication map, and $q_L, p_L \in H^{\otimes 2}$ are defined by

$$q_L := \mathcal{S}(\Phi^{(-1)})\alpha\Phi^{(-2)} \otimes \Phi^{(-3)}, \qquad p_L := \Phi^{(2)}\mathcal{S}^{-1}(\Phi^{(1)}\beta) \otimes \Phi^{(3)}.$$

The family of Frobenius–Schur indicators $\{\chi(\nu_H)\}_\chi$ is an invariant of the k-linear tensor category $\mathrm{Rep}(H, \Phi)$. This means that it is invariant under gauge transformations of (H, Φ).

Also, if α and β are invertible elements of H, then the canonical central element ν_H can be computed as follows [Mason and Ng 2005, Corollary 3.5]:

$$\nu_H = (\Lambda_1\Lambda_2)(\beta\alpha)^{-1} = (\beta\alpha)^{-1}(\Lambda_1\Lambda_2).$$

In analogy with finite group situation, the Frobenius–Schur indicator of the irreducible character $\chi = \chi_V$ satisfies the following:

(i) $\chi(\nu_H) = 0, 1$ or -1, and $\chi(\nu_H) \neq 0$ if and only if $\chi = \chi^*$

(ii) $\chi(\nu_H) = 1$ (respectively -1) if and only if V admits a nondegenerate bilinear form $\langle\ ,\ \rangle : V \otimes V \to k$, with adjoint $\mathcal{S}$, such that $\langle x, y \rangle = \langle y, g^{-1}x \rangle$ (respectively, $\langle x, y \rangle = -\langle y, g^{-1}x \rangle$), where $g \in H$ is the so called *trace element* of H.

3. Group theoretical fusion categories

Group theoretical categories were introduced in [Ostrik 2002, Section 3] and also studied in [Etingof et al. 2002]. In this section we recall their definition and basic properties.

Let G be a finite group, and let $F \subseteq G$ be a subgroup. The identity element of G will be denoted by e. Let the following data be given:

- a normalized 3-cocycle $\omega : G \times G \times G \to k^\times$, meaning that ω satisfies

$$(3\text{–}1) \qquad \omega(ab, c, d)\omega(a, b, cd) = \omega(a, b, c)\omega(a, bc, d)\omega(b, c, d)$$

   and $\omega(e, a, b) = \omega(a, e, b) = \omega(a, b, e) = 1$ for all $a, b, c, d \in G$, and

- a normalized 2-cochain $\alpha : F \times F \to k^\times$,

all subject to the condition

$$(3\text{–}2) \qquad \omega|_{F \times F \times F} = d\alpha.$$

Consider the category Vec_ω^G of finite-dimensional G-graded vector spaces, with associativity constraint given by ω: explicitly, for any three objects U, U' and U'' of Vec_ω^G, we have $a_{U,U',U''} : (U \otimes U') \otimes U'' \to U \otimes (U' \otimes U'')$, given by

$$a_{U,U',U''}((u \otimes u') \otimes u'') = \omega(\|u\|, \|u'\|, \|u''\|)\, u \otimes (u' \otimes u''),$$

on homogeneous elements $u \in U$, $u' \in U'$, $u'' \in U''$, where we use $\| \, \|$ to denote degree of homogeneity. In other words, Vec_ω^G is the category of representations of the quasi-Hopf algebra k^G, with associator $\omega \in (k^G)^{\otimes 3}$.

By (3–2), the twisted group algebra $k_\alpha F$ is an (associative unital) algebra in Vec_ω^G, and one may naturally attach to it a monoidal category. Precisely, the category $\mathscr{C}(G, \omega, F, \alpha)$ is by definition the k-linear monoidal category of $k_\alpha F$-bimodules in Vec_ω^G: the tensor product is $\otimes_{k_\alpha F}$ and the unit object is $k_\alpha F$. This is a fusion category over k with the property that the Frobenius–Perron dimensions of its objects are integers [Etingof et al. 2002, 8.8].

Categories of the form $\mathscr{C}(G, \omega, F, \alpha)$ are called *group-theoretical* [Etingof et al. 2002, Definition 8.46]. By extension, a (quasi)-Hopf algebra A is called group-theoretical if the category $\mathrm{Rep}\, A$ of its finite-dimensional representations is group-theoretical.

Now let $\eta : G \times G \to k^\times$ and $\chi : F \to k^\times$ be normalized cochains, and let $\widetilde{\omega} : G \times G \times G \to k^\times, \widetilde{\alpha} : F \times F \to k^\times$ be given by

$$\widetilde{\omega} = \omega(d\eta), \qquad \widetilde{\alpha} = \alpha(\eta|_{F \times F})(d\chi).$$

Then the categories $\mathscr{C}(G, \omega, F, \alpha)$ and $\mathscr{C}(G, \widetilde{\omega}, F, \widetilde{\alpha})$ are equivalent [Etingof et al. 2002, Remark 8.39].

Remark 3.1. Let G, F, ω and α be as above. Let Q be a set of representatives of the left cosets of F in G such that $e \in Q$; so that every element $g \in G$ writes uniquely

in the form $g = xp$, with $p \in Q$, $x \in F$. Consider the 2-cochain $\eta : G \times G \to k^{\times}$ defined in the form

$$\eta(xp, yq) := \alpha^{-1}(x, y), \qquad p, q \in Q, \quad x, y \in F.$$

Then, taking $\chi = 1$, we obtain $\widetilde{\alpha} = 1$. Therefore the categories $\mathscr{C}(G, \omega, F, \alpha)$ and $\mathscr{C}(G, \widetilde{\omega}, F, 1)$ are equivalent, where $\widetilde{\omega} = \omega(d\eta)$. That is, up to monoidal equivalence, *we may always assume that* $\alpha = 1$.

Note also that the categories $\mathscr{C}(G, \omega, F, 1)$ and $\mathscr{C}(G, \omega(d\eta), F, 1)$ are tensor equivalent for every normalized 2-cochain $\eta : G \times G \to k^{\times}$ such that $\eta|_{F \times F}$ is a *coboundary*.

The fiber functors $\mathscr{C}(G, \omega, F, \alpha) \to \mathrm{Vec}$, in the case they exist, are classified by conjugacy classes of subgroups Γ of G, endowed with a 2-cocycle $\beta \in Z^2(\Gamma, k^{\times})$, such that the class of $\omega|_{\Gamma}$ is trivial; $G = F\Gamma$ and the class of the cocycle $\alpha|_{F \cap \Gamma} \beta^{-1}|_{F \cap \Gamma}$ is nondegenerate [Ostrik 2002, Corollary 3.1].

Remark 3.2. The category $\mathscr{C} = \mathscr{C}(G, \omega, F, \alpha)$ has the property that the Frobenius–Perron dimensions of its objects are integers. A Tannaka–Krein reconstruction argument shows that $\mathscr{C}$ is equivalent to the category of representations of a semisimple quasi-Hopf algebra over k [Etingof et al. 2002, Theorem 8.33].

It follows from [Etingof et al. 2002, 8.8] that duals, opposites, quotient categories, full subcategories, and tensor products of group-theoretical categories are also group-theoretical. Also, by [Etingof et al. 2002, Remark 8.47], the Drinfeld center $\mathscr{Z}(\mathscr{C})$ is group-theoretical if and only if so is $\mathscr{C}$.

However, in [Etingof et al. 2002, Remark 8.48], the authors note that there exist semisimple quasi-Hopf algebras such that their category of representations are *not* group theoretical: an explicit example is quoted there which comes from the construction of Tambara and Yamagami [1998]. The answer to the corresponding question for semisimple Hopf algebras is unknown.

4. Group theoretical quasi-Hopf algebras

The aim of this section is to give an explicit description, up to gauge equivalence, of the structure of group-theoretical quasi-Hopf algebras. This will enable us to explicitly compute the Frobenius–Schur indicators of group-theoretical categories in the next section. The description is based on a result of Schauenburg [2002a, 3.4], which reconstructs a quasi-bialgebra structure from certain monoidal categories of bimodules in a more general context.

Our main new result concerning this description is the explicit construction of the quasi-antipode in the group-theoretical case, which is relevant for our purposes; see Theorem 4.12.

An instance of this quasi-Hopf algebra construction, for the case where $\omega = 1$ and $\alpha = 1$, was studied in [Zhu 2001]. This case was also studied in [Beggs 2003; Beggs and Al-Shomrani 2003], from the point of view of the tensor categories of representations. We shall adopt throughout the notation in [Beggs 2003].

In what follows we shall fix a finite group G and a subgroup $F \subseteq G$. Following [Zhu 2001], we shall also fix a set of simultaneous representatives of the left and right cosets of F in G, $Q \subseteq G$; this is possible since G is finite. Thus every element $g \in G$ has unique factorizations $g = xq = py$, where $x, y \in F$, $q, p \in Q$. We assume that $e \in Q$.

The uniqueness of the factorization $G = FQ$ implies that there are well defined maps

$$\rhd : Q \times F \to F, \qquad \lhd : Q \times F \to Q,$$
$$. : Q \times Q \to Q, \qquad \theta : Q \times Q \to F,$$

determined by the conditions

$$qx = (q \rhd x)(q \lhd x), \quad q \in Q, \ x \in F;$$
$$pq = \theta(p, q)p.q, \qquad p, q \in Q.$$

Lemma 4.1 [Beggs 2003, Proposition 2.4]. *The following identities hold, for all* $p, q, r \in Q, x, y \in G$:

(i) $p \lhd xy = (p \lhd x) \lhd y, \ p \lhd e = p.$

(ii) $(p.q) \lhd x = (p \lhd (q \rhd x)).(q \lhd x).$

(iii) $p \rhd (q \rhd x) = \theta(p, q) ((p.q) \rhd x) \theta((p \lhd (q \rhd x), q \lhd x)^{-1}, e \rhd x = x.$

(iv) $p \rhd xy = (p \rhd x)((p \lhd x) \rhd y).$

(v) $\theta(p, q)\theta(p.q, r) = (p \rhd \theta(q, r))\theta(p \lhd \theta(q, r), q.r).$

(vi) $(p \lhd \theta(q, r)).(q.r) = (p.q).r.$

(vii) $\theta(p, e) = \theta(e, p) = e.$ ∎

Let $\omega : G \times G \times G \to k^\times$ be a normalized 3-cocycle such that $\omega|_{F \times F \times F}$ is trivial. In what follows we shall fix the group-theoretical category $\mathscr{C} = \mathscr{C}(G, \omega, F, 1)$. Thus the cochain $\alpha : F \times F \to k^\times$ of page 358 will be trivial. This is, up to monoidal equivalence, no loss of generality thanks to Remark 3.1.

Proposition 4.2. *There exists a normalized 2-cochain* $\eta : G \times G \to k^\times$ *such that* $\eta|_{F \times F} = 1$ *and* $\omega(d\eta)|_{F \times G \times G} = 1 = \omega(d\eta)|_{F \times F \times Q}.$

Proof. Recall that the coboundary $d\eta : G \times G \times G \to k^\times$ is given by

$$(d\eta)(a, b, c) = \eta(ab, c) \, \eta(a, b) \, \eta(b, c)^{-1} \, \eta(a, bc)^{-1},$$

for all $a, b, c \in G$.

The proof will be done in three steps. First let $\eta_1 : G \times G \to k^\times$ be the normalized cochain given by

$$\eta_1(xp, yq) := \omega(x, y, q), \qquad x, y \in F, \quad p, q \in Q.$$

Then we have $\eta_1|_{G \times F} = 1$ and for all $x, y, z \in F$, $q \in Q$, we have

$$(d\eta_1)(x, y, zq) = \eta_1(xy, zq)\, \eta_1(x, y)\, \eta_1(y, zq)^{-1}\, \eta_1(x, yzq)^{-1}$$
$$= \omega(xy, z, q)\, \omega(y, z, q)^{-1}\, \omega(x, yz, q)^{-1}\, \omega(x, y, z)^{-1}$$
$$= \omega(x, y, zq)^{-1},$$

the second equality holding because $\omega|_{F \times F \times F} = 1$. Thus $\omega(d\eta_1)|_{F \times F \times G} = 1$.

Put now $\omega_0 = \omega(d\eta_1)$ and define $\eta_2 : G \times G \to k^\times$ in the form

$$\eta_2(xp, yq) := \omega_0(x, p, yq)\, \omega_0(p, y, q)^{-1}, \qquad x, y \in F, \quad p, q \in Q.$$

Then $\eta_2|_{F \times G} = 1$ and we have

$$(d\eta_2)(x, yp, zq) = \eta_2(xyp, zq)\, \eta_2(x, yp)\, \eta_2(yp, zq)^{-1}\, \eta_2(x, ypzq)^{-1}$$
$$= \omega_0(xy, p, zq)\, \omega_0(p, z, q)\, \omega_0(p, z, q)^{-1}\, \omega_0(y, p, zq)^{-1}$$
$$= \omega_0(x, yp, zq)\, \omega_0(x, y, p)\, \omega_0(x, y, pzq)^{-1}$$
$$= \omega_0(x, yp, zq),$$

for all $x, y, z \in F$, $p, q \in Q$, where in the third and fourth equalities we have used that $\omega_0|_{F \times F \times G} = 1$. Hence $\omega_0(d\eta_2^{-1})|_{F \times G \times G} = 1$.

Finally, let $\omega_1 = \omega_0(d\eta_2^{-1})$. The condition $\omega_1|_{F \times G \times G} = 1$ is equivalent to $\omega_1(xt, g, h) = \omega_1(t, g, h)$, for all $x \in F$, $t, g, h \in G$. Hence

$$\omega_1(zp, x, yq) = \omega_1(p, x, y)\, \omega_1(p, xy, q)\, \omega_1(px, y, q)^{-1},$$

for all $z \in F$.

Let $\eta_3 : G \times G \to k^\times$ be defined by

$$\eta_3(xp, yq) := \omega_1(xp, y, q), \qquad x, y \in F, \quad p, q \in Q.$$

Then $\eta_3|_{F \times G} = \eta_3|_{G \times F} = 1$, and for all $x, y \in F$, $p, q \in Q$,

$$(d\eta_3)(p, x, yq) = \eta_3(px, yq)\, \eta_3(p, x)\, \eta_3(x, yq)^{-1}\, \eta_3(p, xyq)^{-1}$$
$$= \eta_3(px, yq)\, \eta_3(p, xyq)^{-1}$$
$$= \omega_1(px, y, q)\, \omega_1(p, xy, q)^{-1}$$
$$= \omega_1(p, x, y)\, \omega_1(p, x, yq)^{-1}.$$

In particular, $(d\eta_3)(p, x, y) = 1$, and thus

(4–1) $\omega_1(d\eta_3)(p, x, yq) = \omega_1(d\eta_3)(p, x, y).$

We claim that $\omega_1(d\eta_3)|_{F \times G \times G} = 1$. To see this, let $x, y, z \in F$, $p, q \in Q$. Using the equality $\omega_1|_{F \times G \times G} = 1$, we compute

$$
\begin{aligned}
(d\eta_3)(x, yp, zq) &= \eta_3(xyp, zq)\, \eta_3(x, yp)\, \eta_3(yp, zq)^{-1}\, \eta_3(x, ypzq)^{-1} \\
&= \omega_1(xyp, z, q)\, \omega_1(yp, z, q)^{-1} \\
&= \omega_1(p, z, q)\, \omega_1(p, z, q)^{-1} = 1,
\end{aligned}
$$

which proves the claim.

In view of this, equation (4–1) is equivalent to $\omega_1(d\eta_3)|_{G \times F \times Q} = 1$. This implies the proposition, since by construction $\omega_1(d\eta_3) = \omega(d\eta)$, for a suitable normalized 2-cochain such that $\eta|_{F \times F} = 1$.   □

By Remark 3.1, the property $\eta|_{F \times F} = 1$ in Proposition 4.2 implies that $\mathscr{C}(G, \omega, F, 1)$ is tensor equivalent to $\mathscr{C}(G, \omega(d\eta), F, 1)$. *Then we may and shall assume in what follows that the 3-cocycle* $\omega : G \times G \times G \to k^\times$ *satisfies the normalization conditions*

(4–2) $\omega|_{F \times G \times G} = 1,$

(4–3) $\omega|_{G \times F \times Q} = 1.$

These conditions are necessary in order to apply the results of [Schauenburg 2002a, 3.4]; see Definition 3.3.2 in the same paper.

Lemma 4.3. *Let* $g, h \in G$, $x, y \in F$, $p, q \in Q$.

 (i) $\omega(xp, g, h) = \omega(p, g, h).$

 (ii) $\omega(g, y, xp) = \omega(g, y, x).$

(iii) $\omega(g, x, pq) = \omega(g, x, \theta(p, q)).$

(iv) $\omega(pq, g, h) = \omega(p.q, g, h).$

Proof. Parts (i) and (ii) follow from the cocycle condition (3–1) and the normalization conditions (4–2) and (4–3). Parts (iii) and (iv) are a consequence of parts (i) and (ii), respectively.   □

Let $A = k^Q \#_\sigma kF$ be the crossed product corresponding to the action $\rightharpoonup : kF \otimes k^Q \to k^Q$ given by

$$(x \rightharpoonup f)(p) = f(p \triangleleft x) \quad \text{for } x \in F, \ f \in k^Q, \ p \in Q$$

and the invertible map $\sigma : F \times F \to (k^Q)^\times$ defined by $\sigma(x, y) = \sum_{p \in Q} \sigma_p(x, y)\delta_p$ for $x, y \in F$, where

(4–4) $\sigma_p(x, y) = \omega(p, x, y) \quad \text{for } x, y \in F, \ p \in Q.$

The normalized 3-cocycle condition (3–1) and the normalization assumption (4–2) imply the following normalized 2-cocycle condition for σ:

(4–5)
$$\sigma_{p \triangleleft x}(y, z)\sigma_p(x, yz) = \sigma_p(xy, z)\sigma_p(x, y),$$
$$\sigma_e(x, y) = \sigma_p(e, y) = \sigma_p(x, e) = 1,$$

for all $x, y, z \in F$, $p \in Q$. Thus A is an associative algebra with unit element $\sum_{p \in Q} \delta_p \otimes 1$. For $f \in k^Q$, $x \in F$, the element $f \otimes x \in A$ will be denoted by $f x$. Hence, for all $x, y \in F$, $p, q \in Q$, we have $(\delta_p x).(\delta_q y) := \delta_{p \triangleleft x, q}\sigma_p(x, y)\delta_p xy$.

Consider the (nonassociative) crossed product coalgebra structure on A corresponding to the action $\triangleright$ and the invertible normalized map $\tau : Q \times Q \to (k^F)^\times$ defined by $\tau(p, q) = \sum_{x \in F} \tau_x(p, q)\delta_x$, where

$$\tau_x(p, q) = \omega(p, q, x), \qquad x \in F, \ p, q \in Q.$$

Using (4–2) and again (3–1), we find that τ satisfies the "twisted" normalized 2-cocycle condition

$$\tau_x(p \triangleleft \theta(q,t),q.t)\,\tau_x(q,t)\,\omega(p,q,t)\,\sigma_p(\theta(q,t),q.t \triangleright x)\,\sigma_p\big(q \triangleright (t \triangleright x),\theta(q \triangleleft (t \triangleright x),t \triangleleft x)\big)^{-1}$$
$$= \tau_x(p.q,t)\,\tau_{t \triangleright x}(p,q)\,\omega(p \triangleleft (q \triangleright (t \triangleright x)),q \triangleleft (t \triangleright x),t \triangleleft x)$$

and $\tau_e(p, q) = \tau_x(e, q) = \tau_x(p, e) = 1$, for all $x \in F$, $p, q, t \in Q$.

Explicitly, we have

$$\Delta(\delta_p x) := \sum_{s.t=p} \tau_x(s, t)\,\delta_s(t \triangleright x) \otimes \delta_t x, \qquad p \in Q, \ x \in F.$$

The counit for this coalgebra is $\epsilon \otimes \epsilon$.

Theorem 4.4. *These algebra and coalgebra structures combine into a quasi-bialgebra structure on A^{op} with associator $\Phi \in A^{\otimes 3}$ given by*

$$\Phi = \sum_{p,q,r \in Q} \omega(p, q, r)\,\delta_p\theta(q, r) \otimes \delta_q \otimes \delta_r.$$

There is a monoidal equivalence $\mathrm{Rep}(A^{\mathrm{op}}, \Phi) \sim \mathscr{C}(G, F, \omega, 1)$.

Note that Φ is invertible, with inverse

$$\Phi^{-1} = \sum_{p,q,r \in Q} \omega(p, q, r)^{-1}\sigma_p\big(\theta(q, r), \theta(q, r)^{-1}\big)^{-1}\delta_{p \triangleleft \theta(q,r)}\theta(q, r)^{-1} \otimes \delta_q \otimes \delta_r.$$

Proof. Our definitions are dual to the ones in [Schauenburg 2002a, 3.4.2, 3.4.5]; with the conventions of that paper, Φ is replaced by Φ^{-1} in condition (2–1). Thus A^{op} is a quasi-bialgebra. The monoidal equivalence $\mathrm{Rep}(A^{\mathrm{op}}, \Phi) \sim \mathscr{C}(G, F, \omega, 1)$ follows from Corollary 3.4.4 in the same paper. $\square$

Since $\mathscr{C}(G, \omega, F, 1)$ is a rigid tensor category, it follows from [Schauenburg 2002b] that A^{op} is a quasi-Hopf algebra. We shall give the quasi-antipode in the next subsection.

By Remark 3.1, *every* group-theoretical category is equivalent to one of the form $\mathscr{C}(G, \omega, F, 1)$, for suitable G, F and ω, where ω satisfies (4–2), (4–3), in view of Proposition 4.2. This gives us the following theorem.

Theorem 4.5. *Let (H, ϕ) be a finite-dimensional quasi-Hopf algebra. Then (H, ϕ) is group-theoretical if and only if it is gauge equivalent to a quasi-Hopf algebra of the form (A^{op}, Φ), associated to suitable data G, F, Q and ω satisfying (4–2) and (4–3).* □

We shall use the symbol $\circ$ to denote the multiplication in A^{op}, so that $a \circ b = b.a$ for all $a, b \in A^{\mathrm{op}}$.

Remark 4.6. Using the properties listed in Lemma 4.1 and the normalization conditions (4–2) and (4–3), it is not difficult to check that A^{op} is a quasi-bialgebra. For instance, $\Delta : A \otimes A \to A$ is an algebra map because of parts (ii) and (iv) of Lemma 4.1 and the following relationship between σ and τ:

$$(4\text{–}6) \quad \sigma_{t.s}(x, y)\tau_{xy}(t, s) = \tau_x(t, s)\,\tau_y(t \triangleleft (s \triangleright x), s \triangleleft x)\,\sigma_t(s \triangleright x, (s \triangleleft x) \triangleright y)\,\sigma_s(x, y),$$

for all $s, t \in Q$, $x, y \in F$, which is a consequence of (4–3) and (3–1). Compare with [Masuoka 1999, Proposition 4.7].

Remark 4.7. Identify σ and τ with maps

$$\sigma : Q \times F \times F \to k^{\times}, \qquad \tau : Q \times Q \times F \to k^{\times}.$$

Then the tuple

$$(\Delta_G, 1, 1, . : Q \times Q \to Q, \triangleleft, \triangleright, \theta, \omega|_{Q \times Q \times Q}, \tau, \sigma)$$

constitutes the *skeleton* of (kG, ω) [Schauenburg 2002a, Definition 4.1.1].

We now give the construction of a quasi-antipode for A^{op}. We shall need the existence of inverses for the (nonassociative) multiplication in Q. This is guaranteed by the next lemma.

Lemma 4.8. *The set Q has well-defined left and right inverses with respect to the multiplication . ; that is, for every $p \in Q$ there exist unique p^L, $p^R \in Q$ such that $p^L.p = e = p.p^R$.*

(By definition, $pp^R = \theta(p, p^R)$ and $p^L p = \theta(p^L, p)$.)

Proof. As to left inverses, the lemma is contained in [Beggs 2003, Proposition 2.3]. To prove the statement concerning right inverses, we shall use the assumption that Q is also a set of representatives of the left cosets of F in G.

Let $p \in Q$. Since the factorization $G = QF$ is exact, there exist unique $s \in Q$, $x \in F$, such that $p^{-1} = sx$. Then $e = pp^{-1} = psx = \theta(p, s)(p.s)x$; thus

$$\theta(p, s)^{-1} = (p.s)x.$$

Because $p.s \in Q$ and $\theta(p, s)^{-1}, x \in F$, the exactness of the factorization $G = QF$ implies that $p.s = e$.

We now show the uniqueness of such s, which gives the statement with $s = p^R$. Suppose that $s' \in Q$ is such that $p.s' = e$. Then ps', $ps \in F$, and therefore also $(s')^{-1}s = (ps')^{-1}ps \in F$. This implies that $s' = s$, whence the uniqueness. $\square$

For later use, we give in the next lemma some of the relations between $(\ )^L$, $(\ )^R$ and the actions $\triangleright$, $\triangleleft$. The content of the lemma is part of [Beggs 2003, Section 4].

Lemma 4.9. *The following relations hold, for all $p \in Q$:*

(i) $p^{-1} = \theta(p^L, p)^{-1}p^L = p^R\theta(p, p^R)^{-1}$.

(ii) $p^L \triangleleft \theta(p, p^R) = p^R$ *and* $p^L \triangleright \theta(p, p^R) = \theta(p^L, p)$.

(iii) $p^{LL} = p \triangleleft \theta(p^L, p)^{-1}$.

(iv) $(p \triangleleft x)^L = p^L \triangleleft (p \triangleright x)$.

Proof. This follows from the definitions and Lemma 4.1. $\square$

Definition 4.10. We introduce for notational convenience the map $\succ : Q \times F \to F$ defined by

$$p \succ x = \theta(p, p^R)^{-1} (p \triangleright x) \theta(p \triangleleft x, (p \triangleleft x)^R),$$

for all $p \in Q, x \in F$.

Lemma 4.11. *Let $p \in Q$ and $x, y \in F$.*

(i) $p \succ (p^R \triangleright x) = x = p^R \triangleright (p \succ x)$.

(ii) $(p \triangleleft x)^R \triangleleft (p \succ x)^{-1} = p^R$.

(iii) $p \succ (xy) = (p \succ x)((p \triangleleft x) \succ y)$.

(iv) $p \succ \theta(p^L, p)^{-1} = \theta(p, p^R)^{-1}$.

In particular, for all $p \in Q$ the map $p \succ \underline{\ \ } : F \to F$ is bijective, the inverse being $p^R \triangleright \underline{\ \ } : F \to F$.

Proof. We shall prove part (iv), the proof of (i)–(iii) being straightforward. By Lemma 4.9(iii), we have $p\theta(p^L, p)^{-1} = (p \triangleright \theta(p^L, p)^{-1})p^{LL}$, and on the other hand, $p\theta(p^L, p)^{-1} = (p^L)^{-1} = \theta(p^{LL}, p^L)^{-1}p^{LL}$.

By the exactness of the factorization $G = FQ$, we get

$$\theta(p^{LL}, p^L)^{-1} = p \triangleright \theta(p^L, p)^{-1}.$$

From this and the definitions, we get

$$p \succ \theta(p^L, p)^{-1} = \theta(p, p^R)^{-1} (p \triangleright \theta(p^L, p)^{-1}) \theta(p^{LL}, p^L) = \theta(p, p^R)^{-1} \quad \square$$

Theorem 4.12. *There is a quasi-Hopf algebra structure on A^{op} with quasi-antipode $\mathcal{S} : A^{\mathrm{op}} \to A^{\mathrm{op}}$ given by*

$$\mathcal{S}(\delta_p x) = \tau_{p \succ x}(p, p^R)^{-1} \sigma_{p^R}(p \succ x, (p \succ x)^{-1})^{-1} \delta_{(p \triangleleft x)^R}(p \succ x)^{-1}.$$

We have $\alpha = 1$ and

$$\beta = \sum_{q \in Q} \omega(q^{-1}, q, q^{-1}) \, \delta_q \theta(q^L, q)^{-1} = \sum_{q \in Q} \omega(q, q^{-1}, q)^{-1} \, \delta_q \theta(q^L, q)^{-1}.$$

Compare with the formulas given in [Zhu 2001; Beggs 2003] for the case where $\omega = 1$.

Proof. We use freely the relations in Lemma 4.3 and the normalization conditions (4–2) and (4–3). Using relation (4–6) and Lemma 4.11, it is straightforward to see that $\mathcal{S}$ is an anti-algebra map. The injectivity of $\mathcal{S}$ follows from the injectivity of the map $p \succ \_\_ : F \to F$ and from Lemma 4.11(ii). Therefore $\mathcal{S} : A^{\mathrm{op}} \to A^{\mathrm{op}}$ is an algebra anti-automorphism.

We now check condition (2–2). Let $p \in Q$, $x \in F$, and let $X = \delta_p x \in A^{\mathrm{op}}$. Then

$$\mathcal{S}(X_1) \circ \alpha \circ X_2 = \mathcal{S}(X_1) \circ X_2 = X_2 . \mathcal{S}(X_1)$$

$$= \sum_{s.t=p} \tau_x(s, t) \, \delta_t x \, . \mathcal{S}(\delta_s(t \triangleright x))$$

$$= \sum_{s.t=p} \tau_x(s, t) \, \tau_{s \succ (t \triangleright x)}(s, s^R)^{-1} \, \sigma_{s^R}(s \succ (t \triangleright x), (s \succ (t \triangleright x))^{-1})^{-1}$$
$$\times \delta_t x . \delta_{(s \triangleleft (t \triangleright x))^R}(s \succ (t \triangleright x))^{-1}$$

$$= \sum_{s.t=p} \tau_x(s, t) \, \tau_{s \succ (t \triangleright x)}(s, s^R)^{-1} \, \sigma_{s^R}(s \succ (t \triangleright x), (s \succ (t \triangleright x))^{-1})^{-1}$$
$$\times \delta_{t \triangleleft x, (s \triangleleft (t \triangleright x))^R} \, \sigma_t(x, (s \succ (t \triangleright x))^{-1}) \, \delta_t \, x(s \succ (t \triangleright x))^{-1}.$$

By Lemma 4.9(iv), we have $\delta_{t \triangleleft x, (s \triangleleft (t \triangleright x))^R} = \delta_{s, t^L}$. Hence, using Lemma 4.11(i), the last expression equals

$$\delta_{p,e} \sum_{s \in Q} \tau_x(s, s^R) \, \tau_{s \succ (s^R \triangleright x)}(s, s^R)^{-1} \, \sigma_{s^R}(s \succ (s^R \triangleright x), (s \succ (s^R \triangleright x))^{-1})^{-1}$$
$$\times \sigma_{s^R}(x, (s \succ (s^R \triangleright x))^{-1}) \, \delta_{s^R} \, x(s \succ (s^R \triangleright x))^{-1}$$

$$= \delta_{p,e} \sum_{s \in Q} \delta_{s^R} = \delta_{p,e} 1 = \delta_{p,e} \alpha.$$

This proves the right-hand identity in (2–2). We now compute

$$X_1 \circ \beta \circ \mathcal{S}(X_2)$$
$$= \mathcal{S}(X_2).\beta.X_1$$
$$= \sum_{s.t=p} \tau_x(s,t)\, \mathcal{S}(\delta_t x)\, .\beta\, .\delta_s(t \rhd x)$$
$$= \sum_{s.t=p} \sum_q \omega(q, q^{-1}, q)^{-1}\, \tau_x(s,t)\, \mathcal{S}(\delta_t x)\, .\delta_q \theta(q^L, q)^{-1}\, .\delta_s(t \rhd x)$$
$$= \sum_{s.t=p} \sum_q \omega(q, q^{-1}, q)^{-1}\, \tau_x(s,t)\, \tau_{t \succ x}(t, t^R)^{-1}\, \sigma_{t^R}(t \succ x, (t \succ x)^{-1})^{-1}$$
$$\times \delta_{(t \lhd x)^R}(t \succ x)^{-1}.\delta_q \theta(q^L, q)^{-1}.\delta_s(t \rhd x)$$
$$= \sum_{s.t=p} \sum_q \omega(q, q^{-1}, q)^{-1}\, \tau_x(s,t)\, \tau_{t \succ x}(t, t^R)^{-1}\, \sigma_{t^R}(t \succ x, (t \succ x)^{-1})^{-1}$$
$$\times \delta_{(t \lhd x)^R \lhd (t \succ x)^{-1}, q}\, \sigma_{(t \lhd x)^R}((t \succ x)^{-1}, \theta(q^L, q)^{-1})$$
$$\times \delta_{(t \lhd x)^R}(t \succ x)^{-1}\theta(q^L, q)^{-1}\, \delta_s(t \rhd x).$$

By Lemma 4.11(ii), this equals

$$\sum_{s.t=p} \omega(t^R, (t^R)^{-1}, t^R)^{-1}\, \tau_x(s,t)\, \tau_{t \succ x}(t, t^R)^{-1}\, \sigma_{t^R}(t \succ x, (t \succ x)^{-1})^{-1}$$
$$\times \sigma_{(t \lhd x)^R}((t \succ x)^{-1}, \theta(t, t^R)^{-1})\, \delta_{(t \lhd x)^R}(t \succ x)^{-1}\theta(t, t^R)^{-1}\, \delta_s(t \rhd x),$$

or again

$$\sum_{s.t=p} \omega(t^R, (t^R)^{-1}, t^R)^{-1}\, \tau_x(s,t)\, \tau_{t \succ x}(t, t^R)^{-1}\, \sigma_{t^R}(t \succ x, (t \succ x)^{-1})^{-1}$$
$$\times \sigma_{(t \lhd x)^R}((t \succ x)^{-1}, \theta(t, t^R)^{-1})\, \sigma_{(t \lhd x)^R}((t \succ x)^{-1}\theta(t, t^R)^{-1}, t \rhd x)$$
$$\times \delta_{(t \lhd x)^R \lhd (t \succ x)^{-1}\theta(t, t^R)^{-1}, s}\, \delta_{(t \lhd x)^R}(t \succ x)^{-1}\theta(t, t^R)^{-1}(t \rhd x),$$

which in turn, by Lemma 4.11(ii) and Definition 4.10, is equal to

$$\delta_{p,e} \sum_t \omega(t^R, (t^R)^{-1}, t^R)^{-1}\, \tau_x(t^L, t)\, \tau_{t \succ x}(t, t^R)^{-1}$$
$$\times \sigma_{t^R}(t \succ x, (t \succ x)^{-1})^{-1}\, \sigma_{(t \lhd x)^R}((t \succ x)^{-1}, \theta(t, t^R)^{-1})$$
$$\times \sigma_{(t \lhd x)^R}((t \succ x)^{-1}\theta(t, t^R)^{-1}, t \rhd x)\, \delta_{(t \lhd x)^R}\, \theta(t \lhd x, (t \lhd x)^R)^{-1}.$$

We now evaluate the product

$$\sigma_{t^R}(t \succ x, (t \succ x)^{-1})^{-1}\, \sigma_{(t \lhd x)^R}((t \succ x)^{-1}, \theta(t, t^R)^{-1})\, \sigma_{(t \lhd x)^R}((t \succ x)^{-1}\theta(t, t^R)^{-1}, t \rhd x)$$

appearing in the last sum. We use the cocycle condition (4–5) and Lemma 4.9 and obtain for it

$$\sigma_{t^R}(\theta(t,t^R)^{-1}, t \triangleright x)\,\sigma_{t^R}(t \succ x, \theta(t \triangleleft x, (t \triangleleft x)^R)^{-1})^{-1}$$

$$= \omega(t^R, \theta(t,t^R)^{-1}, t \triangleright x)$$
$$\times \omega(t^R, \theta(t,t^R)^{-1}(t \triangleright x)\theta(t \triangleleft x, (t \triangleleft x)^R), \theta(t \triangleleft x, (t \triangleleft x)^R)^{-1})^{-1}$$
$$= \omega(t^R, \theta(t,t^R)^{-1}, t \triangleright x)\,\omega((t \triangleleft x)^L, \theta(t \triangleleft x, (t \triangleleft x)^R), \theta(t \triangleleft x, (t \triangleleft x)^R)^{-1})^{-1}$$
$$\times \omega(t^R, \theta(t,t^R)^{-1}(t \triangleright x), \theta(t \triangleleft x, (t \triangleleft x)^R))),$$
$$= \omega((t \triangleleft x)^L, \theta(t \triangleleft x, (t \triangleleft x)^R), \theta(t \triangleleft x, (t \triangleleft x)^R)^{-1})^{-1}$$
$$\times \omega(t^L, t \triangleright x, \theta(t \triangleleft x, (t \triangleleft x)^R))\,\omega(t^R, \theta(t,t^R)^{-1}, (t \triangleright x)\theta(t \triangleleft x, (t \triangleleft x)^R))$$
$$= \omega((t \triangleleft x)^L, \theta(t \triangleleft x, (t \triangleleft x)^R), \theta(t \triangleleft x, (t \triangleleft x)^R)^{-1})^{-1}$$
$$\times \omega((t \triangleleft x)^L, t \triangleleft x, (t \triangleleft x)^R)^{-1}\,\omega(t^L, t, x)^{-1}\,\omega(t^L, t, x(t \triangleleft x)^R)$$
$$\times \omega(t^R, \theta(t,t^R)^{-1}, (t \triangleright x)\theta(t \triangleleft x, (t \triangleleft x)^R))),$$

where we have used the definition of σ (4–4), the cocycle condition on ω, and Lemma 4.3(iii).

On the other hand,

$$\omega((t \triangleleft x)^L, \theta(t \triangleleft x, (t \triangleleft x)^R), \theta(t \triangleleft x, (t \triangleleft x)^R)^{-1})^{-1}\,\omega((t \triangleleft x)^L, t \triangleleft x, (t \triangleleft x)^R)^{-1}$$
$$= \omega((t \triangleleft x)^R, \theta(t \triangleleft x, (t \triangleleft x)^R)^{-1}, (t \triangleleft x)(t \triangleleft x)^R)^{-1}\,\omega((t \triangleleft x)^L, t \triangleleft x, (t \triangleleft x)^R)^{-1}$$
$$= \omega((t \triangleleft x)^{-1}, t \triangleleft x, (t \triangleleft x)^R)\,\omega((t \triangleleft x)^R, ((t \triangleleft x)^R)^{-1}, (t \triangleleft x)^R)^{-1}$$
$$\times \omega((t \triangleleft x)^L, t \triangleleft x, (t \triangleleft x)^R)^{-1}$$
$$= \omega((t \triangleleft x)^R, ((t \triangleleft x)^R)^{-1}, (t \triangleleft x)^R)^{-1}.$$

Therefore

$$\omega(t^R, (t^R)^{-1}, t^R)^{-1}\,\tau_x(t^L, t)\,\tau_{t \succ x}(t, t^R)^{-1}\,\sigma_{t^R}(t \succ x, (t \succ x)^{-1})^{-1}$$
$$\times \sigma_{(t \triangleleft x)^R}((t \succ x)^{-1}, \theta(t,t^R)^{-1})\,\sigma_{(t \triangleleft x)^R}((t \succ x)^{-1}\theta(t,t^R)^{-1}, t \triangleright x)$$
$$= \omega(t^R, (t^R)^{-1}, t^R)^{-1}\,\omega(t, t^R, t \succ x)^{-1}\,\omega((t \triangleleft x)^R, ((t \triangleleft x)^R)^{-1}, (t \triangleleft x)^R)^{-1}$$
$$\times \omega(t^{-1}, t, x(t \triangleleft x)^R)\,\omega(t^R, \theta(t,t^R)^{-1}, (t \triangleright x)\theta(t \triangleleft x, (t \triangleleft x)^R))$$
$$= \omega(t^R, (t^R)^{-1}, t^R)^{-1}\,\omega(t, t^R\theta(t,t^R)^{-1}, (t \triangleright x)\theta(t \triangleleft x, (t \triangleleft x)^R))^{-1}$$
$$\times \omega(t, t^R, \theta(t,t^R)^{-1})^{-1}\,\omega((t \triangleleft x)^R, ((t \triangleleft x)^R)^{-1}, (t \triangleleft x)^R)^{-1}$$
$$\times \omega(t^L, t, x(t \triangleleft x)^R)$$
$$= \omega(t^R, (t^R)^{-1}, t^R)^{-1}\,\omega(t, t^{-1}, tx(t \triangleleft x)^R)^{-1}\,\omega(t, t^R, \theta(t,t^R)^{-1})^{-1}$$
$$\times \omega((t \triangleleft x)^R, ((t \triangleleft x)^R)^{-1}, (t \triangleleft x)^R)^{-1}\,\omega(t^{-1}, t, x(t \triangleleft x)^R)$$
$$= \omega(t^R, (t^R)^{-1}, t^R)^{-1}\,\omega(t, t^{-1}, t)^{-1}\,\omega(t^{-1}, t, x(t \triangleleft x)^R)^{-1}\,\omega(t, t^R, \theta(t,t^R)^{-1})^{-1}$$
$$\times \omega((t \triangleleft x)^R, ((t \triangleleft x)^R)^{-1}, (t \triangleleft x)^R)^{-1}\,\omega(t^L, t, x(t \triangleleft x)^R)$$
$$= \omega(t^R, (t^R)^{-1}, t^R)^{-1}\,\omega(t, t^R\theta(t,t^R)^{-1}, t)^{-1}\,\omega(t, t^R, \theta(t,t^R)^{-1})^{-1}$$
$$\times \omega((t \triangleleft x)^R, ((t \triangleleft x)^R)^{-1}, (t \triangleleft x)^R)^{-1}$$

$$= (\omega(t^R, (t^R)^{-1}, t^R) \omega(t, t^R, \theta(t, t^R)^{-1} t) \omega((t \triangleleft x)^R, ((t \triangleleft x)^R)^{-1}, (t \triangleleft x)^R))^{-1}$$

$$= \omega(t^R, (t^R)^{-1}, t^R)^{-1}$$
$$\times \omega((t^R)^{-1}, t^R, (t^R)^{-1})^{-1} \, \omega((t \triangleleft x)^R, ((t \triangleleft x)^R)^{-1}, (t \triangleleft x)^R)^{-1}$$

$$= \omega((t \triangleleft x)^R, ((t \triangleleft x)^R)^{-1}, (t \triangleleft x)^R)^{-1}.$$

Hence we get

$$X_1 \circ \beta \circ \mathcal{S}(X_2) = \delta_{p,e} \sum_t \omega((t \triangleleft x)^R, ((t \triangleleft x)^R)^{-1}, (t \triangleleft x)^R)^{-1} \delta_{(t \triangleleft x)^R} \, \theta(t \triangleleft x, (t \triangleleft x)^R)^{-1}$$
$$= \delta_{p,e} \beta,$$

which gives the right-hand identity in (2–2).

The proof of conditions (2–3) is straightforward, using the properties listed in Lemma 4.9 and the cocycle conditions. This finishes the proof of the theorem. $\square$

5. Frobenius–Schur indicators for $\mathcal{C}(G, \omega, F, 1)$

Let G be a finite group, $F \subseteq G$ a subgroup, and $\omega : G \times G \times G \to k^\times$ a 3-cocycle subject to the normalization conditions (4–2) and (4–3). We keep the notation of the previous sections for the skeleton maps $\triangleright, \triangleleft, \theta, \sigma$ and τ.

We have a k-linear monoidal equivalence $\mathrm{Rep}(A^{\mathrm{op}}, \Phi) \sim \mathcal{C}(G, F, \omega, 1)$, where (A^{op}, Φ) is the quasi-Hopf algebra attached to the data (G, F, ω) in Section 4. Our aim in this section is to give an explicit description of the canonical central element $\nu_{A^{\mathrm{op}}} \in A^{\mathrm{op}}$ and then of the Frobenius–Schur indicators for the quasi-Hopf algebra A^{op}. It follows from gauge invariance of the Frobenius–Schur indicators that these depend only on the fusion category $\mathcal{C}(G, F, \omega, 1)$.

Let $\Lambda_0 := |F|^{-1} \sum_{x \in F} x \in kF$ be the normalized integral. The normalized two-sided integral $\Lambda \in A^{\mathrm{op}}$ has the form

$$(5\text{--}1) \qquad \Lambda = \delta_e \Lambda_0 = |F|^{-1} \sum_{x \in F} \delta_e x.$$

Proposition 5.1. *The element β is invertible with inverse*

$$\beta^{-1} = \sum_{p \in Q} \omega(p^L, p, p^R) \, \delta_{p^L} \theta(p, p^R).$$

In addition, $\mathcal{S}(\beta) = \beta^{-1}$.

Proof. It is not difficult to check that the expression

$$\sum_{p \in Q} \omega(p^R, (p^R)^{-1}, p^R) \, \sigma_{p^R}(\theta(p, p^R)^{-1}, \theta(p, p^R))^{-1} \, \delta_{p^L} \theta(p, p^R)$$

defines an inverse for β. We claim that

$$\omega(p^R, (p^R)^{-1}, p^R)\,\sigma_{p^R}(\theta(p, p^R)^{-1}, \theta(p, p^R))^{-1} = \omega(p^L, p, p^R),$$

for all $p \in Q$. This will imply the claimed expression for β^{-1}.

Letting $q = p^R$, the equality of two lines above is equivalent to

$$\omega(q, q^{-1}, q)\,\sigma_q(\theta(q^L, q)^{-1}, \theta(q^L, q))^{-1} = \omega(q \triangleleft \theta(q^L, q)^{-1}, q^L, q).$$

To establish this latter equality, we note that $\theta(q^L, q) = q^L q \in F$, for all $q \in Q$. Then, applying the cocycle and normalization conditions on ω, we get

$$\begin{aligned}
\omega(q \triangleleft \theta(q^L, q)^{-1}, q^L, q) &= \omega(q, q^{-1}, q)\,\omega(q, \theta(q^L, q)^{-1}, q^L q)^{-1}\\
&= \omega(q, q^{-1}, q)\,\omega(q, \theta(q^L, q)^{-1}, \theta(q^L, q))^{-1},
\end{aligned}$$

proving the claim.

Next, using Lemma 4.11(iv), we get

$$\mathscr{S}(\delta_p\theta(p^L, p)^{-1}) = \tau_{\theta(p, p^R)^{-1}}(p, p^R)^{-1}\,\sigma_{p^R}(\theta(p, p^R)^{-1}, \theta(p, p^R))^{-1}\delta_{p^L}\theta(p, p^R).$$

We now compute

$$\begin{aligned}
\tau_{\theta(p, p^R)^{-1}}(p, p^R) &= \omega(p, p^R, (pp^R)^{-1})\\
&= \omega(pp^R, (p^R)^{-1}, p^{-1})^{-1}\,\omega(p, p^R, (p^R)^{-1})\,\omega(p^R, (p^R)^{-1}, p^{-1})\\
&= \omega(p, p^R, (p^R)^{-1})\,\omega(p^R, (p^R)^{-1}, p^{-1}),
\end{aligned}$$

because $pp^R \in F$. Similarly,

$$\begin{aligned}
\sigma_{p^R}&(\theta(p, p^R)^{-1}, \theta(p, p^R))\\
&= \omega(p^R, (p^R)^{-1}p^{-1}, pp^R)\\
&= \omega(p^R, (p^R)^{-1}, p^{-1})^{-1}\,\omega((p^R)^{-1}, p^{-1}, pp^R)^{-1}\,\omega(p^R, (p^R)^{-1}, p^R)\\
&= \omega(p^R, (p^R)^{-1}, p^R)\,\omega(p, p^{-1}, (pp^R))^{-1}.
\end{aligned}$$

Hence

$$\begin{aligned}
\tau_{\theta(p, p^R)^{-1}}(p, p^R)\,\sigma_{p^R}(\theta(p, p^R)^{-1}, \theta(p, p^R)) &= \omega(p, p^{-1}, p)^{-1}\,\omega(p^{-1}, p, p^R)^{-1}\\
&= \omega(p, p^{-1}, p)^{-1}\,\omega(p^L, p, p^R)^{-1}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathscr{S}(\beta) &= \sum_p \omega(p^{-1}, p, p^{-1})\,\mathscr{S}(\delta_p\theta(p^L, p)^{-1})\\
&= \sum_p \omega(p^L, p, p^R)\,\delta_{p^L}\theta(p, p^R) = \beta^{-1}. \qquad \square
\end{aligned}$$

Theorem 5.2. *The canonical central element $\nu_{A^{\mathrm{op}}}$ is given by*

$$(5\text{–}2) \qquad \nu_{A^{\mathrm{op}}} = |F|^{-1} \sum_{(q \triangleleft x).q=e} \omega(xq, xq, xq)\, \delta_q(xq)^2.$$

Moreover,

$$\nu_{A^{\mathrm{op}}} = |F|^{-1} \sum_{(q \triangleleft x).q=e} \tau_x(q \triangleleft x, q)\, \sigma_q(x, q \triangleright x)\, \omega((q \triangleleft x)^L, q \triangleleft x, q)$$
$$\times \sigma_q(x(q \triangleright x), (q \triangleleft x)q)\, \delta_q(xq)^2.$$

Proof. Since the element β corresponding to the quasi-antipode of A^{op} is invertible and $\alpha = 1$, we have $\nu_{A^{\mathrm{op}}} = (\Lambda_1 \circ \Lambda_2) \circ \beta^{-1} = \beta^{-1} \circ (\Lambda_1 \circ \Lambda_2)$, and thus

$$\nu_{A^{\mathrm{op}}} = \beta^{-1} \circ (\Lambda_1 \circ \Lambda_2) = (\Lambda_1 \circ \Lambda_2).\beta^{-1}.$$

Using formula (5–1) for the integral Λ, we find

$$\Delta(\Lambda) = \Lambda_1 \otimes \Lambda_2 = |F|^{-1} \sum_{x \in F} \sum_{q \in Q} \tau_x(q^L, q) \quad \delta_{q^L}(q \triangleright x) \otimes \delta_q x;$$

thus

$$\Lambda_1 \circ \Lambda_2 = \Lambda_2.\Lambda_1 = |F|^{-1} \sum_{(q \triangleleft x).q=e} \tau_x(q \triangleleft x, q)\, \sigma_q(x, q \triangleright x)\, \delta_q x(q \triangleright x).$$

From these, we compute $\beta^{-1} \circ (\Lambda_1 \circ \Lambda_2)$ and get the second expression for $\nu_{A^{\mathrm{op}}}$. Now, for all $x \in F$ and $q \in Q$ such that $(q \triangleleft x).q = e$, we have

$$\sigma_q(x, q \triangleright x)\, \sigma_q(x(q \triangleright x), (q \triangleleft x)q)$$
$$= \sigma_{q \triangleleft x}(q \triangleright x, (q \triangleleft x)q)\, \sigma_q(x, qxq)$$
$$= \omega(q \triangleleft x, q \triangleright x, (q \triangleleft x)q)\, \omega(q, x, qxq)$$
$$= \omega(q \triangleleft x, q \triangleright x, (q \triangleleft x)q)\, \omega(q \triangleleft x, q, xq)^{-1}\, \omega(q, xq, xq)$$
$$= \omega(q \triangleleft x, q \triangleright x, (q \triangleleft x)q)\, \omega(q \triangleleft x, q, x)^{-1}\, \omega(qx, qx, q)^{-1}\, \omega(q, xq, xq),$$

the last equality because

$$\omega(q \triangleleft x, q, xq) = \omega(q \triangleleft x, q, x)\, \omega(q \triangleleft x, qx, q) = \omega(q \triangleleft x, q, x)\, \omega(qx, qx, q).$$

On the other hand,

$$\omega((q \triangleleft x)^L, q \triangleleft x, q) = \omega(q^L \triangleleft (q \triangleright x), q \triangleleft x, q)$$
$$= \omega(q^L, qx, q)\, \omega(q^L, q \triangleright x, (q \triangleleft x)q)^{-1}$$
$$= \omega(q \triangleleft x, qx, q)\, \omega(q \triangleleft x, q \triangleright x, (q \triangleleft x)q)^{-1}$$
$$= \omega(qx, qx, q)\, \omega(q \triangleleft x, q \triangleright x, (q \triangleleft x)q)^{-1}.$$

Therefore, as needed for (5–2),

$$\tau_x(q \triangleleft x, q)\, \sigma_q(x, q \triangleright x)\, \omega((q \triangleleft x)^L, q \triangleleft x, q)\, \sigma_q(x(q \triangleright x), (q \triangleleft x)q)$$
$$= \omega(q, xq, xq) = \omega(xq, xq, xq). \quad \square$$

Remark 5.3. Computing instead $(\Lambda_1 \circ \Lambda_2) \circ \beta^{-1}$, we get

$$\nu_{A^{\mathrm{op}}} = |F|^{-1} \sum_{(q \triangleleft x).q=e} \tau_x(q \triangleleft x, q)\, \sigma_q(x, q \triangleright x)\, \omega((q \triangleleft x)^L, q \triangleleft x, q)$$
$$\times \omega((q \triangleleft x)^L, (q \triangleleft x)q, x(q \triangleright x))\, \delta_{(q \triangleleft x)^L}((q \triangleleft x)(q \triangleright x))^2.$$

As a consequence of Theorem 5.2, we get the following expression for the Frobenius–Schur indicators. After suitable normalization, this expression allows to compute the Frobenius–Schur indicators for every group-theoretical category.

Corollary 5.4. *Suppose χ is an irreducible character of A^{op}. The Frobenius–Schur indicator of χ is given by*

$$\chi(\nu_{A^{\mathrm{op}}}) = |F|^{-1} \sum_{(q \triangleleft x).q=e} \omega(xq, xq, xq) \quad \chi(\delta_q(xq)^2)$$
$$= |F|^{-1} \sum_{(q \triangleleft x).q=e} \tau_x(q \triangleleft x, q)\, \sigma_q(x, q \triangleright x)\, \omega((q \triangleleft x)^L, q \triangleleft x, q)$$
$$\times \sigma_q(x(q \triangleright x), (q \triangleleft x)q)\, \chi(\delta_q(xq)^2). \quad \square$$

In the remainder of this section we aim to give an explicit description of the irreducible characters (and hence of the indicators) of $\mathscr{C}(G, \omega, F, 1)$ in terms of the groups G and F.

As an algebra $A = k^Q \#_\sigma kF$ is a crossed product (page 362). Hence the irreducible left A-modules can be described using Clifford theory.

On the other hand, to every left A-module V one can associate the left A^{op}-module V^*, the action of $a \in A^{\mathrm{op}}$ being the transpose of the action of $a \in A$ on V. This gives a bijective correspondence between (irreducible) left A-modules V and (irreducible) left A^{op}-modules. Moreover, this bijection preserves characters: $\chi_{V^*} = \chi_V$, for all finite-dimensional left A-module V.

Let $F^p \subseteq F$ denote the isotropy subgroup of $p \in Q$. Then the restriction of σ_p defines a normalized 2-cocycle

$$\sigma_p : F^p \times F^p \to k^\times.$$

Let $k_{\sigma_p} F^p$ denote the corresponding twisted group algebra.

The space of isomorphism classes of irreducible A-modules can be parametrized by the modules $V_{p,W}$, where

$$V_{p,W} = \mathrm{Ind}_{k^Q \#_\sigma kF^p}\, p \otimes W = A \otimes_{k^Q \#_\sigma kF^p} (p \otimes W),$$

where p runs over a set of representatives of the action of F on Q, and W runs over a system of representatives of isomorphism classes of irreducible left $k_{\sigma_p} F^p$-modules. See [Kashina et al. 2002, Section 3].

There is a natural identification between Q and the space $F\backslash G = \{Fg : g \in G\}$ of left cosets of F in G. Under this identification, the action $Q \times F \to Q$ corresponds to the natural action of F on $F\backslash G$ by right multiplication: $Fg.x = F(gx)$, $g \in G$, $x \in F$.

This gives in turn a natural identification between the space of orbits of the action $Q \times F \to Q$ and the space $F\backslash G/F$ of double cosets of F in G. Moreover, the isotropy subgroup of an element $p \in Q$ is $F^p = F \cap p^{-1}Fp$. Hence we get

Proposition 5.5. *The set of isomorphism classes of irreducible A^{op}-modules is parametrized by the modules $U_{p,W}$, where*

$$U_{p,W} = V_{p,W}^* = (\mathrm{Ind}_{k\varrho\#_\sigma kF^p} \, p \otimes W)^*,$$

where p runs over a set of representatives of the double cosets of F in G, $F^p = F \cap p^{-1}Fp$, and W runs over a system of representatives of isomorphism classes of irreducible left $k_{\sigma_p} F^p$-modules.

The character of the irreducible A^{op}-module $U_{p,W}$ is given by the formula

$$(5\text{–}3) \qquad \chi_{p,W}(\delta_q z) = \sum_{y^{-1}zy \in F^p} \delta_{p, q \triangleleft y} \, \sigma_q(z, y) \, \sigma_q(y, y^{-1}zy)^{-1} \, \chi_W(y^{-1}zy),$$

where the sum is over all y running over a set of representatives of the right cosets of F^p in F, and χ_W is the character of W.

Observe that $\dim U_{p,W} = [F : F \cap p^{-1}Fp]\dim W$. So the proposition immediately implies that the dimensions of the irreducible modules of a group-theoretical quasi-Hopf algebra divide its dimension, *i.e.*, that Kaplanksy's conjecture holds in this case.

Proof. We only need to prove the formula for the character. The character of $U_{p,W}$ coincides with the character of $V_{p,W}$. Let Y be a set of representatives of the right cosets of F^p in F. A basis of $V_{p,W}$ is given by $y \otimes p \otimes v$, where (v) is a basis of W, and $y \in Y$.

For all $q \in Q$, $y, z \in F$, we have

$$\begin{aligned}
(\delta_q z).y &= \sigma_q(z, y) \, \delta_q zy \\
&= \sigma_q(z, y) \, \delta_q y(y^{-1}zy) \\
&= \sigma_q(z, y) \, \sigma_q(y, y^{-1}zy)^{-1} \, y.(\delta_{q \triangleleft y} y^{-1}zy).
\end{aligned}$$

Hence, the action of $\delta_q z$ on this basis is

$$
\begin{aligned}
(\delta_q z).y \otimes p \otimes v &= (\delta_q z).y \otimes p \otimes v \\
&= \sigma_q(z, y)\,\sigma_q(y, y^{-1}zy)^{-1}\,y.(\delta_{q \lhd y}\,y^{-1}zy) \otimes p \otimes v.
\end{aligned}
$$

Thus, in order to compute the trace of this action, we only need to consider those basis vectors $y \otimes p \otimes v$, for which $y^{-1}zy \in F^p$; and for such y, we have

$$
\begin{aligned}
(\delta_q z).y \otimes p \otimes v &= \sigma_q(z, y)\,\sigma_q(y, y^{-1}zy)^{-1}\,y(\delta_{q \lhd y}\,y^{-1}zy) \otimes p \otimes v \\
&= \delta_{p,q \lhd y}\,\sigma_q(z, y)\,\sigma_q(y, y^{-1}zy)^{-1}\,y \otimes p \otimes (y^{-1}zy).v.
\end{aligned}
$$

This implies the desired formula.   □

Remark 5.6. The parametrization in Proposition 5.5 yields the statement in [Ostrik 2002, Remark after Proposition 3.1], for the category $\mathscr{C}(G, \omega, F, 1)$. Indeed, the 2-cocycle $\psi^p(x, y) \in Z^2(F^p, k^\times)$ considered there coincides with our $\sigma_p(y^{-1}, x^{-1})$; and this is cohomologous to $\sigma_p(x, y)$ via $d(\gamma)$, where $\gamma(x) = \sigma_p(x^{-1}, x)$, $x \in F$.

6. Examples

We now discuss some special cases of the results in the last two sections.

Abelian extensions. Suppose that $G = FQ$ is an *exact factorization* of the group G; that is, Q is a subgroup of G and (F, Q) is a *matched pair* of finite groups with the actions $\rhd : Q \times F \to F$, $\lhd : Q \times F \to Q$. We refer the reader to [Masuoka 1999; 2002] for the main notions used here, and in particular for the study of the cohomology theory associated to the matched pair (F, Q).

   Fix a representative (τ, σ) of a class in $\mathrm{Opext}(k^G, kF)$; that is,

$$
\sigma : F \times F \to (k^Q)^\times \quad \text{and} \quad \tau : Q \times Q \to (k^F)^\times
$$

are normalized 2-cocycles subject to compatibility conditions. Then consider the 3-cocycle $\omega : G \times G \times G \to k^\times$ given by

$$
\omega(\tau, \sigma)\,(xp, yq, zr) = \tau_z(p \lhd y, q)\,\sigma_p(y, q \rhd z), \quad x, y, z \in F, \; p, q, r \in Q.
$$

The class of the cocycle $\omega = \omega(\tau, \sigma)$ is the image of the class of (τ, σ) in the Kac exact sequence [Schauenburg 2002a; Masuoka 2002].

   It is not difficult to see that σ and τ have the same meaning as in Section 4 (pages 362–363). Note that $\omega|_{Q \times Q \times Q} = 1$.

   There is a bicrossed product Hopf algebra $A := k^G\,{}^\tau\#_\sigma kF$ corresponding to these data. As is well-known, this correspondence gives a bijection between the equivalence classes of Hopf algebra extensions

$$
1 \to k^Q \to A \to kF \to 1,
$$

and the abelian group $\mathrm{Opext}(k^G, kF)$. The Hopf algebra A^{op} coincides with the (quasi-)Hopf algebra corresponding to G, F and ω, as in Theorem 4.4.

Applying Corollary 5.4, we find the following expression for the Frobenius–Schur indicators.

Proposition 6.1. *Let χ be an irreducible character of A^{op}. Then the Frobenius–Schur indicator of χ is given by*

$$\chi(\nu_{A^{\mathrm{op}}}) = |F|^{-1} \sum_{q \triangleleft x = q^{-1}} \tau_x(q^{-1}, q)\, \sigma_q(x, q \triangleright x) \quad \chi(\delta_q x(q \triangleright x))$$

$$= |F|^{-1} \sum_{q \triangleleft x = q^{-1}} \tau_x(q^{-1}, q)\, \sigma_q(x, qxq)\, \chi(\delta_q(xq)^2). \qquad \square$$

This formula coincides with the expression found in [Kashina et al. 2002], where the Frobenius–Schur indicators of cocentral abelian extensions are computed, *i.e.*, extensions giving rise to the trivial action $\triangleright : F \times Q \to Q$. Corollary 5.4 gives also an alternative expression in terms of the 3-cocycle ω attached to σ and τ via the Kac exact sequence.

Twisted quantum doubles. Let G be a finite group and let ω be 3-cocycle on G. Consider the Dijkgraaf–Pasquier–Roche quasi-Hopf algebra $D^\omega G$, also called the *twisted* quantum double of G [Dijkgraaf et al. 1991]. By the results in [Natale 2003], a semisimple quasi-Hopf algebra H is group-theoretical if and only if its quantum double is gauge equivalent to a quasi-Hopf algebra $D^\omega G$. The Frobenius–Schur indicators for $D^\omega G$ have been computed in [Mason and Ng 2005], and seen to coincide in this case with the indicators introduced by Bantay [1997].

It is shown in [Ostrik 2002] that the category $\mathrm{Rep}\, D^\omega G$ is equivalent to $\mathscr{C}(G \times G, \widetilde{\omega}, \Delta(G), 1)$, where $\Delta(G) \simeq G$ is the diagonal subgroup of $G \times G$, and $\widetilde{\omega}$ is the 3-cocycle on $G \times G$ given by $\widetilde{\omega} = p_1^*\omega (p_2^*\omega)^{-1}$; that is,

$$\widetilde{\omega}((a_1, a_2), (b_1, b_2), (c_1, c_2)) = \omega(a_1, b_1, c_1)\, \omega(a_2, b_2, c_2)^{-1}, \quad a_i, b_i \in G.$$

Thus Corollary 5.4 gives an alternative formula for the Frobenius–Schur indicators of $D^\omega G$ in terms of an appropriate normalization of the 3-cocycle $\widetilde{\omega}$.

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