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Let K be any field and G be a finite group. Let G act on the rational function field $K(x_g : g \in G)$ by K-automorphisms defined by $g \cdot x_h = x_{gh}$ for any $g, h \in G$. Denote by $K(G)$ the fixed field $K(x_g : g \in G)^G$. Noether's problem asks whether $K(G)$ is rational (= purely transcendental) over K. A result of Serre shows that $\mathbb{Q}(G)$ is not rational when G is the generalized quaternion group of order 16. We shall prove that $K(G)$ is rational over K if G is any nonabelian group of order 16 except when G is the generalized quaternion group of order 16. When G is the generalized quaternion group of order 16 and $K(\zeta_8)$ is a cyclic extension of K, then $K(G)$ is also rational over K.

1. Introduction

Let K be any field and G be a finite group. Let G act on the rational function field $K(x_g : g \in G)$ by K-automorphisms such that $g \cdot x_h = x_{gh}$ for any $g,\ h \in G$. Denote by $K(G)$ the fixed subfield

$$K(x_g : g \in G)^G = \{f \in K(x_g : g \in G) : \sigma \cdot f = f \ \text{ for any } \ \sigma \in G\}.$$

Noether's problem asks whether $K(G)$ is rational (that is, purely transcendental) over K.

Noether's problem for finite abelian groups has been studied by Swan, Endo and Miyata, Voskresenskii, Lenstra, Colliot-Thélène and Sansuc, among others; see [Swan 1983] and the references therein. But our knowledge about Noether's problem for nonabelian groups is rather incomplete. It is known that $K(G)$ is rational if G is a transitive solvable subgroup of the symmetric group S_p when $p =$ 3, 5, 7, 11 [Furtwängler 1925], the quaternion group of order 8 [Grbner 1934], the alternating group A_5 [Maeda 1989; Kervaire and Vust 1989], $PSL_2(7)$, $PSp_4(3)$ (such that the base fields K contain suitable quadratic fields of $\mathbb{Q}$) [Kemper 1996] or finite reflection groups [Kemper and Malle 1999]. Noether's problem for metabelian groups and dihedral groups is discussed in [Haeuslein 1971; Hajja 1983;

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Kang 2004]. One striking result is Saltman's Theorem [1984], which shows that $\mathbb{C}(G)$ is never rational for a p-group of order p^9. (See [Bogomolov 1987] for p-groups of smaller orders.) If K is a field containing enough roots of unity, then $K(G)$ is rational for any nonabelian group of order p^3 or p^4 [Chu and Kang 2001]. A result of Serre [1995, 3.5] (see also [Garibaldi et al. 2003, Theorem 33.26 and Example 33.27, pp. 89–90]) shows that $\mathbb{Q}(G)$ is not rational if G is the generalized quaternion group of order 16 (see Theorem 1.3 for the definition of this group); in fact, it is shown that, if G is a finite group whose 2-Sylow subgroup is isomorphic to the generalized quaternion group, then $\mathbb{Q}(G)$ is not rational [Garibaldi et al. 2003, Theorem 34.7, p. 92]. Thus it would be interesting to investigate for which fields K and 2-groups G the field $K(G)$ will be rational, at least for groups of small order. It turns out that, if G is a nonabelian group of order 8 or 16, Serre's counterexample is the only exceptional case. See Theorem 1.3.

One motivation to study Noether's problem arises from the inverse Galois problem, in particular, the existence of a generic polynomial for G-extensions over K (equivalently, the existence of a generic Galois G-extension over K). If K is an infinite field and $K(G)$ is rational over K, there exists a generic polynomial for G-extensions over K [Saltman 1982, Theorem 5.1; DeMeyer and McKenzie 2003]. (See also [Hashimoto and Miyake 1999] for the case of dihedral extensions.) For most p-groups G, it is still unknown whether a generic Galois G-extension over K exists [Saltman 1982]. We just mention some relevant results:

Theorem 1.1. *Let K be any infinite field.*

(1) [Black 1999] *There exists a generic Galois G-extension over K if $G = D_4$ or D_8, where D_4 and D_8 are the dihedral groups of order* 8 *and* 16.

(2) [Ledet 2000; 2001] *There exists a generic polynomial for G-extensions over K, if G is*

 (i) *a nonabelian group of order* 8, *or*

 (ii) *a nonabelian group of order* 16 *defined by*

$$G = \langle \sigma,\ \tau : \sigma^8 = \tau^2 = 1,\ \tau^{-1}\sigma\tau = \sigma^a \rangle \quad \text{with } a = 3, 5, 7, \text{ or}$$
$$G = \langle \sigma, \tau, \lambda : \sigma^4 = \tau^2 = \lambda^2 = 1,\ \tau^{-1}\sigma\tau = \lambda^{-1}\sigma\lambda = \sigma,\ \lambda^{-1}\tau\lambda = \sigma^2\tau \rangle.$$

Theorem 1.2 [Chu et al. 2004]. *For any field K, $K(G)$ is rational over K provided that G is*

(i) *a nonabelian group of order* 8, *or*

(ii) *a nonabelian group of order* 16 *defined by*

$$G = \langle \sigma,\ \tau : \sigma^8 = \tau^2 = 1,\ \tau^{-1}\sigma\tau = \sigma^a \rangle \quad \text{with } a = 3, 5 \text{ or } 7.$$

What we will prove in this article completes our knowledge of Noether's problem for groups of order 16:

Theorem 1.3. *For any field K, $K(G)$ is rational over K, if G is any nonabelian group of order* 16 *except possibly the generalized quaternion group defined by*

$$G = \langle \sigma,\ \tau : \sigma^8 = \tau^4 = 1,\ \sigma^4 = \tau^2,\ \tau^{-1}\sigma\tau = \sigma^{-1} \rangle.$$

For this "exceptional" group G, if $K(\zeta_8)$ is cyclic over K where ζ_8 is a primitive 8*-th root of unity* (*in case* char $K \neq 2$), *then $K(G)$ is rational also.*

As mentioned before we cannot improve the "exceptional" group G in Theorem 1.3 if $K = \mathbb{Q}$ because of Serre's Theorem. We will remark that the novelty of Theorem 1.3. is that no "unnecessary" restriction on the field is assumed; it is known that $K(G)$ is always rational provided that G is any nonabelian p-group of order p^3 or p^4 with exponent p^e and K is a field containing a primitive p^e-th root of unity [Chu and Kang 2001, Theorem 1.6].

As an application of the above theorem we obtain the following theorem, thanks to [Saltman 1982, Theorem 5.1].

Theorem 1.4. *For any infinite field K, a generic Galois G-extension over K exists for any nonabelian group G of order* 16, *except possibly the generalized quaternion group of order* 16 *defined in Theorem 1.3. If G is the generalized quaternion group of order* 16 *and $K(\zeta_8)$ is cyclic over K* (*in case* char $K \neq 2$), *then a generic Galois G-extension over K exists also.*

We shall organize this paper as follows. We will recall some preliminaries in Section 2. Theorem 1.3 will be proved in Section 3. We will remark that, since the proof of this theorem is constructive, a transcendental basis of $K(G)$ can be exhibited explicitly. Thus a generic polynomial for G-extensions over K can be found by applying [Kemper and Malle 1999, Proposition 3.1]. Since Noether's problem for finite abelian groups was completely solved by Lenstra [1974], we will concentrate on nonabelian groups.

Notations and terminologies. A field extension L over K is rational if L is purely transcendental over K; L is called stably rational over K if there exist elements $y_1, \ldots, y_N$ which are algebraically independent over L such that $L(y_1, \ldots, y_N)$ is rational over K. ζ_n will denote a primitive n-th root of unity in some extension field of the field K when char $K = 0$ or char $K = p > 0$ with $p \nmid n$. Finally, recall the definition $K(G)$ at the beginning of this section: $K(G) = K(x_g : g \in G)^G$. The representation space of the regular representation of G over K is denoted by $W = \bigoplus_{g \in G} K \cdot x(g)$ where G acts on W by $g \cdot x(h) = x(gh)$ for any $g,\ h \in G$.

2. Generalities

We recall a variant of Hilbert's Theorem 90 that has been used by many people under different guises.

Theorem 2.1 [Hajja and Kang 1995, Theorem 1]. *Let L be a field and G a finite group acting on $L(x_1, \dots, x_m)$, the rational function field of m variables over L. Suppose that*

(i) *for any $\sigma \in G$, $\sigma(L) \subset L$;*

(ii) *the restriction of the action of G to L is faithful;*

(iii) *for any $\sigma \in G$,*

$$\begin{pmatrix} \sigma(x_1) \\ \cdot \\ \cdot \\ \cdot \\ \sigma(x_m) \end{pmatrix} = A(\sigma) \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_m \end{pmatrix} + B(\sigma)$$

where $A(\sigma) \in \mathrm{GL}_m(L)$ and $B(\sigma)$ is an $m \times 1$ matrix over L.

Then there exist $z_1, \dots, z_m \in L(x_1, \dots, x_m)$ such that

$$L(x_1, \dots, x_m)^G = L^G(z_1, \dots, z_m)$$

and $\sigma(z_i) = z_i$ for any $\sigma \in G$ and any $1 \le i \le m$.

Theorem 2.2 [Hajja and Kang 1994, Lemma (2.7)]. *Let K be any field, $a, b \in K \setminus \{0\}$ and $\sigma : K(x, y) \to K(x, y)$ the K-automorphism defined by $\sigma(x) = a/x$, $\sigma(y) = b/y$. Then $K(x, y)^{\langle\sigma\rangle} = K(u, v)$ where*

$$u = \frac{x - \dfrac{a}{x}}{xy - \dfrac{ab}{xy}}, \qquad v = \frac{y - \dfrac{b}{y}}{xy - \dfrac{ab}{xy}}.$$

Moreover, $x + (a/x) = (-bu^2 + av^2 + 1)/v$, $y + (b/y) = (bu^2 - av^2 + 1)/u$, $xy + (ab/(xy)) = (-bu^2 - av^2 + 1)/(uv)$.

Theorem 2.3 [Kuniyoshi 1955; Miyata 1971]. *Let K be a field with $\mathrm{char}\, K = p > 0$ and G be a p-group. Then $K(V)^G$ is rational over K for any representation $\rho : G \to \mathrm{GL}(V)$ where V is a finite-dimensional vector space over K.*

Proof. Since $\mathrm{char}\, K = p > 0$ and $|G| = p^m$, any representation of G can be triangulated. Apply [Hajja and Kang 1994, Theorem (2.2)]. □

3. Proof of Theorem 1.3

Without loss of generality we will assume that K is any field with char $K \neq 2$ throughout this section, because Theorem 2.3 will take care of the case char $K = 2$.

Here is a list of nonabelian groups of order 16, which can be found in [Huppert 1967, p. 349] or in [Chu and Kang 2001, Theorem 3.4]:

(I) $\langle \sigma, \tau : \sigma^8 = \tau^2 = 1,\ \tau^{-1}\sigma\tau = \sigma^{-1} \rangle$,

(II) $\langle \sigma, \tau : \sigma^8 = \tau^4 = 1,\ \sigma^4 = \tau^2,\ \tau^{-1}\sigma\tau = \sigma^{-1} \rangle$,

(III) $\langle \sigma, \tau : \sigma^8 = \tau^2 = 1,\ \tau^{-1}\sigma\tau = \sigma^5 \rangle$,

(IV) $\langle \sigma, \tau : \sigma^8 = \tau^2 = 1,\ \tau^{-1}\sigma\tau = \sigma^3 \rangle$,

(V) $\langle \sigma, \tau, \lambda : \sigma^4 = \tau^2 = \lambda^2 = 1,\ \tau^{-1}\sigma\tau = \lambda^{-1}\sigma\lambda = \sigma,\ \lambda^{-1}\tau\lambda = \sigma^2\tau \rangle$,

(VI) $\langle \sigma, \tau, \lambda : \sigma^4 = \tau^2 = \lambda^2 = 1,\ \tau^{-1}\sigma\tau = \sigma^{-1},\ \lambda^{-1}\sigma\lambda = \sigma,\ \lambda^{-1}\tau\lambda = \tau \rangle$,

(VII) $\langle \sigma, \tau, \lambda : \sigma^4 = \tau^4 = \lambda^2 = 1,\ \sigma^2 = \tau^2,\ \tau^{-1}\sigma\tau = \sigma^{-1},\ \lambda^{-1}\sigma\lambda = \sigma,\ \lambda^{-1}\tau\lambda = \tau \rangle$,

(VIII) $\langle \sigma, \tau : \sigma^4 = \tau^4 = 1,\ \tau^{-1}\sigma\tau = \sigma^{-1} \rangle$,

(IX) $\langle \sigma, \tau, \lambda : \sigma^4 = \tau^2 = \lambda^2 = 1,\ \tau^{-1}\sigma\tau = \sigma,\ \lambda^{-1}\sigma\lambda = \sigma\tau,\ \lambda^{-1}\tau\lambda = \tau \rangle$.

Because we have solved the rationality problem for the groups (I), (III), (IV) in [Chu et al. 2004], we will consider the remaining six groups in this article.

Case 1. The group (V): $G = \langle \sigma, \tau, \lambda : \sigma^4 = \tau^2 = \lambda^2 = 1,\ \tau^{-1}\sigma\tau = \lambda^{-1}\sigma\lambda = \sigma,\ \lambda^{-1}\tau\lambda = \sigma^2\tau \rangle$.

If $\sqrt{-1} \in K$, then $K(G)$ is rational by [Chu and Kang 2001, Theorem 1.6]. Hence we shall assume that $\sqrt{-1} \notin K$ from now on.

Let $W = \bigoplus_{g \in K} K \cdot x(g)$ be the representation space of the regular representation of G. Define

$$x_1 = x(1) + x(\tau) - x(\sigma^2) - x(\sigma^2\tau),$$
$$x_2 = \sigma \cdot x_1, \quad x_3 = \lambda \cdot x_1, \quad x_4 = \lambda\sigma \cdot x_1.$$

Then we find that

$$\sigma : x_1 \mapsto x_2 \mapsto -x_1, \quad x_3 \mapsto x_4 \mapsto -x_3,$$
$$\tau : x_1 \mapsto x_1, \quad x_2 \mapsto x_2, \quad x_3 \mapsto -x_3, \quad x_4 \mapsto -x_4,$$
$$\lambda : x_1 \leftrightarrow x_3, \quad x_2 \leftrightarrow x_4.$$

Moreover, $\bigoplus_{1 \le i \le 4} K \cdot x_i$ is a faithful G-subspace of W. Thus $K(G)$ is rational if $K(x_1, \dots, x_4)^G$ is rational by Theorem 2.1.

Let $\mathrm{Gal}(K\sqrt{-1}/K) = \langle \rho \rangle$ and $\rho : \sqrt{-1} \mapsto -\sqrt{-1}$. We extend the actions of $\sigma, \tau, \lambda, \rho$ to $K(\sqrt{-1})(x_1, \dots, x_4)$ by requiring $\sigma(\sqrt{-1}) = \tau(\sqrt{-1}) = \lambda(\sqrt{-1}) =$

$\sqrt{-1}$, $\rho(x_i)=x_i$ for $1\le i\le 4$. Then

$$K(x_1,\dots,x_4)^{\langle\sigma,\tau,\lambda\rangle}=\{K(\sqrt{-1})(x_1,\dots,x_4)^{\langle\rho\rangle}\}^{\langle\sigma,\tau,\lambda\rangle}$$
$$=K(\sqrt{-1})(x_1,\dots,x_4)^{\langle\sigma,\tau,\lambda,\rho\rangle}.$$

Define

$$y_1=\sqrt{-1}x_1+x_2,\qquad y_2=-\sqrt{-1}x_1+x_2,$$
$$y_3=\sqrt{-1}x_3+x_4,\qquad y_4=-\sqrt{-1}x_3+x_4.$$

Then we get

$$\sigma: y_1\mapsto\sqrt{-1}\,y_1,\quad y_2\mapsto-\sqrt{-1}\,y_2,\quad y_3\mapsto\sqrt{-1}\,y_3,\quad y_4\mapsto-\sqrt{-1}\,y_4,$$
$$\tau: y_1\mapsto y_1,\quad y_2\mapsto y_2,\quad y_3\mapsto -y_3,\quad y_4\mapsto -y_4,$$
$$\lambda: y_1\leftrightarrow y_3,\quad y_2\leftrightarrow y_4,$$
$$\rho: y_1\leftrightarrow y_2,\quad y_3\leftrightarrow y_4.$$

Define

$$z_1=y_1y_2,\quad z_2=y_3y_4,\quad z_3=y_3/y_1,\quad z_4=y_1^4.$$

Then $K(\sqrt{-1})(y_1,\dots,y_4)^{<\sigma>}=K(\sqrt{-1})(z_1,\dots,z_4)$; moreover,

$$\tau: z_1\mapsto z_1,\quad z_2\mapsto z_2,\quad z_3\mapsto -z_3,\quad z_4\mapsto z_4,$$
$$\lambda: z_1\leftrightarrow z_2,\quad z_3\mapsto 1/z_3,\quad z_4\mapsto z_3^4z_4,$$
$$\rho: z_1\mapsto z_1,\quad z_2\mapsto z_2,\quad z_3\mapsto z_2/(z_1z_3),\quad z_4\mapsto z_1^4/z_4.$$

Thus $K(\sqrt{-1})(z_1,\dots,z_4)^{\langle\tau\rangle}=K(\sqrt{-1})(z_1,z_2,z_3^2,z_4)$.

Define

$$u_1=z_1z_2,\quad u_2=z_3^2z_4/(z_1z_2),\quad x=z_1,\quad y=z_1z_3^2/z_2.$$

Then we find that

$$\lambda: u_1\mapsto u_1,\quad u_2\mapsto u_2,\quad x\mapsto a/x,\quad y\mapsto b/y,$$
$$\rho: u_1\mapsto u_1,\quad u_2\mapsto 1/u_2,\quad x\mapsto x,\quad y\mapsto 1/y.$$

where $a=u_1,\ b=1$.

Define

$$u=\frac{x-\dfrac{a}{x}}{xy-\dfrac{ab}{xy}},\qquad v=\frac{y-\dfrac{b}{y}}{xy-\dfrac{ab}{xy}}.$$

By Theorem 2.2, $K(\sqrt{-1})(u_1,u_2,x,y)^{\langle\lambda\rangle}=K(\sqrt{-1})(u_1,u_2,u,v)$.

It is routine to check that

$$\rho : u \mapsto \frac{x - \dfrac{a}{x}}{\dfrac{bx}{y} - \dfrac{ay}{x}}, \qquad v \mapsto -\frac{y - \dfrac{b}{y}}{\dfrac{bx}{y} - \dfrac{ay}{x}}.$$

Define $w = u/v$. Then $\rho(w) = -w$. It is not difficult to verify that

$$\frac{x - \dfrac{a}{x}}{\dfrac{bx}{y} - \dfrac{ay}{x}} = \frac{u}{bu^2 - av^2}. \tag{3–1}$$

In fact, using Theorem 2.2, the right-hand side of (3–1) is equal to $(y + (b/y) - (1/u))^{-1}$. It is very easy to check that the left-hand side of (3–1) is equal to the same quantity.

It follows that $\rho(u) = u/(bu^2 - av^2) = c/u$ where $c = w^2/(bw^2 - a)$.

Define

$$t = u_1, \quad s = \sqrt{-1}w, \quad q = w(1 + u_2)/(1 - u_2).$$

Then

$$\rho : \sqrt{-1} \mapsto -\sqrt{-1}, \quad t \mapsto t, \quad s \mapsto s, \quad q \mapsto q, \quad u \mapsto c/u$$

where $c = w^2/(bw^2 - a) = s^2/(s^2 + t)$.

Define $p = (s^2 + t)u/s$. Then $\rho(p) = A/p$ where $A = s^2 + t$.

It follows that $K(\sqrt{-1})(u_1, u_2, u, v)^{\langle\rho\rangle} = K(\sqrt{-1})(t, s, p, q)^{\langle\rho\rangle} = K(\sqrt{-1})(t, s, p)^{\langle\rho\rangle}(q) = K(t, s, X, Y, q)$ where $X = p + (A/p)$, $Y = \sqrt{-1}(p - (A/p))$. Note that a relation of X and Y is

$$X^2 + Y^2 = 4A = 4(s^2 + t).$$

Hence $t \in K(s, X, Y)$. It follows that $K(t, s, X, Y, q) = K(s, X, Y, q)$ is rational over K.

Case 2. The group (VI): $G = \langle \sigma, \tau, \lambda : \sigma^4 = \tau^2 = \lambda^2 = 1,\ \tau^{-1}\sigma\tau = \sigma^{-1},\ \lambda^{-1}\sigma\lambda = \sigma,\ \lambda^{-1}\tau\lambda = \tau \rangle$.

As before, let $W = \bigoplus_{g \in G} K \cdot x(g)$ be the regular representation of G. Define

$$x_1 = x(1) + x(\tau) - x(\sigma^2) - x(\sigma^2\tau),$$
$$x_2 = \sigma \cdot x_1, \quad x_3 = \lambda \cdot x_1, \quad x_4 = \lambda\sigma \cdot x_1.$$

Then we find that

$$\sigma : x_1 \mapsto x_2 \mapsto -x_1, \quad x_3 \mapsto x_4 \mapsto -x_3,$$
$$\tau : x_1 \mapsto x_1, \quad x_2 \mapsto -x_2, \quad x_3 \mapsto x_3, \quad x_4 \mapsto -x_4,$$
$$\lambda : x_1 \leftrightarrow x_3, \quad x_2 \leftrightarrow x_4.$$

As in Case 1, it suffices to consider the case $\sqrt{-1} \notin K$. Let $\mathrm{Gal}(K(\sqrt{-1}/K) = \langle \rho \rangle$ with $\rho(\sqrt{-1}) = -\sqrt{-1}$.

Define y_i and z_i, for $1 \le i \le 4$, the same way as in Case 1. We find

$$K(\sqrt{-1})(x_1, \dots, x_4)^{\langle \sigma \rangle} = K(\sqrt{-1})(z_1, \dots, z_4).$$

The actions of λ and ρ on $z_1, \dots, z_4$ are the same as in Case 1, while $\tau\rho(\sqrt{-1}) = -\sqrt{-1}$, $\tau\rho(z_i) = z_i$ for $1 \le i \le 4$. Thus

$$K(\sqrt{-1})(z_1, \dots, z_4)^{\langle \tau, \lambda, \rho \rangle} = K(\sqrt{-1})(z_1, \dots, z_4)^{\langle \tau\rho, \lambda, \rho \rangle} = K(z_1, \dots, z_4)^{\langle \lambda, \rho \rangle}.$$

Define

$$u_1 = z_1 z_2, \quad u_2 = z_1, \quad x = z_3/z_2, \quad y = z_3^2 z_4/(z_1 z_2).$$

We find that

$$\lambda ; u_1 \mapsto u_1, \quad u_2 \mapsto u_1/u_2, \quad x \mapsto a/x, \quad y \mapsto y,$$
$$\rho : u_1 \mapsto u_1, \quad u_2 \mapsto u_2, \quad x \mapsto a/x, \quad y \mapsto b/y,$$

where $a = 1/u_1$ and $b = 1$.

Define u, v, and $w = u/v$ the same way as in Case 1. Then $K(u_1, u_2, x, y)^{\langle \rho \rangle} = K(u_1, u_2, w, u)$ and we find that

$$\lambda : u_1 \mapsto u_1, \quad u_2 \mapsto u_1/u_2, \quad w \mapsto -w, \quad u \mapsto c/u,$$

where $c = w^2/(bw^2 - a)$.

Define

$$w_1 = -1/(u_1 w^2), \quad w_2 = w w_1 u_2, \quad w_3 = u/c.$$

Then $K(u_1, u_2, w, u) = K(w_1, w_2, w_3, w)$ and

$$\lambda : w_1 \mapsto w_1, \quad w_2 \mapsto w_1/w_2, \quad w_3 \mapsto (w_1 + 1)/w_3.$$

By Theorem 2.1, it suffices to show that $K(w_1, w_2, w_3)^{\langle \lambda \rangle}$ is rational over K. But this is easy because it is even rational over $K(w_1)$ by Theorem 2.2.

Case 3. The group (VII): $G = \langle \sigma, \tau, \lambda : \sigma^4 = \tau^4 = \lambda^2 = 1,\ \sigma^2 = \tau^2,\ \tau^{-1}\sigma\tau = \sigma^{-1},\ \lambda^{-1}\sigma\lambda = \sigma,\ \lambda^{-1}\tau\lambda = \tau \rangle$.

Define a faithful G-subspace $\bigoplus_{1\le i\le 5} K\cdot x_i$ in the representation space $W=\bigoplus_{g\in G} K\cdot x(g)$ of the regular representation by

$$x_1 = x(1)+x(\lambda)-x(\sigma^2)-x(\sigma^2\lambda),$$
$$x_2=\sigma\cdot x_1,\ x_3=\tau\cdot x_1,\ x_4=\tau\sigma\cdot x_1,$$
$$x_5=\sum_{h\in H} x(h)-\sum_{h\in H} x(h\lambda)$$

where H is the subgroup generated by σ and τ.

We find that

$$\sigma: x_1\mapsto x_2\mapsto -x_1,\quad x_3\mapsto -x_4,\quad x_4\mapsto x_3,\quad x_5\mapsto x_5,$$
$$\tau: x_1\mapsto x_3,\quad x_2\mapsto x_4,\quad x_3\mapsto -x_1,\quad x_4\mapsto -x_2,\quad x_5\mapsto x_5,$$
$$\lambda: x_1\mapsto x_1,\quad x_2\mapsto x_2,\quad x_3\mapsto x_3,\quad x_4\mapsto x_4,\quad x_5\mapsto -x_5.$$

Thus

$$K(x_1,\dots,x_5)^{\langle\lambda\rangle}=K(x_1,\dots,x_5^2),\quad K(x_1,\dots,x_5)^{\langle\sigma,\tau,\lambda\rangle}=K(x_1,\dots,x_4)^{\langle\sigma,\tau\rangle}(x_5^2).$$

However, the fixed field $K(x_1,\dots,x_4)^{\langle\sigma,\tau\rangle}$ is exactly the same as in the proof of [Chu et al. 2004, Theorem 2.6]. (The subgroup $H=\langle\sigma,\tau\rangle$ is the quaternion group of order 8.) Hence the result.

Case 4. The group (VIII): $G=\langle\sigma,\tau:\sigma^4=\tau^4=1,\tau^{-1}\sigma\tau=\sigma^{-1}\rangle$.

Define a faithful G-subspace $\bigoplus_{1\le i\le 4} K\cdot x_i$ in $W=\bigoplus_{g\in G} K\cdot x(g)$ by

$$x_1=x(1)+x(\tau)-x(\sigma^2)-x(\sigma^2\tau),\quad x_2=\sigma\cdot x_1,$$
$$x_3=\sum_{0\le i\le 3} x(\sigma^i)-\sum_{0\le i\le 3} x(\sigma^i\tau^2),\quad x_4=\tau\cdot x_3.$$

Then we find

$$\sigma: x_1\mapsto x_2\mapsto -x_1,\quad x_3\mapsto x_3,\quad x_4\mapsto x_4,$$
$$\tau: x_1\mapsto x_1,\quad x_2\mapsto -x_2,\quad x_3\mapsto x_4\mapsto -x_3.$$

It is clear that $K(x_1,\dots,x_4)^{\langle\sigma^2,\tau^2\rangle}=K(y_1,\dots,y_4)$ where $y_1=x_1x_2,\ y_2=x_1^2,\ y_3=x_3x_4,\ y_4=x_3^2$.

Note that

$$\sigma: y_1\mapsto -y_1,\quad y_2\mapsto y_1^2/y_2,\quad y_3\mapsto y_3,\quad y_4\mapsto y_4,$$
$$\tau: y_1\mapsto -y_1,\quad y_2\mapsto y_2,\quad y_3\mapsto -y_3,\quad y_4\mapsto y_3^2/y_4.$$

Define $u_1=y_2+(y_1^2/y_2),\ u_2=(y_1/y_2)-(y_2/y_1),\ u_3=y_3,\ u_4=y_3/y_4$. Since $[K(u_1,u_2)(y_1):K(u_1,u_2)]\le 2$, it follows that $K(y_1,\dots,y_4)^{\langle\sigma\rangle}=K(u_1,\dots,u_4)$.

Note that

$$\tau : u_1 \mapsto u_1, \quad u_2 \mapsto -u_2, \quad u_3 \mapsto -u_3, \quad u_4 \mapsto -1/u_4.$$

Hence $K(u_1, \dots, u_4)^{\langle\tau\rangle} = K(u_1, u_2u_3, u_3u_4 + (u_3/u_4), u_4 - (1/u_4))$ is rational over K.

Case 5. The group (IX): $G = \langle \sigma, \tau, \lambda : \sigma^4 = \tau^2 = \lambda^2 = 1,\ \tau^{-1}\sigma\tau = \sigma,\ \lambda^{-1}\sigma\lambda = \sigma\tau,\ \lambda^{-1}\tau\lambda = \tau \rangle$.

Define a faithful G-subspace $\bigoplus_{1 \le i \le 4} K \cdot x_i$ in $W = \bigoplus_{g \in G} K \cdot x(g)$ by

$$x_1 = x(1) - x(\sigma^2) - x(\lambda) + x(\sigma^2\lambda),$$
$$x_2 = \sigma \cdot x_1, \quad x_3 = \tau \cdot x_1, \quad x_4 = \tau\sigma \cdot x_1.$$

Then we find that

$$\sigma : x_1 \mapsto x_2 \mapsto -x_1, \quad x_3 \mapsto x_4 \mapsto -x_3,$$
$$\tau : x_1 \leftrightarrow x_3, \quad x_2 \leftrightarrow x_4,$$
$$\lambda : x_1 \mapsto -x_1, \quad x_2 \mapsto -x_4, \quad x_3 \mapsto -x_3, \quad x_4 \mapsto -x_2.$$

Define

$$y_1 = x_1 - x_3, \quad y_2 = x_2 - x_4, \quad y_3 = x_1 + x_3, \quad y_4 = x_2 + x_4.$$

It follows that

$$\sigma : y_1 \mapsto y_2 \mapsto -y_1, \quad y_3 \mapsto y_4 \mapsto -y_3,$$
$$\tau : y_1 \mapsto -y_1, \quad y_2 \mapsto -y_2, \quad y_3 \mapsto y_3, \quad y_4 \mapsto y_4,$$
$$\lambda : y_1 \mapsto -y_1, \quad y_2 \mapsto y_2, \quad y_3 \mapsto -y_3, \quad y_4 \mapsto -y_4.$$

Hence $K(x_1, \dots, x_4)^{\langle\tau\rangle} = K(y_1, \dots, y_4)^{\langle\tau\rangle} = K(z_1, \dots, z_4)$ where $z_1 = y_1^2$, $z_2 = y_1y_2$, $z_3 = y_3$, $z_4 = y_4$. Moreover, it can be verified that

$$\sigma : z_1 \mapsto z_2^2/z_1, \quad z_2 \mapsto -z_2, \quad z_3 \mapsto z_4 \mapsto -z_3,$$
$$\lambda : z_1 \mapsto z_1, \quad z_2 \mapsto -z_2, \quad z_3 \mapsto -z_3, \quad z_4 \mapsto -z_4.$$

Define $u_1 = z_1$, $u_2 = z_2^2$, $u_3 = z_2z_3$, $u_4 = z_3z_4$. Then $K(z_1, \dots, z_4)^{\langle\lambda\rangle} = K(u_1, \dots, u_4)$ and we find

$$\sigma : u_1 \mapsto u_2/u_1, \quad u_2 \mapsto u_2, \quad u_3 \mapsto -u_2u_4/u_3, \quad u_4 \mapsto -u_4.$$

It is easy to check that $K(u_1, \dots, u_4)^{\langle\sigma^2\rangle} = K(u_1, u_2, u_3^2, u_4)$.

Define

$$v_1 = u_2, \quad v_2 = (u_1 - (u_2/u_1))u_4, \quad x = u_1, \quad y = u_3^2/u_4.$$

We find that

$$\sigma : v_1 \mapsto v_1, \quad v_2 \mapsto v_2, \quad x \mapsto v_1/x, \quad y \mapsto -v_1/y.$$

By Theorem 2.2, $K(v_1, v_2, v_3, v_4)^{\langle\sigma\rangle}$ is rational over K.

Case 6. The group (II): $G = \langle \sigma, \tau : \sigma^8 = \tau^4 = 1,\ \sigma^4 = \tau^2,\ \tau^{-1}\sigma\tau = \sigma^{-1}\rangle$ and assume that $K(\zeta_8)$ is cyclic over K.

If $\zeta_8 \in K$, then $K(G)$ is rational over K by [Chu and Kang 2001, Theorem 1.6]. Hence we shall assume that $\zeta_8 \notin K$ from now on.

Because $K(\zeta_8)$ is cyclic over K, it follows that $\mathrm{Gal}(K(\zeta_8)/K) = \langle\rho\rangle$ where $\rho(\zeta_8) = \zeta_8^a$ with $a = 3,\ 5$ or 7.

We will find a faithful G-subspace $\bigoplus_{1\le i\le 8} K \cdot x_i$ of $W = \bigoplus_{g\in G} K \cdot x(g)$ by

$$x_1 = x(1) - x(\sigma^4), \quad x_i = \sigma^{i-1}x_1 \text{ for } 2 \le i \le 4,$$
$$x_j = \tau\sigma^{j-5}x_1 \text{ for } 5 \le j \le 8.$$

We find that

$$\sigma : x_1 \mapsto x_2 \mapsto x_3 \mapsto x_4 \mapsto -x_1, \quad x_8 \mapsto x_7 \mapsto x_6 \mapsto x_5 \mapsto -x_8,$$
$$\tau : x_1 \mapsto x_5, \quad x_2 \mapsto x_6, \quad x_3 \mapsto x_7, \quad x_4 \mapsto x_8, \quad x_5 \mapsto -x_1, \quad x_6 \mapsto -x_2,$$
$$x_7 \mapsto -x_3, \quad x_8 \mapsto -x_4.$$

We shall write ζ for ζ_8 in the sequel and remember $\rho(\zeta) = \zeta^a$ with $a = 3,\ 5$ or 7. We shall extend the actions of $\sigma,\ \tau,\ \rho$ to $K(\zeta)(x_1, \ldots, x_8)$ by requiring $\sigma(\zeta) = \tau(\zeta) = \zeta$ and $\rho(x_i) = x_i$ for $1 \le i \le 8$. It follows that

$$K(x_1, \ldots, x_8)^{\langle\sigma,\tau\rangle} = \{K(\zeta)(x_1, \ldots, x_8)^{\langle\rho\rangle}\}^{\langle\sigma,\tau\rangle} = K(\zeta)(x_1, \ldots, x_8)^{\langle\sigma,\tau,\rho\rangle}.$$

Define $y_1,\ y_3,\ y_5,\ y_7,\ z_1,\ z_3,\ z_5,\ z_7$ by

$$y_1 = (\sigma - \zeta^3)(\sigma - \zeta^5)(\sigma - \zeta^7)\cdot x_1, \quad y_3 = (\sigma - \zeta)(\sigma - \zeta^5)(\sigma - \zeta^7)\cdot x_1$$
$$y_5 = (\sigma - \zeta)(\sigma - \zeta^3)(\sigma - \zeta^7)\cdot x_1, \quad y_7 = (\sigma - \zeta)(\sigma - \zeta^3)(\sigma - \zeta^5)\cdot x_1$$
$$z_1 = (\sigma - \zeta^3)(\sigma - \zeta^5)(\sigma - \zeta^7)\cdot x_8, \quad z_3 = (\sigma - \zeta)(\sigma - \zeta^5)(\sigma - \zeta^7)\cdot x_8$$
$$z_5 = (\sigma - \zeta)(\sigma - \zeta^3)(\sigma - \zeta^7)\cdot x_8, \quad z_7 = (\sigma - \zeta)(\sigma - \zeta^3)(\sigma - \zeta^5)\cdot x_8.$$

It follows that

$$\sigma : y_i \mapsto \zeta^i y_i, \quad z_i \mapsto \zeta^i z_i \quad \text{for } i = 1, 3, 5, 7,$$
$$\tau : y_1 \mapsto -\zeta^7 z_7, \quad y_3 \mapsto -\zeta^5 z_5, \quad y_5 \mapsto -\zeta^3 z_3, \quad y_7 \mapsto -\zeta z_1,$$
$$z_1 \mapsto \zeta^7 y_7, \quad z_3 \mapsto \zeta^5 y_5, \quad z_5 \mapsto \zeta^3 y_3, \quad z_7 \mapsto \zeta y_1.$$

If $\rho(\zeta)=\zeta^a$, then $\rho(y_i)=y_{ai}$ and $\rho(z_i)=z_{ai}$ for $i=1,\ 3,\ 5,\ 7$. (The index ai is understood to be modulo 8.)

Apply Theorem 2.1. Then

$$K(\zeta)(x_1,\dots,x_8)^{\langle\sigma,\tau,\rho\rangle}=K(\zeta)(y_1,y_3,y_5,y_7,z_1,z_3,z_5,z_7)^{\langle\sigma,\tau,\rho\rangle}$$

is rational provided that $K(\zeta)(y_1,y_3,z_5,z_7)^{\langle\sigma,\tau,\rho\rangle}$ be rational when $\rho(\zeta)=\zeta^3$, that $K(\zeta)(y_1,y_5,z_3,z_7)^{\langle\sigma,\tau,\rho\rangle}$ be rational when $\rho(\zeta)=\zeta^5$, and $K(\zeta)(y_1,y_7,z_1,z_7)^{\langle\sigma,\tau,\rho\rangle}$ be rational when $\rho(\zeta)=\zeta^7$.

Subcase 6.1. $\rho(\zeta)=\zeta^3$.

Define

$$u_1=y_1z_7,\quad u_2=y_3z_5,\quad u_3=y_3/y_1^3,\quad u_4=y_1^8.$$

Then $K(\zeta)(y_1,y_3,z_5,z_7)^{\langle\sigma\rangle}=K(\zeta)(u_1,\dots,u_4)$ and the actions of τ and ρ are given by

$$\begin{aligned}\tau&:u_1\mapsto -u_1,\quad u_2\mapsto -u_2,\quad u_3\mapsto u_2/u_1^3u_3,\quad u_4\mapsto u_1^8/u_4,\\ \rho&:u_1\mapsto u_2,\quad u_2\mapsto u_1,\quad u_3\mapsto 1/(u_3^3u_4),\quad u_4\mapsto u_3^8u_4^3.\end{aligned}$$

Define

$$r=u_1,\quad s=u_2/u_1,\quad x=u_1u_3,\quad y=u_3^2u_4/(u_1u_2),\quad t=r(y-(1/y)).$$

Then

$$\begin{aligned}\tau&:r\mapsto -r,\ s\mapsto s,\ x\mapsto -s/x,\ y\mapsto 1/y,\ t\mapsto t,\\ \rho&:r\mapsto rs,\ s\mapsto 1/s,\ x\mapsto 1/xy,\ y\mapsto y,\ t\mapsto st.\end{aligned}$$

Note that $K(\zeta)(u_1,\dots,u_4)=K(\zeta)(r,s,x,y)=K(\zeta)(s,t,x,y)$.

By Theorem 2.2, $K(\zeta)(s,t,x,y)^{\langle\tau\rangle}=K(\zeta)(s,t,u,v)$ where

$$u=\frac{x+\dfrac{s}{x}}{xy+\dfrac{s}{xy}},\qquad v=\frac{y-\dfrac{1}{y}}{xy+\dfrac{s}{xy}}.$$

It is routine to check that $\rho(u)=1/u,\ \rho(v)=sv/u$.

Define

$$\begin{aligned}s'&=(\zeta-\rho(\zeta))(1+s)(1-s)^{-1},\quad t'=(1+s)t,\\ u'&=(\zeta-\rho(\zeta))(1+u)(1-u)^{-1},\quad v'=(1+(s/u))v.\end{aligned}$$

We find that $K(\zeta)(s,t,u,v)=K(\zeta)(s',t',u',v')$ and $\rho(s')=s',\ \rho(t')=t'$, $\rho(u')=u',\ \rho(v')=v'$. Thus $K(\zeta)(s,t,u,v)^{\langle\rho\rangle}=K(s',t',u',v')$ is rational.

Subcase 6.2. $\rho(\zeta)=\zeta^5$.

Define

$$u_1 = y_1 z_7,\ u_2 = y_5 z_3,\ u_3 = y_5/y_1^5,\ u_4 = y_1^8.$$

Then $K(\zeta)(y_1, y_5, z_3, z_7)^{\langle\sigma\rangle} = K(\zeta)(u_1, \dots, u_4)$ and the actions of τ and ρ are given by

$$\tau : u_1 \mapsto -u_1,\ \ u_2 \mapsto -u_2,\ u_3 \mapsto u_2/u_1^5 u_3,\ u_4 \mapsto u_1^8/u_4,$$
$$\rho : u_1 \mapsto u_2,\ u_2 \mapsto u_1,\ u_3 \mapsto 1/(u_3^5 u_4^3),\ u_4 \mapsto u_3^8 u_4^5.$$

Define

$$r = u_1,\ s = u_2/u_1,\ x = u_2^2/(u_3^3 u_4^2),\ y = u_1 u_3^2 u_4/u_2,\ t = r(y - (1/y)).$$

Then

$$\tau : r \mapsto -r,\ s \mapsto s,\ x \mapsto s/x,\ y \mapsto 1/y,\ t \mapsto t,$$
$$\rho : r \mapsto rs,\ s \mapsto 1/s,\ x \mapsto xy/s,\ y \mapsto 1/y,\ t \mapsto -st.$$

By Theorem 2.2, $K(\zeta)(u_1, \dots, u_4)^{\langle\tau\rangle} = K(\zeta)(s, t, x, y)^{\langle\tau\rangle} = K(\zeta)(s, t, u, v)$, where

$$u = \frac{x - \dfrac{s}{x}}{xy - \dfrac{s}{xy}}, \quad v = \frac{y - \dfrac{1}{y}}{xy - \dfrac{s}{xy}}.$$

It is routine to check that $\rho(u) = 1/u,\ \rho(v) = -sv/u$.

Define

$$s' = (\zeta - \rho(\zeta))(1+s)(1-s)^{-1},\ \ t' = (1-s)t,$$
$$u' = (\zeta - \rho(\zeta))(1+u)(1-u)^{-1},\ \ v' = (1 - (s/u))v.$$

We find that $K(\zeta)(s, t, u, v)^{\langle\rho\rangle} = K(s', t', u', v')$ is rational.

Subcase 6.3. $\rho(\zeta) = \zeta^7$.

Define

$$u_1 = y_1 z_7,\ u_2 = y_7 z_1,\ u_3 = y_7/y_1^7,\ u_4 = y_1^8.$$

Then $K(\zeta)(y_1, y_7, z_1, z_7)^{\langle\sigma\rangle} = K(\zeta)(u_1, \dots, u_4)$ and the actions of τ and ρ are given by

$$\tau : u_1 \mapsto -u_1,\ u_2 \mapsto -u_2,\ u_3 \mapsto u_2/u_1^7 u_3,\ u_4 \mapsto u_1^8/u_4,$$
$$\rho : u_1 \mapsto u_2,\ u_2 \mapsto u_1,\ u_3 \mapsto 1/(u_3^7 u_4^6),\ u_4 \mapsto u_3^8 u_4^7.$$

Define

$$r = u_1,\ s = u_2/u_1,\ x = u_3 u_4/u_1,\ y = u_2^2/(u_1^2 u_3^4 u_4^3),\ t = r(y - (1/y)).$$

Then

$$\tau : r \mapsto -r,\ s \mapsto s,\ x \mapsto -s/x,\ y \mapsto 1/y,\ t \mapsto t,$$
$$\rho : r \mapsto rs,\ s \mapsto 1/s,\ x \mapsto x/s,\ y \mapsto 1/y,\ t \mapsto -st.$$

By Theorem 2.2, $K(\zeta)(u_1, \dots, u_4)^{\langle\tau\rangle} = K(\zeta)(s,t,x,y)^{\langle\tau\rangle} = K(\zeta)(s,t,u,v)$, where

$$u = \frac{x - \dfrac{A}{x}}{xy - \dfrac{AB}{xy}}, \quad v = \frac{y - \dfrac{B}{y}}{xy - \dfrac{AB}{xy}}$$

with $A = -s,\ B = 1$.

It is routine to check that

$$\rho : u \mapsto \frac{x - \dfrac{A}{x}}{\dfrac{Bx}{y} - \dfrac{Ay}{x}}, \quad v \mapsto \frac{-s\left(y - \dfrac{B}{y}\right)}{\dfrac{Bx}{y} - \dfrac{Ay}{x}}.$$

Define $w = u/v$. Then $\rho(w) = -w/s$. Note that

$$\frac{x - \dfrac{A}{x}}{\dfrac{Bx}{y} - \dfrac{Ay}{x}} = \frac{u}{bu^2 - Av^2},$$

because this is the same identity as the identity (3–1) we encountered in Case 1.

It follows that $\rho(u) = u/(Bu^2 - Av^2) = C/u$ where $C = w^2/(w^2 + s)$.

Define

$$p = (1 - (1/s))w, \quad q = (1 - s)t.$$

Then $C = p^2/(s + (1/s) + p^2 - 2)$.

It follows that $K(\zeta)(s,t,u,v)^{\langle\rho\rangle} = K(\zeta)(s,q,p,u)^{\langle\rho\rangle}$, where

$$\rho : s \mapsto 1/s, \quad q \mapsto q, \quad p \mapsto p, \quad u \mapsto C/u.$$

Define

$$X = s + (1/s), \quad Y = (\zeta - \rho(\zeta))(s - (1/s)),$$
$$Z = u + (C/u), \quad W = (\zeta - \rho(\zeta))(u - (C/u)).$$

Then $K(\zeta)(s,q,p,u)^{\langle\rho\rangle} = K(\zeta)(s,p,u)^{\langle\rho\rangle}(q) = K(X,Y,Z,W,p,q)$ because $[K(X,Y,Z,W,p)(\zeta) : K(X,Y,Z,W,p)] \le 2$. The relations of $X,\ Y,\ Z,\ W,\ p$ are given by

$$1X^2 - (Y/(\zeta - \rho(\zeta)))^2 = 4 \tag{3–2}$$

and

$$(3\text{–}3)\quad Z^2-(W/(\zeta-\rho(\zeta)))^2 = 4C = 4p^2/(s+(1/s)+p-2) = 4p^2/(X+p^2-2).$$

Define $\eta = 1/(\zeta-\rho(\zeta))^2 \in K$. Then we find, from (3–2), that $(X-2)/Y = \eta Y/(X+2)$. From this we find that $(X-2)/Y$, X and Y all lie in $K((X+2)/Y)$.

Simplify (3–3). We get

$$(3\text{–}4)\qquad (Z(X-p^2-2)/(2p))^2 - \eta(W(X-p^2-2)/(2p))^2 = X+p^2-2.$$

Let $Z_1 = Z(X-p^2-2)/(2p)-p$, $Z_2 = Z(X-p^2-2)/(2p)+p$, and $W_1 = W(X-p^2-2)/(2p)$. Thus (3–4) becomes $Z_1Z_2-\eta W_1^2 = X-2 \in K((X+2)/Y)$. Thus $Z_2, p \in K((X+2)/Y, Z_1, W_1)$; hence $K(X, Y, Z, W, p, q) = K((X+2)/Y, Z_1, W_1, q)$ is rational over K.

References

[Black 1999] E. V. Black, "Deformations of dihedral 2-group extensions of fields", *Trans. Amer. Math. Soc.* **351**:8 (1999), 3229–3241. MR 99m:12004 Zbl 0931.12005

[Bogomolov 1987] F. A. Bogomolov, "The Brauer group of quotient spaces of linear group actions", *Izv. Akad. Nauk SSSR Ser. Mat.* **51**:3 (1987), 485–516, 688. In Russian; translation in *Math. USSR Izv.* **30**:3 (1988), 455–485. MR 88m:16006 Zbl 0679.14025

[Chu and Kang 2001] H. Chu and M.-c. Kang, "Rationality of p-group actions", *J. Algebra* **237**:2 (2001), 673–690. MR 2001k:13008 Zbl 1023.13007

[Chu et al. 2004] H. Chu, S.-J. Hu, and M.-c. Kang, "Noether's problem for dihedral 2-groups", *Comment. Math. Helv.* **79**:1 (2004), 147–159. MR 2004i:12004 Zbl 02055141

[DeMeyer and McKenzie 2003] F. DeMeyer and T. McKenzie, "On generic polynomials", *J. Algebra* **261**:2 (2003), 327–333. MR 2003m:12007 Zbl 1021.12004

[Furtwängler 1925] P. Furtwängler, "Über Minimalbasen für Körper rationalen Funktionen", *Sitzungsber. Akad. Wiss. Wien* **134** (1925), 69–80.

[Garibaldi et al. 2003] S. Garibaldi, A. Merkurjev, and J.-P. Serre, *Cohomological invariants in Galois cohomology*, University Lecture Series **28**, American Mathematical Society, Providence, RI, 2003. MR 2004f:11034 Zbl 01959122

[Gröbner 1934] W. Gröbner, "Minimalbasis der Quaternionengruppe", *Monatshefte Math. Phys.* **41** (1934), 78–84.

[Haeuslein 1971] G. K. Haeuslein, "On the invariants of finite groups having an abelian normal subgroup of prime index", *J. London Math. Soc.* (2) **3** (1971), 355–360. MR 43 #3241 Zbl 0213.31002

[Hajja 1983] M. Hajja, "Rational invariants of meta-abelian groups of linear automorphisms", *J. Algebra* **80**:2 (1983), 295–305. MR 84g:20010 Zbl 0544.20007

[Hajja and Kang 1994] M. Hajja and M.-c. Kang, "Three-dimensional purely monomial group actions", *J. Algebra* **170**:3 (1994), 805–860. MR 95k:12008 Zbl 0831.14003

[Hajja and Kang 1995] M. Hajja and M.-c. Kang, "Some actions of symmetric groups", *J. Algebra* **177**:2 (1995), 511–535. MR 96i:20013 Zbl 0837.20054

[Hashimoto and Miyake 1999] K.-I. Hashimoto and K. Miyake, "Inverse Galois problem for dihedral groups", pp. 165–181 in *Number theory and its applications* (Kyoto, 1997), edited by S.

Kanemitsu and K. Győry, Dev. Math. **2**, Kluwer Acad. Publ., Dordrecht, 1999. MR 2001a:12010 Zbl 0965.12004

[Huppert 1967] B. Huppert, *Endliche Gruppen. I*, Grundlehren der Mathematischen Wissenschaften **134**, Springer, Berlin, 1967. MR 37 #302 Zbl 0217.07201

[Kang 2004] M.-c. Kang, "Introduction to Noether's problem for dihedral groups", *Algebra Colloq.* **11**:1 (2004), 71–78. MR 2058965 Zbl 1061.12005

[Kemper 1996] G. Kemper, "A constructive approach to Noether's problem", *Manuscripta Math.* **90**:3 (1996), 343–363. MR 97d:13005 Zbl 0865.12005

[Kemper and Malle 1999] G. Kemper and G. Malle, "Invariant rings and fields of finite groups", pp. 265–281 in *Algorithmic algebra and number theory* (Heidelberg, 1997), edited by B. H. Matzat et al., Springer, Berlin, 1999. MR 99m:13010 Zbl 0934.13002

[Kervaire and Vust 1989] M. Kervaire and T. Vust, "Fractions rationnelles invariantes par un groupe fini: quelques exemples", pp. 157–179 in *Algebraic Transformation groups and invariant theory*, edited by H. Kraft et al., DMV Sem. **13**, Birkhäuser, Basel, 1989. MR 1044591 Zbl 0703.14006

[Kuniyoshi 1955] H. Kuniyoshi, "On a problem of Chevalley", *Nagoya Math. J.* **8** (1955), 65–67. MR 16,993d Zbl 0065.02602

[Ledet 2000] A. Ledet, "Generic polynomials for quasi-dihedral, dihedral and modular extensions of order 16", *Proc. Amer. Math. Soc.* **128**:8 (2000), 2213–2222. MR 2000k:12003 Zbl 0952.12002

[Ledet 2001] A. Ledet, "Generic polynomials for Q_8-, QC-, and QQ-extensions", *J. Algebra* **237**:1 (2001), 1–13. MR 2001k:12012 Zbl 1004.12002

[Lenstra 1974] H. W. Lenstra, Jr., "Rational functions invariant under a finite abelian group", *Invent. Math.* **25** (1974), 299–325. MR 50 #289 Zbl 0292.20010

[Maeda 1989] T. Maeda, "Noether's problem for A_5", *J. Algebra* **125**:2 (1989), 418–430. MR 91c:12004 Zbl 0697.12018

[Miyata 1971] T. Miyata, "Invariants of certain groups, I", *Nagoya Math. J.* **41** (1971), 69–73. MR 42 #7804 Zbl 0211.06801

[Saltman 1982] D. J. Saltman, "Generic Galois extensions and problems in field theory", *Adv. in Math.* **43**:3 (1982), 250–283. MR 84a:13007 Zbl 0484.12004

[Saltman 1984] D. J. Saltman, "Noether's problem over an algebraically closed field", *Invent. Math.* **77**:1 (1984), 71–84. MR 85m:13006 Zbl 0546.14014

[Serre 1995] J.-P. Serre, "Résumé des cours de 1993–1994", pp. 91–98 in *Annuaire du Collège de France* (1994), 1995. Reprinted as pp. 435–442 in *Œuvres = Collected papers*, vol. IV: 1985–1998, Springer, Berlin, 2000. MR 2001e:01037 Zbl 0933.01034

[Swan 1983] R. G. Swan, "Noether's problem in Galois theory", pp. 21–40 in *Emmy Noether in Bryn Mawr* (Bryn Mawr, PA, 1982), edited by B. Srinivasan and J. Sally, Springer, New York, 1983. MR 84k:12013 Zbl 0538.12012

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