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HYPONORMALITY OF BLOCK TOEPLITZ OPERATORS

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We show that for a block Toeplitz operator T_G to be hyponormal, there is a matrix analogue of Cowen's condition for a scalar hyponormal Toeplitz operator, but an additional condtion, $G^*G = GG^*$, is needed. A detailed analysis is given for the hypornormality of T_G if G is a matrix trigomometric polynomial or G satisfies an extremal condition. Relevant results on kernels of block Hankel operators are also obtained.

1. Introduction

The block Toeplitz operator with matrix symbol $F \in L^\infty(\mathbb{C}^{n\times n})$, denoted T_F, acts on the vector-valued Hardy space of the unit disk, $H^2(\mathbb{C}^n)$ (n is finite), and is defined by $T_F h = P(Fh)$, where P is the projection from $L^2(\mathbb{C}^n)$ to $H^2(\mathbb{C}^n)$. The block Hankel operator with the same symbol, $H_F : H^2(\mathbb{C}^n) \longmapsto H^2(\mathbb{C}^n)$ is defined by $H_F h = J(I-P)Fh$, where J denotes the unitary operator from $H^2(\mathbb{C}^n)^\perp$ to $H^2(\mathbb{C}^n)$ given by $J(e^{-im\theta}) = e^{i(m-1)\theta}$ for $m \geq 1$. Notice that the shift operator S equals T_{zI_n}, where I_n is the $n\times n$ identity matrix.

In fact, a given bounded linear operator T is a Toeplitz operator if and only if $S^*TS = T$, and a given bounded linear operator H is a Hankel operator if and only if $HS = S^*H$. It can be readily verified that $T_F^* = T_{F^*}$, and letting $\widetilde{F} = F^*(\bar{z})$ we have $H_F^* = H_{\widetilde{F}}$. Let $G \in L^\infty(\mathbb{C}^{n\times n})$. There is a basic and important relation between Toeplitz and Hankel operators:

$$T_{FG} - T_F T_G = H_{F^*}^* H_G.$$

The normality, subnormality and hyponormality of a scalar Toeplitz operator have been studied in [Brown and Halmos 1963; Abrahamse 1976; Itô and Wong 1972; Cowen 1988a; 1988b; Hwang and Lee 2002]. Cowen [1988a] gave an elegant characterization of a scalar hyponormal Toeplitz operator: T_φ is hyponormal if and only if there exists a $k \in H^\infty$ with $\|k\|_\infty \leq 1$ satisfying $\varphi - k\bar{\varphi} \in H^2$.

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Here we discuss the hyponormality of a block Toeplitz operator. An extension of Cowen's result to the generalized Toeplitz operators was obtained in [Gu 1994]. Since the symbol of a generalized Toeplitz operator discussed in that reference can be operator-valued, it is assumed to be normal. Because the symbol of a block Toeplitz operator is a matrix-valued function, here we obtain more refined and explicit results than in the general operator-valued case. In particular we show the hyponormality of T_G will force G to be normal, that is, $G^*G = GG^*$. We prove that T_G is hyponormal if and only if $G^*G = GG^*$ and there exists a matrix $K(z) \in H^\infty(\mathbb{C}^{n\times n})$ such that $\|K\|_\infty \le 1$ and $G - KG^* \in H^\infty(\mathbb{C}^{n\times n})$. The extra condition in the block case, $G^*G = GG^*$, indicates, for example, that the hyponormality of a block Toeplitz operator T_G does depend on the constant term $G(0)$, while the hyponormality of a scalar Toeplitz operator T_φ does not depend on $\varphi(0)$.

Efforts have been made to give more explicit conditions for the hyponormality of a scalar Toeplitz operator T_φ. That is, how can we actually check if there is such a $k \in H^\infty$ with $\|k\|_\infty \le 1$ satisfying $\varphi - k\overline{\varphi} \in H^2$? It has been shown in [Zhu 1995; Gu 1994; Gu and Shapiro 2001] that verifying this condition is equivalent to a certain interpolation problem. It is more difficult to find a matrix $K(z) \in H^\infty(\mathbb{C}^{n\times n})$ such that $\|K\|_\infty \le 1$ and $G - KG^* \in H^\infty(\mathbb{C}^{n\times n})$ for the matrix-valued function G. One obvious reason is that matrix multiplication is not commutative. Another lies in the difficulty of factoring or dividing matrix-valued functions. We will show that, as in the scalar case, if $G(z)$ is a trigonometric matrix polynomial with invertible leading coefficient, then finding such a K is equivalent to solving a matrix-valued Carathéodory interpolation problem. We note that detailed analysis of the hyponormality of T_φ with scalar trigonometric or rational symbol φ was done recently in [Farenick and Lee 1996; Hwang and Lee 2002; Gu and Shapiro 2001].

More explicit conditions are given for the hyponormality of T_G with G satisfying an extremal condition. As a pleasant surprise, we obtain a simple characterization of a class of normal block Toeplitz operators.

As alluded to above, Hankel operators play an essential role in our approach. Let $G \in L^\infty(\mathbb{C}^{n\times n})$. Since the kernel of a block Hankel operator is an invariant subspace, by the Beurling–Lax–Halmos Theorem, $\operatorname{Ker} H_G = \Omega(z)H^2(\mathbb{C}^m)$ for some inner matrix $\Omega(z)$ and $m \le n$. In the next section we explore the connection between the symbol G and the inner function $\Omega(z)$. In particular we answer the question for what G, will $\Omega(z)$ be a square inner function (i.e., $m = n$)? For a scalar analytic polynomial φ, it is easy to see that $\operatorname{Ker} H_{\overline{\varphi}} = z^k H^2(\mathbb{C})$ where k is the degree of φ. But for a matrix analytic polynomial F, it is not trivial to identify $\operatorname{Ker} H_{F^*}$. Those discussions appear important for the study of the hyponormality of a single block Toeplitz operator in this paper and for the joint hyponormality of several block Toeplitz opeartors to be discussed in a future article. The results about kernels of block Hankel opearators are of independent interest.

2. The Kernel of a block Hankel operator

An inner matrix $\Omega(z) \in H^2(\mathbb{C}^{n\times m})$ is one satisfying $\Omega(z)^*\Omega(z) = I_m$ for all z on the unit circle. The kernel of a Hankel operator H_F is an invariant subspace. By the Beurling–Lax–Halmos Theorem,

$$\operatorname{Ker} H_F = \Omega(z)H^2(\mathbb{C}^m)$$

for some inner matrix $\Omega(z)$. A question relevant to our work is how the symbol F is related to the inner matrix $\Omega(z)$. In particular, for which symbol F is $\Omega(z)$ a square inner matrix?

Recall a scalar function $f \in L^2$ is of bounded type if f is a quotient p/q for some $p, q \in H^\infty$.

Definition 2.1. Let $F = [f_{ij}] \in L^\infty(\mathbb{C}^{n\times n})$. We say the matrix-valued function F is of bounded type if each entry f_{ij} is of bounded type.

An equivalent definition is that F is of bounded type if $F(z) = P(z)Q(z)^{-1}$ for some $P(z)$, $Q(z) \in H^\infty(\mathbb{C}^{n\times n})$ with $\det Q(z)$ not identically zero.

Theorem 2.2. *Let $F = [f_{ij}] \in L^\infty(\mathbb{C}^{n\times n})$. $\operatorname{Ker} H_F = \Theta(z)H^2(\mathbb{C}^n)$ for some square inner function $\Theta(z)$ if and only if F is of bounded type.*

Proof. Let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis of $\mathbb{C}^n$. If $\operatorname{Ker} H_F = \Theta(z)H^2(\mathbb{C}^n)$ for some square inner function $\Theta(z)$, then

$$(I - P)(F\Theta[\, e_1 \;\; e_2 \;\; \cdots \;\; e_n \,]) = 0.$$

Therefore

$$F(z)\Theta(z) = A(z) \text{ for some } A(z) \in H^\infty(\mathbb{C}^{n\times n}).$$

Multiply both sides by $\operatorname{adj}\Theta(z)$ gives

$$F(z)\det\Theta(z) = A(z)\operatorname{adj}\Theta(z).$$

Since $\operatorname{adj}\Theta(z) \in H^\infty(\mathbb{C}^{n\times n})$ and $\det\Theta(z)$ is a scalar inner function, $F(z)$ is of bounded type.

Now assume f_{ij} is of bounded type. Write $f_{ij} = p_{ij}/\omega_{ij} = p_{ij}\overline{\omega}_{ij}$ where p_{ij}, $\omega_{ij} \in H^\infty$ and ω_{ij} are inner. Thus

$$F(z) = [f_{ij}] = [p_{ij}\overline{\omega}_{ij}] = \overline{\theta(z)}B(z),$$

where $\theta = \prod_{i,j=1}^n \omega_{ij}$ is a scalar inner function and $B(z) \in H^\infty(\mathbb{C}^{n\times n})$. It is clear that

$$\operatorname{Ker} H_F \supset \theta I_n H^2(\mathbb{C}^n).$$

If $\operatorname{Ker} H_F = \Omega(z)H^2(\mathbb{C}^m)$ for some inner matrix $\Omega(z)$, then

$$\Omega(z)H^2(\mathbb{C}^m) \supset \theta I_n H^2(\mathbb{C}^n).$$

Therefore

$$\theta I_n = [\, \theta e_1 \;\; \theta e_2 \;\; \cdots \;\; \theta e_n \,] = \Omega(z) Q(z)$$

for some $Q(z) \in H^\infty(\mathbb{C}^{m\times n})$. Since $m \le n$, this can happen only when $m = n$. The proof is complete. $\square$

As a consequence of the proof we obtain a characterization of matrix-valued functions of bounded type:

Corollary 2.3. *Let $F = [f_{ij}] \in L^2(\mathbb{C}^{n\times n})$. Then F is of bounded type if and only if $F(z) = \overline{\theta(z)} A(z)$ where $A(z) \in H^2(\mathbb{C}^{n\times n})$ and $\theta(z)$ is a scalar inner function.*

Definition 2.4. Let $A(z), B(z) \in H^2(\mathbb{C}^{n\times n})$. We say $A(z)$ and $B(z)$ are not right coprime if there exists a nonconstant inner matrix $\Delta(z)$ such that

$$A(z) = A_1(z)\Delta(z), \;\; B(z) = B_1(z)\Delta(z)$$

for some $A_1(z), B_1(z) \in H^2(\mathbb{C}^{n\times n})$.

Corollary 2.5. *Let $F = [f_{ij}] \in L^\infty(\mathbb{C}^{n\times n})$.* Ker $H_F = \Theta(z)H^2(\mathbb{C}^n)$ *for some square inner matrix $\Theta(z)$ if and only if $F(z) = A(z)\Theta^*(z)$ where $A(z) \in H^\infty(\mathbb{C}^{n\times n})$ and $A(z)$ and $\Theta(z)$ are right coprime.*

Proof. We first prove the sufficiency. Assume $F(z) = A(z)\Theta^*(z)$, where $A(z) \in H^\infty(\mathbb{C}^{n\times n})$ and $A(z)$ and $\Theta(z)$ are right coprime. It is clear that $F(z)$ is of bounded type and Ker $H_F \supset \Theta(z)H^2(\mathbb{C}^n)$. If Ker $H_F = \Omega(z)H^2(\mathbb{C}^n)$ for some square inner matrix $\Omega(z)$, as in the proof of the theorem above, we also have

$$F(z) = A_1(z)\Omega^*(z) \text{ for some } A_1(z) \in H^\infty(\mathbb{C}^{n\times n}).$$

We need to show that $\Omega(z) = \Theta(z)$ (up to a right unitary constant). Since

$$\text{Ker } H_F = \Omega(z)H^2(\mathbb{C}^n) \supset \Theta(z)H^2(\mathbb{C}^n),$$

there exists an inner matrix $\Delta(z)$ such that

$$\Theta(z) = \Omega(z)\Delta(z).$$

Thus

$$F(z) = A_1(z)\Omega^*(z) = A(z)\Theta^*(z) = A(z)\Delta^*(z)\Omega^*(z).$$

Equivalently,

$$A(z) = A_1(z)\Delta(z),$$

that is, $\Delta(z)$ is a common inner (right) factor of $A(z)$ and $\Theta(z)$. Since $A(z)$ and $\Theta(z)$ are right coprime, $\Delta(z)$ is a unitary constant matrix.

The proof of necessity is similar. $\square$

In the scalar case, if $\varphi(e^{i\theta}) = \sum_{m=-k}^{-1} \varphi_m e^{im\theta}$ for some $k \ge 1$ and $\varphi_{-k} \ne 0$, then Ker $H_\varphi = z^k H^2$. For a matrix polynomial F, it is not as trivial to identify Ker H_F. We will deal with the following case frequently in this paper.

Lemma 2.6. *Assume $F \in L^\infty(\mathbb{C}^{n\times n})$ can be written as $F(e^{i\theta}) = \sum_{m=-k}^{-1} F_m e^{im\theta}$.*

(a) $\operatorname{Ker} H_F = z^k H^2(\mathbb{C}^n)$ *if and only if F_{-k} is invertible. In this case* $\operatorname{rank} H_F = nk$.

(b) *Assume that F_{-k} is invertible and that $G \in L^\infty(\mathbb{C}^{n\times n})$ can be written as $G(e^{i\theta}) = \sum_{m=-l}^{-1} G_m e^{im\theta}$ with $l \le k$. Then* $\operatorname{Ker}(H_F^* H_G) = \operatorname{Ker} H_G$ *and rank*$(H_F^* H_G) = \operatorname{rank} H_G$.

Proof. Assume F_{-k} is not invertible. Let $v \in \mathbb{C}^n$ be such that $F_{-k}v = 0$. Note that

$$(I-P)(Fvz^{k-1}) = F_{-k}v\bar{z} = 0.$$

That is, $vz^{k-1} \in \operatorname{Ker} H_F$. Therefore $\operatorname{Ker} H_F \neq z^k H^2(\mathbb{C}^n)$. If F_{-k} is invertible, it is easy to verify that $\operatorname{Ker} H_F = z^k H^2(\mathbb{C}^n)$. Since the codimension of $z^k H^2(\mathbb{C}^n)$ is nk, the rank of H_F is nk.

Now we prove (b). Since $H_G^* = H_{\widetilde{G}}$ and $\widetilde{G}$ is conjugate analytic of degree l, $\operatorname{Ker} H_G^* \supset z^l H^2(\mathbb{C}^n)$. Therefore

$$\operatorname{Range} H_G \subset \operatorname{Ker}(H_G^*)^\perp \subset H^2(\mathbb{C}^n) \ominus z^l H^2(\mathbb{C}^n).$$

If $h \in \operatorname{Ker}(H_F^* H_G)$ or $H_F^* H_G h = 0$, then $H_G h \in \operatorname{Ker} H_F^*$. By part (a), $\operatorname{Ker} H_F^* = \operatorname{Ker} H_{\widetilde{F}} = z^k H^2(\mathbb{C}^n)$. So $H_G h \in \operatorname{Range}(H_G) \cap z^k H^2(\mathbb{C}^n)$. By the assumption $l \le k$, we have $\operatorname{Range}(H_G) \cap z^k H^2(\mathbb{C}^n) = \{0\}$. We conclude $H_G h = 0$. This proves $\operatorname{Ker}(H_F^* H_G) = \operatorname{Ker} H_G$. The proof is complete. ☐

In the case where F_{-k} may be singular, we have the following result for $\operatorname{Ker} H_F$:

Lemma 2.7. *Assume $F \in L^\infty(\mathbb{C}^{n\times n})$ can be written as $F(e^{i\theta}) = \sum_{m=-k}^{-1} F_m e^{im\theta}$ with $F_{-k} \neq 0$. Then*

$$\operatorname{Ker} H_F \supset z^k H^2(\mathbb{C}^n) \quad \textit{and} \quad \operatorname{Ker} H_F \not\supset z^{k-1} H^2(\mathbb{C}^n). \tag{2–1}$$

Set $\operatorname{Ker} H_F = \Theta(z) H^2(\mathbb{C}^n)$ *for some square inner function $\Theta(z)$. Then there exists an inner function $\Omega(z)$ such that*

$$\Theta(z)\Omega(z) = \Omega(z)\Theta(z) = z^k I_n \quad \textit{and} \quad \Omega(0) \neq 0.$$

Proof. Relation (2–1) follows from the definitions. Since

$$\operatorname{Ker} H_F = \Theta(z) H^2(\mathbb{C}^n) \supset z^k H^2(\mathbb{C}^n),$$

there exists $\Omega(z) \in H^\infty(\mathbb{C}^{n\times n})$ such that $\Theta(z)\Omega(z) = z^k I_n$. That is, $\Omega(z) = z^k \Theta^*(z)$ on the unit circle. Thus $\Omega(z)$ is in fact an inner function. We prove $\Omega(0) \neq 0$ by contradiction. Assume $\Omega(z) = z^l \Omega_1(z)$ for some inner function $\Omega_1(z)$ and $l \ge 1$. Then

$$\Theta(z)\Omega_1(z) = z^{k-l} I_n \quad \text{and} \quad \Theta^*(z) = \bar{z}^{k-l}\Omega_1(z).$$

Since $\operatorname{Ker} H_F = \Theta(z) H^2(\mathbb{C}^n)$, by Corollary 2.5, $F(z) = A(z)\Theta^*(z)$. Therefore $F(z) = A(z)\Omega_1(z)\bar{z}^{k-l}$. But this implies that $\operatorname{Ker} H_F \supset z^{k-1} H^2(\mathbb{C}^n)$, contradicting

(2–1). The relation $\Omega(z)\Theta(z) = z^k I_n$ is valid for each fixed nonzero z inside the unit disk because $\Omega(z)$ is the inverse of $\Theta(z)$, and by continuity, $\Omega(z)\Theta(z) = z^k I_n$. The proof is complete. □

Here is an example of $\operatorname{Ker} H_F$, where F_{-k} is singular:

Example 2.8. Set

$$F = \begin{bmatrix} \bar{z}^2+\bar{z} & \bar{z}^2 \\ \bar{z}^2 & \bar{z}^2+\bar{z} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \bar{z}^2 + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \bar{z},$$

$$\Theta(z) = \frac{1}{\sqrt{2}} z \begin{bmatrix} 1 & z \\ -1 & z \end{bmatrix}, \qquad \Omega(z) = z^2\Theta^*(z) = \frac{1}{\sqrt{2}} \begin{bmatrix} z & -z \\ 1 & 1 \end{bmatrix}.$$

Then $\operatorname{Ker} H_F = \Theta(z)H^2(\mathbb{C}^2)$ and $\Theta(z)\Omega(z) = z^2 I_2$.

We end this section with an example of $\operatorname{Ker} H_F = \Omega(z)H^2(\mathbb{C}^m)$ when $\Omega(z)$ is not a square inner matrix.

Example 2.9. Let θ_0, θ_1 and θ_2 be three scalar inner functions such that θ_1 and θ_2 are coprime. Let $q \in L^\infty$ be such that $\operatorname{Ker} H_q = \{0\}$. Set

$$\Omega(z) = \frac{1}{\sqrt{2}} \begin{bmatrix} \theta_0\theta_1 \\ \theta_0\theta_2 \end{bmatrix}, \quad F = \begin{bmatrix} \overline{\theta_0\theta_1} & \overline{\theta_0\theta_2} \\ q\theta_2 & -q\theta_1 \end{bmatrix}.$$

Then $\operatorname{Ker} H_F = \Omega(z)H^2$. For $k = [k_1, k_2]^T \in H^2(\mathbb{C}^2)$, the equality $H_F k = 0$ is equivalent to

$$\overline{\theta_0\theta_1}k_1 + \overline{\theta_0\theta_2}k_2 = p_1 \quad \text{and} \quad q\theta_2 k_1 - q\theta_1 k_2 = p_2 \quad \text{for some } p_1, p_2 \in H^2.$$

Since $\operatorname{Ker} H_q = \{0\}$, we have $\theta_2 k_1 = \theta_1 k_2$. Since θ_1 and θ_2 are coprime, $k_1 = \theta_1 h_1$ and $k_2 = \theta_2 h_1$ for some $h_1 \in H^2$. Now the first equation above becomes $2\overline{\theta_0}h_1 = p_1$. Therefore

$$k = [k_1, k_2]^T = \tfrac{1}{2}[\theta_0\theta_1 p_1, \theta_0\theta_2 p_1]^T = \Omega(z)\sqrt{2}p_1 \in \Omega(z)H^2.$$

3. Hyponormality of a block Toeplitz operator

C. C. Cowen [1988a] characterized the hyponormality of a single Toeplitz operator in terms of its symbol. For a given $\varphi \in L^2$ we may write $\varphi = \varphi_+ + \overline{\varphi}_-$ with $\varphi_+, \varphi_- \in H^2$. Cowen's theorem states that

> *T_φ is hyponormal if and only if there exists a $k \in H^\infty$ with $\|k\|_\infty \le 1$ satisfying $\overline{\varphi}_- - k\overline{\varphi}_+ \in H^2$.*

An equivalent condition in [Nakazi and Takahashi 1993] is that $\varphi - k\overline{\varphi} \in H^\infty$. A generalization of Cowen's result for the hyponormality of generalized Toeplitz operators was obtained in [Gu 1994]. Since the symbol of a generalized Toeplitz operator discussed in that reference can be operator-valued, it is assumed to be normal. Here our contribution is to note that for a block Toeplitz operator T_G, the hyponormality of T_G will force G to be normal. That is, $G^*G = GG^*$. This follows essentially from the fact that a hyponormal constant matrix is normal.

Let $H^2(\mathbb{D})$ be the scalar Hardy space of the unit disk $\mathbb{D}$ and let $\partial\mathbb{D}$ be the unit circle. Let $k_z(w)$ be the normalized reproducing kernel of H^2,

$$k_z(w) = \frac{(1-|z|^2)^{1/2}}{1-\bar{z}w}.$$

For $f \in L^2(\partial\mathbb{D})$ with Fourier series

$$f(e^{i\theta}) = \sum_{m=-\infty}^{\infty} f_m e^{im\theta},$$

the Poisson integral of f, still denoted by f, is

$$f(z) = \langle f(w)k_z(w), k_z(w)\rangle = \sum_{m=-\infty}^{-1} f_m \bar{z}^{|m|} + \sum_{m=0}^{\infty} f_m z^m, \quad z \in \mathbb{D}.$$

The following lemma is probably known. We include a short proof.

Lemma 3.1. *Let $F \in L^\infty(\mathbb{C}^{n\times n})$. If T_F is a positive operator on $H^2(\mathbb{C}^n)$, then $F(z)$ is a positive semidefinite matrix for all $z \in \mathbb{D}$.*

Proof. Without loss of generality, we prove the lemma for $n = 2$. Let $h(w) = [\alpha_1 k_z(w), \alpha_2 k_z(w)]^T$ be a column vector in $H^2(\mathbb{C}^2)$, where α_1 and α_2 are complex numbers. Set $F(w) = [f_{ij}]$. The lemma follows from the following computation:

$$\langle T_F h, h\rangle = \langle Fh, h\rangle = \sum_{i,j=1}^{2} \big\langle f_{ij}(w)k_z(w), k_z(w)\big\rangle \alpha_j\overline{\alpha}_i = \sum_{i,j=1}^{2} f_{ij}(z)\alpha_j\overline{\alpha}_i \geq 0, \quad z \in D.$$

□

We will often make use of the following identities.

Lemma 3.2. *For $G, \Theta \in H^\infty(\mathbb{C}^{n\times n})$ with Θ inner and $F \in L^\infty(\mathbb{C}^{n\times n})$,*

$$H_F T_G = H_{FG}, \quad H_{GF} = T^*_{\widetilde{G}} H_F, \quad \textit{and} \quad H^*_F H_F - H^*_{\Theta F} H_{\Theta F} = H^*_F H_{\Theta^*} H^*_{\Theta^*} H_F.$$

Proof. The identity $H_F T_G = H_{FG}$ follows from the analyticity of G. The identity $H_{GF} = T^*_{\widetilde{G}} H_F$ can be obtained essentially by taking the adjoint of $H_F T_G = H_{FG}$. Note that

$$\begin{aligned} H^*_F H_F - H^*_{\Theta F} H_{\Theta F} &= H^*_F H_F - H^*_F T_{\widetilde{\Theta}} T^*_{\widetilde{\Theta}} H_F \\ &= H^*_F (I - T_{\widetilde{\Theta}} T^*_{\widetilde{\Theta}}) H_F = H^*_F H^*_{\widetilde{\Theta}^*} H_{\widetilde{\Theta}^*} H_F \\ &= H^*_F H_{\Theta^*} H^*_{\Theta^*} H_F. \end{aligned}$$

□

Theorem 3.3. *Let $G \in L^\infty(\mathbb{C}^{n\times n})$. The block Toeplitz operator T_G is hyponormal if and only if the following two conditions hold*:

(1) *G is normal, i.e. $G^*(z)G(z) = G(z)G^*(z)$ for almost every $z \in \partial\mathbb{D}$.*

(2) *There exists a matrix $K(z) \in H^\infty(\mathbb{C}^{n\times n})$ such that $\|K\|_\infty \le 1$ and $G - KG^* \in H^\infty(\mathbb{C}^{n\times n})$.*

Proof. Assuming T_G is hyponormal, we have

$$\begin{aligned} T_G^*T_G - T_GT_G^* &= T_G^*T_G - T_{G^*G} + T_{GG^*} - T_GT_G^* + T_{G^*G-GG^*} \\ &= H_{G^*}^*H_{G^*} - H_G^*H_G + T_{G^*G-GG^*} \ge 0. \end{aligned}$$

Therefore, for all $m \ge 1$, $h \in H^2(\mathbb{C}^n)$,

$$\langle H_{G^*}^*H_{G^*}z^mh, z^mh\rangle - \langle H_G^*H_Gz^mh, z^mh\rangle + \langle T_{G^*G-GG^*}z^mh, z^mh\rangle \ge 0. \tag{3–1}$$

Note that

$$\begin{gathered} \lim_{m\to\infty} H_{G^*}(z^mh) = (I-P)(z^mG^*h) = 0, \\ \langle T_{G^*G-GG^*}z^mh, z^mh\rangle = \langle S^{*m}T_{G^*G-GG^*}S^mh, h\rangle = \langle T_{G^*G-GG^*}h, h\rangle. \end{gathered}$$

Taking the limit in (3–1), we have

$$\langle T_{G^*G-GG^*}h, h\rangle \ge 0.$$

By Lemma 3.1, the Poisson integral of $G^*G - GG^*$ is positive semidefinite for $z \in \mathbb{D}$. The Poisson integral of $G^*G - GG^*$ is in general not equal to $G^*(z)G(z) - G(z)G^*(z)$ for $z \in \mathbb{D}$. By taking nontangential limits, we do know that the limit of the Poisson integral of $G^*G - GG^*$ is the same as $G^*(z)G(z) - G(z)G^*(z)$ for almost every $z \in \partial\mathbb{D}$. Therefore $G^*(z)G(z) - G(z)G^*(z) \ge 0$. But $G(z)$ is a finite matrix, so $G^*G = GG^*$ on $\partial\mathbb{D}$.

Now equation (3–1) becomes

$$H_{G^*}^*H_{G^*} \ge H_G^*H_G.$$

By [Gu 1994, Corollary 2] there exists a contractive co-analytic Toeplitz operator $T_{\tilde{K}}^*$ with $K(z) \in H^\infty(\mathbb{C}^{n\times n})$ and $H_G = T_{\tilde{K}}^*H_{G^*}$. Thus $\|K(z)\|_\infty = \|T_{\tilde{K}}^*\| \le 1$. By Lemma 3.2, $T_{\tilde{K}}^*H_{G^*} = H_{KG^*}$, so $H_G = T_{\tilde{K}}^*H_{G^*} = H_{KG^*}$ and $G - KG^* \in H^\infty(\mathbb{C}^{n\times n})$. Sufficiency follows essentially from the preceding argument. □

Condition (2) in Theorem 3.3 is analogous to the condition in Cowen's theorem for a scalar hyponormal Toeplitz operator. But condition (1) is new for the block case. This implies, for example, that the hyponormality of a block Toeplitz operator T_G does depend on the constant term $G(0)$, while the hyponormality of a scalar Toeplitz operator T_φ does not depend on $\varphi(0)$. If φ is analytic, then the scalar Toeplitz operator T_φ is hyponormal. In the block case we have the following results.

Corollary 3.4. *If $G \in H^\infty(\mathbb{C}^{n\times n})$, then the analytic block Toeplitz operator T_G is hyponormal if and only if $G^*(z)G(z) = G(z)G^*(z)$ for almost every $z \in \partial\mathbb{D}$.*

Corollary 3.5. *Let $G \in L^\infty(\mathbb{C}^{n\times n})$, and assume T_G is hyponormal.*

(1) $\operatorname{Ker} H_G \supset \operatorname{Ker} H_{G^*}$.

(2) *If G^* is of bounded type, so is G.*

Proof. Statement (1) follows from the proof of Theorem 3.3. If T_G is hyponormal, there exists a matrix such that $\|K\|_\infty \le 1$ and

$$G - KG^* = C(z) \quad \text{for some } C(z) \in H^\infty(\mathbb{C}^{n\times n}).$$

If G^* is of bounded type, by Corollary 2.3, $G^* = \overline{\theta(z)}A(z)$ where $\theta(z)$ is scalar inner function and $A(z) \in H^\infty(\mathbb{C}^{n\times n})$, thus

$$G = KG^* + C(z) = \overline{\theta(z)}\big(KA(z) + \theta(z)C(z)\big).$$

That is, G is of bounded type. □

Remark 3.6. In the scalar case, Abrahamse [1976] noted that if φ is not analytic and T_φ is hyponormal, then φ is of bounded type if and only if $\bar\varphi$ is of bounded type, which can also be seen from the above argument. The following example shows T_G is hyponormal, G is not analytic and is of bounded type, but G^* is not of bounded type. Let $f \in H^\infty$ be such that $\bar f$ is not of bounded type and set

$$G = \begin{bmatrix} z+\bar z & 0 \\ 0 & f \end{bmatrix}, \quad G^* = \begin{bmatrix} z+\bar z & 0 \\ 0 & \bar f \end{bmatrix}$$

It is clear that G^* is not of bounded type. Since G is diagonal, $G^*(z)G(z) = G(z)G^*(z)$. Furthermore

$$G - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} G^* = \begin{bmatrix} 0 & 0 \\ 0 & f \end{bmatrix}.$$

By Theorem 3.3, T_G is hyponormal.

Recent efforts have been made to give more explicit conditions for the hyponormality of the scalar Toeplitz operator T_φ. That is, how can we actually check if there is such a $k \in H^\infty$ with $\|k\|_\infty \le 1$ satisfying $\bar\varphi_- - k\bar\varphi_+ \in H^2$? Zhu [1995] showed that verifying this condition for a trigonometric polynomial symbol φ is a Carathéodory interpolation problem. Gu [1994] showed that verifying this condition for a rational symbol φ is a tangential Hermite–Fejér interpolation problem. Gu and Shapiro [2001] showed that verifying this condition for a bounded type symbol φ is a Sarason [1967] interpolation problem. We remark that if T_φ is hyponormal and its symbol φ is not of bounded type, then there exists only one

$k \in H^\infty$ satisfying $\overline{\varphi}_- - k\overline{\varphi}_+ \in H^2$. Because if $k_1, k_2 \in H^\infty$ satisfy $\varphi - k_1\overline{\varphi} = p_1$ and $\varphi - k_2\overline{\varphi} = p_2$ for some $p_1, p_2 \in H^\infty$, then

$$k_1\overline{\varphi} - k_2\overline{\varphi} = p_2 - p_1 \quad \text{and} \quad \overline{\varphi} = (p_2 - p_1)/(k_2 - k_1).$$

So $\overline{\varphi}$ is of bounded type and hence φ is of bounded type, a contradiction. We note that a detailed analysis of the hyponormality of T_φ with trigonometric polynomial symbol φ was done by Lee and his collaborators Farenick and Lee 1996; Hwang et al. 1999; Hwang and Lee 2002.

It is more difficult to verify condition (2) in Theorem 3.3 for $G \in L^\infty(\mathbb{C}^{n\times n})$. One obvious reason is that matrix multiplication is not commutative. Another reason lies in the difficult of factoring or dividing matrix-valued functions. We will show that, as in the scalar case, if $G(z)$ is a trigonometric matrix polynomial with invertible leading coefficient, verifying condition (2) for G amounts to a matrix Carathéodory interpolation problem. This will be done in Section 5. In the next section we try to understand condition (2) for G satisfying an extremal condition.

4. Hyponormality of T_G with $\|G_+\|_2 = \|G_-\|_2$

For a matrix-valued function $M \in L^2(\mathbb{C}^{n\times n})$, let

$$M(e^{i\theta}) = [m_{kl}(e^{i\theta})]_{n\times n} = \sum_{k=-\infty}^{\infty} M_k e^{ik\theta}$$

be the Fourier series of $M(e^{i\theta})$. The 2-norm of M is defined by

$$\|M\|_2^2 := \frac{1}{2\pi}\int_0^{2\pi} \operatorname{tr}\bigl(M(e^{i\theta})^* M(e^{i\theta})\bigr)\, d\theta = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \operatorname{tr}(M_k^* M_k)$$

where $\operatorname{tr} A$ is the trace of matrix A.

The condition $\|\varphi_+\|_2 = \|\varphi_-\|_2$ for the scalar symbol φ was introduced in [Gu and Shapiro 2001]. This condition was inspired by the work of Farenick and Lee [1997] on the hyponormality of T_φ with a circulant trigonometric polynomial symbol φ. Here we view this condition naturally as an extremal condition. Let $G = G_+ + G_0 + G_-^* \in L^\infty(\mathbb{C}^{n\times n})$ where $G_+, G_- \in H^2(\mathbb{C}^{n\times n})$ and G_0 is a constant matrix. By Theorem 3.3, if T_G is hyponormal then there exists a matrix $K(z) \in H^\infty(\mathbb{C}^{n\times n})$ such that $\|K\|_\infty \le 1$ and $G - KG^* \in H^\infty(\mathbb{C}^{n\times n})$. Therefore

$$\begin{aligned} G_-^* &= (I-P)(KG_+^*),\\ \|G_-\|_2 &= \|G_-^*\|_2 = \|(I-P)(KG_+^*)\|_2\\ &\le \|KG_+^*\|_2 \le \|K\|_\infty \|G_+^*\|_2 \le \|G_+^*\|_2 = \|G_+\|_2. \end{aligned} \tag{4–1}$$

We now characterize the hyponormality of T_G in the extremal case $\|G_+\|_2 = \|G_-\|_2$.

Theorem 4.1. *Let $G = G_+ + G_0 + G_-^* \in L^\infty(\mathbb{C}^{n\times n})$. Assume $\|G_+\|_2 = \|G_-\|_2$ and $\det G_+$ is not identically zero. Then T_G is hyponormal if and only if $G^*G = GG^*$ and $G_+ = G_-K$ for some inner matrix $K \in H^\infty(\mathbb{C}^{n\times n})$.*

Proof. We prove necessity. As above, by Theorem 3.3, if T_G is hyponormal, then there exists a matrix $K(z) \in H^\infty(\mathbb{C}^{n\times n})$ such that $\|K\|_\infty \le 1$ and $G_-^* - KG_+^* \in H^\infty(\mathbb{C}^{n\times n})$. Therefore

$$\begin{aligned} G_-^* &= (I-P)(KG_+^*), \\ \|G_-\|_2 &= \|G_-^*\|_2 = \|(I-P)(KG_+^*)\|_2 \\ &\le \|KG_+^*\|_2 \le \|K\|_\infty \|G_+^*\|_2 \le \|G_+^*\|_2 = \|G_+\|_2. \end{aligned}$$

Since $\|G_+\|_2 = \|G_-\|_2$, we must have equality everywhere. But the equality $\|(I-P)(KG_+^*)\|_2 = \|KG_+^*\|_2$ implies that $KG_+^* \in H^2(\mathbb{C}^{n\times n})^\perp$. Therefore

$$G_-^* = (I-P)(KG_+^*) = KG_+^*. \tag{4–2}$$

Now $\|G_+^*\|_2 = \|G_-\|_2 = \|KG_+^*\|_2$ implies that

$$\begin{aligned} \|G_+^*\|_2^2 - \|KG_+^*\|_2^2 &= \frac{1}{2\pi}\int_0^{2\pi} \operatorname{tr}(G_+G_+^*) - \operatorname{tr}(G_+K^*KG_+^*)\, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \operatorname{tr}(G_+(I-K^*K)G_+^*)\, d\theta = \left\|G_+(I-K^*K)^{1/2}\right\|_2^2. \end{aligned}$$

Here we use the fact that $\|K\|_\infty \le 1$ and $I - K^*K$ is positive. Thus $G_+(I-K^*K)^{1/2}$ vanishes. Since $\det G_+$ is analytic and $\det G_+$ is nonzero almost everywhere on ∂D, we have $(I - K^*K) = 0$. That is, K is an inner matrix. Left multiplying (4–2) by K^* and taking adjoints we get $G_-K = G_+$, as desired.

Sufficiency clearly follows from Theorem 3.3. □

Corollary 4.2. *Let $G = G_+ + G_-^* \in L^\infty(\mathbb{C}^{n\times n})$. If $G_+ = G_-K = KG_-$, where K is an inner matrix, then T_G is hyponormal.*

Proof. By the preceding theorem we only need to verify that $G^*G = GG^*$. By assumption

$$\begin{aligned} G &= G_+ + G_-^* = G_-K + G_-^* = (G_- + G_-^*K^*)K = G^*K, \\ G &= G_+ + G_-^* = KG_- + G_-^* = K(G_- + K^*G_-^*) = KG^*. \end{aligned}$$

Therefore

$$G^*G = G^*KG^* = KG^*G^* = GG^*. \qquad \square$$

We now identify normal block Toeplitz operators.

Theorem 4.3. *Let $G = G_+ + G_0 + G_-^* \in L^\infty(\mathbb{C}^{n\times n})$. Assume* $\det G_+$ *is not identically zero. Then T_G is normal if and only if $G^*G = GG^*$ and $G_+ = G_-U$ for some constant unitary matrix U.*

Proof. T_G is normal if and only if both T_G and T_{G^*} are hyponormal. If T_G is hyponormal, then, as in (4–1), we have $\|G_-\|_2 \le \|G_+\|_2$. Similarly, if T_{G^*} is hyponormal, $\|G_+\|_2 \le \|G_-\|_2$. Thus $\|G_+\|_2 = \|G_-\|_2$. By Theorem 4.1, the hyponormality of T_G and T_{G^*} implies that $G_+ = G_-K_1$ and $G_- = G_+K_2$ for some inner matrices K_1 and K_2. Therefore

$$G_+ = G_-K_1 = G_+K_2K_1.$$

Equivalently,

$$G_+(I - K_2K_1) = 0.$$

Since $\det G_+$ is analytic and $\det G_+$ is nonzero almost everywhere on ∂D, we have $(I - K_2K_1) = 0$, and $K_1 = K_2^*$. Therefore $K_1 = K_2^* = U$ for some constant unitary matrix U. □

A criterion for the normality of a block Toeplitz operator T_G was given in [Gu and Zheng 1998, Corollary 8] for a general symbol G, where $\det G_+$ can be identically zero. The characterization there is complicated, and it is a consequence of a result on zero products of block Hankel operators. It is a pleasant surprise that the study of the hyponormality of T_G leads to the simple characterization above for normal block Toeplitz operators. It seems difficult to derive the condition $G_+ = G_-U$ from the criteria given in [Gu and Zheng 1998, Corollary 8] under the assumption that $\det G_+$ is not identically zero.

Corollary 4.4. *Let $G = G_+ + G_-^* \in L^\infty(\mathbb{C}^{n\times n})$. If $G_+ = G_-U = UG_-$ for some constant unitary matrix U, then T_G is normal.*

The characterization of a normal scalar Toeplitz operator in [Brown and Halmos 1963] can be formulated as follows: T_φ is normal if and only if $\varphi_+ = \alpha\varphi_-$ for some unimodular constant α.

5. Interpolation and block Toeplitz operators with trigonometric polynomial symbols

We now show that verifying the hyponormality of T_G for a class of trigonometric symbols G is equivalent to a matrix Carathéodory interpolation problem. Let $G \in L^\infty(\mathbb{C}^{n\times n})$ be a matrix trigonometric polynomial,

$$G = G_+ + G_0 + G_-^* = \sum_{j=-m}^{M} G_j e^{ij\theta}.$$

We will assume that the leading coefficient G_M is invertible and of course that G_{-m} is not zero. It is more convenient to write

$$
\begin{aligned}
G_+^* &= \sum_{j=1}^{M} G_j^* \bar{z}^j = \bar{z}^M A(z) \quad \text{with } A(z) = \sum_{j=0}^{M-1} G_{M-j}^* z^j, \\
G_-^* &= \sum_{j=-m}^{-1} G_j \bar{z}^j = \bar{z}^m B(z) \quad \text{with } B(z) = \sum_{j=0}^{m-1} G_{-m+j} z^j.
\end{aligned}
\tag{5–1}
$$

Assume T_G is hyponormal. By Theorem 3.3, there exists $K(z) \in H^\infty(\mathbb{C}^{n\times n})$ such that $\|K\|_\infty \le 1$ and $G_-^* - KG_+^* \in H^\infty(\mathbb{C}^{n\times n})$. That is,

$$\bar{z}^m B(z) - K(z)\bar{z}^M A(z) = Q_0(z) \quad \text{for some } Q_0(z) \in H^\infty(\mathbb{C}^{n\times n}).$$

If $m > M$, this becomes $B(z) - z^{m-M}K(z)A(z) = z^m Q_0(z)$. This would imply $B(0) = G_{-m}^* = 0$. Therefore $M \ge m$ and

$$z^{M-m}B(z) - K(z)A(z) = z^M Q_0(z). \tag{5–2}$$

By assumption, $A(0) = G_M^*$ is invertible. The equation above implies that

$$K(z) = z^{M-m}\widehat{K}(z). \tag{5–3}$$

Equation (5–2) becomes

$$B(z) - \widehat{K}(z)A(z) = z^m Q_0(z). \tag{5–4}$$

Let $\widehat{K}_p(z)$ be the unique analytic polynomial of degree $m-1$ satisfying (5–4). Set

$$\widehat{K}_p(z) = \sum_{i=0}^{m-1} K_i z^i. \tag{5–5}$$

Using notation as in (5–1), equation (5–4) becomes

$$\sum_{i=0}^{j} K_i G_{M-j+i}^* = G_{-m+j}, \quad j = 0, 1, \ldots, m-1.$$

Equivalently,

$$
[\, K_0 \;\; K_1 \;\; \cdots \;\; K_{m-1} \,]
\begin{bmatrix}
G_M^* & G_{M-1}^* & \cdots & G_{M-m+1}^* \\
0 & G_M^* & \ddots & \vdots \\
\vdots & \ddots & \ddots & G_{M-1}^* \\
0 & \cdots & 0 & G_M^*
\end{bmatrix}
= [\, G_{-m} \;\; G_{-m+1} \;\; \cdots \;\; G_{-1} \,]. \tag{5–6}
$$

The $m \times m$ block matrix on the left side is invertible since G_M^* is invertible. Therefore $\widehat{K}_p(z)$ is uniquely determined by the equation above. Let $\widehat{K}(z) \in H^\infty(\mathbb{C}^{n\times n})$ be another solution of (5–4). Then

$$[\widehat{K}(z) - \widehat{K}_p(z)]A(z) = -z^m Q_1(z) \quad \text{for some } Q_1(z) \in H^\infty(\mathbb{C}^{n\times n}).$$

Again, by assumption, $A(0) = G_M^*$ is invertible, so the preceding equation implies

$$\widehat{K}(z) = \widehat{K}_p(z) - z^m Q(z) \quad \text{for some } Q(z) \in H^\infty(\mathbb{C}^{n\times n}).$$

The existence of some $K(z) = z^{M-m}\widehat{K}(z) \in H^\infty(\mathbb{C}^{n\times n})$ such that $\|K\|_\infty \le 1$ and $G_-^* - KG_+^* \in H^\infty(\mathbb{C}^{n\times n})$ is equivalent to

$$\inf\left\{\|\widehat{K}_p(z) - z^m Q(z)\|_\infty : Q(z) \in H^\infty(\mathbb{C}^{n\times n})\right\} \le 1.$$

This is exactly the matrix Carathéodory interpolation problem. Summarizing the discussion, we have the following result for hyponormal block Toeplitz operators with trigonometric polynomial symbols.

Theorem 5.1. *Let G be a matrix trigonometric polynomial with invertible leading coefficient G_M with notation as above. The following statements are equivalent*:

(a) *T_G is hyponormal.*

(b) *$G^*G = GG^*$ and*

$$\inf\{\|\widehat{K}_p(z) - z^m Q(z)\|_\infty : Q(z) \in H^\infty(\mathbb{C}^{n\times n})\} \le 1.$$

(c) *$G^*G = GG^*$ and the $m \times m$ block Toeplitz matrix*

$$T(\widehat{K}_p) = \begin{bmatrix} K_0 & 0 & 0 \\ \vdots & \ddots & 0 \\ K_{m-1} & \cdots & K_0 \end{bmatrix} \tag{5–7}$$

is a contraction.

Proof. The equivalence of (a) and (b) follows from the preceding discussion. The equivalence of (b) and (c) is a classical result on Carathéodory interpolation; see [Foias and Frazho 1990]. □

Corollary 5.2. *Let $G \in L^\infty(\mathbb{C}^{n\times n})$ be a matrix trigonometric polynomial*,

$$G = G_+ + G_0 + G_-^* = \sum_{j=-m}^{M} G_j e^{ij\theta}.$$

Assume that G_M is invertible, G_{-m} is not zero, and T_G is hyponormal.

(a) *$M \ge m$ and $G_M G_M^* \ge G_{-m}^* G_{-m}$.*

(b) *$n(M-m) \le \operatorname{rank}[T_G^*, T_G] \le nM$.*

(c) *Let* $K_p(z) = z^{M-m}\widehat{K}_p(z)$ *where* $\widehat{K}_p(z)$ *is defined by* (5–5) *and* (5–6). *Then*

$$[T_G^*, T_G] = H_{G^*}^*(I - T_{\widetilde{K}_p}T_{\widetilde{K}_p}^*)H_{G^*}.$$

Proof. Assume T_G is hyponormal. By Theorem 5.1, $T(\widehat{K}_p)$ is a contraction. In particular K_0 is a contraction. So $K_0G_M^* = G_{-m}$ implies that $G_MG_M^* \geq G_{-m}^*G_{-m}$. This proves (a). It follows from

$$[T_G^*, T_G] = H_{G^*}^*H_{G^*} - H_G^*H_G \leq H_{G^*}^*H_{G^*}$$

that $\operatorname{rank}[T_G^*, T_G] \leq \operatorname{rank} H_{G^*}$. But the rank of H_{G^*} is nM, by Lemma 2.6. Since $G - K_p(z)G^* \in H^\infty(\mathbb{C}^{n\times n})$,

$$\begin{aligned}[T_G^*, T_G] &= H_{G^*}^*H_{G^*} - H_G^*H_G = H_{G^*}^*H_{G^*} - H_{K_pG^*}^*H_{K_pG^*}\\ &= H_{G^*}^*H_{G^*} - H_{G^*}^*T_{\widetilde{K}_p}T_{\widetilde{K}_p}^*H_{G^*} = H_{G^*}^*(I - T_{\widetilde{K}_p}T_{\widetilde{K}_p}^*)H_{G^*}.\end{aligned}$$

This is part (c). A subtle point here is that $I - T_{\widetilde{K}_p}T_{\widetilde{K}_p}^*$ may not be positive, but $[T_G^*, T_G]$ is positive. Let $K(z) \in H^\infty(\mathbb{C}^{n\times n})$ be such that $\|K\|_\infty \leq 1$ and $G - KG^* \in H^\infty(\mathbb{C}^{n\times n})$. As in (5–3) above, $K(z) = z^{M-m}\widetilde{Y(z)}$ for some $Y(z) \in H^\infty(\mathbb{C}^{n\times n})$ with $\|Y\|_\infty \leq 1$ and

$$\begin{aligned}[T_G^*, T_G] &= H_{G^*}^*(I - T_{\widetilde{K}}T_{\widetilde{K}}^*)H_{G^*} = H_{G^*}^*(I - T_{z^{M-m}Y(z)}T_{z^{M-m}Y(z)}^*)H_{G^*}\\ &= H_{G^*}^*(I - T_{z^{M-m}I_n}T_{z^{M-m}I_n}^* + T_{z^{M-m}I_n}T_{z^{M-m}I_n}^* - T_{z^{M-m}Y(z)}T_{z^{M-m}Y(z)}^*)H_{G^*}\\ &= H_{G^*}^*(I - T_{z^{M-m}I_n}T_{z^{M-m}I_n}^*)H_{G^*}\\ &\qquad + H_{G^*}^*(T_{z^{M-m}I_n}T_{z^{M-m}I_n}^* - T_{z^{M-m}Y(z)}T_{z^{M-m}Y(z)}^*)H_{G^*},\end{aligned}$$

where I_n is the $n \times n$ identity matrix. Since $\|T_{Y(z)}\| \leq 1$,

$$H_{G^*}^*(I - T_{z^{M-m}I_n}T_{z^{M-m}I_n}^*)H_{G^*} = H_{G^*}^*H_{\bar{z}^{M-m}I_n}H_{\bar{z}^{M-m}I_n}^*H_{G^*},$$

$$T_{z^{M-m}I_n}T_{z^{M-m}I_n}^* - T_{z^{M-m}Y(z)}T_{z^{M-m}Y(z)}^* = T_{z^{M-m}I_n}(I - T_{Y(z)}T_{Y(z)}^*)T_{z^{M-m}I_n}^* \geq 0.$$

Therefore $[T_G^*, T_G] \geq H_{G^*}^*H_{\bar{z}^{M-m}I_n}H_{\bar{z}^{M-m}I_n}^*H_{G^*}$, and

$$\operatorname{rank}[T_G^*, T_G] \geq \operatorname{rank}(H_{G^*}^*H_{\bar{z}^{M-m}I_n}).$$

By Lemma 2.6,

$$\operatorname{rank}(H_{G^*}^*H_{\bar{z}^{M-m}I_n}) = \operatorname{rank}(H_{\bar{z}^{M-m}I_n}) = n(M-m),$$

so $\operatorname{rank}[T_G^*, T_G] \geq n(M-m)$. □

The next result treats the extremal case $G_MG_M^* = G_{-m}^*G_{-m}$.

Corollary 5.3. *Let* $G = G_+ + G_0 + G_-^* = \sum_{j=-m}^{M} G_j e^{ij\theta}$ *be a matrix trigonometric polynomial, where*

$$G_+ = \sum_{j=1}^{M} G_j e^{ij\theta} \quad \textit{and} \quad G_-^* = \sum_{j=-m}^{-1} G_j e^{ij\theta}.$$

Assume that G_M *is invertible and* $G_M G_M^* = G_{-m}^* G_{-m}$. *Then* T_G *is hyponormal if and only if the following two conditions hold*:

(a) $G^*G = GG^*$.

(b) *There exists a constant unitary matrix* U *such that*

$$G_{-(m-j)} = U G_{M-j}, \quad j = 0, 1, \dots, m-1.$$

If this is the case we have

$$[T_G^*, T_G] = H_{G^*}^* H_{\bar{z}^{M-m} I_n} H_{\bar{z}^{M-m} I_n}^* H_{G^*} \quad \textit{and} \quad \operatorname{rank}[T_G^*, T_G] = n(M-m).$$

Proof. Assume T_G is hyponormal. Let U be the constant unitary matrix such that $UG_M^* = G_{-m}$. By (5–6), $K_0 G_M^* = G_{-m}$; thus $K_0 = U$. Since $T(\widehat{K}_p)$ defined by (5–7) is a contraction, we must have $K_i = 0$ for $i = 1, \dots, m-1$. Thus $\widehat{K}_p(z) = K_0 = U$ and $K_p(z) = z^{M-m} U$. Referring back to (5–4), $G_-^* - K_p G_+^* \in H^\infty(\mathbb{C}^{n\times n})$ becomes

$$\sum_{j=-m}^{-1} G_j e^{ij\theta} - e^{i(M-m)\theta} U \sum_{j=1}^{M} G_j^* e^{-ij\theta} = Q_0(z)$$

for some $Q_0(z) \in H^\infty(\mathbb{C}^{n\times n})$. Thus condition (b) holds. The formulas for $[T_G^*, T_G]$ and $\operatorname{rank}[T_G^*, T_G]$ follow from the proof of the previous corollary. The other direction of this result is also clear from the argument above. □

Scalar versions of some results in this section are found in [Farenick and Lee 1996; Hwang and Lee 2002; Zhu 1995].

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