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Let Γ be a Gromov hyperbolic group with a finite set A of generators. We prove that $h_{\mathrm{top}}(\Sigma(\infty)) \leq k_\infty^-(\lambda_A) \leq \mathrm{gr}(\Gamma, A)$, where $\mathrm{gr}(\Gamma, A)$ is the growth entropy, $h_{\mathrm{top}}(\Sigma(\infty))$ is the Coornaert–Papadopoulos topological entropy of the subshift $\Sigma(\infty)$ associated with (Γ, A), and $k_\infty^-(\lambda_A)$ is Voiculescu's numerical invariant, which is an obstruction to the existence of quasicentral approximate units relative to the Macaev norm for a tuple of unitary operators $\lambda_A = (\lambda_a)_{a\in A}$ in the left regular representation of Γ. We also prove that these three quantities are equal for a hyperbolic group splitting over a finite group.

1. Introduction

Let Γ be a finitely generated group with a finite generating set A. We consider the family $\lambda_A = (\lambda_a)_{a\in A}$ of left translation operators on $\ell^2(\Gamma)$, specifically the value of Voiculescu's numerical invariant k_∞^- for this family. Voiculescu introduced this invariant $k_\infty^-(\tau)$, for a tuple τ of Hilbert space operators, in a remarkable series of papers [1979; 1981; 1990; David and Voiculescu 1990] to deal with perturbation problems.

For the case of free groups, Voiculescu gave an estimate for $k_\infty^-(\lambda_A)$; we obtain its exact value. For the case of certain amalgamated free product groups, we proved in [Okayasu 2004] that $k_\infty^-(\lambda_A)$ equals the growth entropy $\mathrm{gr}(\Gamma, A)$ of Γ with respect to A. These groups are Gromov hyperbolic groups in the sense of [Gromov 1987]. In [Okayasu 2004], we showed that if a subshift Σ satisfies a certain condition, then $k_\infty^-(\tau) = h_{\mathrm{top}}(\Sigma)$ for the family τ of creation operators on the Fock space associated with Σ, which is used to define the Matsumoto algebra [1997] associated to Σ. (Here $h_{\mathrm{top}}(\Sigma)$ is the topological entropy of Σ.) This equation holds for every shift of finite type.

M. Coornaert and A. Papadopoulos [2001] have shown the following: Let X be a proper geodesic metric space that is δ-hyperbolic. The class of functions on X called horofunctions (a generalization of Busemann functions) gives a description of the boundary at infinity ∂X. When X is the Cayley graph of a hyperbolic group

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Γ, the space of cocycles associated with horofunctions that take integral values on the vertices is a shift of finite type $\Sigma(\infty)$. (See also [Gromov 1987].)

Continuing this line of investigation, we first determine(Theorem 1.1) a lower bound for $k_\infty^-(\lambda_A)$ in terms of the topological entropy $h_{\text{top}}(\Sigma(\infty))$, for arbitrary hyperbolic groups. We therefore have

$$h_{\text{top}}(\Sigma(\infty)) \le k_\infty^-(\lambda_A) \le \text{gr}(\Gamma, A),$$

since the upper bound was already given in [Okayasu 2004]. We also show here that if a given hyperbolic group Γ splits over a finite group, the equation $h_{\text{top}}(\Sigma(\infty)) = \text{gr}(\Gamma, A)$ holds for a certain finite generating set A of Γ (Corollary 1.2). As a consequence, the inequalities turn into an equalities for such groups:

$$h_{\text{top}}(\Sigma(\infty)) = k_\infty^-(\lambda_A) = \text{gr}(\Gamma, A).$$

It was already known from [Voiculescu 1993] that $k_\infty^-(\lambda_A) \neq 0$ for every nonelementary hyperbolic group Γ, because Γ is nonamenable.

Notation. We denote by $\Sigma(\infty)$ the shift of finite type relative to (Γ, A), constructed in [Coornaert and Papadopoulos 2001].

Theorem 1.1. *Let Γ is a Gromov hyperbolic group with a finite generating set A and λ its left regular representation. Set $\lambda_A = (\lambda_a)_{a\in A}$. Then we have*

$$h_{\text{top}}(\Sigma(\infty)) \le k_\infty^-(\lambda_A) \le \text{gr}(\Gamma, A).$$

Corollary 1.2. *Let Γ is a nonelementary hyperbolic group with a finite generating set A, λ its left regular representation and $\lambda_A = (\lambda_a)_{a\in A}$. Suppose that either*

(1) *Γ can be written nontrivially as a free product $G_1 * G_2$ and $A = F_1 \cup F_2$ for some finite generating sets F_1, F_2 of G_1, G_2; or*

(2) *Γ has a form of a free product $G_1 *_H G_2$ with finite amalgamated subgroup H, which is properly contained in both factors and of index greater than 2 in at least one factor, and $A = F_1 \cup F_2$ for some finite generating sets F_1, F_2 of G_1, G_2, containing H; or*

(3) *Γ is an HNN extension*

$$G *_H \theta = \langle G, x \mid hx = x\theta(h) \text{ for } h \in H\rangle,$$

where H is a proper finite subgroup of G and $A = F \cup \{x, x^{-1}\}$ for some finite generating set F of G, which contains both H and $\theta(H)$.

Then $k_\infty^-(\lambda_A) = \text{gr}(\Gamma, A) = h_{\text{top}}(\Sigma(\infty))$.

2. Preliminaries

Voiculescu's perturbation theory. Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space and let $\mathbb{B}(\mathcal{H})$, $\mathbb{K}(\mathcal{H})$ denote, respectively, the spaces of bounded linear operators and compact operators on $\mathcal{H}$. A *symmetrically normed ideal* $(\mathfrak{S}, \|\cdot\|_{\mathfrak{S}})$ is an ideal $\mathfrak{S}$ of $\mathbb{K}(\mathcal{H})$ which is a Banach space endowed with the norm $\|\cdot\|_{\mathfrak{S}}$ satisfying

$$\|XTY\|_{\mathfrak{S}} \leq \|X\| \cdot \|T\|_{\mathfrak{S}} \cdot \|Y\|$$

for $T \in \mathfrak{S}$ and $X, Y \in \mathbb{B}(\mathcal{H})$, where $\|\cdot\|$ is the operator norm on $\mathbb{B}(\mathcal{H})$.

It is well-known that the Schatten p-classes $\mathscr{C}_p(\mathcal{H})$ are symmetrically normed ideals. So are the ideals $\mathscr{C}_p^-(\mathcal{H})$ defined for $1 \leq p \leq \infty$ by the norm

$$\|T\|_p^- = \sum_{j=1}^{\infty} \lambda_j j^{-1+1/p}$$

(where $\lambda_1 \geq \lambda_2 \geq \cdots$ are the eigenvalues of $(T^*T)^{1/2}$); they are important for perturbation theory. The particular case $\mathscr{C}_\infty^-(\mathcal{H})$ is also known as the *Macaev ideal*. Note that $\mathscr{C}_1^-(\mathcal{H}) = \mathscr{C}_1(\mathcal{H})$ but

$$\mathscr{C}_p^-(\mathcal{H}) \subsetneq \mathscr{C}_p(\mathcal{H}) \subsetneq \mathscr{C}_q^-(\mathcal{H}) \quad \text{if } 1 < p < q.$$

The dual $\mathfrak{S}^*$, where the duality is given by the bilinear form $(X, Y) \mapsto \mathrm{Tr}(XY)$, is again a normed ideal. We have $\mathscr{C}_p(\mathcal{H})^* = \mathscr{C}_q(\mathcal{H})$, where $p > 1$ and $1/p + 1/q = 1$. Moreover $\mathscr{C}_p^-(\mathcal{H})^* = \mathscr{C}_q^+(\mathcal{H})$, where $\mathscr{C}_q^+(\mathcal{H})$ consists of all $T \in \mathbb{K}(\mathcal{H})$ such that

$$\|T\|_q^+ = \sup_k \frac{\sum_{j=1}^k \lambda_j}{\sum_{j=1}^k j^{-1/q}} < \infty.$$

Let $\mathfrak{S}$ be a symmetrically normed ideal of $\mathbb{K}(\mathcal{H})$. For an N-tuple $\tau = (T_1, \ldots, T_N)$ of bounded linear operators on $\mathcal{H}$, we define

$$k_{\mathfrak{S}}(\tau) = \liminf_{A \in \mathbb{F}(H)_1^+} \max_{1 \leq i \leq N} \|[A, T_i]\|_{\mathfrak{S}},$$

where the inferior limit is taken with respect to the natural order on

$$\mathbb{F}(H)_1^+ = \{T \in \mathbb{K}(\mathcal{H}) \mid T : \text{finite rank}, \ 0 \leq T \leq I\}$$

and $[A, B] = AB - BA$. We write $k_p^-(\tau)$ when $\mathfrak{S} = \mathscr{C}_p^-(\mathcal{H})$.

We see from the definition that $k_{\mathfrak{S}}(\tau)$ measures the obstruction to the existence of a sequence $\{A_n\}_{n=1}^\infty \subseteq \mathbb{F}(\mathcal{H})_1^+$ such that $A_n \nearrow I$ and $\lim_{n\to\infty} \|[A_n, T_i]\|_\Phi = 0$ for $1 \leq i \leq N$. If such a sequence exists, it is called a *quasicentral approximate unit* for τ relative to $\mathfrak{S}$.

Proposition 2.1 [Voiculescu 1990, Proposition 2.1]. *Let* $\tau = (T_1, \dots, T_N) \in \mathbb{B}(\mathcal{H})^N$ *and* $X_i \in \mathcal{C}_1^+(\mathcal{H})$ *for* $i = 1, \dots, N$. *If*

$$\sum_{i=1}^{N} [X_i, T_i] \in \mathcal{C}_1(\mathcal{H}) + \mathbb{B}(\mathcal{H})_+,$$

then

$$\left| \operatorname{Tr}\left(\sum_{i=1}^{N} [X_i, T_i] \right) \right| \le k_\infty^-(\tau) \sum_{a=1}^{N} \|X_i\|_1^{\widetilde{+}},$$

where $\|X_i\|_1^{\widetilde{+}} = \inf_{Y \in \mathbb{F}(\mathcal{H})} \|X_i - Y\|_1^+$.

Proposition 2.2 [Gohberg and Kreĭn 1969, Theorem 14.1]. *For* $T \in \mathcal{C}_1^+(\mathcal{H})$, *we have*

$$\|T\|_1^{\widetilde{+}} = \limsup_{n \to \infty} \frac{\sum_{j=1}^n s_j(T)}{\sum_{j=1}^n 1/j}.$$

Subshifts. We briefly define the necessary concepts from symbolic dynamics; see [Lind and Marcus 1995] for a more leisurely introduction.

Let $\mathcal{A}$ be a finite alphabet and $\mathcal{A}^{\mathbb{N}}$ the one-sided infinite product space $\prod_{i=0}^{\infty} \mathcal{A}$ with the product topology (of discrete topologies). The *shift map* σ on $\mathcal{A}^{\mathbb{N}}$ is given by $(\sigma(x))_i = x_{i+1}$ for $i \in \mathbb{N}$. A *word* over $\mathcal{A}$ is a finite sequence $w = (a_1, \dots, a_n)$ with $a_i \in \mathcal{A}$. For $x \in \mathcal{A}^{\mathbb{N}}$ and a word $w = (a_1, \dots, a_n)$, we say that w *occurs in* x if there is an index i such that $x_i = a_1, \dots, x_{i+n-1} = a_n$. For a collection $\mathcal{F}$ of words over $\mathcal{A}^{\mathbb{N}}$, we define the (*one-sided*) *subshift* $X = X_{\mathcal{F}}$ to be the subset of sequences in $\mathcal{A}^{\mathbb{N}}$ in which *no* word in $\mathcal{F}$ occurs.

Let X be a subshift of $\mathcal{A}^{\mathbb{N}}$. We denote by $\mathcal{W}_n(X)$ the set of all words with length n that occur in X and we set

$$\mathcal{W}(X) = \bigcup_{n=0}^{\infty} \mathcal{W}_n(X).$$

Let $\varphi : \mathcal{W}_{m+n+1}(X) \to \mathcal{A}$ be a map, which we call a *block map*. The extension of φ from X to $\mathcal{A}^{\mathbb{N}}$ is defined by $(x_i)_{i \in \mathbb{N}} \mapsto (y_i)_{i \in \mathbb{N}}$, where

$$y_i = \varphi((x_{i-m}, x_{i-m+1}, \dots, x_{i+n})).$$

We also denote this extension by φ and call it a *sliding block code*.

The *topological entropy* of a subshift X is defined by

$$h_{\text{top}}(X) = \lim_{n \to \infty} \frac{1}{n} \log \operatorname{card} \mathcal{W}_n(X).$$

A simple class of subshifts is that of *shifts of finite type* (SFT), those that can be described by a finite set of forbidden words. Let $M = [M(a, b)]_{a, b \in \mathcal{A}}$ be a 0–1 matrix. Then

$$\Sigma_M := \{(x_i)_{i=0}^{\infty} \in \mathscr{A}^{\mathbb{N}} \mid M(x_i, x_{i+1}) = 1\}$$

is called the *one-sided topological Markov shift by M* and it is a shift of finite type.

Gromov hyperbolic groups. For basic facts about Gromov hyperbolic spaces and groups, see [Gromov 1987] and [Coornaert and Papadopoulos 1993].

Let $(X, |\ |)$ be a metric space which is proper, geodesic and δ-hyperbolic for some $\delta \geq 0$. A function $f : X \to \mathbb{R}$ is *ε-convex*, where $\varepsilon \geq 0$, if for any geodesic segment $[x_0, x_1]$ in X and any $t \in [0, 1]$, we have

$$f(x_t) \leq (1-t) f(x_0) + t f(x_1) + \varepsilon,$$

where x_t is the point on $[x_0, x_1]$ satisfying $|x_0 - x_t| = t\,|x_0 - x_1|$.

Definition 2.3. Let $\varepsilon \geq 0$. An *ε-horofunction* on X is an ε-convex function $h : X \to \mathbb{R}$ satisfying $h(x) - \lambda = \operatorname{dist}(x, h^{-1}(\lambda))$ for every $x \in X$ and $\lambda \in \mathbb{R}$ such that $h(x) \geq \lambda$.

Definition 2.4. Let $r : [0, \infty) \to X$ be a geodesic ray. The associated *Busemann function h_r* : $X \to \mathbb{R}$ is defined by

$$h_r(x) = \lim_{t \to \infty} |x - r(t)| - t.$$

A Busemann function on a δ-hyperbolic X is a 4δ-horofunction [Coornaert and Papadopoulos 2001, Proposition 2.5]. Thus Busemann functions form an important class of horofunctions.

Definition 2.5. A function $\varphi : X \times X \to \mathbb{R}$ is called an *ε-cocycle* if there is an ε-horofunction $h : X \to \mathbb{R}$ such that

$$\varphi(x, y) = h(x) - h(y)$$

for every $x, y \in X$. We call such a function h a *primitive* for φ. (If h is a primitive for φ, so is $h + c$ for any constant c.)

Proposition 2.6 [Coornaert and Papadopoulos 2001, Proposition 2.7]. *Let φ be a cocycle on X. For x, y, z and $w \in X$, we have*

(1) $\varphi(x, x) = 0$,

(2) $\varphi(x, y) = -\varphi(y, x)$,

(3) $\varphi(x, y) = \varphi(x, z) + \varphi(z, y)$,

(4) $|\varphi(x, y)| \leq |x - y|$,

(5) $|\varphi(x, y) - \varphi(z, w)| \leq |x - z| + |y - w|$.

Let γ be an isometry of X, $h : X \to \mathbb{R}$ an ε-horofunction and $\varphi : X \times X \to \mathbb{R}$ an ε-cocycle. The functions γh and $\gamma\varphi$ defined by

$$\gamma h(x) = h(\gamma^{-1}x), \quad \gamma\varphi(x, y) = \varphi(\gamma^{-1}x, \gamma^{-1}y),$$

for $x, y \in X$, are an ε-horofunction and an ε-cocycle, respectively. If φ is the cocycle of h, then $\gamma\varphi$ is the cocycle of γh. Let Φ be the set of ε-cocycles on X for all possible values of $\varepsilon \geq 0$. We equip Φ with the topology of uniform convergence on compact sets.

Definition 2.7. Let φ be a cocycle on X. A *φ-gradient arc* is a path $g : I \to X$, parameterized by arclength, satisfying

$$\varphi(g(t), g(t')) = t' - t$$

for every $t, t' \in I$. If $I = \mathbb{R}$ or $I = [0, \infty)$, we say that g is a *φ-gradient line* or *ray*, respectively. If $g(0) = x$, we say that g starts at x.

Lemma 2.8 [Coornaert and Papadopoulos 2001, Lemma 2.9]. *Let φ be a cocycle on X and $I \subseteq \mathbb{R}$ an interval with $a \in I$, $I_1 = I \cap (-\infty, a]$ and $I_2 = I \cap [a, \infty)$. If $g : I \to X$ is a path whose restrictions to I_1 and I_2 are φ-gradient arcs, then g is itself a φ-gradient arc.*

Proposition 2.9 [Coornaert and Papadopoulos 2001, Proposition 2.10]. *Let φ be a cocycle on X.*

(1) *Any φ-gradient arc $g : I \to X$ is a geodesic.*

(2) *If $x, y \in X$ satisfying $\varphi(x, y) = |x - y|$, and if $g : [a, b] \to X$ is a geodesic joining x and y, then g is a φ-gradient arc.*

Proposition 2.10 [Coornaert and Papadopoulos 2001, Proposition 2.13]. *For every cocycle φ on X and for every $x \in X$, there is a φ-gradient ray $g : [0, \infty) \to X$ starting at x.*

Let φ be a cocycle on X and $g : [0, \infty) \to X$ a φ-gradient ray. By Proposition 2.9, part (1), g is a geodesic and so converges to a well-defined point $g(\infty) \in \partial X$.

Proposition 2.11 [Coornaert and Papadopoulos 2001, Proposition 3.1]. *Let φ be a cocycle on X and let $g, g' : [0, \infty) \to X$ be φ-gradient rays. Then $g(\infty) = g'(\infty)$.*

Definition 2.12. We define a map $\pi : \Phi \to \partial X$ by setting $\Phi(\varphi) = g(\infty) \in \partial X$, where $g : [0, \infty) \to X$ is a φ-gradient ray.

Let $\mathrm{Isom}(X)$ denote the group of isometries of X. The action of $\mathrm{Isom}(X)$ on Φ defined by $(\gamma, \varphi) \mapsto \gamma\varphi$ is continuous.

Proposition 2.13 [Coornaert and Papadopoulos 2001, Proposition 3.3]. *The map $\pi : \Phi \to \partial X$ is continuous, surjective, and commutes with the actions of* $\mathrm{Isom}(X)$ *on Φ and ∂X.*

For any cocycle φ, any geodesic ray $r : [0, \infty) \to X$ satisfying $r(\infty) = \pi(\varphi)$, and any $t \geq 0$, we set

$$R_{\varphi,t} = \{x \in X \mid \varphi(x, r(t)) = 0\} \cap B(r(t), 16\delta).$$

Proposition 2.14 [Coornaert and Papadopoulos 2001, Proposition 3.4]. *For $\varphi \in \Phi$, let $r : [0, \infty) \to X$ be a geodesic ray such that $r(\infty) = \pi(\varphi)$. For all $x \in X$ and $t \in \mathbb{R}$ satisfying $t > |x - r(0)| + 16\delta$, we have*

$$\varphi(x, r(t)) = \operatorname{dist}(x, R_{\varphi,t}).$$

In all that follows, Γ is a δ-hyperbolic group with respect to a finite set of generators A and X is the Cayley graph associated to the pair (Γ, A). We denote by $X^0 = \Gamma$ the set of vertices and by X^1 the set of edges of X. For $x \in \Gamma$, we denote by $|x|$ the word length of x with respect to A.

Definition 2.15. A horofunction $h : X \to \mathbb{R}$ is said to be *integral* if $h(x) \in \mathbb{Z}$ for every $x \in X^0$. A cocycle having an integral horofunction as a primitive is called an *integral* cocycle.

Every integral cocycle is completely determined by its values on $\Gamma \times \Gamma$, by [Coornaert and Papadopoulos 2001, Corollary 4.4]. Thus we can regard an integral cocycle on X as a function from $\Gamma \times \Gamma$ to $\mathbb{Z}$. Let $\Phi_0 \subseteq \Phi$ be the space of integral cocycles on X. The topology induced on Φ_0 by Φ is the topology of pointwise convergence on $\Gamma \times \Gamma$. For simplicity, we denote by $\pi : \Phi_0 \to \partial\Gamma$ the restriction of the map $\pi : \Phi \to \partial\Gamma$.

Proposition 2.16 [Coornaert and Papadopoulos 2001, Proposition 4.5]. *The map $\pi : \Phi_0 \to \partial\Gamma$ is continuous, Γ-equivalent, surjective and uniformly finite to one. In fact, for every $\xi \in \partial\Gamma$ we have*

$$\operatorname{card}\{\varphi \in \Phi_0 \mid \pi(\varphi) = \xi\} \leq (2N_0 + 1)^{N_1},$$

where N_0 is the integral part of $16\delta + 1$ and N_1 is the number of elements in Γ contained in the closed ball of radius N_0 centered at the identity.

Lemma 2.17 [Coornaert and Papadopoulos 2001, Lemma 5.1]. *For every $\varphi \in \Phi_0$ and $x \in X^0$, there is $a \in A$ such that $\varphi(x, xa) = 1$.*

Now we fix a total order relation on the finite generating set A. Let $\varphi \in \Phi_0$ and $x \in X^0$. The lexicographic order on $A^{\mathbb{N}}$ induces a total order on the set of φ-gradient rays starting at x.

Proposition 2.18 [Coornaert and Papadopoulos 2001, Proposition 5.2]. *Let $\varphi \in \Phi_0$ and $x \in X^0$. The set of φ-gradient rays starting at x has a smallest element.*

Definition 2.19. We define a map $\alpha : \Phi_0 \to \Phi_0$ by $\alpha(\varphi) = a^{-1}\varphi$, where $\varphi \in \Phi_0$ and a is the smallest element in A satisfying $\varphi(e, a) = 1$.

Proposition 2.20 [Coornaert and Papadopoulos 2001, Proposition 5.6]. *The map* $\alpha : \Phi_0 \to \Phi_0$ *is continuous.*

Proposition 2.21 [Coornaert and Papadopoulos 2001, Proposition 5.7]. *Let* $\varphi \in \Phi_0$ *and* $g : [0, \infty) \to X$ *be the smallest* φ*-gradient ray starting at* e*. For* $n \in \mathbb{N}$*, let* $a_n \in A$ *be the label of the oriented edge from* $g(n)$ *to* $g(n+1)$ *and* $g_n : [0, \infty) \to X$ *the smallest* $\alpha^n(\varphi)$*-gradient ray starting at* e*.*

(1) $\alpha^n(\varphi) = g(n)^{-1}\varphi$.

(2) $g_n(t) = g(n)^{-1}g(t+n)$ *for any* $t \in [0, \infty)$.

(3) *For every* $k \in \mathbb{N}$*, the label of oriented edge from* $g_n(k)$ *to* $g_n(k+1)$ *is* a_{k+n}.

Next we introduce the shift of finite type $(\Sigma(\infty), T)$ and the conjugacy P from (Φ_0, α) to $(\Sigma(\infty), T)$. We take integers $R \geq 100\delta + 1$ and $L \geq 2R + 32\delta + 1$. For a subset $Y \subseteq X$ and $\varepsilon \geq 0$, we set

$$N(Y, \varepsilon) = \{x \in X \mid \operatorname{dist}(x, Y) \leq \varepsilon\}.$$

For $\varphi \in \Phi_0$, let $g : [0, \infty) \to X$ be the smallest φ-gradient ray starting at e. Set

$$V(\varphi) = N(g([0, L]), R).$$

$V(\varphi)$ is contained in the closed ball $B(e, L+R)$ of radius $L+R$ centered at e.

For each $\varphi \in \Phi_0$, we define a function $\rho(\varphi) : V(\varphi) \to \mathbb{R}$ by

$$\rho(\varphi)(x) = \varphi(x, e)$$

for $x \in V(\varphi)$. Note that $\rho(\varphi)$ is the restriction to $V(\varphi)$ of the primitive h of φ with $h(e) = 0$. We set

$$S = \{\rho(\varphi) : V(\varphi) \to \mathbb{R} \mid \varphi \in \Phi_0\}.$$

Lemma 2.22 [Coornaert and Papadopoulos 2001, Lemma 6.2]. *The set* S *is finite.*

Definition 2.23. Let Σ be the set of sequences $(\sigma_n)_{n\geq 0}$ with $\sigma_n \in S$ for $n \geq 0$, and give it the product topology (of discrete topologies on copies of S). The map $T : \Sigma \to \Sigma$ is the shift map. Define a map $P : \Phi_0 \to \Sigma$ by

$$\Phi_0 \ni \varphi \mapsto (\sigma_n)_{n\geq 0} \in \Sigma,$$

where $\sigma_n = \rho(\alpha^n(\varphi))$ for $n \geq 0$.

Let $s \in S$. We denote by $V(s)$ the domain of the function s. Since $R \geq 1$, the domain $V(s)$ contains the closed unit ball $B(e, 1)$. Hence the value $s(a)$ is well-defined for all $a \in A$. Since the finite generating set A is equipped with a fixed total order relation, we can define $w(s)$ to be the smallest element $a \in A$ satisfying $s(a) = -1$. (Such an a exists because of Lemma 2.17.)

Let $\sigma = (\sigma_n)_{n\geq 0} \in \Sigma$. We define a sequence $(\gamma_n(\sigma))_{n\geq 0}$ by setting

$$\gamma_0(\sigma) = e, \qquad \gamma_n(\sigma) = w(\sigma_0)\cdots w(\sigma_{n-1}) \quad \text{for } n \geq 1.$$

For $n \geq 0$, we set

$$V_n(\sigma) = \gamma_n(\sigma)V(\sigma_n).$$

This depends only on the first $n+1$ coordinates of σ. We also define functions $f_n(\sigma) : V_n(\sigma) \to \mathbb{R}$ by $f_n(\sigma)(x) = \sigma_n(\gamma_n(\sigma)^{-1}x) - n$ for $x \in V_n(\sigma)$.

Lemma 2.24 [Coornaert and Papadopoulos 2001, Lemma 6.5]. *For $\varphi \in \Phi_0$, take $\sigma = P(\varphi)$ and let $g : [0,\infty) \to X$ be the smallest φ-gradient ray starting at e. Assume $n \geq 0$.*

(1) $\gamma_n(\sigma) = g(n)$.

(2) $V_n(\sigma) = N(g([n, n+L]), R)$.

(3) *$f_n(\sigma)$ is the restriction to $V_n(\sigma)$ of the primitive h of φ with $h(e) = 0$.*

Definition 2.25. Let $\sigma \in \Sigma$. We say that σ is *consistent* if for all $i, j \geq 0$, we have

$$f_i(\sigma)(x) = f_j(\sigma)(x)$$

for all $x \in V_i(\sigma) \cap V_j(\sigma)$. We denote by $\Sigma(\infty)$ the set of all consistent sequences.

Lemma 2.26 [Coornaert and Papadopoulos 2001, Lemma 6.8]. $P(\Phi_0) \subseteq \Sigma(\infty)$.

Theorem 2.27 [Coornaert and Papadopoulos 2001, Theorem 7.18]. *The set of consistent sequences $\Sigma(\infty)$ is a shift of finite type. Moreover (Φ_0, α) and $(\Sigma(\infty), T)$ are conjugate via the map P.*

3. The topological entropy of $\Sigma(\infty)$

Let Γ be a Gromov hyperbolic group with a finite generating set A on which we fix a total order relation. Let $\Sigma(\infty)$ the corresponding SFT.

For $n \in \mathbb{N}$, we denote by W_n the set of all words with length n that occur in $\Sigma(\infty)$ and by A_n the set of all elements in Γ with word length n with respect to the finite generating set A (as a particular case, $A_0 = \{e\}$). We set $D_n = \bigcup_{1\leq k\leq n} W_k$ and $B_n = \bigcup_{0\leq k\leq n} A_k$. For each $s \in S$, we set

$$W_n(s) = \{(\sigma_0, \dots, \sigma_{n-1}) \in W_n \mid \sigma_0 = s\},$$

and for each $a \in A$,

$$A_n(a) = \{a\gamma \in A_n \mid \gamma \in A_{n-1}\}.$$

We write $D_n(s) = \bigcup_{1\leq k\leq n} W_k(s)$ and $B_n(a) = \bigcup_{1\leq k\leq n} A_n(a)$.

We denote by $\mathrm{gr}(\Gamma, A)$ the *growth entropy* of Γ with respect to A:

$$\mathrm{gr}(\Gamma, A) = \lim_{n\to\infty} \frac{1}{n} \log \operatorname{card} A_n.$$

We also define

$$\overline{A}_n = \{\gamma \in A_n \mid \gamma = w(\sigma_0)\cdots w(\sigma_{n-1}) \text{ for some } (\sigma_0, \ldots, \sigma_{n-1}) \in W_n\},$$
$$\overline{B}_n = \bigcup_{1\le k\le n} \overline{A}_n,$$
$$\overline{A}_n(w(s)) = \{\gamma \in A_n(w(s)) \mid \gamma = w(\sigma_0)\cdots w(\sigma_{n-1}) \text{ for some } (\sigma_0, \ldots, \sigma_{n-1}) \in W_n(s)\},$$
$$\overline{B}_n(w(s)) = \bigcup_{1\le k\le n} \overline{A}_n(w(s)).$$

Lemma 3.1. *There is a constant $K > 0$ such that*

$$\operatorname{card}\{(\sigma_0, \ldots, \sigma_{n-1}) \in W_n \mid w(\sigma_0)\ldots w(\sigma_{n-1}) = \gamma\} \le K,$$

for every $n \ge 1$ and every $\gamma \in A_n$.

Proof. Let $\varphi, \varphi' \in \Phi_0$ and g, g' their smallest gradient rays starting at e such that $g(n) = g'(n) = \gamma \in A_n$. Note that $\varphi(\gamma, e) = \varphi'(\gamma, e) = -n$. We denote $\sigma = P(\varphi)$ and $\sigma' = P(\varphi')$. By Lemma 2.24, we have $\gamma_n(\sigma) = \gamma_n(\sigma') = \gamma$.

We first claim that $g = g'$ on $[0, n]$. We now assume that $g \ne g'$ on $[0, n]$. We may assume that $g' < g$ in the lexicographic order on $A^{\mathbb{N}}$ without loss of generality.

Note that $\varphi(e, \gamma) = \varphi(g(0), g(n)) = n = |e - \gamma|$, and $g' : [0, n] \to X$ is a geodesic joining e and γ. From Proposition 2.9(2) it follows that $g' : [0, n] \to X$ is a φ-gradient arc. Then we define the path $\bar{g} : [0, \infty) \to X$ by

$$\bar{g}(k) = \begin{cases} g'(k) & \text{for } 0 \le k \le n, \\ g(k) & \text{for } n \le k. \end{cases}$$

By Lemma 2.8, the path $\bar{g}$ is φ-gradient ray starting at e such that $\bar{g} < g$ in the lexicographic order on $A^{\mathbb{N}}$. Therefore g would be not the smallest φ-gradient ray. Hence we have $g = g'$ on $[0, n]$.

Let h, h' be primitives for φ, φ' satisfying $h(e) = h'(e) = 0$, respectively. We set $B = B(\gamma, L + R)$.

We secondly claim that if $h = h'$ on B, then $h = h'$ on $N(g([0, n + L]), R)$. Notice that $R > 16\delta$ and $L > 2R$. Let $k \in [0, n]$ satisfying $n - k \le 2R$. Since $N(g([k, n + L]), R) \subseteq B$, we have $h = h'$ on $N(g([k, n + L]), R)$. Next let $k \ge 0$ satisfying $n - k > 2R$. For $x \in B(g(k), R)$, we have

$$\begin{aligned} n &= |g(0) - g(n)| = |g(0) - g(k)| + |g(k) - g(n)| \\ &\ge |g(0) - x| - |x - g(k)| + |g(k) - g(n)| \ge |g(0) - x| - R + (n - k) \\ &> |g(0) - x| + R > |g(0) - x| + 16\delta. \end{aligned}$$

By Proposition 2.14, we have $\varphi(x, g(n)) = \operatorname{dist}(x, R_{\varphi,n})$. Recall that

$$R_{\varphi,n} = \{x \in X \mid \varphi(x, g(n)) = 0\} \cap B(g(n), 16\delta).$$

Hence $h(x)+n = \mathrm{dist}(x, R_{\varphi,n})$. This shows that the value of $h(x)$ depends only on the restriction of h on $B(g(n), 16\delta) \subseteq B$. Namely we obtain our claim.

We now assume that $h = h'$ on B. In this case, we remark that $g = g'$ on $[0, n+L]$. By Proposition 2.21, we have $V(\alpha^k(\varphi)) = N(g(k)^{-1}g([k, k+L]), R) = V(\alpha^k(\varphi'))$ for $0 \le k \le n$. For each $x \in V(\alpha^k(\varphi))$, since

$$N(g(k)^{-1}g([k, k+L]), R) = g(k)^{-1}N(g([k, k+L]), R),$$

there is $y \in N(g([k, k+L]), R)$ such that $x = g(k)^{-1}y$. Then

$$\rho(\alpha^k(\varphi))(x) = g(k)^{-1}\varphi(x, e) = \varphi(y, g(k)) = h(y) - h(g(k)) = h(y) + k.$$

Similarly we also obtain $\rho(\alpha^k(\varphi))(x) = h'(y) + k$. Hence if $h = h'$ on B, then it follows from the second claim that

$$\rho(\alpha^k(\varphi))(x) = h(y) + k = h'(y) + k = \rho(\alpha^k(\varphi')).$$

Therefore $\rho(\alpha^k(\varphi)) = \rho(\alpha^k(\varphi'))$; that is, $\sigma_k = \sigma'_k$ for all $0 \le k \le n$.

Hence it suffices to set $K = (2(L+R)+1)^b$, where $b = \mathrm{card}\, B = \mathrm{card}\, B(e, L+R)$. Indeed, for every $x \in B$ we have, using Proposition 2.6,

$$|h(x)+n| = |h(x) - h(\gamma)| = |\varphi(x, \gamma)| \le |x - \gamma| \le L + R.$$

This easily leads to the assertion. □

Corollary 3.2. $h_{\mathrm{top}}(\Sigma(\infty)) \le \mathrm{gr}(\Gamma, A)$.

Proof. For each $n \ge 0$, the map $W_n \ni (\sigma_0, \dots, \sigma_{n-1}) \mapsto w(\sigma_0) \cdots w(\sigma_{n-1}) \in A_n$ is uniformly finite-to-one by Lemma 3.1. Thus

$$\mathrm{card}\, W_n \le K \,\mathrm{card}\, A_n.$$

The assertion follows immediately. □

Remark 3.3. A fundamental theorem of J. Stallings [1971] shows that a finitely generated group Γ has infinitely many ends if and only if it has a form of either (2) or (3) of Corollary 1.2. In particular, a torsion-free group has the form (1).

4. Proof of main results

Proof of Corollary 1.2. In view of Corollary 3.2, we just need to show that $h_{\mathrm{top}}(\Sigma(\infty)) \ge \mathrm{gr}(\Gamma, A)$ if one of the conditions (1)–(3) of Corollary 1.2 is satisfied. Remark 3.3 shows that it suffices to check cases (2) and (3); but we check case (1) explicitly as well because it is very simple.

Case (1): It suffices to show that the map $(\sigma_0, \dots, \sigma_{n-1}) \mapsto w(\sigma_0) \cdots w(\sigma_{n-1})$ from W_n to A_n is surjective. Let $\gamma \in A_n$. There is the smallest geodesic segment $r : [0, n] \to X$ from e to γ. We can take g to be a geodesic ray extending r, meaning that $r(k) = g(k)$ for all $0 \le k \le n$. Indeed, by assumption, we have $\Gamma = G_1 * G_2$. Then γ is written as a reduced word $g_1 \cdots g_m$, where $g_k \in G_{i_k}$ with $i_k \neq i_{k+1}$ for $1 \le k \le m-1$. Hence for $l \ge 1$, it is enough to set

$$g(n+2l) = \gamma \cdot \underbrace{ab \cdots ab}_{2l}, \quad g(n+2l-1) = \gamma \cdot \underbrace{ab \cdots ba}_{2l-1},$$

for some $a \in F_i$ and $b \in F_{i_m}$ with $i \neq i_m$ and $a, b \neq e$.

We consider the cocycle φ_g having the Busemann function h_g as a primitive. It is clear that g is a φ_g-gradient ray. Moreover by definition, g is, in fact, the smallest φ_g-gradient ray starting at e. Hence $\gamma = w(P(\varphi_g)) \cdots w(P(\alpha^{n-1}(\varphi_g)))$. It follows that the map above is surjective. Thus card $A_n \le$ card W_n, and $\mathrm{gr}(\Gamma, A) \le h_{\mathrm{top}}(\Sigma(\infty))$ as needed.

Case (2): Now we assume that $\Gamma = G_1 *_H G_2$. Let $\gamma \in A_n$ with $n \ge 2$. We express the element γ by the reduced word $g_1 \cdots g_m$, where $g_k \in G_{i_k} \setminus H$ with $i_k \neq i_{k+1}$ for $1 \le k \le m-1$. We take a sequence $(g_{m+1}, g_{m+2}, \dots)$ such that $g_k \in F_{i_k} \setminus H$ with $i_{k-1} \neq i_k$ for all $k \ge m+1$. We define a sequence $(g(k))_{k=1}^{\infty}$ in X by $g(k) = g_1 \dots g_k$ for $k \ge 1$. Let $\langle y, z\rangle = \frac{1}{2}(|y| + |z| - |y - z|)$ be the Gromov product based at e. For $l \ge k \ge m$,

$$\begin{aligned} 2\langle g(k), g(l)\rangle &= |g(k)| + |g(l)| - |g(k)^{-1}g(l)| \\ &\ge k + l - |g(k)^{-1}g(l)| = k + l - |g_{k+1} \cdots g_l| \\ &\ge k + l - (l - k) = 2k \end{aligned}$$

tends to ∞ with k; thus there exists $\xi \in \partial X$ such that the sequence $(g(k))_{k=1}^{\infty}$ converges to ξ. Let $r : [0, \infty) \to X$ be a geodesic ray starting at e with $r(\infty) = \xi$. We denote by φ_r the cocycle with respect to the Busemann function h_r. Let $g' : [0, \infty) \to X$ be the smallest φ_r-gradient ray starting at e. Because r is also a φ_r-gradient ray, it follows from Proposition 2.11 that $g'(\infty) = \xi$. We can express g' by the infinite reduced word $(g'_1, g'_2, \dots)$ with $g'_k \in G_{j_k} \setminus H$ and $j_k \neq j_{k+1}$ for $k \ge 1$. Since $g'(\infty) = \xi$, we have $i_k = j_k$ for all $k \ge 1$. Moreover we obtain $\gamma = g(m) = g_1 \cdots g_m = g'_1 \cdots g'_m h$ for some $h \in H$. Let $k_m \ge 1$ such that $g'(k_m) = g'_1 \cdots g'_m$. Then we have $|g(m) - g'(k_m)| \le 1$. Note that $n - 1 \le k_m \le n + 1$. Hence we have proved that for any $\gamma \in A_n$, there is $\gamma' \in A_n$ such that $\gamma' \in B(\gamma, 2)$ and $\gamma' = w(\sigma_0) \cdots w(\sigma_{n-1})$ for some $(\sigma_0, \dots, \sigma_{n-1}) \in W_n$. Therefore card $A_n \le$ card $B(e, 2) \cdot$ card W_n, and the assertion follows.

Case (3): We assume that $\Gamma = G *_H \theta$. Let $\gamma \in A_n$. The element γ can be represented by either (i) $g_0 \in G$ or (ii) a reduced word $g_0 x^{\varepsilon_0} \cdots g_{m-1} x^{\varepsilon_{m-1}} g_m$, where $g_k \in G$ and $\varepsilon_k \in \{1, -1\}$ for all $0 \le k \le m$. In case (i), we set $g_k = e$ for $k \ge 1$ and $\varepsilon_k = 1$ for $k \ge 0$. In case (ii), we set $g_k = e$ for $k \ge m+1$ and $\varepsilon_k = \varepsilon_{m-1}$

for $k \geq m$. Then we define the sequence $(g(k))_{k=0}^{\infty}$ in X by $g(k) = g_0 x^{\varepsilon_0} \cdots g_k x^{\varepsilon_k}$ for all $k \geq 0$. Again, for $l \geq k \geq m$,

$$2\langle g(k), g(l)\rangle = |g(k)| + |g(l)| - |g(k)^{-1} g(l)| \geq k + l - |x^{\varepsilon_{k+1}} \cdots x^{\varepsilon_l}| = 2k$$

goes to ∞ with k; hence $(g(k))_{k=0}^{\infty}$ converges to some $\xi \in \partial\Gamma$. Let $r : [0, \infty) \to X$ be a geodesic ray with $r(0) = e$ and $r(\infty) = \xi$. We denote by φ_r the cocycle with respect to the Busemann function h_r. Let $g' : [0, \infty) \to X$ be the smallest φ_r-gradient ray starting at e. We can also represent the geodesic ray g' as the infinite reduced word $(g'_0 x^{\delta_0}, g'_1 x^{\delta_1}, \dots)$. Since $g'(\infty) = \xi$, we have $\varepsilon_i = \delta_i$ for all $i \geq 0$. Moreover we obtain $\gamma = g_0 x^{\varepsilon_0} \cdots g_m = g'_0 x^{\varepsilon_0} \cdots g'_m g$, for some either $g \in H$ if $\varepsilon_m = 1$, or $g \in \theta(H)$ if $\varepsilon_m = -1$. Let $k_m \geq 1$ such that $g'(k_m) = g'_0 x^{\varepsilon_0} \cdots g'_m$. Then we have $|\gamma - g(k_m)| \leq 1$. Note that $n-1 \leq k_m \leq n+1$. Hence we have shown that for each $\gamma \in A_n$, there is $\gamma' \in A_n$ such that $\gamma' \in B(\gamma, 2)$ and $\gamma' = w(\sigma_0) \cdots w(\sigma_{n-1})$ for some $(\sigma_0, \dots, \sigma_{n-1}) \in W_n$. Therefore $\operatorname{card} A_n \leq \operatorname{card} B(e, 2) \cdot \operatorname{card} W_n$, and $h_{\mathrm{top}}(\Sigma(\infty)) \leq \operatorname{gr}(\Gamma, A)$ as needed. □

Remark 4.1. It is easy to check that the topological entropy $h_{\mathrm{top}}(\Sigma(\infty))$ does not depend on the choice of total order relations on A.

Proof of Theorem 1.1. It suffices to show that $h_{\mathrm{top}}(\Sigma(\infty)) \leq k_{\infty}^{-}(\lambda_A)$, because the inequality $k_{\infty}^{-}(\lambda_A) \leq \operatorname{gr}(\Gamma, A)$ has been proved in [Okayasu 2004, Proposition 4.1]. Let $\lambda_{w(S)} = \{\lambda_{w(s)} \mid s \in S\}$. Note that $k_{\infty}^{-}(\lambda_{w(S)}) \leq k_{\infty}^{-}(\lambda_A)$.

Since $\Sigma(\infty)$ is an SFT, there are $N \in \mathbb{N}$ and $W \subseteq S^{N+1}$ such that

$$\Sigma(\infty) = \{(\sigma_n)_{n \geq 0} \in \Sigma \mid (\sigma_n, \dots, \sigma_{n+N}) \in W \text{ for any } n \geq 0\}.$$

Let $I = S^N$ and $\beta_N : \Sigma(\infty) \to I^{\mathbb{N}}$ be the N-th higher block code. Then the subshift $\beta_N(\Sigma(\infty))$ is the Markov shift Σ_M for some matrix $M = [M(i, j)]_{i,j \in I}$. Let μ be the maximal measure on $\Sigma(\infty)$, i.e., $h_{\mathrm{top}}(\Sigma(\infty)) = h_{\mu}(T|_{\Sigma(\infty)})$. For simplicity, we denote by h the topological entropy of $\Sigma(\infty)$. We denote by $[\sigma_0, \dots, \sigma_{n-1}]$ the cylinder set at 0-th coordinate. For $(\sigma_0, \dots, \sigma_{n-1}) \in W_n$ with $n \geq N$, we have

$$\mu([\sigma_0, \dots, \sigma_{n-1}]) = \frac{l_i r_j}{e^{(n-N)h}},$$

where $i = (\sigma_0, \dots, \sigma_{N-1})$, $j = (\sigma_{n-N}, \dots, \sigma_{n-1}) \in I$ and l, r are the left and right Perron vectors of M with $\sum_{i \in I} l_i r_i = 1$ (see [Kitchens 1998]).

For each $n \geq 0$, denote by P_n the projection onto the subspace

$$\overline{\operatorname{span}}\{\delta_\gamma \in \ell^2(\Gamma) \mid |\gamma| = n\}.$$

For $a \in A$, define the partial isometry $T_a \in \mathbb{B}(\ell^2(\Gamma))$ [Okayasu 2002; 2004] by

$$T_a = \sum_{n \geq 0} P_{n+1} \lambda_a P_n.$$

For each $s \in S$, we define X_s by

$$\sum_{n\geq 1}\sum_{\substack{(\sigma_0,\sigma_1,\dots,\sigma_{n-1})\\ \in W_n(s)}} \mu([\sigma_0,\sigma_1,\dots\sigma_{n-1}])T_{w(\sigma_1)}\cdots T_{w(\sigma_{n-1})}P_0T^*_{w(\sigma_{n-1})}\cdots T^*_{w(\sigma_1)}T^*_{w(\sigma_0)}.$$

Then $\sum_{s\in S}[X_s,\lambda_{w(s)}]=P_0$, because

$$\begin{aligned}\sum_{s\in S}\lambda_{w(s)}X_s &= \sum_{n\geq 1}\sum_{s\in S}\sum_{\substack{(\sigma_0,\dots,\sigma_{n-1})\\ \in W_n(s)}} \mu([\sigma_0,\dots\sigma_{n-1}])T_{w(\sigma_0)}\cdots T_{w(\sigma_{n-1})}P_0T^*_{w(\sigma_{n-1})}\cdots T^*_{w(\sigma_0)}\\ &= \sum_{n\geq 1}\sum_{\substack{(\sigma_0,\dots,\sigma_{n-1})\\ \in W_n}} \mu([\sigma_0,\dots\sigma_{n-1}])T_{w(\sigma_0)}\cdots T_{w(\sigma_{n-1})}P_0T^*_{w(\sigma_{n-1})}\cdots T^*_{w(\sigma_0)}\end{aligned}$$

and

$$\begin{aligned}\sum_{s\in S}X_s\lambda_{w(s)} &= \sum_{n\geq 1}\sum_{s\in S}\sum_{\substack{(\sigma_0,\dots,\sigma_{n-1})\\ \in W_n(s)}} \mu([\sigma_0,\sigma_1,\dots\sigma_{n-1}])T_{w(\sigma_1)}\cdots T_{w(\sigma_{n-1})}P_0T^*_{w(\sigma_{n-1})}\cdots T^*_{w(\sigma_1)}\\ &= P_0+\sum_{n\geq 1}\sum_{\substack{(\sigma_0,\dots,\sigma_{n-1})\\ \in W_n}} \mu([\sigma_0,\sigma_1,\dots\sigma_{n-1}])T_{w(\sigma_0)}\cdots T_{w(\sigma_{n-1})}P_0T^*_{w(\sigma_{n-1})}\cdots T^*_{w(\sigma_0)}.\end{aligned}$$

Next we give an estimate of $\|X_s\|_1^{\widetilde{+}}$. For $n\in\mathbb{N}$ and $\gamma\in\overline{A}_n(w(s))$, we define

$$s_\gamma=\sum_{\substack{(\sigma_0,\dots,\sigma_{n-1})\in W_n(s)\\ \gamma=w(\sigma_0)\cdots w(\sigma_{n-1})}}\mu([\sigma_0,\dots,\sigma_{n-1}]).$$

This sum is uniformly finite by Lemma 3.1. Thus

$$C_1e^{-nh}\leq s_\gamma\leq C_2e^{-nh}$$

for constants $C_1, C_2>0$, independent of n and γ.

Let $s_1\geq s_2\geq\cdots$ be the eigenvalues of $(X_s^*X_s)^{1/2}$. For each $j\in\mathbb{N}$, there is $\gamma_j\in\overline{A}_{n_j}(w(s))$ such that $s_j=s_{\gamma_j}$.

Let $\varepsilon>0$. Recall that

$$\|X_s\|_1^{\widetilde{+}}=\inf_{Y\in\mathbb{F}(\ell^2(\Gamma))_1^+}\|X_s-Y\|_1^+.$$

By doing finite rank perturbations if necessary, we may assume that for all $j\geq 1$,

$$e^{-n_j(h+\varepsilon)}\leq s_j\leq e^{-n_j(h-\varepsilon)}.$$

Let $N \in \mathbb{N}$ with $e^{-N\varepsilon} \leq C_1$ and $n \geq N$. If there is $m > n$ such that $j \leq \operatorname{card} \overline{B}_n(w(s))$ and $\gamma_j \in \overline{A}_m(w(s))$, we have

$$e^{-m(h-\varepsilon)} \geq e^{-n(h+\varepsilon)}.$$

For otherwise we would have

$$s_j \leq e^{m(h-\varepsilon)} < e^{-n(h+\varepsilon)} \leq e^{-n\varepsilon}\frac{s_\gamma}{C_1} \leq s_\gamma$$

for all $\gamma \in \overline{B}_n(w(s))$ and this is a contradiction. Therefore $e^{m(h-\varepsilon)} \geq e^{-n(h+\varepsilon)}$, namely

$$m \leq n\frac{h+\varepsilon}{h-\varepsilon}.$$

We put

$$k = \max\left\{m \in \mathbb{N} \;\middle|\; m \leq n\frac{h+\varepsilon}{h-\varepsilon}\right\}.$$

Since

$$\mu([s]) = \sum_{(\sigma_0,\ldots,\sigma_{n-1})\in W_n(s)} \mu([\sigma_0,\ldots,\sigma_{n-1}]) \leq \operatorname{card} W_n(s)\cdot Ce^{-nh},$$

for some $C > 0$, we obtain

$$\frac{\mu([s])e^{nh}}{C} \leq \operatorname{card} W_n(s).$$

Hence

$$\begin{aligned}
\|X_s\|_1^{\widetilde{+}} &\leq \limsup_{n\to\infty} \frac{\sum_{j=1}^{\operatorname{card}\overline{B}_n(w(s))} s_j}{\sum_{j=1}^{\operatorname{card}\overline{B}_n(w(s))} j^{-1}} \\
&\leq \limsup_{n\to\infty} \frac{\sum_{l=1}^{k}\sum_{\gamma\in\overline{A}_l(w(s))}\sum_{w(\sigma_0)\cdots w(\sigma_{l-1})=\gamma} \mu([\sigma_0,\ldots,\sigma_{l-1}])}{\log\operatorname{card}\overline{B}_n(w(s))} \\
&= \limsup_{n\to\infty} \frac{\sum_{l=1}^{k}\mu([s])}{\log\operatorname{card}\overline{B}_n(w(s))} \\
&\leq \limsup_{n\to\infty} \frac{n}{\log\operatorname{card}\overline{A}_n(w(s))}\frac{h+\varepsilon}{h-\varepsilon}\mu([s]) \\
&\leq \limsup_{n\to\infty} \frac{n}{\log\operatorname{card} W_n(s) - \log K}\frac{h+\varepsilon}{h-\varepsilon}\mu([s]) \\
&\leq \limsup_{n\to\infty} \frac{n}{\log\mu([s]) + nh - \log C - \log K}\frac{h+\varepsilon}{h-\varepsilon}\mu([s]) \\
&= \frac{h+\varepsilon}{h(h-\varepsilon)}\mu([s]).
\end{aligned}$$

Here we have used that card $W_n(s) \leq K \operatorname{card} \bar{A}_n(w(s))$ (Lemma 3.1). Since $\varepsilon > 0$ is arbitrary, we have

$$\|X_s\|_1^{\widetilde{+}} \leq \frac{1}{h}\,\mu([s]).$$

Thanks to Proposition 2.1, we obtain

$$h = h_{\mathrm{top}}(\Sigma(\infty)) \leq k_\infty^-(\lambda_{w(S)}) \leq k_\infty^-(\lambda_A) \leq \mathrm{gr}(\Gamma, A). \qquad \square$$

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